Research Article

Fixed Point Results of a General Class of Monotone Nonexpansive Mappings in Hyperbolic Metric Spaces

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In this paper, we introduce and study some properties of a new class of generalized monotone nonexpansive mappings in hyperbolic metric spaces. Further, we give some results on the convergence of the Mann iterative process for this class of mappings in hyperbolic ordered metric spaces with some interesting examples.

1. Introduction

Nonexpansive mappings appear in many nonlinear problems modeled by integral equations arising from different scientific fields as mechanics, electricity, and population dynamics. Recall that a nonexpansive mapping is 1-Lipschitz mapping.

The study of the existence of fixed points for nonexpansive mappings was initiated by Browder [1, 2], Gôhde [3], and Kirk [4]. In the last few years, a number of extensions and generalizations of nonexpansive mappings in different spaces were obtained by many mathematicians. In 2008, Suzuki [5] introduced a generalized definition of nonexpansive mappings, named condition (C) or Suzuki generalize nonexpansive mapping, as follows:

Definition 1. Let K be a subset of a Banach space X . A mapping $T : K \rightarrow K$ is said to satisfy condition (C) if for all $x, y \in K$,

$$\frac{1}{2} \|x - Tx\| \leq \|x - y\| \text{ implies } \|Tx - Ty\| \leq \|x - y\|. \quad (1)$$

He proved also some existence and convergence results for such type of mappings. Patir et al. [6] defined the following general class of nonexpansive-type mappings, and

they presented some fixed point results for this class of operators.

Definition 2. Let K be a nonempty subset of a Banach space X . Let $\gamma \in [0, 1]$ and $\mu \in [0, 1/2]$ such that $2\mu \leq \gamma$. A mapping $T : K \rightarrow K$ is said to satisfy the condition $B_{\gamma, \mu}$ on K if for all $x, y \in K$,

$$\begin{aligned} \gamma \|x - Tx\| &\leq \|x - y\| + \mu \|y - Ty\| \text{ implies} \\ \|Tx - Ty\| &\leq (1 - \gamma) \|x - y\| + \mu (\|x - Ty\| + \|Tx - y\|). \end{aligned} \quad (2)$$

Otherwise, the structure of spaces plays an important role in the study of the existence of fixed points. It is well known in the literature that Banach spaces have been studied extensively, because Banach spaces always have convex structures. However, metric spaces do not naturally benefit from this structure. Therefore, there is a need to introduce and define convex structures.

The study of fixed point theory for nonexpansive mappings in the framework of hyperbolic metric spaces was initiated by Takahashi [7]. He formulated some fixed point results in convex metric spaces. Goebel and Kirk [8] used hyperbolic-type spaces, which contain hyperbolic metric spaces (see also [9]).

Reich and Shafrir [10] introduced hyperbolic metric spaces as an appropriate background to study the existence and the approximation of fixed points for nonexpansive mappings. Accommodating previous definitions of hyperbolic metric spaces, Kohlenbach [11] introduced a more general definition of hyperbolic metric spaces and he proved the existence of fixed points for nonexpansive mappings. This class of spaces includes all normed linear spaces and Busemann spaces [12]. Leustean [13] showed that CAT(0) spaces are uniformly convex hyperbolic metric spaces. In 2016, Bin Dehaish and Khamsi [14] gave an analogue theorem to Browder and Göhde's fixed point theorem for monotone nonexpansive mappings in uniformly convex hyperbolic metric spaces. Recently, Shukla et al. [15] present some existence and convergence results for a monotone mapping satisfying condition (C) in partially ordered hyperbolic metric spaces.

Motivated by all these facts, we introduce in this paper a general class of nonexpansive-type mappings. We prove some fixed point existence results in uniformly convex hyperbolic metric spaces. Further, we study the approximation of fixed points for such type of mapping using the Mann iterative process defined in hyperbolic metric spaces with some illustrative examples. The results obtained in this work extend and generalize corresponding results on uniformly convex Banach spaces and CAT(0) spaces and many other results in this direction.

2. Preliminaries

Let (X, d) be a metric space. A path joining $x \in X$ to $y \in X$ is a map c from a closed interval $[0, l] \subset \mathbb{R}$ to X such that $c(0) = x$, $c(l) = y$, and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$. In particular, c is an isometry and $d(x, y) = l$. The image of c is called a metric segment joining x and y . When it is unique, the metric segment is denoted by $[x, y]$. We shall denote by $(1 - \beta)x \oplus \beta y$ the unique point z of $[x, y]$ which satisfies

$$\begin{aligned} d(x, z) &= \lambda d(x, y), \\ d(z, y) &= (1 - \lambda)d(x, y). \end{aligned} \quad (3)$$

Such metric spaces are usually called convex metric spaces (see Takahashi [7]).

Let us recall the following definition which is due to Kohlenbach [11].

Definition 3 (see [11]). A triplet (X, d, H) is said to be a hyperbolic metric space if (X, d) is a metric space and $H : X \times X \times [0, 1] \rightarrow X$ is a function such that for all $x, y, w, z \in X$ and $\beta, \gamma \in [0, 1]$, the following hold:

- (i) $d(z, H(x, y, \beta)) \leq (1 - \beta)d(z, x) + \beta d(z, y)$
- (ii) $d(H(x, y, \beta), H(x, y, \gamma)) = |\beta - \gamma| d(x, y)$
- (iii) $H(x, y, \beta) = H(y, x, 1 - \beta)$
- (iv) $d(H(x, y, \beta), H(w, z, \beta)) \leq (1 - \beta)d(x, w) + \beta d(y, z)$

The set $[x, y] = \{H(x, y, \beta), \beta \in [0, 1]\}$ is called the metric segment with endpoints x and y . Now onwards, we write $H(x, y, \beta) = (1 - \beta)x \oplus \beta y$. A subset Y of X is said to be convex if $(1 - \beta)x \oplus \beta y \in Y$ for all $x, y \in Y$ and $\beta \in [0, 1]$.

When there is no ambiguity, we write (X, d) for (X, d, H) .

Let X be a partially ordered set with a partial order " $\leq$ ", and let $(X, d, \leq)$ be a partially ordered hyperbolic metric space. We say that $x, y \in X$ are comparable whenever $x \leq y$ or $y \leq x$. Throughout, we will assume that order intervals are closed convex subsets of hyperbolic metric space X . We denote these intervals as follows:

$$\begin{aligned} [a, \longrightarrow) &= \{x \in X, a \leq x\}, \\ (\longleftarrow, b] &= \{x \in X, x \leq b\}, \end{aligned} \quad (4)$$

for any $a, b \in X$.

Definition 4 (see [16, 17]). Let (X, d) be a hyperbolic metric space. We say that X is uniformly convex if for any $a \in X$, $r > 0$ and $\varepsilon > 0$:

$$\begin{aligned} \delta(r, \varepsilon) &= \inf \left\{ 1 - \frac{1}{r} d\left(\frac{1}{2}x \oplus \frac{1}{2}y, a\right) ; \right. \\ &\quad \left. d(x, a) \leq r, d(y, a) \leq r, d(x, y) \geq r\varepsilon \right\} > 0. \end{aligned} \quad (5)$$

Proposition 5 (see [18]). Let (X, d) be a uniformly convex hyperbolic metric space and $(A_n)_n$ be a decreasing sequence of nonempty closed, bounded, and convex subsets of X . Then, $\bigcap_{n \geq 1} A_n \neq \emptyset$.

Lemma 6 (see [19]). Let (E, d) be a uniformly convex hyperbolic metric space with a monotone modulus of uniform convexity δ . Let $z \in E$ and $(\alpha_n)_n$ be a sequence such that $0 < a \leq \alpha_n \leq b < 1$. If $(u_n)_n$ and $(v_n)_n$ are two sequences in E such that $\limsup_{n \rightarrow \infty} d(u_n, z) \leq r$, $\limsup_{n \rightarrow \infty} d(v_n, z) \leq r$, and $\lim_{n \rightarrow \infty} d(\alpha_n v_n \oplus (1 - \alpha_n)u_n, z) = r$, for some $r \geq 0$, then

$$\lim_{n \rightarrow \infty} d(u_n, v_n) = 0. \quad (6)$$

Definition 7. Let $(X, d, \leq)$ be a metric space endowed with a partial order. A mapping $T : X \rightarrow X$ is said to be monotone nondecreasing, if T satisfies

$$T(x) \leq T(y) \quad \text{whenever } x \leq y, \quad (7)$$

for all $x, y \in X$.

Definition 8 (see [14]). Let $(X, d, \leq)$ be a partially ordered metric space. A mapping $T : X \rightarrow X$ is called monotone nonexpansive, if T is monotone and

$$d(Tx, Ty) \leq d(x, y), \quad (8)$$

for all $x, y \in X$ such that x and y are comparable.

Definition 9 (see [14]). Let K be a nonempty subset of a hyperbolic metric space (X, d) . A function $\tau : K \rightarrow [0, \infty)$ is said to be a type function, if there exists a bounded sequence $(x_n)_n$ in X such that

$$\tau(x) = \limsup_{n \rightarrow \infty} d(x_n, x), \quad (9)$$

for any $x \in K$.

Lemma 10 (see [14]). Let $(X, d, \leq)$ be a uniformly convex hyperbolic metric space and K a nonempty closed convex subset of X . Let $\tau : K \rightarrow [0, \infty)$ be a type function. Then, τ is continuous. Moreover, there exists a unique minimum point $z \in K$ such that $\tau(z) = \inf \{\tau(x) ; x \in K\}$.

3. Main Results

Before we give the existence and convergence theorems, we present a class of monotone nonexpansive mappings and some auxiliary results.

Definition 11. Let $(X, d, \leq)$ be a partially ordered metric space, and let $\gamma \in [0, 1]$ and $\mu \in [0, 1/2]$ such that $2\mu \leq \gamma$. Let $T : X \rightarrow X$ be a monotone mapping. Then, T is said to satisfy the condition $B_{\gamma, \mu}$ on X if for all $x, y \in X$ such that $x \leq y$,

$$\begin{aligned} \gamma d(x, Tx) &\leq d(x, y) + \mu d(y, Ty) \text{ implies} \\ d(Tx, Ty) &\leq (1 - \gamma)d(x, y) + \mu(d(x, Ty) + d(Tx, y)). \end{aligned} \quad (10)$$

Clearly, this class includes the class of monotone nonexpansive mappings (for $\gamma = \mu = 0$).

Example 1. Let $X = \mathbb{R}$ be equipped with the usual ordering and the distance $d(x, y) = |x - y|$. Let $K = [0, 1]$ and $T : K \rightarrow K$ be a mapping defined by

$$T(x) = \begin{cases} x + \frac{3}{5}, & \text{if } x \in \left[0, \frac{1}{5}\right], \\ \frac{x+4}{5}, & \text{if } x \in \left[\frac{1}{5}, 1\right], \end{cases} \quad (11)$$

- (1) T is not a monotone nonexpansive mapping
- (2) T is monotone and satisfies condition $B_{\gamma, \mu}$

Setting $x = 18/100$ and $y = 1/5$, we obtain

$$d(Tx, Ty) = \frac{6}{100} > \frac{2}{100} = d(x, y). \quad (12)$$

However, T is not a nonexpansive mapping. Now, we show that T is monotone and satisfies condition $B_{\gamma, \mu}$ with $\gamma = 3/4$ and $\mu = 3/8$. In fact, we have the following three cases:

- (i) Let $x, y \in [0, 1/5)$ such that $x \leq y$, and we have

$$\begin{aligned} (1 - \gamma)d(x, y) + \mu(d(Tx, y) + d(x, Ty)) \\ = \frac{1}{4}|x - y| + \frac{3}{8} \left(\left| x + \frac{3}{5} - y \right| + \left| x - y - \frac{3}{5} \right| \right) \\ \geq \frac{1}{4}|x - y| + \frac{3}{8}|2x - 2y| = |x - y| = d(Tx, Ty). \end{aligned} \quad (13)$$

- (ii) Let $x, y \in [1/5, 1]$ such that $x \leq y$, and we have

$$\begin{aligned} (1 - \gamma)d(x, y) + \mu(d(Tx, y) + d(x, Ty)) \\ = \frac{1}{4}|x - y| + \frac{3}{8} \left(\left| \frac{x}{5} + \frac{4}{5} - y \right| + \left| x - \frac{y}{5} - \frac{4}{5} \right| \right) \\ \geq \frac{1}{4}|x - y| + \frac{3}{8} \left| \frac{6x}{5} - \frac{6y}{5} \right| \\ \geq \left(\frac{1}{4} + \frac{9}{4} \right) \left| \frac{x}{5} - \frac{y}{5} \right| = \frac{10}{4} \left| \frac{x}{5} - \frac{y}{5} \right| \\ \geq \left| \frac{x}{5} - \frac{y}{5} \right| = d(Tx, Ty). \end{aligned} \quad (14)$$

- (iii) Let $x \in [0, 1/5)$ and $y \in [1/5, 1]$, and we have

$$\begin{aligned} (1 - \gamma)d(x, y) + \mu(d(x, y) + d(x, Ty)) \\ = \frac{1}{4}|x - y| + \frac{3}{8} \left(\left| x + \frac{3}{5} - y \right| + \left| \frac{y}{5} + \frac{4}{5} - x \right| \right) \\ \geq \frac{1}{4}|x - y| + \frac{3}{8} \left| \frac{7}{5} - \frac{4y}{5} \right| = -\frac{1}{4}x - \frac{1}{20}y + \frac{21}{40}. \end{aligned} \quad (15)$$

Otherwise, we have

$$d(Tx, Ty) = \left| x + \frac{3}{5} - \frac{y}{5} - \frac{4}{5} \right| = \frac{1}{5} - x + \frac{y}{5}, \quad (16)$$

since $x - (y/5) - (1/5) \leq 0$. From (19) and (20), we have

$$\begin{aligned} d(Tx, Ty) - (1 - \gamma)d(x, y) - \mu(d(x, y) + d(x, Ty)) \\ = \frac{1}{5} - x + \frac{y}{5} + \frac{1}{4}x + \frac{1}{20}y - \frac{21}{40} = -\frac{13}{40} - \frac{3}{4}x + \frac{1}{4}y. \end{aligned} \quad (17)$$

Since $-(11/40) \leq (1/4)y - (13/40) \leq -(3/40)$, then

$$d(Tx, Ty) \leq (1 - \gamma)d(x, y) + \mu(d(x, y) + d(x, Ty)). \quad (18)$$

Note that there is no relationship between the condition (C) and the class of mapping satisfying the condition $B_{\gamma, \mu}$, which contradicts with the statement of Patir et al. in the paper [6]. To justify this, we give the following example.

Example 2. Define the mapping $T : [0, 3] \rightarrow [0, 3]$ by

$$T(x) = \begin{cases} 0 & \text{if } x \neq 3, \\ 1 & \text{if } x = 3. \end{cases} \quad (19)$$

The monotone mapping T satisfies the condition (C) (for details, see [5]) but does not satisfy the condition $B_{\gamma,\mu}$ for $\gamma = \mu = 0$. In fact, for $x = 5/2$ and $y = 3$, we have

$$d(Tx, Ty) = 1 > d(x, y) = \frac{1}{2}. \quad (20)$$

Therefore, the condition $B_{\gamma,\mu}$ is not satisfied.

The Mann [20] iterative scheme has been used to study the approximation of fixed points of nonexpansive mappings by many authors [21–26]. Following the same line, we define the Mann process in the framework of hyperbolic metric spaces in order to prove its convergence to fixed points for monotone mapping satisfying condition $B_{\gamma,\mu}$.

Definition 12. Let $(X, d, \leq)$ be an ordered hyperbolic metric space. Let K be a nonempty convex subset of X and $T : K \rightarrow K$ a nondecreasing mapping on $(X, \leq)$. Fix $x_0 \in K$. The Mann iteration process is the sequence $(x_n)_{n \in \mathbb{N}}$ defined by

$$x_{n+1} = (1 - t_n)x_n \oplus t_n T(x_n), \quad (21)$$

for all $n \geq 0$, where $(t_n)_{n \geq 0} \subset [0, 1]$.

Proposition 13 (see [8]). Let $(X, d, \leq)$ be an ordered hyperbolic metric space. Let $(x_n)_{n \geq 0}$ and $(y_n)_{n \geq 0}$ be two sequences in X and $t_n \in [0, 1]$ which satisfy for all $n \in \mathbb{N}$,

- (i) $x_{n+1} \in [x_n, y_n]$ with $d(x_n, x_{n+1}) = t_n d(x_n, y_n)$
- (ii) $d(y_{n+1}, y_n) \leq d(x_{n+1}, x_n)$

Then, for all $i \geq 0$ and $n \geq 1$,

$$\left(1 + \sum_{s=i}^{i+n-1} t_s\right) d(y_i, x_i) \leq d(y_{i+n}, x_i) + \prod_{s=i}^{i+n-1} (1 - t_s)^{-1} [d(y_i, x_i) - d(y_{i+n}, x_{i+n})]. \quad (22)$$

The next lemma is a technical tool which will be used to prove that $(x_n)_n$ is an almost fixed point sequence.

Lemma 14. Let K be a nonempty closed convex subset of a partially ordered hyperbolic metric space $(X, d, \leq)$ and $T : K \rightarrow K$ be a monotone mapping satisfying the condition $B_{\gamma,\mu}$ on K , with $2\mu < \gamma < 1$. Assume that there exists $x_0 \in K$ such that $x_0 \leq T(x_0)$. Let $(x_n)_{n \in \mathbb{N}}$ be the Mann iterative sequence defined by (25) such that $(t_n)_{n \geq 0} \subset [a, b]$ with $a = \sup \{\gamma, 2\mu/\gamma\}$ and $b < 1$. Then, for any $i \geq 0$ and $n \geq 1$, we have

$$\begin{aligned} \left(1 + \sum_{s=i}^{i+n-1} t_s\right) d(x_i, Tx_i) &\leq d(x_i, Tx_{i+n}) \\ &+ \prod_{s=i}^{i+n-1} (1 - t_s)^{-1} [d(x_i, Tx_i) \\ &- d(x_{i+n}, Tx_{i+n})]. \end{aligned} \quad (23)$$

Proof. Let $x_0 \in K$ such that $x_0 \leq Tx_0$. Using the convexity of the order interval $[x_0, Tx_0]$ and the monotonicity of T , we obtain

$$x_0 \leq x_1 \leq Tx_0 \leq Tx_1. \quad (24)$$

Thus, by induction, we get

$$x_n \leq x_{n+1} \leq Tx_n \leq Tx_{n+1}, \quad (25)$$

for any $n \in \mathbb{N}$. On the other hand, we have

$$d(x_n, x_{n+1}) = d(x_n, (1 - t_n)x_n \oplus t_n Tx_n) = t_n d(x_n, Tx_n). \quad (26)$$

Since $\gamma \leq t_n$ for all $n \in \mathbb{N}$, we have

$$\gamma d(x_n, Tx_n) \leq t_n d(x_n, Tx_n) = d(x_n, x_{n+1}). \quad (27)$$

By the condition $B_{\gamma,\mu}$, for $x = x_n$ and $y = x_{n+1}$, we have

$$\begin{aligned} d(Tx_n, Tx_{n+1}) &\leq (1 - \gamma)d(x_n, x_{n+1}) \\ &+ \mu(d(x_n, Tx_{n+1}) + d(Tx_n, x_{n+1})) \\ &\leq t_n(1 - \gamma)d(x_n, Tx_n) + \mu(d(x_n, Tx_n) \\ &+ d(Tx_n, Tx_{n+1}) + (1 - t_n)d(Tx_n, x_n)) \\ &\leq \left(\frac{t_n(1 - \gamma) + \mu(2 - t_n)}{1 - \mu}\right) d(x_n, Tx_n) \\ &= \left(\frac{t_n(1 - \gamma) + \mu(2 - t_n)}{t_n(1 - \mu)}\right) d(x_n, x_{n+1}). \end{aligned} \quad (28)$$

Since $t_n \geq 2\mu/\gamma$, then $((t_n(1 - \gamma) + \mu(2 - t_n))/(t_n(1 - \mu))) \leq 1$. Hence,

$$d(Tx_n, Tx_{n+1}) \leq d(x_n, x_{n+1}), \quad (29)$$

for all $n \in \mathbb{N}$. According to (29), (30), and (33), it is concluded that all the assumptions of Proposition 13 are satisfied, where $y_n = T(x_n)$, which implies

$$\begin{aligned}
\left(1 + \sum_{s=i}^{i+n-1} t_s\right) d(x_i, Tx_i) &\leq d(x_i, Tx_{i+n}) \\
&+ \prod_{s=i}^{i+n-1} (1 - t_s)^{-1} [d(x_i, Tx_i) \\
&- d(x_{i+n}, Tx_{i+n})],
\end{aligned} \tag{30}$$

for any $i \geq 0$ and $n \geq 1$.

Theorem 15. Let X be as above. Let K be a nonempty, closed, convex, and bounded subset of X and $T : K \rightarrow K$ be a monotone mapping satisfying the condition $B_{\gamma, \mu}$ on K , with $2\mu < \gamma < 1$. Assume that there exists $x_0 \in K$ such that $x_0 \leq T(x_0)$ and the sequence $(x_n)_n$ defined by (25) such that $(t_n)_{n \geq 0} \subset [a, b]$, with $a = \sup \{\gamma, 2\mu/\gamma\}$ and $b < 1$. Then, we have $\lim_{n \rightarrow \infty} d(x_n, Tx_n) = 0$.

Proof. First we show that the sequence $(d(x_n, Tx_n))_{n \geq 0}$ is decreasing. Indeed, by the triangle inequality, we have

$$\begin{aligned}
d(x_{n+1}, Tx_{n+1}) &\leq d(x_{n+1}, Tx_n) + d(Tx_n, Tx_{n+1}) \\
&= d((1 - t_n)x_n \oplus t_n Tx_n, Tx_n) + d(Tx_n, Tx_{n+1}) \\
&= (1 - t_n)d(x_n, Tx_n) + d(Tx_n, Tx_{n+1}).
\end{aligned} \tag{31}$$

From the proof of Lemma 14, we have

$$d(Tx_n, Tx_{n+1}) \leq d(x_n, x_{n+1}), \tag{32}$$

for all $n \in \mathbb{N}$. Hence,

$$\begin{aligned}
d(x_{n+1}, Tx_{n+1}) &\leq (1 - t_n)d(x_n, Tx_n) + d(Tx_n, Tx_{n+1}) \\
&\leq d(x_n, Tx_n),
\end{aligned} \tag{33}$$

for all $n \in \mathbb{N}$. Consequently, $(d(x_n, Tx_n))_n$ is a decreasing and bounded sequence in $\mathbb{R}_+$. Thus, there exists $r \geq 0$ such that

$$\lim_{n \rightarrow \infty} d(x_n, Tx_n) = r \geq 0. \tag{34}$$

On the other hand, we have

$$\begin{cases} 1 + n\gamma \leq 1 + \sum_{s=i}^{i+n-1} t_s, \\ \prod_{s=i}^{i+n-1} (1 - t_s)^{-1} \leq (1 - b)^{-n}, \\ d(x_i, Tx_{i+n}) \leq \delta(K) = \sup \{d(x, y) : x, y \in K\}, \end{cases} \tag{35}$$

since $\sup \{\gamma, 2\mu/\gamma\} \leq t_n$ and $t_n \leq b < 1$ for all $n \geq 0$. Therefore, the inequality obtained in Lemma 14 implies

$$(1 + n\gamma)d(x_i, Tx_i) \leq \delta(K) + (1 - b)^{-n}[d(x_i, Tx_i) - d(x_{i+n}, Tx_{i+n})], \tag{36}$$

for all $i \geq 0$ and $n \geq 1$. If we let $i \rightarrow +\infty$, we get $(1 + n\gamma)r \leq \delta(K)$, for any $n \geq 1$. Then,

$$r \leq \frac{\delta(K)}{1 + n\gamma}, \tag{37}$$

holds for all $n \geq 1$. Clearly, this is possible only for $r = 0$. Thus,

$$\lim_{n \rightarrow \infty} d(x_n, Tx_n) = 0. \tag{38}$$

Lemma 16. Let $(X, d, \leq)$ be a partially ordered hyperbolic metric space and K a nonempty subset of E . Let $T : K \rightarrow K$ be a monotone mapping which satisfies the $B_{\gamma, \mu}$ condition. If z is a fixed point of T , then for all $x \in K$, $x \leq z$,

$$d(z, Tx) \leq d(z, x). \tag{39}$$

Proof. Let z be a fixed point of T and $x \in K$ such that $x \leq z$.

We have

$$0 = d(z, Tz) \leq d(x, z) \leq d(x, z) + \mu d(x, Tx). \tag{40}$$

By the condition $B_{\gamma, \mu}$,

$$d(Tx, Tz) = d(z, Tx) \leq (1 - \gamma)d(x, z) + \mu d(x, Tz) + \mu d(z, Tx). \tag{41}$$

Hence,

$$(1 - \mu)d(z, Tx) \leq (1 - \gamma + \mu)d(z, x), \tag{42}$$

which implies

$$d(z, Tx) \leq \frac{1 - \gamma + \mu}{1 - \mu} d(z, x). \tag{43}$$

Since $2\mu \leq \gamma$, then

$$d(z, Tx) \leq d(z, x). \tag{44}$$

In the sequel, we will use the fixed point set with the partial orders $\mathcal{F}_{x_0}^{\geq}(T)$ given by

$$\mathcal{F}_{x_0}^{\geq}(T) = \{p \in K : Tp = p, x_0 \leq p\}. \tag{45}$$

Definition 17. Let $(X, \leq)$ be a partially ordered set and $(x_n)_n$ be a sequence in X . The sequence $(x_n)_n$ is said to be $\leq$ -bounded if there exists a point $w \in X$ such that $x_n \leq w$, for all $n \geq 0$.

Lemma 18. Let K be a nonempty closed and convex subset of a partially ordered hyperbolic metric space $(X, d \leq)$. Let $T : K \rightarrow K$ be a monotone mapping satisfying the condition $B_{\gamma, \mu}$ on K , for $x_0 \in K$ such that $x_0 \leq Tx_0$ and $\mathcal{F}_{x_0}^{\geq}(T)$

is nonempty. Let $(x_n)_n$ be a sequence defined by (25). Then, the following assertions hold:

- (i) The sequence $(x_n)_n$ is $\leq$ -bounded
- (ii) $\lim_{n \rightarrow \infty} d(x_n, z)$ exists for $z \in \mathcal{F}_{x_0}^{\geq}(T)$, and $\lim_{n \rightarrow \infty} \text{dist}(x_n, \mathcal{F}_{x_0}^{\geq}(T))$ exists, where $\text{dist}(x, \mathcal{F}_{x_0}^{\geq}(T))$ denotes the distance from x to $\mathcal{F}_{x_0}^{\geq}(T)$

Proof.

- (i) Let $\mathcal{F}_{x_0}^{\geq}(T) \neq \emptyset$, and fix $z \in \mathcal{F}_{x_0}^{\geq}(T)$. From the proof of Lemma 14, we have

$$x_n \leq x_{n+1} \leq Tx_n, \quad \text{for all } n \in \mathbb{N}. \quad (46)$$

Since T is monotone and $Tx_0 \leq Tz = z$, we have

$$x_0 \leq x_1 \leq Tx_0 \leq z. \quad (47)$$

So, $x_1 \leq z$. By monotonicity of T ,

$$x_1 \leq x_2 \leq Tx_1 \leq z. \quad (48)$$

Applying the same argument, we get

$$x_n \leq Tx_n \leq z, \quad \text{for all } n \geq 0. \quad (49)$$

Thus, $(x_n)_n$ is $\leq$ -bounded.

- (ii) By Lemma 16, we have

$$\begin{aligned} d(x_{n+1}, z) &= d((1 - t_n)x_n \oplus t_n Tx_n, z) \\ &\leq (1 - t_n)d(x_n, z) + t_n d(Tx_n, z) \\ &\leq (1 - t_n)d(x_n, z) + t_n d(x_n, z) \\ &= d(x_n, z). \end{aligned} \quad (50)$$

Hence,

$$d(x_{n+1}, z) \leq d(x_n, z), \quad (51)$$

for all $n \geq 0$. Consequently, $(d(x_n, z))_n$ is a decreasing and bounded sequence in $\mathbb{R}_+$. Thus, $\lim_n d(x_n, z)$ exists. For each $z \in \mathcal{F}_{x_0}^{\geq}(T)$ and $n \in \mathbb{N}$ we have

$$d(x_{n+1}, z) \leq d(x_n, z). \quad (52)$$

Taking infimum over all $z \in \mathcal{F}_{x_0}^{\geq}(T)$, we get

$$\text{dist}(x_{n+1}, \mathcal{F}_{x_0}^{\geq}(T)) \leq \text{dist}(x_n, \mathcal{F}_{x_0}^{\geq}(T)), \quad (53)$$

for all $n \in \mathbb{N}$. So the sequence $(\text{dist}(x_n, \mathcal{F}_{x_0}^{\geq}(T)))_n$ is bounded and decreasing. Therefore, $\lim_{n \rightarrow \infty} \text{dist}(x_n, \mathcal{F}_{x_0}^{\geq}(T))$ exists.

Theorem 19. Let K be a nonempty closed convex and bounded subset of a hyperbolic ordered metric space X and $T : K \rightarrow K$ be a monotone mapping satisfying the condition $B_{\gamma, \mu}$ on K , with $2\mu < \gamma < 1$. Assume that there exists $x_0 \in K$ such that $x_0 \leq T(x_0)$ and $\mathcal{F}_{x_0}^{\geq}(T)$ is nonempty. Let $(x_n)_n$ be a sequence defined by (25) where $(t_n)_{n \geq 0} \subset [a, b]$, with $a = \sup \{\gamma, 2\mu/\gamma\}$ and $b < 1$. Then, $(x_n)_n$ converges to a fixed point of T .

Proof. Assume that $\mathcal{F}_{x_0}^{\geq}(T)$ is nonempty, and let $z \in \mathcal{F}_{x_0}^{\geq}(T)$. Then, from the proof of Lemma 18, we have

$$x_n \leq Tx_n \leq z, \quad \text{for all } n \in \mathbb{N}. \quad (54)$$

By Lemma 18, there exists $r \geq 0$ such that

$$\lim_{n \rightarrow \infty} d(x_n, z) = r \geq 0. \quad (55)$$

Assume that $r > 0$. By Theorem 15, one has

$$\lim_{n \rightarrow \infty} d(x_n, Tx_n) = 0. \quad (56)$$

This leads to $d(x_n, Tx_n) \leq d(x_n, z)$, for large sufficient $n \geq 0$. Then,

$$\gamma d(x_n, Tx_n) \leq d(x_n, z) + \mu d(z, Tz). \quad (57)$$

By the condition $B_{\gamma, \mu}$, we get

$$\begin{aligned} d(Tx_n, Tz) &\leq (1 - \gamma)d(x_n, z) + \mu(d(Tx_n, z) + d(x_n, z)) \\ &\leq (1 - \gamma + \mu)d(x_n, z) + \mu d(Tx_n, z) \\ &\leq \frac{1 - \gamma + \mu}{1 - \mu} d(x_n, z). \end{aligned} \quad (58)$$

Otherwise,

$$\begin{aligned} d(x_n, z) &\leq d(x_n, Tx_n) + d(Tx_n, z) \\ &\leq d(x_n, Tx_n) + \frac{1 - \gamma + \mu}{1 - \mu} d(x_n, z). \end{aligned} \quad (59)$$

Taking limit as n goes to infinity, we get

$$r \leq \frac{1 - \gamma + \mu}{1 - \mu} r. \quad (60)$$

Since $((1 - \gamma + \mu)/(1 - \mu)) < 1$, we get a contradiction. Therefore, $(x_n)_n$ converges to z .

Example 3. Let $X = C^0([0, 1]) \times C^0([0, 1])$. Define $d : X \times X \rightarrow \mathbb{R}_+$ by

$$d((f_1, f_2), (g_1, g_2)) = \|f_2 - g_2\|_{\infty} + \|f_1 F_2 - g_1 G_2\|_{\infty}, \quad (61)$$

such that $F_2(x) = 1 + \int_0^x f_2(t) dt$ and $G_2(x) = 1 + \int_0^x g_2(t) dt$, for all $x \in [0, 1]$. It is obvious to see that (X, d) is a metric space.

First step. Let $K = \{(f_1, f_2) \in X / (1/2) \leq f_1(x) \leq 1, 0 \leq f_2(x) \leq 1, \text{ for all } x \in [0, 1]\} \subset X$. We can easily verify that K is a non-empty, bounded, and convex subset of X . We show that K is closed in X . Indeed, let $(f_n = (f_{1,n}, f_{2,n}))_{n \in \mathbb{N}}$ be a sequence in X and $f = (f_1, f_2) \in X$ such that

$$(f_{1,n}, f_{2,n}) \xrightarrow[n \rightarrow +\infty]{d} (f_1, f_2). \quad (62)$$

Then, for all $\varepsilon > 0$, there exists an $N_\varepsilon \in \mathbb{N}$ such that

$$d(f_n, f) = \|f_{2,n} - f_2\|_\infty + \|f_{1,n}F_{2,n} - f_1F_2\|_\infty \leq \varepsilon, \quad \forall n \geq N_\varepsilon. \quad (63)$$

This implies

$$\begin{aligned} \|f_{2,n} - f_2\|_\infty &\leq \varepsilon, \\ \|f_{1,n}F_{2,n} - f_1F_2\|_\infty &\leq \varepsilon, \quad \forall n \geq N_\varepsilon. \end{aligned} \quad (64)$$

Thus,

$$f_{2,n} \xrightarrow[n \rightarrow +\infty]{\text{unif.}} f_2 \in C^0([0, 1]), \quad (65)$$

$$f_{1,n}F_{2,n} \xrightarrow[n \rightarrow +\infty]{\text{unif.}} f_1F_2 \in C^0([0, 1]). \quad (66)$$

Using (69), it is easy to verify that

$$F_{2,n} \xrightarrow[n \rightarrow +\infty]{\text{unif.}} F_2 \in C^0([0, 1]). \quad (67)$$

On the other hand, fix $x \in [0, 1]$, and we have

$$\begin{aligned} |f_{1,n}(x) - f_1(x)| &\leq |f_{1,n}(x) - f_1(x)F_2(x)F_{2,n}^{-1}(x)| \\ &\quad + |f_1(x) - f_1(x)F_2(x)F_{2,n}^{-1}(x)| \\ &\leq |F_{2,n}^{-1}(x)| |f_{1,n}(x)F_{2,n}(x) - f_1(x)F_2(x)| \\ &\quad + |f_1(x)F_2(x)| |F_2^{-1}(x) - F_{2,n}^{-1}(x)| \\ &\leq |F_{2,n}^{-1}(x)| |f_{1,n}(x)F_{2,n}(x) - f_1(x)F_2(x)| \\ &\quad + \frac{|f_1(x)F_2(x)|}{|F_{2,n}(x)F_2(x)|} |F_{2,n}(x) - F_2(x)|. \end{aligned} \quad (68)$$

Since $1 \leq F_{2,n}(x) \leq 2$ and $1/2 \leq f_{1,n}(x) \leq 1$, for all $n \in \mathbb{N}$, then

$$\|f_{1,n} - f_1\|_\infty \leq \|f_{1,n}F_{2,n} - f_1F_2\|_\infty + 2\|F_{2,n} - F_2\|_\infty, \quad \forall n \geq N_\varepsilon. \quad (69)$$

By (70) and (71), we conclude that

$$f_{1,n} \xrightarrow[n \rightarrow +\infty]{\text{unif.}} f_1 \in C^0([0, 1]). \quad (70)$$

Therefore, $(f_1, f_2) \in K$.

Second step. Define a function $H : X \times X \times [0, 1] \rightarrow X$ by

$$H(f, g, \beta) = \left(\frac{(1-\beta)f_1F_2 + \beta g_1G_2}{(1-\beta)F_2 + \beta G_2}, (1-\beta)f_2 + \beta g_2 \right), \quad (71)$$

for all $f = (f_1, f_2)$ and $g = (g_1, g_2)$ in X . We show that (X, d, H) is a hyperbolic metric space. For $f = (f_1, f_2)$, $g = (g_1, g_2)$, $h = (h_1, h_2)$, $v = (v_1, v_2) \in X$, and $\beta, \lambda \in [0, 1]$, we have the following:

(i)

$$\begin{aligned} d(h, H(f, g, \beta)) &= \|h_2 - (1-\beta)f_2 - \beta g_2\|_\infty + \|h_1H_2 \\ &\quad - (1-\beta)f_1F_2 - \beta g_1G_2\|_\infty \\ &\leq (1-\beta)\|h_2 - f_2\|_\infty + \beta\|h_2 \\ &\quad - g_2\|_\infty + (1-\beta)\|h_1H_2 - f_1F_2\|_\infty \\ &\quad + \beta\|h_1H_2 - g_1G_2\|_\infty \\ &= (1-\beta)d(h, f) + \beta d(h, g) \end{aligned}$$

(ii)

$$\begin{aligned} d(H(f, g, \beta), H(f, g, \lambda)) &= d\left(\left(\frac{(1-\beta)f_1F_2 + \beta g_1G_2}{(1-\beta)F_2 + \beta G_2}, (1-\beta)f_2\right.\right. \\ &\quad \left.\left.+ \beta g_2\right), \left(\frac{(1-\lambda)f_1F_2 + \lambda g_1G_2}{(1-\lambda)F_2 + \lambda G_2}, (1-\lambda)f_2 + \lambda g_2\right)\right) \\ &= \|(\lambda - \beta)f_2 + (\beta - \lambda)g_2\|_\infty + \|(\lambda - \beta)f_1F_2 + (\beta - \lambda)g_1G_2\|_\infty \\ &= |\lambda - \beta|d(f, g) \end{aligned}$$

(iii)

$$\begin{aligned} H(f, g, \beta) &= \left(\frac{(1-\beta)f_1F_2 + \beta g_1G_2}{(1-\beta)F_2 + \beta G_2}, (1-\beta)f_2 + \beta g_2 \right) \\ &= \left(\frac{\beta g_1G_2 + (1-\beta)f_1F_2}{\beta G_2 + (1-\beta)F_2}, \beta g_2 + (1-\beta)f_2 \right) \\ &= H(g, f, (1-\beta)) \end{aligned}$$

(iv)

$$\begin{aligned} d(H(f, g, \beta), H(h, v, \beta)) &= \|(1-\beta)(f_2 - h_2) \\ &\quad + \beta(g_2 - v_2)\|_\infty \\ &\quad + \|(1-\beta)(f_1F_2 - h_1H_2) \\ &\quad + \beta(g_1G_2 - v_1V_2)\|_\infty \\ &\leq (1-\beta)d(f, h) + \beta d(g, v) \end{aligned}$$

Consider the partial ordering defined on X as follows:

$$\begin{aligned} f \leq g &\Leftrightarrow f_1(t) \geq g_1(t), \\ f_2(t) &\geq g_2(t), \quad \forall t \in [0, 1], \end{aligned} \quad (72)$$

where $f = (f_1, f_2)$ and $g = (g_1, g_2)$ in X .

Thus, (X, d, H) is a partially ordered hyperbolic metric space, but not a normed space.

Third step. Let $T : X \longrightarrow X$ be a mapping defined by

$$T((f_1, f_2)) = \begin{cases} (y_1, y_2) & \text{if } (f_1, f_2) \neq \left(\frac{1}{2}, 0\right), \\ (z_1, z_2) & \text{if } (f_1, f_2) = \left(\frac{1}{2}, 0\right), \end{cases} \quad (73)$$

where (y_1, y_2) and (z_1, z_2) designate constant functions, such that

$$\begin{aligned} (y_1(t), y_2(t)) &= \left(\frac{1}{2}, \frac{1}{2}\right), \\ (z_1(t), z_2(t)) &= \left(\frac{1}{2}, \frac{5}{11}\right) \quad \text{for all } t \in [0, 1]. \end{aligned} \quad (74)$$

It is obvious that the mapping T is monotone.

(i) T is not nonexpansive. In fact, for $f = (1/2, 1/33)$ and $g = (1/2, 0)$, we have

$$d(T(f), T(g)) = \frac{3}{44} > \frac{1}{22} = d(f, g). \quad (75)$$

(ii) T satisfies the condition $B_{\gamma, \mu}$ for $\gamma = 10/21$ and $\mu = 5/44$. In fact, for $f = (f_1, f_2) \neq (1/2, 0)$ and $g = (g_1, g_2) = (1/2, 0)$ such that $f \leq g$, we have

$$\begin{aligned} d(T(f), T(g)) &= d\left(\left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{5}{11}\right)\right) = \frac{3}{44}, \\ d(T(f), g) &= \left\| \frac{1}{2} \right\|_{\infty} + \frac{1}{2} \left\| \frac{Id}{2} \right\|_{\infty} = \frac{3}{4}. \end{aligned} \quad (76)$$

Therefore,

$$\begin{aligned} (1 - \gamma)d(f, g) + \mu(d(T(f), g) + d(T(g), f)) \\ = \frac{11}{21}d(f, g) + \frac{15}{176} + \frac{5}{44}d(T(g), f) \\ \geq \frac{15}{176} > \frac{3}{44} = d(Tf, Tg). \end{aligned} \quad (77)$$

Otherwise, if $f = (f_1, f_2)$ and $g = (g_1, g_2) \neq (1/2, 0)$ (or equal to $(1/2, 0)$), it is quite easy to verify that the mapping T satisfies the condition $B_{\gamma, \mu}$ for $\gamma = 10/21$ and $\mu = 5/44$.

Therefore, T is monotone and satisfies the condition $B_{\gamma, \mu}$ and T has a unique fixed point $h^* = (1/2, 1/2)$.

Fourth step. Suppose that $x_0 = (1, 1/2)$ and $t_n = 2\mu/\gamma = 21/44 \geq \gamma$, then $x_0 \leq Tx_0$ and

$$x_n = \left(\frac{1}{2} + \frac{(1 - \alpha)^n}{2}, \frac{1}{2}\right), \quad (78)$$

where $\alpha = 21/44$, for all $n \in \mathbb{N}$. Indeed, assume that (82) is true for n and let us prove it for $n + 1$. We have

$$x_{n+1} = (1 - \alpha)x_n + \alpha Tx_n. \quad (79)$$

Since $x_n \neq (1/2, 0)$, then

$$\begin{aligned} x_{n+1} &= (1 - \alpha)x_n + \alpha \left(\frac{1}{2}, \frac{1}{2}\right) \\ &= \left(\frac{1}{2}(1 - \alpha)^{n+1} + \frac{1}{2}(1 - \alpha), \frac{1}{2}(1 - \alpha)\right) + \left(\frac{1}{2}\alpha, \frac{1}{2}\alpha\right) \\ &= \left(\frac{1}{2}(1 - \alpha)^{n+1} + \frac{1}{2}, \frac{1}{2}\right). \end{aligned} \quad (80)$$

Since T is monotone and satisfies condition $B_{\gamma, \mu}$ and there exists $x_0 \in K$ such that $x_0 \leq T(x_0)$, then all the conditions of Theorem 19 are satisfied. Consequently, it is obvious to see that

$$\lim_{n \rightarrow \infty} d\left(x_n, \left(\frac{1}{2}, \frac{1}{2}\right)\right) = 0. \quad (81)$$

Thus, $(x_n)_n$ converges to the fixed point $h^* = (1/2, 1/2)$.

Theorem 20. Let K be a nonempty closed convex and bounded subset of a uniformly convex partially ordered hyperbolic metric space $(X, d, \leq)$ and $T : K \longrightarrow K$ be a monotone mapping satisfying the condition $B_{\gamma, \mu}$ on K , with $2\mu < \gamma < 1$. Assume that there exists $x_0 \in K$ such that $x_0 \leq T(x_0)$ and the sequence $(x_n)_n$ defined by (25) be $\leq$ -bounded such that $(t_n)_{n \geq 0} \subseteq [a, b]$ with $a = \sup \{\gamma, 2\mu/\gamma\}$ and $b < 1$. Then, T has a fixed point $z \in K$, $x_0 \leq z$, and $\lim_{n \rightarrow \infty} d(x_n, z) = 0$.

Proof. As $(x_n)_n$ is $\leq$ -bounded, then there exists a point $w \in X$ such that $x_n \leq w$, for all $n \geq 0$.

Let $K_n = \{t \in K : x_n \leq t\}$ for all $i \in \mathbb{N}$. It is clear that K_i is nonempty, since $w \in K_n$, and is a closed and convex subset of K , for all $n \in \mathbb{N}$, and the sequence $(K_n)_n$ is decreasing, since $x_0 \leq x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$. Hence, by Proposition 5,

$$K_{\infty} = \bigcap_{n \geq 0} K_n, \quad (82)$$

is a nonempty, closed, and convex subset of K . Let $p \in K_{\infty}$; then, by the definition of K_{∞} and the monotonicity of T , we get

$$x_n \leq Tx_n \leq Tp. \quad (83)$$

Hence, $T(K_{\infty}) \subseteq K_{\infty}$. Consider the type function $\tau : K_{\infty} \longrightarrow [0, \infty)$ generated by $(x_n)_n$, that is, $\tau(v) = \limsup_{n \rightarrow \infty} d(x_n, v)$, for all $v \in K_{\infty}$. From Lemma 10, there exists a unique element $z \in K_{\infty}$ such that

$$\tau(z) = \inf \{\tau(v) : v \in K_{\infty}\}. \quad (84)$$

Moreover,

$$\tau(Tz) = \limsup_{n \rightarrow \infty} d(x_n, Tz). \quad (85)$$

We assert that $Tz = z$. Assume that this is not the case. By Theorem 15, we have $\lim_{n \rightarrow \infty} d(x_n, Tx_n) = 0$, then

$$d(x_n, Tx_n) \leq \mu d(z, Tz), \quad \text{for large integers } n. \quad (86)$$

Since $z \in K_\infty$, then $x_n \leq z$ for all $n \geq 1$. By the condition $B_{\gamma, \mu}$, for $x = x_n$ and $y = z$, we have

$$\begin{aligned} d(Tx_n, Tz) &\leq (1 - \gamma)d(x_n, z) + \mu(d(x_n, Tz) + d(z, Tx_n)) \\ &\leq (1 - \gamma + \mu)d(x_n, z) + \mu(d(Tx_n, Tz) + 2d(x_n, Tx_n)) \\ &\leq \frac{1 - \gamma + \mu}{1 - \mu}d(x_n, z) + \frac{2\mu}{1 - \mu}d(x_n, Tx_n). \end{aligned} \quad (87)$$

By the triangle inequality,

$$d(x_n, Tz) \leq d(x_n, Tx_n) + d(Tx_n, Tz), \quad (88)$$

we obtain

$$d(x_n, Tz) \leq \frac{1 - \gamma + \mu}{1 - \mu}d(x_n, z) + \frac{1 + \mu}{1 - \mu}d(x_n, Tx_n), \quad (89)$$

which implies

$$\limsup_{n \rightarrow \infty} d(x_n, Tz) \leq \frac{1 - \gamma + \mu}{1 - \mu} \limsup_{n \rightarrow \infty} d(x_n, z). \quad (90)$$

Hence,

$$\tau(Tz) \leq \tau(z). \quad (91)$$

By the uniqueness of the minimum point, we conclude a contradiction with our hypothesis $Tz \neq z$. Then, we have $Tz = z$.

$\mathcal{F}_{x_0}^{\geq}(T)$ is nonempty, because it contains z . Following the same proof of Theorem 15, we obtain $\lim_{n \rightarrow \infty} d(x_n, z) = 0$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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