

Research Article

Certain Subclasses of β -Uniformly q -Starlike and β -Uniformly q -Convex Functions

Eman S. A. AbuJarad,¹ Mohammed H. A. AbuJarad,² Thabet Abdeljawad ,^{3,4,5} and Fahd Jarad⁶

¹Mathematics Department, Gaza University, State of Palestine

²Statistics Department, Gaza University, State of Palestine

³Department of Mathematics and General Sciences, Prince Sultan University, P.O. Box 66833, 11586 Riyadh, Saudi Arabia

⁴Department of Medical Research, China Medical University, 40402 Taichung, Taiwan

⁵Department of Computer Science and Information Engineering, Asia University, Taichung, Taiwan

⁶Department of Mathematics, Çankaya University, 06790 Ankara, Turkey

Correspondence should be addressed to Thabet Abdeljawad; tabdeljawad@psu.edu.sa

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In this paper, the authors introduced certain subclasses β -uniformly q -starlike and β -uniformly q -convex functions of order α involving the q -derivative operator defined in the open unit disc. Coefficient bounds were also investigated.

1. Introduction

The q -analysis is a generalization of the ordinary analysis. The application of the q -calculus was first introduced by Jackson [1–3]. In geometric function theory, the q -hypergeometric functions were first used by Srivastava [4]. The q -calculus provides valuable tools that have been used to define several subclasses of the normalized analytic function in the open unit disk \mathbb{U} . Ismail et al. [5] were the first to study a certain class \mathcal{S}^* of starlike functions by using the q -derivative operator. Recently, new subclasses of analytic functions associated with q -derivative operators are introduced and discussed, see for example [4, 6–18]. Motivated by the importance of q -analysis, in this paper, we introduce the classes of β -uniformly q -starlike and β -uniformly q -convex functions defined by the q -derivative operator in the open unit disc, as a generalization of β -uniformly starlike and β -uniformly convex functions.

First, we recall some basic notations and definitions from q -calculus, which are used in this paper. The q -derivative of the function f is defined as follows [1–3]:

$$D_q f(z) = \frac{f(z) - f(qz)}{(1-q)z}, \quad (z \neq 0, 0 < q < 1). \quad (1)$$

From equation (1), it is clear that if f and g are the two functions, then

$$D_q(f(z) + g(z)) = D_q f(z) + D_q g(z), \quad (2)$$

$$D_q(cf(z)) = cD_q f(z), \quad (3)$$

where c is a constant. We note that $D_q f(z) \rightarrow f'(z)$ as $q \rightarrow 1^-$, where f' is the ordinary derivative of the function f .

In particular, using equation (1), the q -derivative of the function $h(z) = z^n$ is as follows:

$$D_q h(z) = [n]_q z^{n-1}, \quad (4)$$

where $[n]_q$ denotes the q -number and is given as follows:

$$[n]_q = \frac{1 - q^n}{1 - q}, \quad (0 < q < 1). \tag{5}$$

Since we note that $[n]_q \rightarrow n$ as $q \rightarrow 1^-$, therefore, in view of equation (4), $D_q h(z) \rightarrow h'(z)$ as $q \rightarrow 1^-$, where $h'(z)$ denotes the ordinary derivative of the function $h(z)$ with respect to z .

In this paper, we consider the classes \mathcal{A} and \mathcal{T} of the functions, analytic in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$, of the following forms, respectively:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{6}$$

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad (a_n \geq 0). \tag{7}$$

Also, using equations (2), (3), (4), and (6), we get the following q -derivatives of the function f :

$$D_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}, \quad (0 < q < 1), \tag{8}$$

$$D_q (zD_q f(z)) = 1 + \sum_{n=2}^{\infty} [n]_q^2 a_n z^{n-1}, \quad (0 < q < 1), \tag{9}$$

where $[n]_q$ is given by equation (5).

The classes of starlike functions of order α ($0 \leq \alpha < 1$) and convex functions of order α ($0 \leq \alpha < 1$), denoted by $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$, respectively, are defined as follows [19]:

$$\mathcal{S}^*(\alpha) = \left\{ f \in \mathcal{A} : \Re \left(\frac{zf'(z)}{f(z)} \right) > \alpha \right\}, \tag{10}$$

$$\mathcal{K}(\alpha) = \left\{ f \in \mathcal{A} : \Re \left(1 + \frac{zf''(z)}{f'(z)} \right) > \alpha \right\}. \tag{11}$$

It is clear that $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ are the subclasses of the class \mathcal{A} .

The classes of β -uniformly starlike functions of order α and β -uniformly convex functions of order α , denoted by $\mathcal{SD}(\alpha, \beta)$ and $\mathcal{KD}(\alpha, \beta)$, respectively, are defined as follows [20]:

$$\mathcal{SD}(\alpha, \beta) = \left\{ f \in \mathcal{A} : \Re \left(\frac{zf'(z)}{f(z)} - \alpha \right) > \beta \left| \frac{zf'(z)}{f(z)} - 1 \right| \right\}, \tag{12}$$

$$\mathcal{KD}(\alpha, \beta) = \left\{ f \in \mathcal{A} : \Re \left(\frac{zf''(z)}{f'(z)} - \alpha \right) > \beta \left| \frac{zf''(z)}{f'(z)} - 1 \right| \right\}, \tag{13}$$

where $z \in \mathbb{U}$, $0 \leq \alpha < 1$, and $\beta \geq 0$.

The class of q -starlike functions of order μ , denoted by $\mathcal{S}_q^*(\mu)$, is defined as follows [13]:

$$\mathcal{S}_q^*(\mu) = \left\{ f \in \mathcal{A} : \Re \left(\frac{zD_q f(z)}{f(z)} \right) > \mu, \quad (z \in \mathbb{U}; 0 \leq \mu < 1) \right\}. \tag{14}$$

Also, the class of q -convex functions of order μ , denoted by $C_q(\mu)$, is defined as [13]:

$$C_q(\mu) = \left\{ f \in \mathcal{A} : \Re \left(\frac{D^q(zD_q f(z))}{D_q f(z)} \right) > \mu, \quad (z \in \mathbb{U}; 0 \leq \mu < 1) \right\}. \tag{15}$$

The analytic function g is said to be subordinate to the analytic function f in \mathbb{U} [21], represented as follows:

$$g(z) \prec f(z) \text{ or } g \prec f, \tag{16}$$

if there exists a Schwarz function w , which is analytic in \mathbb{U} with

$$\begin{aligned} w(0) &= 0, \\ |w(z)| &< 1, \end{aligned} \tag{17}$$

such that

$$g(z) = f(w(z)), \quad (z \in \mathbb{U}). \tag{18}$$

In the next section, we introduce the classes of β -uniformly q -starlike and β -uniformly q -convex functions of order α , denoted by $\mathcal{S}_q(\alpha, \beta)$ and $\mathcal{UCV}_q(\alpha, \beta)$, respectively. Also, we obtain the coefficient bounds of the functions belonging to these classes.

2. Coefficient Bounds

Since the q -derivative is a generalized form of the ordinary derivative, therefore, in view of definitions of $\mathcal{SD}(\alpha, \beta)$ and $\mathcal{KD}(\alpha, \beta)$, we define the classes of β -uniformly q -starlike and β -uniformly q -convex functions of order α , denoted by $S_q(\alpha, \beta)$ and $\mathcal{UCV}_q(\alpha, \beta)$, respectively, by replacing the ordinary derivative with the q -derivative in equations (12) and (13).

We provide the respective definitions of the classes $S_q(\alpha, \beta)$ and $\mathcal{UCV}_q(\alpha, \beta)$.

Definition 1. The function $f \in \mathcal{A}$ is said to be β -uniformly q -starlike of order α , if it satisfies the following inequality:

$$\Re \left(\frac{zD_q(f(z))}{f(z)} - \alpha \right) > \beta \left| \frac{zD_q(f(z))}{f(z)} - 1 \right|, \quad (19)$$

where $0 < q < 1$, $\beta \geq 0$, $0 \leq \alpha < 1$, and $z \in \mathbb{U}$.

Definition 2. The function $f \in \mathcal{A}$ is said to be β -uniformly q -convex of order α , if it satisfies the following inequality:

$$\Re \left(\frac{D_q(zD_q f(z))}{D_q f(z)} - \alpha \right) > \beta \left| \frac{D_q(zD_q f(z))}{D_q f(z)} - 1 \right|, \quad (20)$$

where $0 < q < 1$, $\beta \geq 0$, $0 \leq \alpha < 1$, and $z \in \mathbb{U}$.

Further, we define the classes $\mathcal{T}\mathcal{S}_q(\alpha, \beta)$ and $\mathcal{UC}\mathcal{T}_q(\alpha, \beta)$ containing functions with negative coefficients and satisfying inequalities (19) and (20), respectively, as follows:

$$\begin{aligned} \mathcal{T}\mathcal{S}_q(\alpha, \beta) &= \mathcal{S}_q(\alpha, \beta) \cap \mathbb{T}, \\ \mathcal{UC}\mathcal{T}_q(\alpha, \beta) &= \mathcal{UC}\mathcal{V}_q(\alpha, \beta) \cap \mathbb{T}. \end{aligned} \quad (21)$$

Remark 3. We note that

- (i) $\lim_{q \rightarrow 1^-} \mathcal{S}_q(\alpha, \beta) = \mathcal{SD}(\alpha, \beta)$ and $\lim_{q \rightarrow 1^-} \mathcal{UC}\mathcal{V}_q(\alpha, \beta) = \mathcal{K}\mathcal{D}(\alpha, \beta)$
- (ii) $\mathcal{S}_q(\alpha, \beta) = \mathcal{S}_q^* \mathcal{D}(\alpha)$ and $\mathcal{UC}\mathcal{V}_q(\alpha, 0) = \mathcal{C}_q(\alpha)$ (see [8]).
- (iii) $\lim_{q \rightarrow 1^-} \mathcal{S}_q(\alpha, 0) = \mathcal{S}^*(\alpha)$ and $\lim_{q \rightarrow 1^-} \mathcal{UC}\mathcal{V}_q(\alpha, 0) = \mathcal{K}(\alpha)$

Now, the relation between the subclasses $\mathcal{S}_q^*(\mu)$ and $\mathcal{S}_q(\alpha, \beta)$ is given by the following result.

Theorem 4. Let $f \in \mathcal{S}_q(\alpha, \beta)$, then $f \in \mathcal{S}_q^*((\alpha + \beta)/(1 + \beta))$, where $\beta \geq 0$, $0 \leq \alpha < 1$, and $0 < q < 1$.

Proof. If $f \in \mathcal{S}_q(\alpha, \beta)$, then in view of Definition 1 and using the fact that $-\Re < (z) \leq |z|$, we get

$$\Re \left(\frac{zD_q(f(z))}{f(z)} - \alpha \right) > \beta \left| \frac{zD_q(f(z))}{f(z)} - 1 \right| \geq \beta \Re \left(\frac{zD_q(f(z))}{f(z)} - 1 \right), \quad (22)$$

which implies that

$$\Re \left(\frac{zD_q(f(z))}{f(z)} \right) - \alpha > \beta \Re \left(\frac{zD_q(f(z))}{f(z)} \right) + \beta, \quad (23)$$

then

$$\Re \left(\frac{zD_q(f(z))}{f(z)} \right) > \frac{\alpha + \beta}{1 + \beta}. \quad (24)$$

Since $\beta \geq 0$ and $0 \leq \alpha < 1$, then $0 \leq (\alpha + \beta)/(1 + \beta) < 1$. Hence, in view of equation (14), we obtain $f \in \mathcal{S}_q^*((\alpha + \beta)/(1 + \beta))$.

Also, the relation between the subclasses $\mathcal{C}_q(\alpha)$ and $\mathcal{UC}\mathcal{V}_q(\alpha, \beta)$ is given by the following result.

Theorem 5. Let $f \in \mathcal{UC}\mathcal{V}_q(\alpha, \beta)$, then $f \in \mathcal{C}_q((\alpha + \beta)/(1 + \beta))$, where $\beta \geq 0$, $0 \leq \alpha < 1$, and $0 < q < 1$.

Proof. If $f \in \mathcal{UC}\mathcal{V}_q(\alpha, \beta)$, then in view of Definition 2 and using the fact that $-\Re < (z) \leq |z|$, we get

$$\begin{aligned} \Re \left(\frac{D_q(zD_q f(z))}{D_q f(z)} - \alpha \right) &> \beta \left| \frac{D_q(zD_q f(z))}{D_q f(z)} - 1 \right| \\ &\geq -\beta \Re \left(\frac{D_q(zD_q f(z))}{D_q f(z)} - 1 \right), \end{aligned} \quad (25)$$

which implies that

$$\Re \left(\frac{D_q(zD_q f(z))}{D_q f(z)} \right) - \alpha > -\beta \Re \left(\frac{D_q(zD_q f(z))}{D_q f(z)} \right) + \beta, \quad (26)$$

then

$$\Re \left(\frac{D_q(zD_q f(z))}{D_q f(z)} \right) > \frac{\alpha + \beta}{1 + \beta}, \quad (27)$$

since $\beta \geq 0$ and $0 \leq \alpha < 1$, then $0 \leq (\alpha + \beta)/(1 + \beta) < 1$. Hence, in view of equation (15), we obtain $f \in \mathcal{C}_q((\alpha + \beta)/(1 + \beta))$.

Next, the coefficient bound of the class $\mathcal{S}_q(\alpha, \beta)$ is given by the following result.

Theorem 6. A function $f \in \mathcal{A}$ belongs to the class $\mathcal{S}_q(\alpha, \beta)$ if

$$\sum_{n=2}^{\infty} \left([n]_q(1 + \beta) - (\alpha + \beta) \right) |a_n| \leq 1 - \alpha, \quad (28)$$

where $0 < q < 1$, $\beta \geq 0$, $0 \leq \alpha < 1$, and $[n]_q$ denotes the q -number.

Proof. Now, using the fact that $-\Re(z) \leq |z|$, we have

$$\begin{aligned} \beta \left| \frac{zD_q(f(z))}{f(z)} - 1 \right| - \Re \left(\frac{zD_q(f(z))}{f(z)} - 1 \right) \\ \leq (1 + \beta) \left| \frac{zD_q(f(z))}{f(z)} - 1 \right|. \end{aligned} \quad (29)$$

Using equations (6) and (8) in the right hand side of inequality (29), we get

$$(1 + \beta) \left| \frac{zD_q(f(z))}{f(z)} - 1 \right| = (1 + \beta) \left| \frac{\sum_{n=2}^{\infty} ([n]_q - 1) a_n z^{n-1}}{1 + \sum_{n=2}^{\infty} a_n z^{n-1}} \right|. \tag{30}$$

Since $|z| < 1$, therefore, from the above inequality, we get

$$(1 + \beta) \left| \frac{zD_q(f(z))}{f(z)} - 1 \right| < \frac{(1 + \beta) \sum_{n=2}^{\infty} ([n]_q - 1) |a_n|}{1 - \sum_{n=2}^{\infty} |a_n|}. \tag{31}$$

Combining inequalities (29) and (31), we get

$$\begin{aligned} \beta \left| \frac{zD_q(f(z))}{f(z)} - 1 \right| - \Re \left(\frac{zD_q(f(z))}{f(z)} - 1 \right) \\ < \frac{(1 + \beta) \sum_{n=2}^{\infty} ([n]_q - 1) |a_n|}{1 - \sum_{n=2}^{\infty} |a_n|}. \end{aligned} \tag{32}$$

If $(1 + \beta) \sum_{n=2}^{\infty} ([n]_q - 1) |a_n| / (1 - \sum_{n=2}^{\infty} |a_n|) < 1 - \alpha$, which is equivalent to inequality (28), then from inequality (32) we get

$$\beta \left| \frac{zD_q(f(z))}{f(z)} - 1 \right| - \Re \left(\frac{zD_q(f(z))}{f(z)} - 1 \right) \leq 1 - \alpha, \tag{33}$$

which is equivalent to inequality (19). Thus, in view of Definition 1, the function $f \in \mathcal{S}_q(\alpha, \beta)$.

Also, we obtain the coefficient bound for $f \in \mathcal{T}\mathcal{S}_q(\alpha, \beta)$ in the following result.

Theorem 7. *The function $f \in \mathcal{T}$ belongs to the class $\mathcal{T}\mathcal{S}_q(\alpha, \beta)$, if and only if*

$$\sum_{n=2}^{\infty} ([n]_q (1 + \beta) - (\alpha + \beta)) a_n \leq 1 - \alpha, \tag{34}$$

where $0 < q < 1$, $\beta \geq 0$, $0 \leq \alpha < 1$, and $[n]_q$ denotes the q -number.

Proof. Since \mathcal{T} is a subclass of class \mathcal{A} , therefore in view of Theorem 6, the sufficient condition of our result holds. Now, we need to prove only the necessary condition. Let $f \in \mathcal{T}\mathcal{S}_q(\alpha, \beta)$ and taking z real, then from inequality (19), we have

$$\frac{zD_q(f(z))}{f(z)} - \alpha > \beta \left| \frac{zD_q(f(z))}{f(z)} - 1 \right|. \tag{35}$$

Now, using equations (7) and (8) in inequality (35), we get

$$\frac{1 - \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}}{1 - \sum_{n=2}^{\infty} a_n z^{n-1}} - \alpha > \frac{\sum_{n=2}^{\infty} \beta ([n]_q a_n z^{n-1})}{1 - \sum_{n=2}^{\infty} a_n z^{n-1}}, \tag{36}$$

then, letting $z \rightarrow 1$ along the real axis, inequality (36), gives the condition (34).

The coefficient bound of the class $UCV_q(\alpha, \beta)$ is given by the following result.

Theorem 8. *A function $f \in \mathcal{A}$ belongs to the class $UCV_q(\alpha, \beta)$ if*

$$\sum_{n=2}^{\infty} [n]_q ([n]_q (1 + \beta) - (\alpha + \beta)) |a_n| \leq 1 - \alpha, \tag{37}$$

where $0 < q < 1$, $\beta \geq 0$, $0 \leq \alpha < 1$, and $[n]_q$ denotes the q -number.

Proof. Now, using the fact that $-\Re < (z) \leq |z|$, we have

$$\begin{aligned} \beta \left| \frac{D_q(zD_q f(z))}{D_q f(z)} - 1 \right| - \Re \left(\frac{D_q(zD_q f(z))}{D_q f(z)} - 1 \right) \\ \leq (1 + \beta) \left| \frac{D_q(zD_q f(z))}{D_q f(z)} - 1 \right|. \end{aligned} \tag{38}$$

Using equations (8) and (9) in the right hand side of inequality (38), we get

$$(1 + \beta) \left| \frac{D_q(zD_q f(z))}{D_q f(z)} - 1 \right| = (1 + \beta) \left| \frac{1 + \sum_{n=2}^{\infty} [n]_q^2 a_n z^{n-1}}{1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}} - 1 \right|. \tag{39}$$

Since $|z| < 1$, therefore, from the above inequality, we get

$$(1 + \beta) \left| \frac{D_q(zD_q f(z))}{D_q f(z)} - 1 \right| < \frac{(1 + \beta) \sum_{n=2}^{\infty} ([n]_q^2 - [n]_q) |a_n|}{1 - \sum_{n=2}^{\infty} [n]_q |a_n|}. \tag{40}$$

Combining inequalities (38) and (40), we get

$$\begin{aligned} \beta \left| \frac{D_q(D_q f(z))}{D_q f(z)} - 1 \right| - \Re \left(\frac{D_q(D_q f(z))}{D_q f(z)} - 1 \right) \\ < \frac{(1 + \beta) \sum_{n=2}^{\infty} ([n]_q^2 - [n]_q) |a_n|}{1 - \sum_{n=2}^{\infty} [n]_q |a_n|}. \end{aligned} \tag{41}$$

If $((1 + \beta) \sum_{n=2}^{\infty} ([n]_q^2 - [n]_q) |a_n|) / (1 - \sum_{n=2}^{\infty} [n]_q |a_n|) < 1 - \alpha$, which is equivalent to inequality (37), then from inequality (41), we get

$$\beta \left| \frac{D_q(zD_q f(z))}{D_q f(z)} - 1 \right| - \Re \left(\frac{D_q(D_q f(z))}{D_q f(z)} - 1 \right) \leq 1 - \alpha, \quad (42)$$

which is equivalent to inequality (20). Thus, in view of Definition 2, the function $f \in \mathcal{UCV}_q(\alpha, \beta)$.

The coefficient bound for $f \in \mathcal{UCST}_q(\alpha, \beta)$ is given by the following result.

Theorem 9. *The function $f \in \mathcal{T}$ belongs to the class $\mathcal{UCST}_q(\alpha, \beta)$ if and only if*

$$\sum_{n=2}^{\infty} [n]_q \left([n]_q(1 + \beta) - (\alpha + \beta) \right) a_n \leq 1 - \alpha, \quad (43)$$

where $0 < q < 1$, $\beta \geq 0$, $0 \leq \alpha < 1$, and $[n]_q$ denotes the q -number.

Proof. Since \mathcal{T} is a subclass of class \mathcal{A} , therefore, in view of Theorem 8, the sufficient condition holds. Now, we need to prove only the necessary condition. Let f belong to the class $\mathcal{UCST}_q(\alpha, \beta)$ and taking z real, then from inequality (20), we have

$$\frac{D_q(zD_q f(z))}{D_q f(z)} - \alpha > \beta \left| \frac{D_q(zD_q f(z))}{D_q f(z)} - 1 \right|. \quad (44)$$

Now, using equations (8) and (9) in inequality (44), we get

$$\frac{1 - \sum_{n=2}^{\infty} [n]_q^2 a_n z^{n-1}}{1 - \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}} - \alpha > \frac{\sum_{n=2}^{\infty} \beta \left([n]_q^2 - [n]_q a_n z^{n-1} \right)}{1 - \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}}, \quad (45)$$

then letting $z \rightarrow 1$ along real axis, inequality (45) gives condition (43).

We note that, $q \rightarrow 1^-$ in Theorems 6 and 8, we get the coefficient bounds for the functions belonging to the classes $\mathcal{SD}(\alpha, \beta)$ and $\mathcal{KSD}(\alpha, \beta)$ in [20], respectively.

In the next section, we obtain the extreme points for the functions belonging to the classes $\mathcal{TS}_q(\alpha, \beta)$ and $\mathcal{UCST}_q(\alpha, \beta)$.

3. Extreme Points

The extreme points of $f \in \mathcal{TS}_q(\alpha, \beta)$ are given by the following result.

Theorem 10. *Let $\{f_n(z)\}_{n \in \mathbb{N}}$ be sequences of functions such that*

$$\begin{aligned} f_1(z) &= z, \\ f_n(z) &= z - \frac{1 - \alpha}{[n]_q(1 + \beta) - (\alpha + \beta)} z^n, \end{aligned} \quad (n \geq 2, 0 < q < 1, \beta \geq 0, 0 \leq \alpha < 1), \quad (46)$$

where $[n]_q$ denotes the q -number. Then f belongs to $\mathcal{TS}_q(\alpha, \beta)$ if and only if f can be expressed as the form

$$f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z), \quad (47)$$

where $\lambda_n \geq 0 (n \geq 1)$ and $\sum_{n=1}^{\infty} \lambda_n = 1$.

Proof. Let $f \in \mathcal{TS}_q(\alpha, \beta)$, then in view of Theorem 7, inequality (34) holds. Since $a_n \geq 0 (n \geq 1)$ and $0 \leq \alpha < 1$, therefore from inequality (34), we have

$$\left([n]_q(1 + \beta) - (\alpha + \beta) \right) a_n \leq 1 - \alpha, \quad n \geq 2. \quad (48)$$

Thus, if we take

$$\lambda_n \frac{[n]_q(1 + \beta) - (\alpha + \beta)}{1 - \alpha} a_n, \quad n \geq 2, \quad (49)$$

since $\lambda_1 \geq 0$, then, $\lambda_n \geq 0 (n \geq 1)$.

Substituting a_n from equation (49) with a_n from equation (7), we get:

$$f(z) = z - \sum_{n=2}^{\infty} \frac{1 - \alpha}{[n]_q(1 + \beta) - (\alpha + \beta)} \lambda_n z^n. \quad (50)$$

Since $\sum_{n=1}^{\infty} \lambda_n = 1$, therefore, we have

$$\begin{aligned} f(z) &= \lambda_1 z + \sum_{n=2}^{\infty} \lambda_n z - \sum_{n=2}^{\infty} \frac{1 - \alpha}{[n]_q(1 + \beta) - (\alpha + \beta)} \lambda_n z^n \\ &= \lambda_1 z + \sum_{n=2}^{\infty} \lambda_n \left(z - \frac{1 - \alpha}{[n]_q(1 + \beta) - (\alpha + \beta)} z^n \right), \end{aligned} \quad (51)$$

since $f_1(z) = z$ and $f_n(z)$ is given by equation (46). Therefore, from equation (51), we get the assertion (47). Conversely, let f be expressible in the form (47), which on using equation (46), gives

$$f(z) = z - \sum_{n=2}^{\infty} \frac{1 - \alpha}{[n]_q(1 + \beta) - (\alpha + \beta)} \lambda_n z^n, \quad (52)$$

which can be expressed as follows:

$$f(z) = z - \sum_{n=2}^{\infty} \eta_n z^n, \quad (53)$$

where

$$\eta_n = \frac{1 - \alpha}{[n]_q(1 + \beta) - (\alpha + \beta)} \lambda_n, \quad n \geq 2. \quad (54)$$

Now, to prove that the function f , given by equation (53), belongs to the class $\mathcal{T}\mathcal{S}_q(\alpha, \beta)$, we need to show that the coefficients η_n ($n \geq 2$) satisfy the inequality (34).

Since $\lambda_1 \geq 0$ and $\sum_{n=1}^{\infty} \lambda_n = 1$, therefore from equation (54), we have

$$\begin{aligned} & \sum_{n=2}^{\infty} \frac{[n]_q(1 + \beta) - (\alpha + \beta)}{1 - \alpha} \eta_n \\ &= \sum_{n=2}^{\infty} \frac{[n]_q(1 + \beta) - (\alpha + \beta)}{1 - \alpha} \frac{1 - \alpha}{[n]_q(1 + \beta) - (\alpha + \beta)} \lambda_n \\ &= \sum_{n=2}^{\infty} \lambda_n = 1 - \lambda_1 \leq 1. \end{aligned} \quad (55)$$

Thus, we get

$$\sum_{n=2}^{\infty} \left([n]_q(1 + \beta) - (\alpha + \beta) \eta_n \leq 1 - \alpha \right). \quad (56)$$

Therefore, in view of Theorem 7 and the above inequality, we proved that the function f , given by equation (53), belongs to the class $\mathcal{T}\mathcal{S}_q(\alpha, \beta)$.

Also, the extreme points of $f \in \mathcal{UC}\mathcal{T}_q(\alpha, \beta)$ are given by the following result.

Theorem 11. Let $\{f_n(z)\}_{n \in \mathbb{N}}$ be a sequence of functions such that

$$\begin{aligned} f_1(z) &= z, \\ f_n &= z - \frac{1 - \alpha}{[n]_q \left([n]_q(1 + \beta) - (\alpha + \beta) \right)} z^n, \end{aligned} \quad (57)$$

where $n \geq 2$, $0 < q < 1$, $\beta \geq 0$, and $0 \leq \alpha < 1$. Then, f belongs to $\mathcal{UC}\mathcal{T}_q(\alpha, \beta)$ if and only if f can be expressed in the form given by equation (47) in terms of functions f_n ($n \geq 2$), given by equation (57), and $\lambda_n \geq 0$ ($n \geq 1$), $\sum_{n=1}^{\infty} \lambda_n = 1$.

Proof. Let $f \in \mathcal{UC}\mathcal{T}_q(\alpha, \beta)$, then from inequality (43), we have

$$\left([n]_q \left([n]_q(1 + \beta) - (\alpha + \beta) \right) \right) a_n \leq 1 - \alpha \quad (n \geq 2). \quad (58)$$

If we set

$$\lambda_n = \frac{[n]_q \left([n]_q(1 + \beta) - (\alpha + \beta) \right)}{1 - \alpha} a_n \quad (n \geq 2), \quad (59)$$

since $\lambda_1 = 1$, then $\lambda_n \geq 0$ ($n \geq 1$). Then, substituting a_n from equation (59) with a_n equation (7), we get

$$f(z) = z - \sum_{n=2}^{\infty} \frac{1 - \alpha}{[n]_q \left([n]_q(1 + \beta) - (\alpha + \beta) \right)} \lambda_n z^n. \quad (60)$$

Since $\sum_{n=1}^{\infty} \lambda_n = 1$, therefore, we have

$$f(z) = \lambda_1 z + \sum_{n=2}^{\infty} \left(\frac{1 - \alpha}{[n]_q \left([n]_q(1 + \beta) - (\alpha + \beta) \right)} z^n \right), \quad (61)$$

since $f_1(z) = z$ and $f_n(z)$ is given by equation (57). Therefore, from equation (61), we get assertion (47).

Conversely, let f be expressible in the form (47), which on using equation (60), gives

$$f(z) = z - \sum_{n=2}^{\infty} \frac{1 - \alpha}{[n]_q \left([n]_q(1 + \beta) - (\alpha + \beta) \right)} \lambda_n z^n, \quad (62)$$

which can be expressed as

$$f(z) = z - \sum_{n=2}^{\infty} \eta_n z^n, \quad (63)$$

where

$$\eta_n = \frac{1 - \alpha}{[n]_q \left([n]_q(1 + \beta) - (\alpha + \beta) \right)} \lambda_n, \quad n \geq 2. \quad (64)$$

Now, to prove that function f is given by equation (63) and belongs to the class $\mathcal{UC}\mathcal{T}_q(\alpha, \beta)$, we need to show that the coefficient η_n ($n \geq 2$) satisfies inequality (43). Since $\lambda_1 \geq 0$ and $\sum_{n=1}^{\infty} \lambda_n = 1$, therefore from equation (64), we have

$$\sum_{n=2}^{\infty} \frac{[n]_q \left([n]_q(1 + \beta) - (\alpha + \beta) \right)}{1 - \alpha} \eta_n = \sum_{n=2}^{\infty} \lambda_n = 1 - \lambda_1 \leq 1. \quad (65)$$

Thus, we get

$$\sum_{n=2}^{\infty} [n]_q \left([n]_q(1 + \beta) - (\alpha + \beta) \right) \eta_n < 1 - \alpha. \quad (66)$$

Therefore, in view of Theorem 9 and the above inequality, we proved that function f , given by equation (63), belongs to the class $\mathcal{UC}\mathcal{T}_q(\alpha, \beta)$.

4. Partial Sums

The sequence of partial sums of the function $f(z) \in \mathcal{A}$, is defined as [22].

$$f_k(z) = z + \sum_{n=2}^k a_n z^n \quad (k \in \mathbb{N}; z \in \mathbb{U}). \quad (67)$$

Now, we find the bounds of the real part of the ratio of the complex valued function $f \in \mathcal{A}$ to its partial sums f_k ($k \in \mathbb{N}$), for the function to be in the class $\mathcal{S}_q(\alpha, \beta)$ in the following result.

Theorem 12. *Let $f(z) \in \mathcal{A}$ in the form (6) and suppose that*

$$\sum_{n=2}^{\infty} c_n |a_n| \leq 1, \quad (68)$$

where

$$c_n = \frac{[n]_q(1 + \beta) + (\alpha + \beta)}{1 - \alpha} \quad (n \geq 2; 0 < q < 1, \beta \geq 0, 0 \leq \alpha < 1), \quad (69)$$

then $f(z) \in \mathcal{S}_q(\alpha, \beta)$. Further, the following inequalities hold:

$$\operatorname{Re} \left(\frac{f(z)}{f_k(z)} \right) \geq 1 - \frac{1}{c_{k+1}}, \quad (70)$$

$$\operatorname{Re} \left(\frac{f_k(z)}{f(z)} \right) \geq \frac{c_{k+1}}{1 + c_{k+1}}, \quad (71)$$

where

$$c_n \geq \begin{cases} 1, & \text{if } n = 2, 3, \dots, k, \\ c_{k+1}, & \text{if } n = k + 1, k + 2, \dots, \end{cases} \quad (72)$$

Proof. Since $\{[n]_q\}_{n \geq 2}$ is increasing and $\beta \geq 0, \alpha < 1$, therefore, in view of equation (69), $\{c_n\}_{n \geq 2}$ is an increasing sequence. Then, $c_{n+1} \geq c_n, \forall n$ and

$$c_n \geq c_2 \frac{[2]_q(1 + \beta) - (\alpha + \beta)}{1 + \alpha} \geq \frac{[1]_q(1 + \beta) - (\alpha + \beta)}{1 - \alpha}. \quad (73)$$

Since $[1]_q = 1$, therefore, we have

$$c_n \geq 1, \quad \forall n. \quad (74)$$

Thus, for the particular value k of n , condition (72) holds. In view of the first inequality of condition (72), we have

$$\sum_{n=2}^k |a_n| + c_{k+1} \sum_{n=k+1}^{\infty} |a_n| \leq \sum_{n=2}^{\infty} c_n |a_n|, \quad (75)$$

which in view of inequality (68), gives

$$\sum_{n=2}^k |a_n| + c_{k+1} \sum_{n=k+1}^{\infty} |a_n| \leq 1, \quad (76)$$

or, equivalently

$$c_{k+1} \sum_{n=k+1}^{\infty} |a_n| \leq 1 - \sum_{n=2}^k |a_n|. \quad (77)$$

Now, for some fixed positive integer k , we define

$$h_1(z) := 1 + \frac{c_{k+1}(f(z) - f_k(z))}{f_k(z)}. \quad (78)$$

Now, using equations (6) and (67), equation (78) gives

$$h_1(z) = 1 + \frac{c_{k+1} \left(\sum_{n=k+1}^{\infty} a_n z^{n-1} \right)}{1 + \sum_{n=2}^k a_n z^{n-1}}. \quad (79)$$

From equation (79), we have

$$\begin{aligned} \left| \frac{h_1(z) - 1}{h_1(z) + 1} \right| &= \left| \frac{c_{k+1} \sum_{n=k+1}^{\infty} a_n z^{n-1}}{2 + 2 \sum_{n=2}^k a_n z^{n-1} + c_{k+1} \sum_{n=k+1}^{\infty} a_n z^{n-1}} \right| \\ &\leq \frac{c_{k+1} \sum_{n=k+1}^{\infty} |a_n|}{2 - 2 \sum_{n=2}^k |a_n| - c_{k+1} \sum_{n=k+1}^{\infty} |a_n|}. \end{aligned} \quad (80)$$

In view of inequality (77), the above inequality gives $|(h_1(z) - 1)/(h_1(z) + 1)| \leq 1$, which implies

$$\operatorname{Re}(h_1(z)) \geq 0. \quad (81)$$

Since each $c_n \in \mathbb{R}$, therefore, using equation (79) in inequality (81), we get assertion (70).

Again, since $\{c_n\}_{n \geq 2}$ is an increasing function and $c_n \geq 1, \forall n \geq 2$, therefore, we have

$$\sum_{n=2}^{\infty} |a_n| \leq \sum_{n=2}^{\infty} c_n |a_n|, \quad (82)$$

which in view of inequality (68), gives

$$\sum_{n=2}^{\infty} |a_n| \leq 1. \quad (83)$$

Now, we define the function $h_2(z)$ as follows:

$$h_2(z) = (1 + c_{k+1}) \left(\frac{f_k(z)}{f(z)} \right) - c_{k+1}. \quad (84)$$

Using equations (6) and (67) in equation (84), we get

$$h_2(z) = 1 - \frac{(c_{k+1} + 1) \sum_{n=k+1}^{\infty} a_n z^{n-1}}{1 + \sum_{n=2}^{\infty} a_n z^{n-1}}. \quad (85)$$

From equation (85), we have

$$\begin{aligned} \left| \frac{h_2(z) - 1}{h_2(z) + 1} \right| &= \left| \frac{-(c_{k+1} + 1) \sum_{n=k+1}^{\infty} a_n z^{n-1}}{2 + 2 \sum_{n=2}^{\infty} a_n z^{n-1} - (c_{k+1} + 1) \sum_{n=k+1}^{\infty} a_n z^{n-1}} \right| \\ &= \left| \frac{-(c_{k+1} + 1) \sum_{n=k+1}^{\infty} a_n z^{n-1}}{2 + 2 \sum_{n=2}^k a_n z^{n-1} + (1 - c_{k+1}) \sum_{n=k+1}^{\infty} a_n z^{n-1}} \right| \\ &\leq \frac{(c_{k+1} + 1) \sum_{n=k+1}^{\infty} |a_n|}{2 - 2 \sum_{n=2}^k |a_n| - (1 - c_{k+1}) \sum_{n=k+1}^{\infty} |a_n|} \\ &\leq \frac{(c_{k+1} + 1) \sum_{n=k+1}^{\infty} |a_n|}{2 - 2 \sum_{n=2}^{\infty} |a_n| + (c_{k+1} + 1) \sum_{n=k+1}^{\infty} |a_n|}, \end{aligned} \tag{86}$$

using inequality (83) in inequality (86), we get $|(h_2(z) - 1)/(h_2(z) + 1)| \leq 1$, which implies

$$\operatorname{Re}(h_2(z)) \geq 0. \tag{87}$$

Therefore, using equation (84) in inequality (87), we get assertion (71).

Now, we find the bounds of the real part of the ratio of the complex valued function $f \in \mathcal{A}$ to its partial sums $f_k (k \in \mathbb{N})$, for the function to be in the class $\mathcal{UCV}_q(\alpha, \beta)$ in the following result.

Theorem 13. Let $f(z) \in \mathcal{A}$ be in the form given by equation (6) and

$$\sum_{n=2}^{\infty} s_n |a_n| \leq 1, \tag{88}$$

where

$$s_n = \frac{[n]_q \left([n]_q (1 + \beta) - (\alpha + \beta) \right)}{1 - \alpha} \quad (n \geq 2; \beta \geq 0, 0 \leq \alpha < 1, 0 < q < 1). \tag{89}$$

Then, $f(z) \in \mathcal{UCV}_q(\alpha, \beta)$. Further, the following inequalities hold:

$$\operatorname{Re} \left(\frac{f(z)}{f_k(z)} \right) > 1 - \frac{1}{s_{k+1}}, \tag{90}$$

$$\operatorname{Re} \left(\frac{f_k(z)}{f(z)} \right) > \frac{s_{k+1}}{1 + s_{k+1}}, \tag{91}$$

where

$$s_n \geq \begin{cases} 1, & \text{if } n = 2, 3, \dots, k, \\ s_{k+1}, & \text{if } n = k + 1, k + 2, \dots. \end{cases} \tag{92}$$

Proof. Using Theorem 6 and following the same steps involved in the proof of Theorem 12, we get assertion (90) and (91).

In the next section, we discuss the integral means inequality for the functions belonging to the classes $\mathcal{TS}_q(\alpha, \beta)$ and $\mathcal{UCV}_q(\alpha, \beta)$.

5. Integral Means Inequality

Silverman [23] has been using the subordination principle to show that the integral $\int_0^{2\pi} |f(re^{i\theta})|^\sigma d\theta$ ($\sigma > 0, 0 < r < 1$) attains its maximum value in class \mathcal{T} , when $f_2(z) = z - (z^2/2)$. Then, he applied that principle to solve the integral means inequality $\int_0^{2\pi} |f(re^{i\theta})|^\sigma d\theta \leq \int_0^{2\pi} |f_2(re^{i\theta})|^\sigma d\theta$. Also, he found the integral means inequality for the classes $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ with negative coefficients.

First, we need to mention the following lemma [24].

Lemma 14. If f and g are two analytic functions in \mathbb{U} in the form \mathcal{T} and $f \prec g$, then

$$\int_0^{2\pi} |f(re^{i\theta})|^\sigma d\theta \leq \int_0^{2\pi} |g(re^{i\theta})|^\sigma d\theta, \tag{93}$$

where $\sigma > 0, 0 < r < 1$, and $z = re^{i\theta}$.

Now, we establish the integral means inequality for the functions belonging to the class $\mathcal{TS}_q(\alpha, \beta)$.

Theorem 15. Let f be of the form given by equation (7) that belongs to the class $\mathcal{TS}_q(\alpha, \beta)$ and $f_2(z)$ be defined as follows:

$$f_2(z) = 1 \frac{1 - \alpha}{[2]_q(1 + \beta) - (\alpha + \beta)} z, \tag{94}$$

then, for $z = re^{i\theta} (0 < r < 1)$, we have

$$\int_0^{2\pi} |f(z)|^\sigma d\theta \leq \int_0^{2\pi} |f_2(z)|^\sigma d\theta \quad (\sigma > 0). \tag{95}$$

Proof. We define the function $w_1(z)$ as follows:

$$w_1(z) = \sum_{n=2}^{\infty} \frac{[2]_q(1 + \beta) - (\alpha + \beta)}{1 - \alpha} a_n z^{n-1}. \tag{96}$$

From the above equation, we have

$$w_1(0) = 0. \tag{97}$$

Again, from equation (96), we have

$$\begin{aligned} |w_1(z)| &= \sum_{n=2}^{\infty} \frac{[2]_q(1 + \beta) - (\alpha + \beta)}{1 - \alpha} a_n z^{n-1} \\ &\leq \sum_{n=2}^{\infty} \frac{[2]_q(1 + \beta) - (\alpha + \beta)}{1 - \alpha} a_n |z|^{n-1}, \end{aligned} \tag{98}$$

since $z = re^{i\theta} (0 < r < 1)$ implies $|z| = |r| < 1$, and using inequality (37), therefore, from the above inequality, we have

$$|w_1| \leq \sum_{n=2}^{\infty} \frac{[n]_q(1+\beta) - (\alpha + \beta)}{1-\alpha} a_n \leq 1. \tag{99}$$

From equation (96), we have

$$1 - \sum_{n=2}^{\infty} a_n z^{n-1} = 1 - \frac{1-\alpha}{[2]_q(1+\beta) - (\alpha + \beta)} w_1(z). \tag{100}$$

Since w_1 is analytic in \mathbb{U} , therefore in view of equations (18), (96), (97), and (100); inequality (99); and the subordination principle, we have

$$1 - \sum_{n=2}^{\infty} a_n z^{n-1} < 1 - \frac{1-\alpha}{[2]_q(1+\beta) - (\alpha + \beta)} z. \tag{101}$$

Since, the function on the both sides of the above relation are analytic in \mathbb{U} , therefore, in view of Lemma 14 and equation (94), we get assertion (95).

Next, we establish the integral means inequality for the functions belonging to the class $\mathcal{UC}\mathcal{T}_q(\alpha, \beta)$ with the positive coefficients.

Theorem 16. *Let f belong to the class $\mathcal{UC}\mathcal{T}_q(\alpha, \beta)$ and $f_3(z)$ is defined by*

$$f_3(z) = 1 - \frac{1-\alpha}{[2]_q([2]_q(1+\beta) - (\alpha + \beta))} z, \tag{102}$$

then, for $z = re^{i\theta}$ ($0 < r < 1$), we have

$$\int_0^{2\pi} |f(z)|^\sigma d\theta \leq \int_0^{2\pi} |f_3(z)|^\sigma d\theta \quad (\sigma > 0). \tag{103}$$

Proof. We define the function $w_2(z)$ as follows:

$$w_2(z) = \sum_{n=2}^{\infty} \frac{[2]_q([2]_q(1+\beta) - (\alpha + \beta))}{1-\alpha} a_n z^{n-1}. \tag{104}$$

From the above equation, we have

$$w_2(0) = 0. \tag{105}$$

Again, from equation (104), we have

$$\begin{aligned} |w_2(z)| &= \left| \sum_{n=2}^{\infty} \frac{[2]_q([2]_q(1+\beta) - (\alpha + \beta))}{1-\alpha} a_n z^{n-1} \right| \\ &\leq \sum_{n=2}^{\infty} \frac{[2]_q([2]_q(1+\beta) - (\alpha + \beta))}{1-\alpha} a_n |z|^{n-1}, \end{aligned} \tag{106}$$

since $z = re^{i\theta}$, then $|z| = |r| < 1$ and using inequality (103), therefore, from the above inequality, we have

$$|w_2(z)| \leq \sum_{n=2}^{\infty} \frac{[n]_q([n]_q(1+\beta) - (\alpha + \beta))}{1-\alpha} a_n < 1. \tag{107}$$

From equation (104), we have

$$1 - \sum_{n=2}^{\infty} a_n z^{n-1} = 1 - \frac{1-\alpha}{[2]_q([2]_q(1+\beta) - (\alpha + \beta))} w_2(z). \tag{108}$$

Since w_2 is analytic in \mathbb{U} , therefore, in view of equations (18), (104), (105), (108); inequality (107); and the subordination principle, we have

$$1 - \sum_{n=2}^{\infty} a_n z^{n-1} < - \frac{1-\alpha}{[2]_q([2]_q(1+\beta) - (\alpha + \beta))} z. \tag{109}$$

Since, the function on the both sides of the above relation are analytic in \mathbb{U} , therefore, in view of Lemma 14 and equation (102), we get assertion (103).

Data Availability

Data sharing is not applicable to this article as no data sets were generated or analysed during the current study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors equally contributed to this manuscript and approved the final version.

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