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Research Article

Internal Perturbation Projection Algorithm for the Extended Split Equality Problem and the Extended Split Equality Fixed Point Problem

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In this article, we study the extended split equality problem and extended split equality fixed point problem, which are extensions of the convex feasibility problem. For solving the extended split equality problem, we present two self-adaptive stepsize algorithms with internal perturbation projection and obtain the weak and the strong convergence of the algorithms, respectively. Furthermore, based on the operators being quasinonexpansive, we offer an iterative algorithm to solve the extended split equality fixed point problem. We introduce a way of selecting the stepsize which does not need any prior information about operator norms in the three algorithms. We apply our iterative algorithms to some convex and nonlinear problems. Finally, several numerical results are shown to confirm the feasibility and efficiency of the proposed algorithms.

1. Introduction

Let H_1, H_2, and H_3 be three real Hilbert spaces and $C \subset H_1$ and $Q \subset H_2$ be two nonempty, closed, and convex sets. The split feasibility problem (SFP) is formulated

$$\text{to find } x \in C \text{ such that } Ax \in Q, \tag{1}$$

where $A : H_1 \longrightarrow H_2$ is a bounded linear operator. The SFP was first introduced by Censor and Elfving [1], which was used in modeling various inverse problems arising from phase retrievals and medical image reconstruction and further studied by many researchers. See, for instance, [2–10].

Moudafi [11, 12] introduced the following split equality feasibility problem (SEFP), which is

$$\text{to find } x \in C, y \in Q \text{ such that } Ax = By, \tag{2}$$

where $A : H_1 \longrightarrow H_3$ and $B : H_2 \longrightarrow H_3$ are two bounded linear operators. Obviously, if $B = I$ and $H_3 = H_2$, then (2) reduces to (1). The split equality feasibility problem (2) allows asymmetric and partial relations between the variables x and y. In order to solve SEFP, many researchers proposed their suggestions, such as [13–17] and references therein. Moudafi [11] introduced the following iterative method:

$$\begin{cases} x_{k+1} = P_C(x_k - \lambda_k A^*(Ax_k - By_k)), \\ y_{k+1} = P_Q(y_k + \lambda_k B^*(Ax_{k+1} - By_k)). \end{cases} \tag{3}$$

Under some suitable conditions, he proved that the sequence $\{(x_n, y_n)\}$ weakly converges to the solution of (2) in Hilbert spaces. In addition, Yu and Wang [18] proposed the following iterative algorithm:

$$\begin{cases} x_{k+1} = x_k - \lambda[(x_k - P_C(x_k)) + A^*(Ax_k - By_k)], \\ y_{k+1} = y_k - \lambda[(y_k - P_C(y_k)) - B^*(Ax_{k+1} - By_k)], \end{cases} \tag{4}$$

where $0 < \lambda < (1 + c)^{-1}$ with $c = \max\{\|A\|^2, \|B\|^2\}$. They studied the weak convergence of scheme (4).

Recently, Che et al. [19] proposed the following extended split equality problem (ESEP) which is an extension of the convex feasibility problem. Let H be a real Hilbert space. For $i = 1, 2, \cdots, n$, assume C_i are nonempty closed convex subsets of real Hilbert spaces H_i, respectively. The extended split equality problem is

$$\text{to find } x_1 \in C_1, x_2 \in C_2, \cdots, x_n \in C_n \text{ such that } A_1x_1 = A_2x_2 = \cdots = A_nx_n, \tag{5}$$

where $A_i : H_i \longrightarrow H$ are linear operators. They presented the following simultaneous iterative algorithm:

$$\begin{cases} \omega_k = \dfrac{\sum_{i=1}^n A_i x_i^k}{n}, \\ x_1^{k+1} = P_{C_1}\left(x_1^k - \lambda_k A_1^*\left(A_1x_1^k - \omega_k\right)\right), \\ x_2^{k+1} = P_{C_2}\left(x_2^k - \lambda_k A_2^*\left(A_2x_2^k - \omega_k\right)\right), \\ \cdots \\ x_n^{k+1} = P_{C_n}\left(x_n^k - \lambda_k A_n^*\left(A_nx_n^k - \omega_k\right)\right). \end{cases} \tag{6}$$

Under some suitable conditions, they obtained the weak convergence of (6).

In order to avoid using the projection, Moudafi [11] introduced and studied the following problem. Let $S : H_1 \longrightarrow H_1$ and $T : H_2 \longrightarrow H_2$ be nonlinear operators such that $\text{Fix}(S) \neq \emptyset$ and $\text{Fix}(T) \neq \emptyset$, where $\text{Fix}(S)$ and $\text{Fix}(T)$ denote the sets of fixed points of S and T, respectively. If $C = \text{Fix}(S)$ and $Q = \text{Fix}(T)$, then SEFP (2) reduces

$$\text{to find } x \in \text{Fix}(S), y \in \text{Fix}(T) \text{ such that } Ax = By, \tag{7}$$

which is called the split equality fixed point problem (SEFPP). Many scholars have studied this issue, such as [20–22].

To solve problem (7), Che and Li [23] proposed the following iterative algorithm:

$$\begin{cases} u_k = x_k - \lambda_k A^*(Ax_k - By_k), \\ x_{k+1} = \alpha_k x_k + (1-\alpha_k)Su_k, \\ v_k = y_k + \lambda_k B^*(Ax_k - By_k), \\ y_{k+1} = \alpha_k y_k + (1-\alpha_k)Tv_k. \end{cases} \tag{8}$$

They established the weak convergence of scheme (8) under the conditions that the operators S and T are quasinonexpansive mappings.

Similarly, Che et al. [19] proposed the following extended split equality fixed point problem (ESEFPP), which is

$$\text{to find } x_1 \in \text{Fix}(G_1), x_2 \in \text{Fix}(G_2), \cdots, x_n \in \text{Fix}(G_n) \text{ such that } A_1x_1 = A_2x_2 = \cdots = A_nx_n, \tag{9}$$

and presented the following simultaneous iterative algorithm:

$$\begin{cases} \omega_k = \dfrac{\sum_{i=1}^n A_i x_i^k}{n}, \\ u_1^k = x_1^k - \lambda_k A_1^*\left(A_1x_1^k - \omega_k\right), \\ x_1^{k+1} = \alpha_k u_1^k + (1-\alpha_k)G_1\left(u_1^k\right), \\ u_2^k = x_2^k - \lambda_k A_2^*\left(A_2x_2^k - \omega_k\right), \\ x_2^{k+1} = \alpha_k u_2^k + (1-\alpha_k)G_2\left(u_2^k\right), \\ \cdots \\ u_n^k = x_n^k - \lambda_k A_n^*\left(A_nx_n^k - \omega_k\right), \\ x_n^{k+1} = \alpha_k u_n^k + (1-\alpha_k)G_n\left(u_n^k\right), \end{cases} \tag{10}$$

where G_i are the G-mapping generated by $\{T_i\}_{i=1}^N$ which is a finite family of k_i-strictly pseudononspreading. They obtained the weak convergence of (10).

Motivated by the works mentioned above, we continue to study the ESEP (5) and ESEFPP (9) with internal perturbation projection and do not need any prior information about the operator norms. The paper is organized as follows. In Section 2, we introduce some preliminaries to be employed in the subsequent analysis. In Section 3, we present two simultaneous iterative algorithms to solve ESEP (5) and establish the weak and the strong convergence of the proposed algorithms, respectively. We propose a simultaneous iterative algorithm to solve ESEFPP (9) and obtain the weak convergence of the proposed algorithm in Section 4. In Section 5, we apply our iterative algorithms to some convex and nonlinear problems. In the concluding section, several numerical results are shown to confirm the effectiveness of our algorithms.

2. Preliminaries

In this paper, we use $\longrightarrow$ and $\rightharpoonup$ to denote the strong convergence and the weak convergence, respectively. We use $\omega_\omega(x_i^k) = \{x : \exists x_i^{k_j} \rightharpoonup x\}$ to stand for the weak ω-limit set of $\{x_i^k\}$. For any $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \forall y \in C. \tag{11}$$

It is well known that P_C is nonexpansive and firmly nonexpansive. And P_C has the following well-known properties.

Lemma 1. *Let $C \subset H$ be a nonempty, closed, and convex set. The following conclusions hold:*

(1) $\langle x - P_C(x), z - P_C(x)\rangle \leq 0, \forall x \in H, \forall z \in C$

(2) $\|P_C(x) - P_C(y)\|^2 \leq \langle P_C(x) - P_C(y), x - y\rangle, \forall x, y \in H$

(3) $\langle P_C(x) - x, y - x\rangle \geq \|P_C(x) - x\|^2, \forall x \in H, \forall y \in C$

Definition 2. A mapping $T : C \longrightarrow C$ is said to be

(1) nonexpansive if $\|Tx - Ty\| \le \|x - y\|, \forall x, y \in C$

(2) quasinonexpansive if $\|Tx - x^*\| \le \|x - x^*\|, \forall x \in C, x^* \in \mathrm{Fix}(T)$

(3) firmly nonexpansive if $\|Tx - Ty\|^2 \le \|x - y\|^2 - \|(I - T)x - (I - T)y\|^2, \forall x, y \in C$

(4) firmly quasinonexpansive if $\|Tx - x^*\|^2 \le \|x - x^*\|^2 - \|(I - T)x\|^2, \forall x \in C, x^* \in \mathrm{Fix}(T)$

(5) k-strictly pseudononspreading if $\|Tx - Ty\|^2 \le \|x - y\|^2 + k\|(x - Tx) - (y - Ty)\|^2 + 2\langle x - Tx, y - Ty\rangle, \exists k \in [0, 1), \forall x, y \in C$

Lemma 3 [24]. *In the real Hilbert space H, for $\forall x, y \in H$, the following relations hold:*

(1) $\|x + y\|^2 \le \|x\|^2 + 2\langle y, x + y\rangle$

(2) $\|x + y\|^2 = \|x\|^2 + \|y\|^2 + 2\langle x, y\rangle$

(3) $\|\alpha x + (1 - \alpha)y\|^2 = \alpha\|x\|^2 + (1 - \alpha)\|y\|^2 - \alpha(1 - \alpha)\|x - y\|^2, \alpha \in [0, 1]$

(4) $\|\sum_{i=1}^{m}\alpha_i x_i\|^2 = \sum_{i=1}^{m}\alpha_i\|x_i\|^2 - \sum_{i\ne j}\alpha_i\alpha_j\|x_i - x_j\|^2,$ *for* $\sum_{i=1}^{m}\alpha_i = 1, \alpha_i \in [0, 1], i = \{1, 2, \cdots, m\}$

Definition 4 [25]. A mapping $T : C \longrightarrow C$ is said to be demi-closed at 0 if, for any sequence $\{x_n\} \subset C$ which converges weakly to x and with $\|x_n - T(x_n)\| \longrightarrow 0, T(x) = x$.

Lemma 5 [26]. *Let $\{\Gamma_k\}$ be a sequence of real numbers that does not decrease at infinity in the sense that there exists a subsequence $\{\Gamma_{k_j}\}$ of $\{\Gamma_k\}$ which satisfies $\Gamma_{k_j} \le \Gamma_{k_j+1}$ for all $j \in N$. Define the integer sequence $\{\tau(k)\}$, for $k > k_0$ (such that $\Gamma_{k_0} \le \Gamma_{k_0+1}$) is as follows:*

$$\tau(k) = \max\{l \in N \mid k_0 \le l < k, \Gamma_l \le \Gamma_{l+1}\}. \tag{12}$$

Then, there hold the following properties:

(1) $\tau(k_0) \le \tau(k_0 + 1) \le \cdots$ *and* $\tau(k) \longrightarrow \infty$

(2) $\Gamma_{\tau(k)} \le \Gamma_{\tau(k)+1}$ *and* $\Gamma_k \le \Gamma_{\tau(k)+1}, \forall k \ge k_0$

Lemma 6 [27]. *Assume that $\{a_k\}$ is a sequence of nonnegative real numbers such that*

$$a_{k+1} \le (1 - \alpha_k)a_k + \alpha_k\delta_k, \forall k \ge 0, \tag{13}$$

where $\{\alpha_k\}$ is a sequence in $(0, 1)$ and $\{\delta_k\}$ is a sequence in $\mathfrak{R}$ such that

(1) $\sum_{k=1}^{\infty}\alpha_k = \infty$ *and* $\lim_{k\to\infty}\alpha_k = 0$

(2) $\limsup_{k\to\infty}\delta_k \le 0$ *or* $\sum_{k=1}^{\infty}\alpha_k \mid \delta_k \mid < \infty$

Then, $\lim_{k\to\infty} a_k = 0$.

Definition 7 [19]. Let C be a nonempty closed and convex subset of real Banach space. Let $\{T_i\}_{i=1}^N$ be a finite family of mappings of C into itself. For $i = 1, 2, \cdots, N$, let $\pi_i = (\alpha_i, \beta_i, \gamma_i, \delta_i)$, where $\alpha_i, \beta_i, \gamma_i, \delta_i \in [0, 1]$ and $\alpha_i + \beta_i + \gamma_i + \delta_i = 1$. We define a mapping $G : C \longrightarrow C$ as follows:

$$\begin{cases} U_0 = I, \\ U_1 = \alpha_1 T_1^2 U_0 + \beta_1 T_1 U_0 + \gamma_1 U_0 + \delta_1 I, \\ U_2 = \alpha_2 T_2^2 U_1 + \beta_2 T_2 U_1 + \gamma_2 U_1 + \delta_2 I, \\ U_3 = \alpha_3 T_3^2 U_2 + \beta_3 T_3 U_2 + \gamma_3 U_2 + \delta_3 I, \\ \cdots \\ U_{N-1} = \alpha_{N-1} T_{N-1}^2 U_{N-2} + \beta_{N-1} T_{N-1} U_{N-2} + \gamma_{N-1} U_{N-2} + \delta_{N-1} I, \\ G = U_N = \alpha_N T_N^2 U_{N-1} + \beta_N T_N U_{N-1} + \gamma_N U_{N-1} + \delta_N I. \end{cases} \tag{14}$$

Such a mapping G is called the G-mapping generated by $T_1, T_2, \cdots, T_N$ and $\pi_1, \pi_2, \cdots, \pi_N$.

Lemma 8 [19]. *Let C be a nonempty closed and convex subset of real Banach space. Let $\{T_i\}_{i=1}^N$ be a finite family of ρ_i-strictly pseudononspreading mappings of C into itself with $\bigcap_{i=1}^N Fix(T_i) \ne \varnothing$ and $\rho = \max_{i=1,2,\cdots,N}\{\rho_i\}$. For $i = 1, 2, \cdots, N$, let $\pi_i = (\alpha_i, \beta_i, \gamma_i, \delta_i)$, where $\alpha_i, \beta_i, \gamma_i, \delta_i \in [0, 1]$ and $\alpha_i + \beta_i + \gamma_i + \delta_i = 1$. Assume that $\rho \le \beta_i$ for $i = 1, 2, \cdots, N$, and $(\alpha_i + \beta_i)\rho \le \beta_i\gamma_i$ for $i = 2, 3, \cdots, N$, and $(\alpha_1 + \beta_1)\rho \le \beta_1(\gamma_1 + \delta_1)$. If G is the G-mapping generated by $T_1, T_2, \cdots, T_N$ and $\pi_1, \pi_2, \cdots, \pi_N$, then $Fix(G) = \bigcap_{i=1}^N Fix(T_i)$ and G is a quasinonexpansive mapping.*

3. Iterative Algorithms for ESEP

In this section, we introduce two simultaneous iterative algorithms with internal perturbation projection to solve ESEP (5) and define the solution set of ESEP (5) as Ω_1.

Algorithm 9. **Initialization:** take $(x_1^1, x_2^1, \cdots, x_n^1) \in H_1 \times H_2 \times \cdots \times H_n$ arbitrary.

Iteration step: for a given current iterate $(x_1^k, x_2^k, \cdots, x_n^k) \in H_1 \times H_2 \times \cdots \times H_n$, we calculate the next iterate $(x_1^{k+1}, x_2^{k+1}, \cdots, x_n^{k+1})$ by

$$\begin{cases} \omega_k = \dfrac{\sum_{i=1}^{n} A_i x_i^k}{n}, \\ x_1^{k+1} = x_1^k - \lambda_k \left(\left(x_1^k - P_{C_1}\left(x_1^k\right) \right) + A_1^* \left(A_1 x_1^k - \omega_k \right) \right), \\ x_2^{k+1} = x_2^k - \lambda_k \left(\left(x_2^k - P_{C_2}\left(x_2^k\right) \right) + A_2^* \left(A_2 x_2^k - \omega_k \right) \right), \\ \cdots \\ x_n^{k+1} = x_n^k - \lambda_k \left(\left(x_n^k - P_{C_n}\left(x_n^k\right) \right) + A_n^* \left(A_n x_n^k - \omega_k \right) \right), \end{cases} \quad (15)$$

where the stepsize λ_k is chosen in such a way that if $k \in \Omega$, then

$$\lambda_k \in \left(\varepsilon, \rho_k \frac{2\sum_{i=1}^{n} \left\| x_i^k - P_{C_i}\left(x_i^k\right) \right\|^2 + \sum_{i=1}^{n} \left\| A_i x_i^k - \omega_k \right\|^2}{\sum_{i=1}^{n} \left\| \left(x_i^k - P_{C_i}\left(x_i^k\right) \right) + A_i^* \left(A_i x_i^k - \omega_k \right) \right\|^2} \right), \quad (16)$$

where the index set $\Omega = \{k : A_i x_i^k - \omega_k \neq 0, \exists i \in \{1, 2, \cdots, n\}\}$, small enough $\varepsilon > 0$ and $0 < \delta_1 < \rho_k < \delta_2 < 1$, set $k := k + 1$ and go to (15). Otherwise, $A_i x_i^k - \omega_k = 0$, for $i = 1, 2, \cdots, n$, the iteration stops.

Remark 10. Note that in (16), the choice of the stepsize $\{\lambda_k\}$ is independent of the norm $\|A_i\|$, for $i = 1, 2, \cdots, n$. Furthermore, we will show from Lemma 11 that $\{\lambda_k\}$ is well defined.

Lemma 11. *Assume the solution set Ω_1 of ESEP (5) is nonempty, then $\{\lambda_k\}$ defined by (16) is well defined.*

Proof. Let $(z_1, z_2, \cdots, z_n) \in \Omega_1$, then $A_1 z_1 = A_2 z_2 = \cdots = A_n z_n$. Noting that for $i = 1, 2, \cdots, n$, we have

$$\begin{aligned} &\left\langle \left(x_i^k - P_{C_i}\left(x_i^k\right) \right) + A_i^* \left(A_i x_i^k - \omega_k \right), x_i^k - z_i \right\rangle \\ &= \left\langle x_i^k - P_{C_i}\left(x_i^k\right), x_i^k - z_i \right\rangle + \left\langle A_i x_i^k - \omega_k, A_i x_i^k - A_i z_i \right\rangle \\ &= \left\langle x_i^k - P_{C_i}\left(x_i^k\right), x_i^k - z_i \right\rangle + \left\langle A_i x_i^k - \omega_k, A_i x_i^k - \omega_k \right\rangle \\ &\quad + \left\langle A_i x_i^k - \omega_k, \omega_k - A_i z_i \right\rangle. \end{aligned} \quad (17)$$

As a result,

$$\begin{aligned} &\left\langle \left(x_1^k - P_{C_1}\left(x_1^k\right) \right) + A_1^* \left(A_1 x_1^k - \omega_k \right), x_1^k - z_1 \right\rangle \\ &= \left\langle x_1^k - P_{C_1}\left(x_1^k\right), x_1^k - z_1 \right\rangle + \left\langle A_1 x_1^k - \omega_k, A_1 x_1^k - \omega_k \right\rangle \\ &\quad + \left\langle A_1 x_1^k - \omega_k, \omega_k - A_1 z_1 \right\rangle, \\ &\left\langle \left(x_2^k - P_{C_2}\left(x_2^k\right) \right) + A_2^* \left(A_2 x_2^k - \omega_k \right), x_2^k - z_2 \right\rangle \\ &= \left\langle x_2^k - P_{C_2}\left(x_2^k\right), x_2^k - z_2 \right\rangle + \left\langle A_2 x_2^k - \omega_k, A_2 x_2^k - \omega_k \right\rangle \\ &\quad + \left\langle A_2 x_2^k - \omega_k, \omega_k - A_2 z_2 \right\rangle, \\ &\left\langle \left(x_n^k - P_{C_n}\left(x_n^k\right) \right) + A_n^* \left(A_n x_n^k - \omega_k \right), x_n^k - z_n \right\rangle \\ &= \left\langle x_n^k - P_{C_n}\left(x_n^k\right), x_n^k - z_n \right\rangle + \left\langle A_n x_n^k - \omega_k, A_n x_n^k - \omega_k \right\rangle \\ &\quad + \left\langle A_n x_n^k - \omega_k, \omega_k - A_n z_n \right\rangle. \end{aligned} \quad (18)$$

Summing the above equalities and applying Lemma 1 (3) as well as the condition $A_1 z_1 = A_2 z_2 = \cdots = A_n z_n$, one has

$$\begin{aligned} &\sum_{i=1}^{n} \left\| x_i^k - P_{C_i}\left(x_i^k\right) \right\|^2 + \sum_{i=1}^{n} \left\| A_i x_i^k - \omega_k \right\|^2 \\ &\leq \sum_{i=1}^{n} \left\langle \left(x_i^k - P_{C_i}\left(x_i^k\right) \right) + A_i^* \left(A_i x_i^k - \omega_k \right), x_i^k - z_i \right\rangle \\ &\leq \sum_{i=1}^{n} \left\| \left(x_i^k - P_{C_i}\left(x_i^k\right) \right) + A_i^* \left(A_i x_i^k - \omega_k \right) \right\| \left\| x_i^k - z_i \right\|. \end{aligned} \quad (19)$$

Consequently, for $k \in \Omega$, we have $\sum_{i=1}^{n} \|A_i x_i^k - \omega_k\|^2 \neq 0$, then $\sum_{i=1}^{n} \|(x_i^k - P_{C_i}(x_i^k)) + A_i^* (A_i x_i^k - \omega_k)\| \neq 0$, which leads that λ_k is well defined.

Theorem 12. *Let H be a real Hilbert space. For $i = 1, 2, \cdots, n$, assume that C_i are nonempty closed convex subsets of real Hilbert spaces H_i, and $A_i : H_i \longrightarrow H$ are bounded linear operators with their adjoint operators A_i^*. Then, the sequence $\{(x_1^k, x_2^k, \cdots, x_n^k)\}$ generated by Algorithm 9 converges weakly to a solution of ESEP (5). Furthermore, for $i = 1, 2, \cdots, n$, $\|x_i^k - P_{C_i}(x_i^k)\| \longrightarrow 0$, $\|A_i x_i^k - \omega_k\| \longrightarrow 0$ and $\|x_i^{k+1} - x_i^k\| \longrightarrow 0$ as $k \longrightarrow \infty$.*

Proof. Let $(z_1, z_2, \cdots, z_n) \in \Omega_1$ and

$$z = A_1 z_1 = A_2 z_2 = \cdots = A_n z_n. \quad (20)$$

For $i = 1, 2, \cdots, n$, from (15) and Lemma 3, we have

$$\begin{aligned} &\left\| x_i^{k+1} - z_i \right\|^2 \\ &= \left\| x_i^k - \lambda_k \left(\left(x_i^k - P_{C_i}\left(x_i^k\right) \right) + A_i^* \left(A_i x_i^k - \omega_k \right) \right) - z_i \right\|^2 \\ &= \left\| x_i^k - z_i \right\|^2 + \lambda_k^2 \left\| \left(x_i^k - P_{C_i}\left(x_i^k\right) \right) + A_i^* \left(A_i x_i^k - \omega_k \right) \right\|^2 \\ &\quad - 2\lambda_k \left\langle x_i^k - z_i, \left(x_i^k - P_{C_i}\left(x_i^k\right) \right) + A_i^* \left(A_i x_i^k - \omega_k \right) \right\rangle \\ &= \left\| x_i^k - z_i \right\|^2 + \lambda_k^2 \left\| \left(x_i^k - P_{C_i}\left(x_i^k\right) \right) + A_i^* \left(A_i x_i^k - \omega_k \right) \right\|^2 \\ &\quad - 2\lambda_k \left\langle x_i^k - z_i, x_i^k - P_{C_i}\left(x_i^k\right) \right\rangle \\ &\quad - 2\lambda_k \left\langle x_i^k - z_i, A_i^* \left(A_i x_i^k - \omega_k \right) \right\rangle. \end{aligned} \quad (21)$$

According to Lemma 1 and

$$\begin{aligned} &-2\left\langle x_i^k - z_i, A_i^* \left(A_i x_i^k - \omega_k \right) \right\rangle \\ &= -2\left\langle A_i x_i^k - A_i z_i, A_i x_i^k - \omega_k \right\rangle \\ &= -\left\| A_i x_i^k - A_i z_i \right\|^2 - \left\| A_i x_i^k - \omega_k \right\|^2 + \left\| A_i z_i - \omega_k \right\|^2, \end{aligned} \quad (22)$$

(21) can be written as

$$\begin{aligned}\left\|x_i^{k+1}-z_i\right\|^2 \le{} & \left\|x_i^k-z_i\right\|^2 \\ & + \lambda_k^2\left\|\left(x_i^k-P_{C_i}\left(x_i^k\right)\right)+A_i^*\left(A_ix_i^k-\omega_k\right)\right\|^2 \\ & - 2\lambda_k\left\|x_i^k-P_{C_i}\left(x_i^k\right)\right\|^2-\lambda_k\left\|A_ix_i^k-A_iz_i\right\|^2 \\ & - \lambda_k\left\|A_ix_i^k-\omega_k\right\|^2+\lambda_k\|A_iz_i-\omega_k\|^2.\end{aligned} \tag{23}$$

Note that

$$\begin{aligned}\|A_iz_i-\omega_k\|^2 &= \left\|A_iz_i-\frac{\sum_{i=1}^n A_ix_i^k}{n}\right\|^2=\left\|\frac{\sum_{i=1}^n\left(A_iz_i-A_ix_i^k\right)}{n}\right\|^2 \\ &\le \frac{\sum_{i=1}^n\left\|A_iz_i-A_ix_i^k\right\|^2}{n}.\end{aligned} \tag{24}$$

Summing (23) for i from 1 to n, we can obtain

$$\begin{aligned}\sum_{i=1}^n\left\|x_i^{k+1}-z_i\right\|^2 \le{} & \sum_{i=1}^n\left\|x_i^k-z_i\right\|^2+\lambda_k^2\sum_{i=1}^n\left\|\left(x_i^k-P_{C_i}\left(x_i^k\right)\right)\right. \\ & \left.+A_i^*\left(A_ix_i^k-\omega_k\right)\right\|^2-2\lambda_k\sum_{i=1}^n\left\|x_i^k-P_{C_i}\left(x_i^k\right)\right\|^2 \\ & -\lambda_k\sum_{i=1}^n\left\|A_ix_i^k-\omega_k\right\|^2 \le \sum_{i=1}^n\left\|x_i^k-z_i\right\|^2+\lambda_k\rho_k \\ & \cdot\left(2\sum_{i=1}^n\left\|x_i^k-P_{C_i}\left(x_i^k\right)\right\|^2+\sum_{i=1}^n\left\|A_ix_i^k-\omega_k\right\|^2\right) \\ & -\lambda_k\left(2\sum_{i=1}^n\left\|x_i^k-P_{C_i}\left(x_i^k\right)\right\|^2+\sum_{i=1}^n\left\|A_ix_i^k-\omega_k\right\|^2\right) \\ ={} & \sum_{i=1}^n\left\|x_i^k-z_i\right\|^2-\lambda_k(1-\rho_k) \\ & \cdot\left(2\sum_{i=1}^n\left\|x_i^k-P_{C_i}\left(x_i^k\right)\right\|^2+\sum_{i=1}^n\left\|A_ix_i^k-\omega_k\right\|^2\right).\end{aligned} \tag{25}$$

Let $\Gamma_k(z_1,z_2,\cdots,z_n)=\sum_{i=1}^n\|x_i^k-z_i\|^2$, we have

$$\Gamma_{k+1}(z_1,z_2,\cdots,z_n)\le\Gamma_k(z_1,z_2,\cdots,z_n). \tag{26}$$

Therefore, the sequence $\{\Gamma_k(z_1,z_2,\cdots,z_n)\}$ is nonincreasing and lower bounded by 0. Hence, $\{\Gamma_k(z_1,z_2,\cdots,z_n)\}$ converges to some finite limit, suppose as $l(z_1,z_2,\cdots,z_n)$, and sequence $\{x_i^k\}$ is bounded. Letting $k\longrightarrow\infty$ and taking the limit in the two sides of (25), for $0<\delta_1<\rho_k<\delta_2<1$, we can obtain that

$$\lambda_k(1-\rho_k)\left(2\sum_{i=1}^n\left\|x_i^k-P_{C_i}\left(x_i^k\right)\right\|^2+\sum_{i=1}^n\left\|A_ix_i^k-\omega_k\right\|^2\right)\longrightarrow 0, \tag{27}$$

which implies

$$2\lambda_k\sum_{i=1}^n\left\|x_i^k-P_{C_i}\left(x_i^k\right)\right\|^2\longrightarrow 0,\quad \lambda_k\sum_{i=1}^n\left\|A_ix_i^k-\omega_k\right\|^2\longrightarrow 0. \tag{28}$$

Since $\lambda_k>\varepsilon$, then

$$\left\|x_i^k-P_{C_i}\left(x_i^k\right)\right\|\longrightarrow 0,\left\|A_ix_i^k-\omega_k\right\|\longrightarrow 0. \tag{29}$$

From the definition of λ_k, we have

$$\begin{aligned}&\lambda_k^2\sum_{i=1}^n\left\|\left(x_i^k-P_{C_i}\left(x_i^k\right)\right)+A_i^*\left(A_ix_i^k-\omega_k\right)\right\|^2 \\ &\quad\le\lambda_k\rho_k\left(2\sum_{i=1}^n\left\|x_i^k-P_{C_i}\left(x_i^k\right)\right\|^2+\sum_{i=1}^n\left\|A_ix_i^k-\omega_k\right\|^2\right)\longrightarrow 0,\end{aligned} \tag{30}$$

then,

$$\lambda_k^2\sum_{i=1}^n\left\|\left(x_i^k-P_{C_i}\left(x_i^k\right)\right)+A_i^*\left(A_ix_i^k-\omega_k\right)\right\|^2\longrightarrow 0, \tag{31}$$

which implies

$$\lambda_k\left\|\left(x_i^k-P_{C_i}\left(x_i^k\right)\right)+A_i^*\left(A_ix_i^k-\omega_k\right)\right\|\longrightarrow 0. \tag{32}$$

From (32), for $k\longrightarrow\infty$, we can get

$$\begin{aligned}\left\|x_i^{k+1}-x_i^k\right\| &= \left\|\lambda_k\left(\left(x_i^k-P_{C_i}\left(x_i^k\right)\right)+A_i^*\left(A_ix_i^k-\omega_k\right)\right)\right\| \\ &= \lambda_k\left\|x_i^k-P_{C_i}\left(x_i^k\right)+A_i^*\left(A_ix_i^k-\omega_k\right)\right\|\longrightarrow 0.\end{aligned} \tag{33}$$

Consequently, the sequence $\{x_i^k\}$ is asymptotically regular.

Let $(\bar{x}_1,\bar{x}_2,\cdots,\bar{x}_n)\in\omega_\omega(x_1^k,x_2^k,\cdots,x_n^k)$, then for $i=1,2,\cdots,n$, there exists a subsequence of $\{x_i^k\}$ which converges weakly to $\bar{x}_i$. From (29) and the lower semicontinuity of the squared norm, we have

$$\begin{aligned}0 &\le \left\|\bar{x}_i-P_{C_i}(\bar{x}_i)\right\|\le\liminf_{j\to\infty}\left\|x_i^{k_j}-P_{C_i}x_i^{k_j}\right\| \\ &= \lim_{k\to\infty}\left\|x_i^k-P_{C_i}x_i^k\right\|=0,\end{aligned} \tag{34}$$

which implies $\bar{x}_i\in C_i$.

Furthermore, it follows from (29) and the lower semicontinuity of the squared norm

$$\|A_i\bar{x}_i - \bar{\omega}\|^2 \le \liminf_{k\to\infty} \left\|A_i x_i^k - \omega_k\right\|^2 = 0, \tag{35}$$

where $\bar{\omega} = (\sum_{i=1}^n A_i\bar{x}_i)/n$, which means that $A_i\bar{x}_i - \bar{\omega} = 0$. That is,

$$\begin{aligned} A_2\bar{x}_2 + A_3\bar{x}_3 + \cdots + A_n\bar{x}_n &= (n-1)A_1\bar{x}_1, \\ A_1\bar{x}_1 + A_3\bar{x}_3 + \cdots + A_n\bar{x}_n &= (n-1)A_2\bar{x}_2, \\ &\cdots \\ A_1\bar{x}_1 + A_2\bar{x}_2 + \cdots + A_{n-1}\bar{x}_{n-1} &= (n-1)A_n\bar{x}_n. \end{aligned} \tag{36}$$

Obviously,

$$A_1\bar{x}_1 = A_2\bar{x}_2 = \cdots = A_n\bar{x}_n. \tag{37}$$

Hence, $(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n) \in \Omega_1$.

In the sequel, we will show the uniqueness of the weak cluster points of $\{x_i^k\}$. Assume that x_i^* is another weak cluster point of $\{x_i^k\}$. It follows from the definition of Γ_k that

$$\begin{aligned} \Gamma_k(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n) &= \sum_{i=1}^n \left\|x_i^k - \bar{x}_i\right\|^2 = \sum_{i=1}^n \left\|x_i^k - x_i^*\right\|^2 \\ &\quad + \sum_{i=1}^n \|x_i^* - \bar{x}_i\|^2 + 2\sum_{i=1}^n \left\langle x_i^k - x_i^*, x_i^* - \bar{x}_i \right\rangle \\ &= \Gamma_k(x_1^*, x_2^*, \cdots, x_n^*) + \sum_{i=1}^n \|x_i^* - \bar{x}_i\|^2 \\ &\quad + 2\sum_{i=1}^n \left\langle x_i^k - x_i^*, x_i^* - \bar{x}_i \right\rangle. \end{aligned} \tag{38}$$

Without loss of generality, we can suppose $x_i^k \rightharpoonup x_i^*$, and then

$$l(\bar{x}_1, \bar{x}_2, \cdots \bar{x}_n) = l(x_1^*, x_2^*, \cdots, x_n^*) + \sum_{i=1}^n \|x_i^* - \bar{x}_i\|^2. \tag{39}$$

Reversing the role of $(\bar{x}_1, \bar{x}_2, \cdots \bar{x}_n)$ and $(x_1^*, x_2^*, \cdots, x_n^*)$, we can obtain

$$l(x_1^*, x_2^*, \cdots, x_n^*) = l(\bar{x}_1, \bar{x}_2, \cdots \bar{x}_n) + \sum_{i=1}^n \|x_i^* - \bar{x}_i\|^2. \tag{40}$$

Adding (39) and (40), we deduce

$$\sum_{i=1}^n \|x_i^* - \bar{x}_i\|^2 = 0, \tag{41}$$

which yields that $x_i^* = \bar{x}_i$. Hence, the sequence $\{x_i^k\}$ weakly converges to a solution of ESEP (5), which completes the proof.

In the following, we introduce another simultaneous iterative algorithm with internal perturbation projection to solve ESEP (5) and prove the strong convergence of the algorithm.

Algorithm 13. **Initialization:** let $(x_1^1, x_2^1, \cdots, x_n^1) \in H_1 \times H_2 \times \cdots \times H_n$ arbitrary.

Iteration step: for a given current iterate $(x_1^k, x_2^k, \cdots, x_n^k) \in H_1 \times H_2 \times \cdots \times H_n$, we calculate the next iterate $(x_1^{k+1}, x_2^{k+1}, \cdots, x_n^{k+1})$ by

$$\begin{cases} \omega_k = \dfrac{\sum_{i=1}^n A_i x_i^k}{n}, \\ u_1^k = x_1^k - \lambda_k\left(\left(x_1^k - P_{C_1}\left(x_1^k\right)\right) + A_1^*\left(A_1x_1^k - \omega_k\right)\right), \\ x_1^{k+1} = \alpha_k v_1 + (1-\alpha_k)u_1^k, \\ u_2^k = x_2^k - \lambda_k\left(\left(x_2^k - P_{C_2}\left(x_2^k\right)\right) + A_2^*\left(A_2x_2^k - \omega_k\right)\right), \\ x_2^{k+1} = \alpha_k v_2 + (1-\alpha_k)u_2^k, \\ \cdots \\ u_n^k = x_n^k - \lambda_k\left(\left(x_n^k - P_{C_n}\left(x_n^k\right)\right) + A_n^*\left(A_nx_n^k - \omega_k\right)\right), \\ x_n^{k+1} = \alpha_k v_n + (1-\alpha_k)u_n^k, \end{cases} \tag{42}$$

where $\{\lambda_k\}$ is the same as Algorithm 9, $(v_1, v_2, \cdots, v_n) \in H_1 \times H_2 \times \cdots \times H_n$.

Theorem 14. *Let H be a real Hilbert space. For $i = 1, 2, \cdots, n$, let C_i be nonempty closed convex subsets of real Hilbert spaces H_i, respectively. $A_i : H_i \longrightarrow H$ are bounded linear operators with their adjoint operators A_i^*, respectively. If $\alpha_k \in (0,1)$, $\lim_{k\to\infty} \alpha_k = 0$ and $\sum_{k=1}^\infty \alpha_k = \infty$, then the sequence $\{(x_1^k, x_2^k, \cdots, x_n^k)\}$ generated by Algorithm 13 converges strongly to a solution of ESEP (5) denoted by $(z_1, z_2, \cdots, z_n) = P_{\Omega_1}(v_1, v_2, \cdots v_n)$. Furthermore, for $i = 1, 2, \cdots, n$, $\|x_i^k - P_{C_i}(x_i^k)\| \longrightarrow 0$, $\|A_ix_i^k - \omega_k\| \longrightarrow 0$ and $\|x_i^{k+1} - x_i^k\| \longrightarrow 0$ as $k \longrightarrow \infty$.*

Proof. Let $(z_1, z_2, \cdots, z_n) = P_{\Omega_1}(v_1, v_2, \cdots v_n)$, which means $(z_1, z_2, \cdots, z_n) \in \Omega_1$. Let

$$z = A_1z_1 = A_2z_2 = \cdots = A_nz_n. \tag{43}$$

Similar to the proof of the Theorem 12, we have the following inequality

$$\begin{aligned} \sum_{i=1}^n \left\|u_i^k - z_i\right\|^2 &\le \sum_{i=1}^n \left\|x_i^k - z_i\right\|^2 - \lambda_k(1-\rho_k) \\ &\quad \cdot \left(2\sum_{i=1}^n \left\|x_i^k - P_{C_i}\left(x_i^k\right)\right\|^2 + \sum_{i=1}^n \left\|A_ix_i^k - \omega_k\right\|^2\right). \end{aligned} \tag{44}$$

From Lemma 3, we have

$$\begin{aligned}\sum_{i=1}^{n}\left\|x_i^{k+1}-z_i\right\|^2&=\sum_{i=1}^{n}\left\|\alpha_k v_i+(1-\alpha_k)u_i^k-z_i\right\|^2\\&\le \alpha_k\sum_{i=1}^{n}\|v_i-z_i\|^2+(1-\alpha_k)\sum_{i=1}^{n}\left\|u_i^k-z_i\right\|^2\\&\le \alpha_k\sum_{i=1}^{n}\|v_i-z_i\|^2+(1-\alpha_k)\\&\quad\cdot\left(\sum_{i=1}^{n}\left\|x_i^k-z_i\right\|^2-\lambda_k(1-\rho_k)\right.\\&\quad\left.\cdot\left(2\sum_{i=1}^{n}\left\|x_i^k-P_{C_i}\left(x_i^k\right)\right\|^2+\sum_{i=1}^{n}\left\|A_ix_i^k-\omega_k\right\|^2\right)\right)\\&\le \alpha_k\sum_{i=1}^{n}\|v_i-z_i\|^2+(1-\alpha_k)\sum_{i=1}^{n}\left\|x_i^k-z_i\right\|^2\\&\le \max\left\{\sum_{i=1}^{n}\|v_i-z_i\|^2,\ \sum_{i=1}^{n}\left\|x_i^k-z_i\right\|^2\right\}.\end{aligned}\tag{45}$$

By induction, we can get

$$\sum_{i=1}^{n}\left\|x_i^{k+1}-z_i\right\|^2\le \max\left\{\sum_{i=1}^{n}\|v_i-z_i\|^2,\ \sum_{i=1}^{n}\left\|x_i^1-z_i\right\|^2\right\},\tag{46}$$

which implies that sequence $\{\Gamma_k\}$ is bounded by setting $\Gamma_k=\sum_{i=1}^{n}\|x_i^k-z_i\|^2$. In the sequel, we divide the proof into two cases.

Case 1. Assume that there exists integer $k_0>0$ such that $\{\Gamma_k\}$ is decreasing sequence for all $k>k_0$, which implies $\lim_{k\to\infty}\Gamma_k$ exists. Consequently, (45) can be rewritten as

$$\begin{aligned}&\lambda_k(1-\rho_k)\left(2\sum_{i=1}^{n}\left\|x_i^k-P_{C_i}\left(x_i^k\right)\right\|^2+\sum_{i=1}^{n}\left\|A_ix_i^k-\omega_k\right\|^2\right)\\&\quad\le-\sum_{i=1}^{n}\left\|x_i^{k+1}-z_i\right\|^2+(1-\alpha_k)\sum_{i=1}^{n}\left\|x_i^k-z_i\right\|^2\\&\qquad+\alpha_k\sum_{i=1}^{n}\|v_i-z_i\|^2.\end{aligned}\tag{47}$$

Similar to the proof of Theorem 12, taking limit in the two sides of (47) for $k\longrightarrow\infty$, $i=1,2,\cdots n$, we can obtain

$$\begin{aligned}&\left\|x_i^k-P_{C_i}\left(x_i^k\right)\right\|\longrightarrow 0\,;\left\|A_ix_i^k-\omega_k\right\|\longrightarrow 0,\\&\lambda_k\left\|x_i^k-P_{C_i}\left(x_i^k\right)+A^*\left(A_ix_i^k-\omega_k\right)\right\|\longrightarrow 0.\end{aligned}\tag{48}$$

For another, since $\{x_i^k\}$ is bounded, $\{x_i^k\}$ has a convergent subsequence. Without loss of generality, we have $(\bar{x}_1,\bar{x}_2,\cdots,\bar{x}_n)\in\omega_\omega(x_1^k,x_2^k,\cdots,x_n^k)$. From (42), for $k\longrightarrow\infty$, we have

$$\begin{aligned}\left\|u_i^k-x_i^k\right\|&=\left\|\lambda_k\left(\left(x_i^k-P_{C_i}\left(x_i^k\right)\right)+A_i^*\left(A_ix_i^k-\omega_k\right)\right)\right\|\\&=\lambda_k\left\|x_i^k-P_{C_i}\left(x_i^k\right)+A_i^*\left(A_ix_i^k-\omega_k\right)\right\|\longrightarrow 0,\end{aligned}\tag{49}$$

which implies $\{u_i^k\}$ is bounded. Since $\lim_{k\to\infty}\alpha_k=0$, for $k\longrightarrow\infty$, we can infer

$$\begin{aligned}\left\|x_i^{k+1}-u_i^k\right\|&=\left\|\alpha_k v_i+(1-\alpha_k)u_i^k-u_i^k\right\|\\&=\left\|\alpha_k v_i-\alpha_k u_i^k\right\|=\alpha_k\left\|v_i-u_i^k\right\|\longrightarrow 0.\end{aligned}\tag{50}$$

Combining (49) and (50), we have

$$\left\|x_i^{k+1}-x_i^k\right\|\le\left\|x_i^{k+1}-u_i^k\right\|+\left\|u_i^k-x_i^k\right\|\longrightarrow 0.\tag{51}$$

Similar to the proof of Theorem 12, we can get $(\bar{x}_1,\bar{x}_2,\cdots,\bar{x}_n)\in\Omega_1$. Since $\{x_i^k\}$ is asymptotically regular, we can get $(\bar{x}_1,\bar{x}_2,\cdots,\bar{x}_n)\in\omega_\omega(x_1^{k+1},x_2^{k+1},\cdots,x_n^{k+1})$. Depending on the property of the projection, we have

$$\begin{aligned}&\limsup_{k\to\infty}\left\langle v_i-z_i,x_i^{k+1}-z_i\right\rangle\\&\quad=\lim_{j\to\infty}\left\langle v_i-z_i,x_i^{k_j+1}-z_i\right\rangle\\&\quad=\langle v_i-z_i,\bar{x}_i-z_i\rangle\le 0.\end{aligned}\tag{52}$$

$$\begin{aligned}\sum_{i=1}^{n}\left\|x_i^{k+1}-z_i\right\|^2&=\sum_{i=1}^{n}\left\|\alpha_k v_i+(1-\alpha_k)u_i^k-z_i\right\|^2\\&\le(1-\alpha_k)^2\sum_{i=1}^{n}\left\|u_i^k-z_i\right\|^2\\&\quad+2\alpha_k\sum_{i=1}^{n}\left\langle v_i-z_i,x_i^{k+1}-z_i\right\rangle\\&\le(1-\alpha_k)\sum_{i=1}^{n}\left\|x_i^k-z_i\right\|^2\\&\quad+2\alpha_k\sum_{i=1}^{n}\left\langle v_i-z_i,x_i^{k+1}-z_i\right\rangle,\end{aligned}\tag{53}$$

which combining with Lemma 6 and (52), we conclude the sequence $\{(x_1^k,x_2^k,\cdots,x_n^k)\}$ converges strongly to $(z_1,z_2,\cdots,z_n)$.

Case 2. Assume $\{\Gamma_k\}$ is not a decreasing sequence. Then suppose that there exists a subsequence $\{\Gamma_{k_j}\}$ of $\{\Gamma_k\}$, for $j\ge 0$, such that $\Gamma_{k_j}\le\Gamma_{k_j+1}$. For all $k>k_0$ (such that $\Gamma_{k_0}\le\Gamma_{k_0+1}$), define a integer sequence as follows:

$$\tau(k)=\max\{l\in N\mid k_0\le l<k,\Gamma_l\le\Gamma_{l+1}\}.\tag{54}$$

It is clear that $\tau(k)$ is a nondecreasing sequence satisfying $\lim_{k\to\infty} \tau(k) = \infty$, and $\Gamma_{\tau(k)} \leq \Gamma_{\tau(k)+1}$.

Similar to the proof of Case 1, we can obtain

$$\begin{aligned}\lambda_{\tau(k)}\left(1-\rho_{\tau(k)}\right)&\left(2\sum_{i=1}^{n}\left\|x_i^{\tau(k)}-P_{C_i}\left(x_i^{\tau(k)}\right)\right\|^2\right.\\&\left.+\sum_{i=1}^{n}\left\|A_i x_i^{\tau(k)}-\omega_{\tau(k)}\right\|^2\right)\\&\leq -\Gamma_{\tau(k)+1}+\left(1-\alpha_{\tau(k)}\right)\Gamma_{\tau(k)}+\alpha_{\tau(k)}\sum_{i=1}^{n}\|v_i-z_i\|^2.\end{aligned} \tag{55}$$

Taking the limit in the two sides of (55) for $k \longrightarrow \infty$, according to Case 1, we have

$$\begin{aligned}\left\|x_i^{\tau(k)}-P_{C_i}\left(x_i^{\tau(k)}\right)\right\| &\longrightarrow 0,\\ \left\|A_i x_i^{\tau(k)}-\omega_{\tau(k)}\right\| &\longrightarrow 0,\\ \lambda_{\tau(k)}\left\|x_i^{\tau(k)}-P_{C_i}\left(x_i^{\tau(k)}\right)+A^*\left(A_i x_i^{\tau(k)}-\omega_{\tau(k)}\right)\right\| &\longrightarrow 0,\\ \left\|x_i^{\tau(k)+1}-x_i^{\tau(k)}\right\| &\longrightarrow 0.\end{aligned} \tag{56}$$

That implies $(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n) \in \omega_\omega(x_1^{\tau(k)+1}, x_2^{\tau(k)+1}, \cdots, x_n^{\tau(k)+1})$. Depending on the property of the projection, we have

$$\limsup_{\tau(k)\to\infty}\left\langle v_i - z_i, x_i^{\tau(k)+1} - z_i\right\rangle \leq 0. \tag{57}$$

From (53), we can get

$$\Gamma_{\tau(k)+1} \leq \left(1-\alpha_{\tau(k)}\right)\Gamma_{\tau(k)} + 2\alpha_{\tau(k)}\sum_{i=1}^{n}\left\langle v_i - z_i, x_i^{\tau(k)+1} - z_i\right\rangle. \tag{58}$$

Since $\Gamma_{\tau(k)} \leq \Gamma_{\tau(k)+1}$, we have

$$\Gamma_{\tau(k)} \leq 2\sum_{i=1}^{n}\left\langle v_i - z_i, x_i^{\tau(k)+1} - z_i\right\rangle, \tag{59}$$

which implies $\limsup_{\tau(k)\to\infty}\Gamma_{\tau(k)} \leq 0$, that is,

$$\lim_{\tau(k)\to\infty}\Gamma_{\tau(k)} = 0. \tag{60}$$

By (58), we obtain

$$\limsup_{\tau(k)\to\infty}\Gamma_{\tau(k)+1} \leq \limsup_{\tau(k)\to\infty}\Gamma_{\tau(k)} \leq 0, \tag{61}$$

which implies

$$\lim_{\tau(k)\to\infty}\Gamma_{\tau(k)+1} = 0. \tag{62}$$

By Lemma 5, we have $\Gamma_k \leq \Gamma_{\tau(k)+1}$, which means the sequence $\{(x_1^k, x_i^k, \cdots, x_n^k)\}$ converges strongly to $(z_1, z_2, \cdots, z_n)$. Thus, we complete the proof.

4. Iterative Algorithm for ESEFPP

In this section, we introduce a simultaneous iterative algorithm with internal perturbation projection to solve ESEFPP (9) and define the solution set of ESEFPP (9) as Ω_2.

Algorithm 15. **Initialization:** let $0 < a < \alpha_k < b < 1$, $(x_1^1, x_2^1, \cdots, x_n^1) \in H_1 \times H_2 \times \cdots \times H_n$ arbitrary.

Iteration step: for a given current iterate $(x_1^k, x_2^k, \cdots, x_n^k) \in H_1 \times H_2 \times \cdots \times H_n$, we calculate the next iterate $(x_1^{k+1}, x_2^{k+1}, \cdots, x_n^{k+1})$ by

$$\begin{cases}\omega_k = \dfrac{\sum_{i=1}^{n} A_i x_i^k}{n},\\ u_1^k = x_1^k - \lambda_k\left(\left(x_1^k - G_1\left(x_1^k\right)\right) + A_1^*\left(A_1 x_1^k - \omega_k\right)\right),\\ x_1^{k+1} = \alpha_k u_1^k + (1-\alpha_k)G_1\left(u_1^k\right),\\ u_2^k = x_2^k - \lambda_k\left(\left(x_2^k - G_2\left(x_2^k\right)\right) + A_2^*\left(A_2 x_2^k - \omega_k\right)\right),\\ x_2^{k+1} = \alpha_k u_2^k + (1-\alpha_k)G_2\left(u_2^k\right),\\ \cdots\\ u_n^k = x_n^k - \lambda_k\left(\left(x_n^k - G_n\left(x_n^k\right)\right) + A_n^*\left(A_n x_n^k - \omega_k\right)\right),\\ x_n^{k+1} = \alpha_k u_n^k + (1-\alpha_k)G_n\left(u_n^k\right),\end{cases} \tag{63}$$

where the stepsize λ_k is chosen in such a way that if $k \in \Omega$, then

$$\lambda_k \in \left(\varepsilon, \rho_k \frac{\sum_{i=1}^{n}\left\|x_i^k - G_i\left(x_i^k\right)\right\|^2 + \sum_{i=1}^{n}\left\|A_i x_i^k - \omega_k\right\|^2}{\sum_{i=1}^{n}\left\|\left(x_i^k - G_i\left(x_i^k\right)\right) + A_i^*\left(A_i x_i^k - \omega_k\right)\right\|^2}\right), \tag{64}$$

where the index set $\Omega = \{k : A_i x_i^k - \omega_k \neq 0, \exists i \in \{1, 2, \cdots, n\}\}$, small enough $\varepsilon > 0$ and $0 < \delta_1 < \rho_k < \delta_2 < 1$, set $k := k + 1$ and go to (63). Otherwise, $A_i x_i^k - \omega_k = 0$, for $i = 1, 2, \cdots, n$, the iteration stops.

Theorem 16. *Let H be a real Hilbert space. For $i = 1, 2, \cdots, n$, let $A_i : H_i \longrightarrow H$ be bounded linear operators with their adjoint operators A_i^*, respectively. And $G_i : H_i \longrightarrow H_i$ are quasinonexpansive mappings and demiclosed at 0. Then, the sequence $\{(x_1^k, x_2^k, \cdots, x_n^k)\}$ generated by Algorithm 15 weakly converges to a solution of ESEFPP (9). Furthermore, for $i = 1, 2, \cdots, n$, $\|x_i^k - G_i(x_i^k)\| \longrightarrow 0$, $\|A_i x_i^k - \omega_k\| \longrightarrow 0$, $\|G_i(u_i^k) - u_i^k\| \longrightarrow 0$ and $\|x_i^{k+1} - x_i^k\| \longrightarrow 0$ as $k \longrightarrow \infty$.*

Proof. Let $(z_1, z_2, \cdots, z_n) \in \Omega_2$ and

$$z = A_1 z_1 = A_2 z_2 = \cdots = A_n z_n. \quad (65)$$

For $i = 1, 2, \cdots, n$, from (63) and Lemma 3 (2), we have

$$\begin{aligned}\left\|u_i^k - z_i\right\|^2 &= \left\|x_i^k - \lambda_k\left(\left(x_i^k - G_i\left(x_i^k\right)\right)\right.\right.\\ &\quad \left.\left.+ A_i^*\left(A_i x_i^k - \omega_k\right)\right) - z_i\right\|^2\\ &= \left\|x_i^k - z_i\right\|^2 + \lambda_k^2\left\|\left(x_i^k - G_i\left(x_i^k\right)\right)\right.\\ &\quad \left.+ A_i^*\left(A_i x_i^k - \omega_k\right)\right\|^2\\ &\quad - 2\lambda_k\left\langle x_i^k - z_i, \left(x_i^k - G_i\left(x_i^k\right)\right)\right.\\ &\quad \left.+ A_i^*\left(A_i x_i^k - \omega_k\right)\right\rangle\\ &= \left\|x_i^k - z_i\right\|^2 + \lambda_k^2\left\|\left(x_i^k - G_i\left(x_i^k\right)\right)\right.\\ &\quad \left.+ A_i^*\left(A_i x_i^k - \omega_k\right)\right\|^2\\ &\quad - 2\lambda_k\left\langle x_i^k - z_i, x_i^k - G_i\left(x_i^k\right)\right\rangle\\ &\quad - 2\lambda_k\left\langle x_i^k - z_i, A_i^*\left(A_i x_i^k - \omega_k\right)\right\rangle.\end{aligned} \quad (66)$$

It follows from

$$\begin{aligned}&-2\left\langle x_i^k - z_i, x_i^k - G_i\left(x_i^k\right)\right\rangle\\ &\quad = -2\left\|x_i^k - z_i\right\|^2 - 2\left\langle x_i^k - z_i, z_i - G_i\left(x_i^k\right)\right\rangle\\ &\quad = -2\left\|x_i^k - z_i\right\|^2 + \left\|x_i^k - z_i\right\|^2 + \left\|G_i\left(x_i^k\right) - z_i\right\|^2\\ &\qquad - \left\|x_i^k - G_i\left(x_i^k\right)\right\|^2 \le -\left\|x_i^k - G_i\left(x_i^k\right)\right\|^2, \quad (67)\\ &-2\left\langle x_i^k - z_i, A_i^*\left(A_i x_i^k - \omega_k\right)\right\rangle\\ &\quad = -2\left\langle A_i x_i^k - A_i z_i, A_i x_i^k - \omega_k\right\rangle\\ &\quad = \|A_i z_i - \omega_k\|^2 - \left\|A_i x_i^k - A_i z_i\right\|^2 - \left\|A_i x_i^k - \omega_k\right\|^2,\end{aligned}$$

then (66) can be rewritten as

$$\begin{aligned}\left\|u_i^k - z_i\right\|^2 &\le \left\|x_i^k - z_i\right\|^2 + \lambda_k^2\left\|\left(x_i^k - G_i\left(x_i^k\right)\right)\right.\\ &\quad \left.+ A_i^*\left(A_i x_i^k - \omega_k\right)\right\|^2\\ &\quad - \lambda_k\left\|x_i^k - G_i\left(x_i^k\right)\right\|^2 + \lambda_k\|A_i z_i - \omega_k\|^2\\ &\quad - \lambda_k\left\|A_i x_i^k - A_i z_i\right\|^2 - \lambda_k\left\|A_i x_i^k - \omega_k\right\|^2.\end{aligned} \quad (68)$$

Since G_i are quasinonexpansive mappings, by (68), we have

$$\begin{aligned}\left\|x_i^{k+1} - z_i\right\|^2 &= \left\|\alpha_k u_i^k + (1 - \alpha_k) G_i\left(u_i^k\right) - z_i\right\|^2\\ &= \alpha_k\left\|u_i^k - z_i\right\|^2 + (1 - \alpha_k)\left\|G_i\left(u_i^k\right) - z_i\right\|^2\\ &\quad - \alpha_k(1 - \alpha_k)\left\|G_i\left(u_i^k\right) - u_i^k\right\|^2\\ &\le \left\|u_i^k - z_i\right\|^2 - \alpha_k(1 - \alpha_k)\left\|G_i\left(u_i^k\right) - u_i^k\right\|^2\\ &\le \left\|x_i^k - z_i\right\|^2 + \lambda_k^2\left\|\left(x_i^k - G_i\left(x_i^k\right)\right)\right.\\ &\quad \left.+ A_i^*\left(A_i x_i^k - \omega_k\right)\right\|^2 - \lambda_k\left\|x_i^k - G_i\left(x_i^k\right)\right\|^2\\ &\quad + \lambda_k\|A_i z_i - \omega_k\|^2 - \lambda_k\left\|A_i x_i^k - A_i z_i\right\|^2\\ &\quad - \lambda_k\left\|A_i x_i^k - \omega_k\right\|^2 - \alpha_k(1 - \alpha_k)\left\|G_i\left(u_i^k\right) - u_i^k\right\|^2.\end{aligned} \quad (69)$$

Note that

$$\begin{aligned}\|A_i z_i - \omega_k\|^2 &= \left\|A_i z_i - \frac{\sum_{i=1}^n A_i x_i^k}{n}\right\|^2 = \left\|\frac{\sum_{i=1}^n\left(A_i z_i - A_i x_i^k\right)}{n}\right\|^2\\ &\le \frac{\sum_{i=1}^n\left\|A_i z_i - A_i x_i^k\right\|^2}{n}.\end{aligned} \quad (70)$$

Summing (69) for i from 1 to n, we can obtain

$$\begin{aligned}\sum_{i=1}^n\left\|x_i^{k+1} - z_i\right\|^2 &\le \sum_{i=1}^n\left\|x_i^k - z_i\right\|^2 + \lambda_k^2\sum_{i=1}^n\left\|\left(x_i^k - G_i\left(x_i^k\right)\right)\right.\\ &\quad \left.+ A_i^*\left(A_i x_i^k - \omega_k\right)\right\|^2 - \lambda_k\sum_{i=1}^n\left\|x_i^k - G_i\left(x_i^k\right)\right\|^2\\ &\quad - \lambda_k\sum_{i=1}^n\left\|A_i x_i^k - \omega_k\right\|^2 - \alpha_k(1 - \alpha_k)\sum_{i=1}^n\\ &\quad \cdot\left\|G_i\left(u_i^k\right) - u_i^k\right\|^2 \le \sum_{i=1}^n\left\|x_i^k - z_i\right\|^2 + \lambda_k\rho_k\\ &\quad \cdot\left(\sum_{i=1}^n\left\|x_i^k - G_i\left(x_i^k\right)\right\|^2 + \sum_{i=1}^n\left\|A_i x_i^k - \omega_k\right\|^2\right)\\ &\quad - \lambda_k\left(\sum_{i=1}^n\left\|x_i^k - G_i\left(x_i^k\right)\right\|^2 + \sum_{i=1}^n\left\|A_i x_i^k - \omega_k\right\|^2\right)\\ &\quad - \alpha_k(1 - \alpha_k)\sum_{i=1}^n\left\|G_i\left(u_i^k\right) - u_i^k\right\|^2\\ &= \sum_{i=1}^n\|x_i^k - z_i\|^2 - \lambda_k(1 - \rho_k)\\ &\quad \cdot\left(\sum_{i=1}^n\left\|x_i^k - G_i\left(x_i^k\right)\right\|^2 + \sum_{i=1}^n\left\|A_i x_i^k - \omega_k\right\|^2\right)\\ &\quad - \alpha_k(1 - \alpha_k)\sum_{i=1}^n\left\|G_i\left(u_i^k\right) - u_i^k\right\|^2.\end{aligned} \quad (71)$$

Let $\Gamma_k(z_1, z_2, \cdots, z_n) = \sum_{i=1}^n\|x_i^k - z_i\|^2$. According to the definition of $\{\lambda_k\}$, we have

$$\Gamma_{k+1}(z_1, z_2, \cdots, z_n) \le \Gamma_k(z_1, z_2, \cdots, z_n). \quad (72)$$

Therefore, the sequence $\{\Gamma_k(z_1, z_2, \cdots, z_n)\}$ is a nonincreasing sequence and lower bounded by 0. As a result $\{\Gamma_k (z_1, z_2, \cdots, z_n)\}$ converges to some finite limit, suppose as $l(z_1, z_2, \cdots, z_n)$, and sequence $\{x_i^k\}$ is bounded. Letting $n \longrightarrow \infty$ and taking limit in the two sides of (71) for $k \longrightarrow \infty$, we obtain that

$$\lim_{k\to\infty} \lambda_k \sum_{i=1}^n \left\| x_i^k - G_i\left(x_i^k\right) \right\|^2 = 0, \lim_{k\to\infty} \lambda_k \sum_{i=1}^n \left\| A_i x_i^k - \omega_k \right\|^2 = 0, \lim_{k\to\infty} \left\| G_i\left(u_i^k\right) - u_i^k \right\| = 0. \quad (73)$$

Since $\lambda_k > \varepsilon$, we have

$$\lim_{k\to\infty} \left\| x_i^k - G_i\left(x_i^k\right) \right\| = 0, \lim_{k\to\infty} \left\| A_i x_i^k - \omega_k \right\| = 0. \quad (74)$$

From the definition of λ_k, we have

$$\lambda_k^2 \left(\sum_{i=1}^n \left\| \left(x_i^k - G_i\left(x_i^k\right)\right) + A_i^*\left(A_i x_i^k - \omega_k\right) \right\|^2 \right) \le \lambda_k \rho_k \left(\sum_{i=1}^n \left\| x_i^k - G_i\left(x_i^k\right) \right\|^2 + \sum_{i=1}^n \left\| A_i x_i^k - \omega_k \right\|^2 \right) \longrightarrow 0, \quad (75)$$

which implies

$$\lambda_k \left\| \left(x_i^k - G_i\left(x_i^k\right)\right) + A_i^*\left(A_i x_i^k - \omega_k\right) \right\| \longrightarrow 0. \quad (76)$$

Now, let us prove that $\{x_i^k\}$ is asymptotically regular. Indeed, from

$$\begin{aligned} \left\| u_i^k - x_i^k \right\| &= \left\| x_i^k - \lambda_k \left(\left(x_i^k - G_i\left(x_i^k\right)\right) + A_i^*\left(A_i x_i^k - \omega_k\right) \right) - x_i^k \right\| \\ &= \lambda_k \left\| \left(x_i^k - G_i\left(x_i^k\right)\right) + A_i^*\left(A_i x_i^k - \omega_k\right) \right\|, \end{aligned} \quad (77)$$

we have

$$\lim_{k\to\infty} \left\| u_i^k - x_i^k \right\| = 0. \quad (78)$$

Consequently,

$$\begin{aligned} \lim_{k\to\infty} \left\| x_i^{k+1} - x_i^k \right\| &= \lim_{k\to\infty} \left\| \alpha_k u_i^k + (1-\alpha_k) G_i\left(u_i^k\right) - x_i^k \right\| \\ &\le \lim_{k\to\infty} \left(\alpha_k \left\| u_i^k - x_i^k \right\| + (1-\alpha_k) \left\| G_i\left(u_i^k\right) - x_i^k \right\| \right) \\ &\le \lim_{k\to\infty} \left(\alpha_k \left\| u_i^k - x_i^k \right\| + (1-\alpha_k) \left\| G_i\left(u_i^k\right) - u_i^k \right\| + (1-\alpha_k) \left\| u_i^k - x_i^k \right\| \right) = 0, \end{aligned} \quad (79)$$

which yields $\{x_i^k\}$ is asymptotically regular.

Let $(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n) \in \omega_\omega(x_1^k, x_2^k, \cdots, x_n^k)$, then there exists a subsequence of $\{x_i^k\}$ which weakly converges to $\bar{x}_i$. For $\lim_{k\to\infty} \|u_i^k - x_i^k\| = 0$, we can obtain $\bar{x}_i \in \omega_\omega(u_i^k)$. Due to the demiclosednesses of G_i at 0 and $\lim_{k\to\infty} \|G_i(u_i^k) - u_i^k\| = 0$, it yields $G_i \bar{x}_i = \bar{x}_i$, which implies $\bar{x}_i \in \mathrm{Fix}(G_i)$. Furthermore, it follows from (74) and the lower semicontinuity of the squared norm, we have

$$\|A_i \bar{x}_i - \bar{\omega}\|^2 \le \liminf_{k\to\infty} \left\| A_i x_i^k - \omega_k \right\|^2 = 0, \quad (80)$$

where $\bar{\omega} = (\sum_{i=1}^n A_i \bar{x}_i)/n$, which means that $A_i \bar{x}_i - \bar{\omega} = 0$. That is,

$$\begin{aligned} A_2 \bar{x}_2 + A_3 \bar{x}_3 + \cdots + A_n \bar{x}_n &= (n-1) A_1 \bar{x}_1, \\ A_1 \bar{x}_1 + A_3 \bar{x}_3 + \cdots + A_n \bar{x}_n &= (n-1) A_2 \bar{x}_2, \\ &\cdots \\ A_1 \bar{x}_1 + A_2 \bar{x}_2 + \cdots + A_{n-1} \bar{x}_{n-1} &= (n-1) A_n \bar{x}_n. \end{aligned} \quad (81)$$

Obviously,

$$\begin{aligned} &A_1 \bar{x}_1 = A_2 \bar{x}_2 = \cdots = A_n \bar{x}_n, \\ &(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n) \in \Omega_2. \end{aligned} \quad (82)$$

Finally, following the proof of Theorem 12, we can prove the uniqueness of the weak cluster points of $\{x_i^k\}$. Thus, we complete the proof.

Similar to the proof of Theorem 16, from Definition 7 and Lemma 8, we have the following corollary.

Corollary 17. *Let H be a real Hilbert space. For $i = 1, 2, \cdots, n$, let $A_i : H_i \longrightarrow H$ be bounded linear operators with their adjoint operators A_i^* and C_i be nonempty closed convex subsets of real Hilbert spaces H_i. Let $\{T_j^i\}_{i=1}^N$ be a finite family of ρ_j^i-strictly pseudononspreading mappings of C_i into itself with $\bigcap_{i=1}^N Fix(T_j^i) \ne \emptyset$ and $\rho^i = \max_{j=1,2,\cdots,N} \{\rho_j^i\}$. For $i = 1, 2, \cdots, N$, let $\pi_j^i = (\alpha_j^i, \beta_j^i, \gamma_j^i, \delta_j^i)$, where $\alpha_j^i, \beta_j^i, \gamma_j^i, \delta_j^i \in [0, 1]$ and $\alpha_j^i + \beta_j^i + \gamma_j^i + \delta_j^i = 1$. Assume that $\rho^i \le \beta_j^i$ for $j = 1, 2, \cdots N$, and $(\alpha_j^i + \beta_j^i) \rho^i \le \beta_j^i \gamma_j^i$ for $j = 2, 3, \cdots, N$, and $(\alpha_1^i + \beta_1^i)\rho^i \le \beta_1^i(\gamma_1^i + \delta_1^i)$. If G_i are the G-mapping generated by $T_1^i, T_2^i, \cdots, T_N^i$ and $\pi_1^i, \pi_2^i, \cdots, \pi_N^i$ and demiclosed at 0, then the sequence $\{(x_1^k, x_2^k, \cdots, x_n^k)\}$ generated by Algorithm 15 weakly converges to a solution of ESEFPP (9). Furthermore, for $i = 1, 2, \cdots, n$, $\|x_i^k - G_i(x_i^k)\| \longrightarrow 0$, $\|A_i x_i^k - \omega_k\| \longrightarrow 0$, $\|G_i(u_i^k) - u_i^k\| \longrightarrow 0$, and $\|x_i^{k+1} - x_i^k\| \longrightarrow 0$ as $k \longrightarrow \infty$.*

5. Application

In this section, we apply our iterative algorithms to some convex and nonlinear problems, for example, [11, 28].

5.1. Split Equality Feasibility Problem. Note that ESEP (5) reduces to SEFP (2) as $n = 2$. Then, we can obtain the following algorithm to study SEFP (2) from Algorithm 9.

Algorithm 18. **Initialization:** let $(x_1^1, x_2^1) \in H_1 \times H_2$ arbitrary.

Iteration step: for a given current iterate $(x_1^k, x_2^k) \in H_1 \times H_2$, we calculate the next iterate (x_1^{k+1}, x_2^{k+1}) by

$$\begin{cases} \omega_k = \dfrac{A_1 x_1^k + A_2 x_2^k}{2}, \\ x_1^{k+1} = x_1^k - \lambda_k\left(\left(x_1^k - P_{C_1}\left(x_1^k\right)\right) + A_1^*\left(A_1 x_1^k - \omega_k\right)\right), \\ x_2^{k+1} = x_2^k - \lambda_k\left(\left(x_2^k - P_{C_2}\left(x_2^k\right)\right) + A_2^*\left(A_2 x_2^k - \omega_k\right)\right), \end{cases} \tag{83}$$

where the stepsize $\{\lambda_k\}$ is chosen as Algorithm 9. Set $k \coloneqq k + 1$ and go to (83). Otherwise, $A_i x_i^k - \omega_k = 0$, for $i = 1, 2$, iteration stops.

5.2. Split Equality Fixed Point Problem. Note that ESEFPP (9) reduces to SEFPP (7) as $n = 2$. Then, we can obtain the following algorithm to study SEFPP (7) from Algorithm 15.

Algorithm 19. **Initialization:** let $0 < a < \alpha_k < b < 1$, $(x_1^1, x_2^1) \in H_1 \times H_2$ arbitrary.

Iteration step: for a given current iterate $(x_1^k, x_2^k) \in H_1 \times H_2$, we calculate the next iterate (x_1^{k+1}, x_2^{k+1}) by

$$\begin{cases} \omega_k = \dfrac{A_1 x_1^k + A_2 x_2^k}{2}, \\ u_1^k = x_1^k - \lambda_k\left(\left(x_1^k - G_1\left(x_1^k\right)\right) + A_1^*\left(A_1 x_1^k - \omega_k\right)\right), \\ x_1^{k+1} = \alpha_k u_1^k + (1-\alpha_k) G_1\left(u_1^k\right), \\ u_2^k = x_2^k - \lambda_k\left(\left(x_2^k - G_2\left(x_2^k\right)\right) + A_2^*\left(A_2 x_2^k - \omega_k\right)\right), \\ x_2^{k+1} = \alpha_k u_2^k + (1-\alpha_k) G_2\left(u_2^k\right), \end{cases} \tag{84}$$

where the stepsize $\{\lambda_k\}$ is chosen as Algorithm 15. Set $k \coloneqq k + 1$ and go to (84). Otherwise, $A_i x_i^k - \omega_k = 0$, for $i = 1, 2$, iteration stops.

5.3. Extend Split Equation Problem. Similar to Algorithm 15, we have another algorithm to solve ESEP (5).

Algorithm 20. **Initialization:** let $0 < a < \alpha_k < b < 1$, $(x_1^1, x_2^1, \cdots, x_n^1) \in H_1 \times H_2 \times \cdots \times H_n$ arbitrary.

Iteration step: for a given current iterate $(x_1^k, x_2^k, \cdots, x_n^k) \in H_1 \times H_2 \times \cdots \times H_n$, we calculate the next iterate $(x_1^{k+1}, x_2^{k+1}, \cdots, x_n^{k+1})$ by

$$\begin{cases} \omega_k = \dfrac{\sum_{i=1}^n A_i x_i^k}{n}, \\ u_1^k = x_1^k - \lambda_k\left(\left(x_1^k - P_{C_1}\left(x_1^k\right)\right) + A_1^*\left(A_1 x_1^k - \omega_k\right)\right), \\ x_1^{k+1} = \alpha_k u_1^k + (1-\alpha_k) P_{C_1}\left(u_1^k\right), \\ u_2^k = x_2^k - \lambda_k\left(\left(x_2^k - P_{C_2}\left(x_2^k\right)\right) + A_2^*\left(A_2 x_2^k - \omega_k\right)\right), \\ x_2^{k+1} = \alpha_k u_2^k + (1-\alpha_k) P_{C_2}\left(u_2^k\right), \\ \cdots \\ u_n^k = x_n^k - \lambda_k\left(\left(x_n^k - P_{C_n}\left(x_n^k\right)\right) + A_n^*\left(A_n x_n^k - \omega_k\right)\right), \\ x_n^{k+1} = \alpha_k u_n^k + (1-\alpha_k) P_{C_n}\left(u_n^k\right), \end{cases} \tag{85}$$

where the stepsize $\{\lambda_k\}$ is chosen as Algorithm 9. Set $k \coloneqq k + 1$ and go to (85). Otherwise, $A_i x_i^k - \omega_k = 0$, for $i = 1, 2, \cdots, n$, iteration stops.

6. Numerical Examples

We are in a position to show some numerical examples to demonstrate the performance and convergence of Algorithms 9 and 13. The whole programs are written in MATLAB 2017b. All the numerical results are carried out on a personal Lenovo computer with Intel(R) Core(TM) i7-7500U CPU 2.70 GHz RAM 4.00 GB. We denote the vector with all elements 1 by e in what follows.

Example 21. Let

$$\begin{aligned} A &= \begin{pmatrix} 0.694828622975817 & 0.950222048838355 & 0.438744359656398 \\ 0.317099480060861 & 0.0344460805029088 & 0.381558457093008 \end{pmatrix}, \\ B &= \begin{pmatrix} 0.765516788149002 & 0.186872604554379 \\ 0.795199901137063 & 0.489764395788231 \end{pmatrix}, \\ C &= \begin{pmatrix} 0.445586200710900 & 0.709364830858073 & 0.276025076998578 & 0.655098003973841 \\ 0.646313010111265 & 0.754686681982361 & 0.679702676853675 & 0.162611735194631 \end{pmatrix}. \end{aligned} \tag{86}$$

Table 1: The numerical results of Example 22.

	P	M	N	Q	Algorithm 9		Algorithm 13		ECCL	
					n	s	n	s	n	s
Case 1	10	20	9	25	191	0.016657	144	0.012685	206	0.027401
	25	35	30	35	475	0.019802	248	0.023921	555	0.044554
	50	30	40	50	1684	0.201384	857	0.103873	2675	0.202256
	10	20	9	25	184	0.019802	113	0.009622	230	0.026396
Case 2	25	35	30	35	367	0.048778	146	0.0017966	433	0.022374
	50	30	40	50	2147	0.264869	758	0.089149	2533	0.179859
	10	20	9	25	230	0.019426	170	0.015182	232	0.011593
Case 3	25	35	30	35	347	0.038227	269	0.028535	433	0.039384
	50	30	40	50	1935	0.235992	1446	0.175387	2087	0.131646
	10	20	9	25	192	0.016825	121	0.009010	213	0.014185
Case 4	25	35	30	35	361	0.047210	167	0.018008	467	0.040100
	50	30	40	50	2065	0.256091	562	0.066533	2307	0.180630

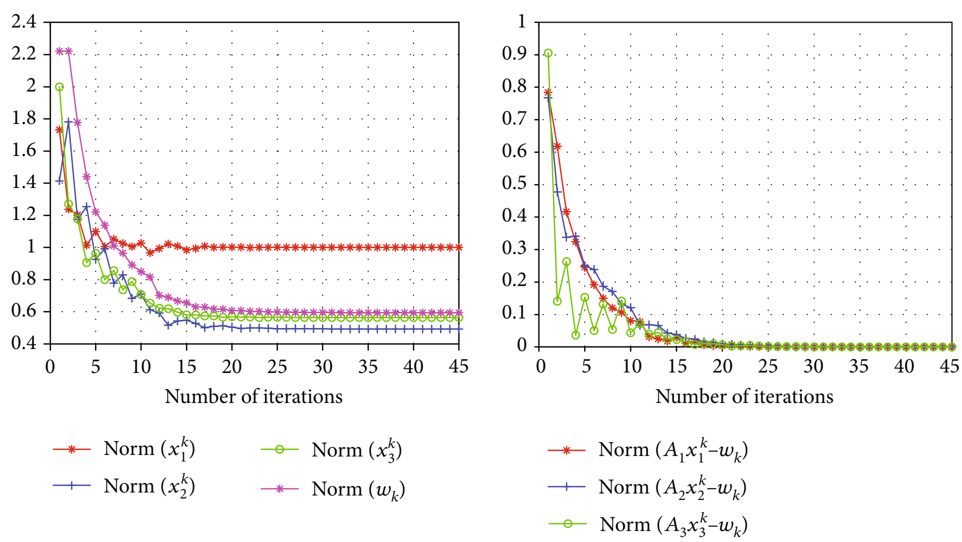

Figure 1: The behaviors of $\{x_i^k\}$, $\{\omega_k\}$, and $\{A_i x_i^k - \omega_k\}$ of Algorithm 9 for Example 21.

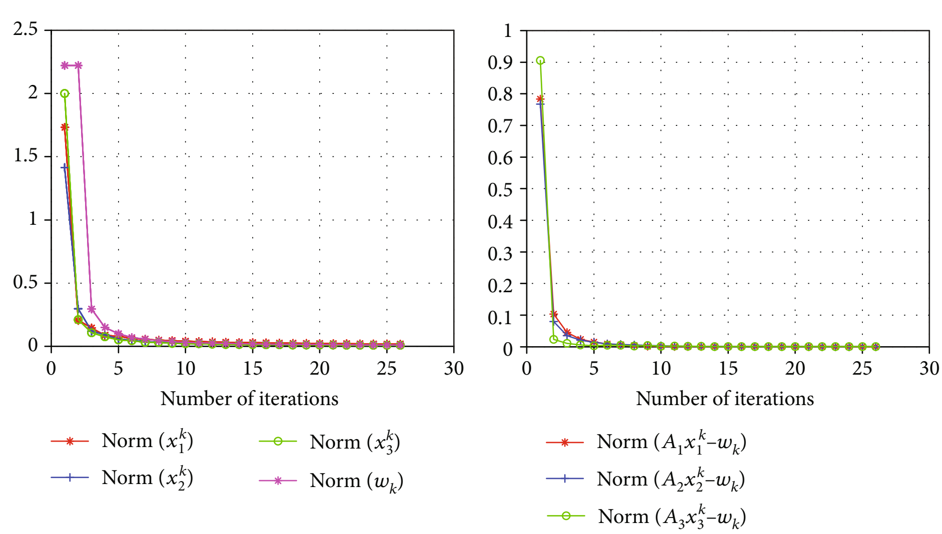

Figure 2: The behaviors of $\{x_i^k\}$, $\{\omega_k\}$, and $\{A_i x_i^k - \omega_k\}$ of Algorithm 13 for Example 21.

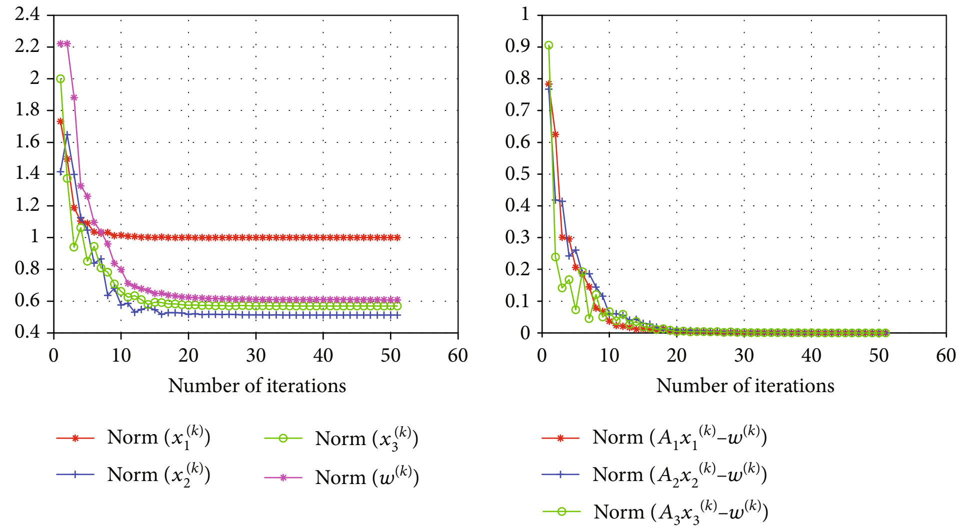

FIGURE 3: The behaviors of $\{x_i^k\}$, $\{\omega_k\}$, and $\{A_i x_i^k - \omega_k\}$ of ECCL for Example 21.

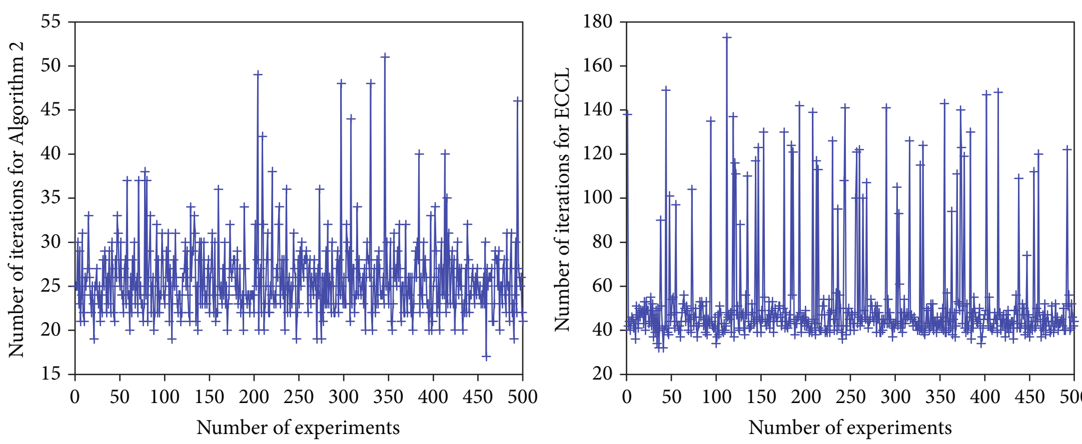

FIGURE 4: The iterative number of Algorithm 13 and ECCL for Case A.

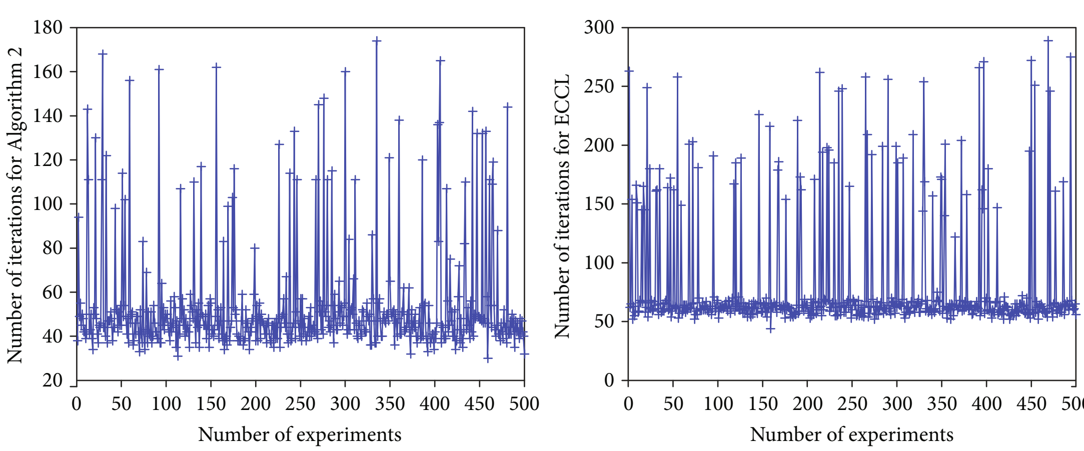

FIGURE 5: The iterative number of Algorithm 13 and ECCL for Case B.

$C_1 = \{x_1 \in R^3 : \|x_1\| \le 1\}$, $C_2 = \{x_2 \in R^2 : -3e \le x_2 \le e\}$, and $C_3 = \{x_3 \in R^4 : -2e \le x_3 \le 2e\}$, to find $(x, y, z) \in C_1 \times C_2 \times C_3$ such that $Ax = By = Cz$.

Example 22. Let $A_1 = (a_{ij})_{P\times M}$, $A_2 = (b_{ij})_{P\times N}$, $A_3 = (c_{ij})_{P\times Q}$, $C_1 = \{x_1 \in R^M : \|x_1\| \le 1\}$, $C_2 = \{x_2 \in R^N : \|x_2\| \le 2\}$, and $C_3 = \{x_3 \in Q : \|x_3\| \le 3\}$, where $a_{ij}, b_{ij}, c_{ij} \in [0, 1]$ are all generated randomly.

In the Example 22, we consider $P = 10, M = 20, N = 9, Q = 25$; $P = 25, M = 35, N = 30, Q = 35$; $P = 50, M = 30, N = 40, Q = 50$, and four initial values.

Case 1. $x = \text{ones}(M, 1), y = \text{ones}(N, 1)$, and $z = 10 * \text{ones}(Q, 1)$.

Case 2. $x = -10 * \text{ones}(M, 1), y = 0 * \text{ones}(N, 1)$, and $z = 10 * \text{ones}(Q, 1)$.

Case 3. $x = 100 * \text{ones}(N, 1), y = \text{ones}(M, 1)$, and $z = -10 * \text{ones}(Q, 1)$.

Case 4. $x = 10 * \text{ones}(N, 1), y = -10 * \text{ones}(M, 1)$, and $z = \text{ones}(Q, 1)$.

We take $\alpha_k = 5/6k$, $\rho_k = 1/2 + 1/10^k$, $(v_1, v_2, \cdots, v_n) = (0, 0, \cdots, 0)$ in Algorithms 9 and 13. In the following tables and figures, we denote Algorithm 5.4 in reference [19] by ECCL. And we set "n" and "s" to express the number of iteration and CPU time in seconds, respectively. We use $\sum_{i=1}^{3} \|A_i x_i^k - \omega_k\| \le 10^{-4}$ as the stop criterion, where $\omega_k = (\sum_{i=1}^{3} A_i x_i^k)/3$. The numerical results can be seen from Table 1 and Figures 1–5.

We take the initial point $x = \text{ones}(3, 1), y = \text{ones}(2, 1)$, and $z = \text{ones}(4, 1)$ in the Example 21. Figures 1–3 express the behaviors of $\{x_i^k\}, \{\omega_k\}$, and $\{A_i x_i^k - \omega_k\}$, for $i = 1, 2, 3$, of Algorithms 9 and 13 and ECCl for Example 21, respectively.

Furthermore, for testing the stationary property of iterative number, we carry 500 experiments for different initial points which are presented randomly, such as

$$\begin{aligned} &\text{Case A } x = \text{rand } (3, 1), y = \text{rand } (2, 1), z = \text{rand } (4, 1), \\ &\text{Case B } x = 1000 * \text{rand } (3, 1), y \\ &= 1000 * \text{rand } (2, 1), z = \text{rand } (4, 1), \end{aligned} \tag{87}$$

in Example 21 for Algorithm 13 and ECCL, the results can be found in Figures 4 and 5, respectively.

Table 1 shows the numerical results of Example 22 when Algorithms 9 and 13 and ECCL select different initial points and matrix dimensions.

From Figures 1–3 and Table 1, we can see that the iterative numbers of Algorithms 9 and 13 are less than ECCL. From Figures 4 and 5 and Table 1, we can see that the stationary property of the iterative number and CPU time of Algorithm 13 are exceeded by ECCL. And Algorithm 2 is better than Algorithm 13 in these aspects.

Data Availability

All data generated or analysed during this study are included in this manuscript.

Conflicts of Interest

The authors declare that they have no competing interests regarding the present manuscript.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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