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Research Article

On Some Integral Inequalities in Quantum Calculus

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The objective of this paper is to establish q-analogue of some well-known inequalities in analysis, namely, Poincaré-type inequalities, Sobolev-type inequalities, and Lyapunov-type inequalities. Our obtained results may serve as a useful source of inspiration for future works in quantum calculus.

1. Introduction and Preliminaries

Mathematical inequalities play a crucial role in the development of various branches of mathematics as well as other disciplines of science. In particular, integral inequalities involving the function and its gradient provide important tools in the proof of regularity of solutions to differential and partial differential equations, stability, boundedness, and approximations. One of these categories of inequalities is the Poincaré-type inequality. Namely, if Ω is a bounded (or bounded at least in one direction) domain of $\mathbb{R}^N$, then, there exists a constant $C = C(\Omega) > 0$ such that for all $u \in H_0^1(\Omega)$,

$$\int_\Omega |u(x)|^2\,dx \le C\int_\Omega |\nabla u(x)|^2 dx. \tag{1}$$

For a smooth bounded domain Ω, the best constant C satisfying the above inequality is equal to $\lambda(\Omega)^{-1}$, where $\lambda(\Omega)$ is the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$, and Δ is the Laplacian operator (see, e.g., [1–5]). Due to the importance of Poincaré inequality in the qualitative analysis of partial differential equations and also in numerical analysis, numerous contributions dealing with generalizations and extensions of this inequality appeared in the literature (see, e.g., [6–17] and the references therein). Another important inequality involving the function and its gradient is the Sobolev inequality (see [18, 19]). Namely, if u is a smooth function of compact support in $\mathbb{R}^2$, then

$$\int_{\mathbb{R}^2} u^4(x)\,dx \le \frac{\kappa}{2}\left(\int_{\mathbb{R}^2} u^2(x)\,dx\right)\left(\int_{\mathbb{R}^2} |\nabla u(x)|^2\,dx\right), \tag{2}$$

where $\kappa > 0$ is a dimensionless constant and ∇u denotes the gradient of u. For further results related to Sobolev-type inequalities and their applications, see, for example, [20–26].

Lyapunov's inequality is one of the important results in analysis. It was shown that this inequality is very useful in the study of spectral properties of differential equations, namely, stability of solutions, eigenvalues, and disconjugacy criteria. More precisely, consider the second order differential equation

$$-\vartheta''(t) = f(t)\vartheta(t),\ m_1 < t < m_2, \tag{3}$$

under the Dirichlet boundary conditions

$$\vartheta(m_i) = 0, \quad i = 1, 2, \tag{4}$$

where $f \in C([m_1, m_2])$. Obviously, the trivial function $\vartheta \equiv 0$ is a solution to (3)–(4). Lyapunov's inequality provides a necessary criterion for the existence of a nontrivial solution. Namely, if $\vartheta \in C^1([m_1, m_2])$ is a nontrivial solution to (3)–(4), then (see Lyapunov [27] and Borg

[28])

$$\int_{m_1}^{m_2} |f(t)|\, dt > \frac{4}{m_1 - m_2}. \quad (5)$$

Since the appearance of the above result, numerous contributions related to Lyapunov-type inequalities have been published (see, e.g., [18, 29–32] and the references therein).

On the other hand, because of its usefulness in several areas of physics (thermostatistics, conformal quantum mechanics, nuclear and high energy physics, black holes, etc.), the theory of quantum calculus received a considerable attention by many researchers from various disciplines (see, e.g., [33–35]).

In this paper, motivated by the abovementioned contributions, our goal is to derive q-analogs of some Poincaré-type inequalities, Sobolev-type inequalities, and Lyapunov-type inequalities. Notice that only the one dimensional case is considered in this work.

We recall below some notions and properties related to q -calculus (see, e.g., [36–51] and the references therein).

We first fix $q \in (0, 1)$. Let $\mathbb{N}$ be the set of positive natural numbers, i.e., $\mathbb{N} = \{1, 2, 3, \cdots\}$, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Definition 1. The q -derivative of a function $\vartheta \in C^1([0, T])$ ($T > 0$) is defined by

$$D_q\vartheta(t) = \begin{cases} \dfrac{\vartheta(t) - \vartheta(qt)}{(1-q)t} & \text{if } 0 < t \leq T, \\ \vartheta'(0) & \text{if } t = 0. \end{cases} \quad (6)$$

Remark 2. Using L'Hospital's rule, one obtains

$$\lim_{t \to 0^+} D_q\vartheta(t) = \vartheta'(0), \quad (7)$$

which shows that $D_q\vartheta \in C([0, T])$ for all $\vartheta \in C^1([0, T])$.

Remark 3. It can be easily seen that

$$\lim_{q \to 1^-} D_q\vartheta(t) = \vartheta'(t), \quad 0 \leq t \leq T. \quad (8)$$

Lemma 4 (see [45]). *Let $\vartheta, \rho \in C^1([0, T])$. Then*

$$D_q(\vartheta\rho)(t) = \vartheta(qt)D_q\rho(t) + \rho(t)D_q\vartheta(t). \quad (9)$$

Definition 5. The q-integral of a function $\vartheta \in C([0, T])$ is defined by

$$\int_0^t \vartheta(\xi)d_q\xi = (1-q)t\sum_{\sigma=0}^{\infty} q^\sigma \vartheta(q^\sigma t), \quad 0 \leq t \leq T, \quad (10)$$

and

$$\int_s^t \vartheta(\xi)d_q\xi = \int_0^t \vartheta(\xi)d_q\xi - \int_0^s \vartheta(\xi)d_q\xi, \quad 0 < s \leq t \leq T. \quad (11)$$

Remark 6. Obviously, if $\vartheta \in C([0, T])$, then

$$\int_0^t \vartheta(\xi)d_q\xi<\infty, \int_s^t \vartheta(\xi)d_q\xi<\infty, \quad 0 < s \leq t \leq T. \quad (12)$$

Lemma 7 (see [39]). *Let $\vartheta, \rho \in C([0, T])$, $0 \leq t \leq T$, $p > 1$ and $p' = p/p - 1$. Then*

(i) $|\int_0^t \vartheta(\xi)d_q\xi| \leq \int_0^t |\vartheta(\xi)|d_q\xi$

(ii) For all $\sigma \in \mathbb{N}_0$, $\vartheta(q^\sigma t) \leq \rho(q^\sigma t)\int_0^t \vartheta(\xi)\, d_q\xi \leq \int_0^t \rho(\xi)\, d_q\xi$

(iii) $\int_0^T |\vartheta(\xi)||\rho(\xi)|d_q\xi \leq (\int_0^T |\vartheta(\xi)|^p\, d_q\xi)^{1/p} (\int_0^T |\rho(\xi)|^{p'}\, d_q\xi)^{1/p'}$.

Lemma 8 (see [45]). *Let $\vartheta \in C^1([0, T])$. Then*

(i) $\int_s^t D_q\vartheta(\xi)d_q\xi = \vartheta(t) - \vartheta(s), 0 \leq s \leq t \leq T$

(ii) $D_q\int_0^t \vartheta(\xi)d_q\xi = \vartheta(t), 0 < t \leq T$

Remark 9. Notice that in general, for $0 < s < t \leq T$,

$$\left|\int_s^t \vartheta(\xi)d_q\xi\right| \leq \int_s^t |\vartheta(\xi)|d_q\xi. \quad (13)$$

Namely, following [40], consider the function $\vartheta : [0, 1] \to \mathbb{R}$ defined by

$$\vartheta(\xi) = \begin{cases} \dfrac{1}{1-q}(4q^{-n}\xi - 1 - 3q) & \text{if } q^{n+1} \leq \xi \leq \dfrac{q^n(1+q)}{2}, n \in \mathbb{N}_0, \\ \dfrac{4}{1-q}(-\xi q^{-n} + 1) - 1 & \text{if } \dfrac{q^n(1+q)}{2} \leq \xi \leq q^n, n \in \mathbb{N}_0, \\ 0 & \text{if } \xi = 0. \end{cases} \quad (14)$$

Then, one has

$$\vartheta(q^n) = -1 \quad \text{and} \quad \vartheta\left(\frac{q^n(1+q)}{2}\right) = 1, \quad \text{for all } n \in \mathbb{N}_0. \quad (15)$$

Therefore, an elementary calculation shows that

$$\int_{1+q/2}^{1} \vartheta(\xi) d_q \xi = -\frac{3+q}{2} \quad \text{and} \quad \int_{1+q/2}^{1} |\vartheta(\xi)| d_q \xi = \frac{1-q}{2}. \tag{16}$$

Hence, one has

$$\left| \int_{1+q/2}^{1} \vartheta(\xi)\, d_q \xi \right| > \int_{1+q/2}^{1} |\vartheta(\xi)| d_q \xi. \tag{17}$$

We have the following integration by parts rule.

Lemma 10 (see [45]). *Let $\vartheta_i \in C^1([0, T])$, $i = 1, 2$. Then*

$$\int_0^T \vartheta_1(\xi) \left(D_q \vartheta_2\right)(\xi)\, d_q \xi = [\vartheta_1(\xi)\vartheta_2(\xi)]_{\xi=0}^{T} - \int_0^T \vartheta_2(q\xi) \left(D_q \vartheta_1\right)(\xi)\, d_q \xi. \tag{18}$$

Let us introduce the set

$$\Lambda_q = \{q^n : n \in \mathbb{N}\} \cup \{0\}. \tag{19}$$

Let $T \in \Lambda_q$, $T > 0$, i.e.,

$$T = q^k, \text{for some } k \in \mathbb{N}, \tag{20}$$

and $I_q = [0, T] \cap \Lambda_q$, i.e.,

$$I_q = \left\{ q^{i+k} : i \in \mathbb{N}_0 \right\} \cup \{0\}. \tag{21}$$

Let $s, t \in I_q$ be such that $0 < s < t$, i.e., $t = q^{i+k}$ for some $i \in \mathbb{N}_0$ and $s = q^{i+k+j}$ for some $j \in \mathbb{N}$. In this case, for $\vartheta \in C([0, T])$, by Definition 5, one has

$$\begin{aligned} \int_s^t \vartheta(\xi)\, d_q \xi &= \int_0^t \vartheta(\xi)\, d_q \xi - \int_0^s \vartheta(\xi)\, d_q \xi = (1-q)t \sum_{\sigma=0}^{\infty} q^\sigma \vartheta(q^\sigma t) \\ &- (1-q)s \sum_{\sigma=0}^{\infty} q^\sigma \vartheta(q^\sigma s) = (1-q) \left(\sum_{\sigma=0}^{\infty} t q^\sigma \vartheta(q^\sigma t) \right. \\ &\left. - \sum_{\sigma=0}^{\infty} s q^\sigma \vartheta(q^\sigma s) \right) = (1-q) \left(\sum_{\sigma=0}^{\infty} q^{\sigma+i+k} \vartheta\left(q^{\sigma+i+k}\right) \right. \\ &\left. - \sum_{\sigma=0}^{\infty} q^{\sigma+i+k+j} \vartheta\left(q^{\sigma+i+k+j}\right) \right) = (1-q) \\ &\left(\sum_{n=i+k}^{\infty} q^n \vartheta(q^n) - \sum_{n=i+k+j}^{\infty} q^n \vartheta(q^n) \right) \\ &= (1-q) \sum_{n=i+k}^{i+k+j-1} q^n \vartheta(q^n), \end{aligned} \tag{22}$$

which is a finite sum. Hence, one deduces the following property.

Lemma 11. *Let $\vartheta, \rho \in C([0, T])$, where $T \in \Lambda_q$, $T > 0$. Let $s, t \in I_q$ be such that $0 < s < t$. Then*

$$\left| \int_s^t \vartheta(\xi)\, d_q \xi \right| \leq \int_s^t |\vartheta(\xi)|\, d_q \xi. \tag{23}$$

2. Poincaré and Sobolev Type Inequalities

Let $q \in (0, 1)$ be fixed.

Theorem 12. *Let $p > 1$ and $T \in \Lambda_q$, $T > 0$. Let $\vartheta \in C^1([0, T])$ be such that*

$$\vartheta(0) = \vartheta(T) = 0. \tag{24}$$

Then

$$\int_0^T |\vartheta(\xi)|^p d_q \xi \leq \left(\frac{T}{2}\right)^p \int_0^T \left|D_q \vartheta(\xi)\right|^p d_q \xi. \tag{25}$$

Proof. Let $t = q^\sigma T$, where $\sigma \in \mathbb{N}$. Notice that since $T \in \Lambda_q$, then $t \in I_q$. By property (i) of Lemma 8, one has

$$\vartheta(t) - \vartheta(0) = \int_0^t D_q \vartheta(\xi) d_q \xi. \tag{26}$$

Since $\vartheta(0) = 0$, it holds that

$$\vartheta(t) = \int_0^t D_q \vartheta(\xi) d_q \xi. \tag{27}$$

Next, by property (i) of Lemma 7, one obtains

$$|\vartheta(t)| \leq \int_0^t \left|D_q \vartheta(\xi)\right| d_q \xi. \tag{28}$$

Again, using property (i) of Lemma 8, and the fact that $\vartheta(T) = 0$, one obtains

$$-\vartheta(t) = \int_t^T D_q \vartheta(\xi) d_q \xi. \tag{29}$$

Hence, by Lemma 11, one deduces that

$$|\vartheta(t)| \leq \int_t^T \left|D_q \vartheta(\xi)\right| d_q \xi. \tag{30}$$

Combining (28) with (30), it holds that

$$|\vartheta(t)| \leq \frac{1}{2} \int_0^T \left|D_q \vartheta(\xi)\right| d_q \xi. \tag{31}$$

On the other hand, by Hölder's inequality (see property

(iii) of Lemma 7), one has

$$\int_0^T \left|D_q\vartheta(\xi)\right| d_q\xi \le \left(\int_0^T \left|D_q\vartheta(\xi)\right|^p d_q\xi\right)^{1/p} \left(\int_0^T 1 d_q\xi\right)^{p-1/p}. \tag{32}$$

Notice that

$$\int_0^T 1 d_q\xi = (1-q)T\sum_{n=0}^{\infty} q^n = T. \tag{33}$$

Therefore,

$$\int_0^T \left|D_q\vartheta(\xi)\right| d_q\xi \le T^{p-1/p}\left(\int_0^T \left|D_q\vartheta(\xi)\right|^p d_q\xi\right)^{1/p}. \tag{34}$$

Combining (31) with (34), one deduces that

$$|\vartheta(t)| \le \frac{T^{p-1/p}}{2}\left(\int_0^T \left|D_q\vartheta(\xi)\right|^p d_q\xi\right)^{1/p}, \tag{35}$$

which yields

$$|\vartheta(t)|^p \le \frac{T^{p-1}}{2^p}\int_0^T \left|D_q\vartheta(\xi)\right|^p d_q\xi, \tag{36}$$

i.e.,

$$|\vartheta(q^\sigma T)|^p \le \frac{T^{p-1}}{2^p}\int_0^T \left|D_q\vartheta(\xi)\right|^p d_q\xi, \quad \sigma \in \mathbb{N}. \tag{37}$$

Notice that since $\vartheta(T) = 0$, the above inequality is also true for $\sigma = 0$. Hence, by property (ii) of Lemma 7, one deduces that

$$\int_0^T |\vartheta(\xi)|^p d_q\xi \le \frac{T^{p-1}}{2^p}\int_0^T \left|D_q\vartheta(\xi)\right|^p d_q\xi \int_0^T 1\, d_q\xi. \tag{38}$$

Finally, (25) follows from (33) and (38).

Remark 13. Inequality (25) is the one dimensional q-analog of the Poincaré-type inequality derived by Pachpatte [11].

Theorem 14. *Let $p_1, p_2 > 1$ and $T \in \Lambda_q$, $T > 0$. Let $\vartheta_1, \vartheta_2 \in C^1([0, T])$ be such that*

$$\vartheta_i(0) = \vartheta_i(T) = 0, i = 1, 2. \tag{39}$$

Then

$$\int_0^T |\vartheta_1(\xi)|^{p_1}|\vartheta_2(\xi)|^{p_2} d_q\xi \le \frac{1}{2}\left(\frac{T}{2}\right)^{p_1+p_2} \int_0^T \left(\left|D_q\vartheta_1(\xi)\right|^{2p_1} + \left|D_q\vartheta_2(\xi)\right|^{2p_2}\right) d_q\xi. \tag{40}$$

Proof. From (37) and (70), one has

$$\left|\vartheta_1(q^\sigma T)\mid^{p_1} \le \frac{T^{p_1-1}}{2^{p_1}}\int_0^T \left|D_q\vartheta_1(\xi)\right|^{p_1} d_q\xi, \sigma \in \mathbb{N}_0,\right. \tag{41}$$

$$|\vartheta_2(q^\sigma T)|^{p_2} \le \frac{T^{p_2-1}}{2^{p_2}}\int_0^T \left|D_q\vartheta_2(\xi)\right|^{p_2} d_q\xi, \quad \sigma \in \mathbb{N}_0. \tag{42}$$

Multiplying (41) by (42), one obtains

$$|\vartheta_1(q^\sigma T)|^{p_1}|\vartheta_2(q^\sigma T)|^{p_2} \le \frac{T^{p_1+p_2-2}}{2^{p_1+p_2}}\left(\int_0^T \left|D_q\vartheta_1(\xi)\right|^{p_1} d_q\xi\right) \cdot \left(\int_0^T \left|D_q\vartheta_2(\xi)\right|^{p_2} d_q\xi\right).$$

Next, using the inequality $2AB \le A^2 + B^2$, $A, B \in \mathbb{R}$, one deduces that

$$|\vartheta_1(q^\sigma T)|^{p_1}|\vartheta_2(q^\sigma T)|^{p_2} \le \frac{T^{p_1+p_2-2}}{2^{p_1+p_2+1}}\left[\left(\int_0^T \left|D_q\vartheta_1(\xi)\right|^{p_1} d_q\xi\right)^2 + \left(\int_0^T \left|D_q\vartheta_2(\xi)\right|^{p_2} d_q\xi\right)^2\right]. \tag{44}$$

On the other hand, by Hölder's inequality (see property (iii) of Lemma 7), for $i = 1, 2$, one has

$$\left(\int_0^T \left|D_q\vartheta_i(\xi)\right|^{p_i} d_q\xi\right)^2 \le T\int_0^T \left|D_q\vartheta_i(\xi)\right|^{2p_i} d_q\xi. \tag{45}$$

Hence, combining (44) with (45), it holds that

$$|\vartheta_1(q^\sigma T)|^{p_1}|\vartheta_2(q^\sigma T)|^{p_2} \le \frac{T^{p_1+p_2-1}}{2^{p_1+p_2+1}}\int_0^T \cdot \left(\left|D_q\vartheta_1(\xi)\right|^{2p_1} + \left|D_q\vartheta_2(\xi)\right|^{2p_2}\right) d_q\xi. \tag{46}$$

Since the above inequality holds for all $\sigma \in \mathbb{N}_0$, by property (ii) of Lemma 7, one deduces that

$$\int_0^T |\vartheta_1(\xi)|^{p_1}|\vartheta_2(\xi)|^{p_2} d_q\xi \le \frac{T^{p_1+p_2-1}}{2^{p_1+p_2+1}}\int_0^T \cdot \left(\left|D_q\vartheta_1(\xi)\right|^{2p_1} + |D_q\vartheta_2(\xi)\mid\right) d_q\xi \int_0^T 1\, d_q\xi. \tag{47}$$

Finally, (40) follows from (33) and (47).

Remark 15. Inequality (40) is the one dimensional q-analog of the Poincaré-type inequality derived by Pachpatte [10].

Theorem 16. *Let $p > 1$, $m > (p/2(p-1))$, $N \in \mathbb{N}$ and $T \in \Lambda_q$, $T > 0$. Let $\vartheta_i \in C^1([0, T])$, $i = 1, 2, \cdots, N$ be such that*

$$\vartheta_i(0) = \vartheta_i(T) = 0. \tag{48}$$

Then

$$\left[\int_0^T \left(\sum_{i=1}^N |\vartheta_i(\xi)|^2\right)^{p/p-1} d_q\xi\right]^{2m(p-1)/p} \cdot \leq \frac{1}{N}\left(\frac{N}{4}\right)^{2m} T^{(6m-1)p-2m/p} \sum_{i=1}^N \int_0^T \left|D_q\vartheta_i(\xi)\right|^{4m} d_q\xi. \quad (49)$$

Proof. Let $t = q^\sigma T$, where $\sigma \in \mathbb{N}$. From (31), one has

$$|\vartheta_i(t)| \leq \frac{1}{2}\int_0^T |D_q\vartheta_i(\xi)|\, d_q\xi, i = 1, 2, \cdots, N. \quad (50)$$

On the other hand, by Hölder's inequality (see property (iii) of Lemma 7) and (33), one has

$$\left(\int_0^T |\, D_q\vartheta_i(\xi)\,|\ d_q\xi\right)^2 \leq T\int_0^T \left|D_q\vartheta_i(\xi)\right|^2 d_q\xi. \quad (51)$$

Hence, by (50), one deduces that

$$|\vartheta_i(t)|^2 \leq \frac{T}{4}\int_0^T \left|D_q\vartheta_i(\xi)\right|^2 d_q\xi, \quad i = 1, 2, \cdots, N, \quad (52)$$

which yields

$$\left(\sum_{i=1}^N |\vartheta_i(t)|^2\right)^{p/p-1} \leq \left(\frac{T}{4}\right)^{p/p-1} \left(\sum_{i=1}^N \int_0^T \left|D_q\vartheta_i(\xi)\right|^2 d_q\xi\right)^{p/p-1}. \quad (53)$$

Next, using the discrete version of Hölder's inequality, one obtains

$$\begin{aligned}&\left(\sum_{i=1}^N |\vartheta_i(t)|^2\right)^{p/p-1}\\ &\leq \left(\frac{T}{4}\right)^{p/p-1} \left(\sum_{i=1}^N 1\right)^{1/p-1} \sum_{i=1}^N \left(\int_0^T \left|D_q\vartheta_i(\xi)\right|^2 d_q\xi\right)^{p/p-1}\\ &= N^{1/p-1}\left(\frac{T}{4}\right)^{p/p-1} \sum_{i=1}^N \left(\int_0^T \left|D_q\vartheta_i(\xi)\right|^2 d_q\xi\right)^{p/p-1}.\end{aligned} \quad (54)$$

On the other hand, by Hölder's inequality (see property (iii) of Lemma 7) and (33), one has

$$\begin{aligned}&\int_0^T \left|D_q\vartheta_i(\xi)\right|^2 d_q\xi\\ &\leq \left(\int_0^T 1\, d_q\xi\right)^{1/p} \left(\int_0^T \left|D_q\vartheta_i(\xi)\right|^{2p/p-1} d_q\xi\right)^{p-1/p}\\ &= T^{1/p}\left(\int_0^T \left|D_q\vartheta_i(\xi)\right|^{2p/p-1} d_q\xi\right)^{p-1/p}.\end{aligned} \quad (55)$$

Therefore, by (54), one deduces that

$$\left(\sum_{i=1}^N |\vartheta_i(t)|^2\right)^{p/p-1} \leq N^{1/p-1}\left(\frac{T}{4}\right)^{p/p-1} T^{1/p-1} \sum_{i=1}^N \int_0^T \left|D_q\vartheta_i(t)\right|^{2p/p-1} d_q\xi. \quad (56)$$

Since the above inequality is true for all $\sigma \in \mathbb{N}_0$ (recall that $t = q^\sigma T$), by property (ii) of Lemma 7, and using (33), one deduces that

$$\int_0^T \left(\sum_{i=1}^N |\vartheta_i(\xi)|^2\right)^{p/p-1} d_q\xi \leq N^{1/p-1}\left(\frac{T^2}{4}\right)^{p/p-1} \sum_{i=1}^N \int_0^T \left|D_q\vartheta_i(t)\right|^{2p/p-1} d_q\xi, \quad (57)$$

which yields

$$\left[\int_0^T \left(\sum_{i=1}^N |\vartheta_i(\xi)|^2\right)^{p/p-1} d_q\xi\right]^{2m(p-1)/p} \cdot \leq N^{2m/p}\left(\frac{T^2}{4}\right)^{2m} \left(\sum_{i=1}^N \int_0^T \left|D_q\vartheta_i(t)\right|^{2p/p-1} d_q\xi\right)^{2m(p-1)/p}. \quad (58)$$

Next, using Hölder's inequality with exponents $2m(p-1)/2m(p-1)-p$ and $2m(p-1)/p$ (notice that $2m(p-1)/p > 1$ by assumption), one obtains

$$\begin{aligned}&\int_0^T \left|D_q\vartheta_i(t)\right|^{2p/p-1} d_q\xi\\ &\leq \left(\int_0^T 1\, d_q\xi\right)^{2m(p-1)-p/2m(p-1)} \left(\int_0^T \left|D_q\vartheta_i(t)\right|^{4m} d_q\xi\right)^{p/2m(p-1)}\\ &= T^{2m(p-1)-p/2m(p-1)}\left(\int_0^T \left|D_q\vartheta_i(t)\right|^{4m} d_q\xi\right)^{p/2m(p-1)},\end{aligned} \quad (59)$$

which yields

$$\begin{aligned}&\sum_{i=1}^N \int_0^T \left|D_q\vartheta_i(t)\right|^{2p/p-1} d_q\xi \leq T^{2m(p-1)-p/2m(p-1)}\\ &\quad \sum_{i=1}^N \left(\int_0^T \left|D_q\vartheta_i(t)\right|^{4m} d_q\xi\right)^{p/2m(p-1)}.\end{aligned} \quad (60)$$

Furthermore, the discrete Hölder's inequality shows that

$$\begin{aligned}&\sum_{i=1}^N \left(\int_0^T \left|D_q\vartheta_i(t)\right|^{4m} d_q\xi\right)^{p/2m(p-1)} \leq N^{2m(p-1)-p/2m(p-1)}\\ &\quad \cdot \left(\sum_{i=1}^N \int_0^T \left|D_q\vartheta_i(t)\right|^{4m} d_q\xi\right)^{p/2m(p-1)}.\end{aligned} \quad (61)$$

Hence, by (60), one deduces that

$$\sum_{i=1}^{N}\int_0^T \left|D_q\vartheta_i(t)\right|^{2p/p-1} d_q\xi \le T^{2m(p-1)-p/2m(p-1)}N^{2m(p-1)-p/2m(p-1)} \cdot \left(\sum_{i=1}^{N}\int_0^T \left|D_q\vartheta_i(t)\right|^{4m} d_q\xi\right)^{p/2m(p-1)}. \tag{62}$$

Finally, combining (58) with (62), (49) follows.

Remark 17. Inequality (49) is the one dimensional q-analog of the Poincaré-type inequality derived by Pachpatte [12].

Theorem 18. *Let $T \in \Lambda_q$, $T > 0$. Let $\vartheta \in C^1([0,T])$ be such that*

$$\vartheta(0) = \vartheta(T) = 0. \tag{63}$$

Then

$$\int_0^T \vartheta^2(\xi)\, d_q\xi \le \frac{T}{2}\left(\int_0^T \left(|\vartheta(q\xi)| + |\vartheta(\xi)|\right)^2 d_q\xi\right)^{1/2}\left(\int_0^T \left|D_q\vartheta(\xi)\right|^2 d_q\xi\right)^{1/2}. \tag{64}$$

Proof. Let $t = q^\sigma T$, where $\sigma \in \mathbb{N}$. By Lemma 4, property (i) of Lemma 8, and using the boundary conditions, one has

$$\vartheta^2(t) = \int_0^t \left(\vartheta(q\xi) + \vartheta(\xi)\right) D_q\vartheta(\xi)\, d_q\xi \tag{65}$$

and

$$-\vartheta^2(t) = \int_t^T \left(\vartheta(q\xi) + \vartheta(\xi)\right) D_q\vartheta(\xi)\, d_q\xi. \tag{66}$$

Combining (65) with (66), it holds that

$$\vartheta^2(t) \le \frac{1}{2}\int_0^T \left(|\vartheta(q\xi)| + |\vartheta(\xi)|\right)\left|D_q\vartheta(\xi)\right| d_q\xi. \tag{67}$$

Using Hölder's inequality, one obtains

$$\vartheta^2(t) \le \frac{1}{2}\left(\int_0^T \left(|\vartheta(q\xi)| + |\vartheta(\xi)|\right)^2 d_q\xi\right)^{1/2}\left(\int_0^T \left|D_q\vartheta(\xi)\right|^2 d_q\xi\right)^{1/2}. \tag{68}$$

Since the above inequality holds for all $\sigma \in \mathbb{N}_0$, using property (ii) of Lemma 7, integrating over $(0,T)$, and using (33), (64) follows.

Remark 19. Inequality (64) is the one dimensional q-analog of the Sobolev-type inequality derived by Pachpatte [11].

3. Lyapunov-Type Inequalities

We fix $q \in (0,1)$ and $T \in \Lambda_q$, $T > 0$. Consider the second order q-difference equation

$$-D_q\left(D_q\vartheta\right)(t/q) + a(t)D_q\vartheta(t) = f(t)\varphi(\vartheta(t)), \quad 0 < t < T, \tag{69}$$

under the boundary conditions

$$\vartheta(0) = \vartheta(T) = 0, \tag{70}$$

where $a, f \in C([0,T])$ and $\varphi : \mathbb{R} \to \mathbb{R}$. We suppose that there exists a constant $L_\varphi > 0$ such that

$$|\varphi(x)| \le L_\varphi |x|, \quad x \in \mathbb{R}. \tag{71}$$

Obviously, from (71), one has $\varphi(0) = 0$. Hence, $\vartheta \equiv 0$ is a trivial solution to (69) and (70). The following theorem provides a necessary condition for the existence of a nontrivial solution to (69) and (70) satisfying $\vartheta(t) \neq 0$, $0 < t < T$.

Theorem 20. *Suppose that $\vartheta \in C^1([0,T])$ is a solution to (69) and (70) satisfying*

$$\vartheta(t) \neq 0, \quad 0 < t < T. \tag{72}$$

Then

$$1 \le L_\varphi \int_0^T \sqrt{\frac{\xi(T-\xi)}{4}}|f(\xi)|d_q\xi + \left(\int_0^T \sqrt{\frac{\xi(T-\xi)}{4}}|a(\xi)|^2 d_q\xi\right)^{1/2}. \tag{73}$$

Proof. Let $s = q^\sigma T$, where $\sigma \in \mathbb{N}$. Since $\vartheta(0) = 0$, using property (i) of Lemma 8, one has

$$\vartheta(s) = \int_0^s D_q\vartheta(\xi) d_q\xi. \tag{74}$$

By Hölder's inequality (see property (iii) of Lemma 7) and (33), one obtains

$$|\vartheta(s)| \le \sqrt{s}\left(\int_0^s \left|D_q\vartheta(\xi)\right|^2 d_q\xi\right)^{1/2}, \tag{75}$$

which yields

$$|\vartheta(s)|^2 \le s\int_0^s \left|D_q\vartheta(\xi)\right|^2 d_q\xi. \tag{76}$$

Similarly, since $\vartheta(T) = 0$, one has

$$-\vartheta(s) = \int_s^T D_q\vartheta(\xi) d_q\xi, \tag{77}$$

which implies that (see Lemma 11)

$$|\vartheta(s)| \le \int_s^T |D_q\vartheta(\xi)| d_q\xi. \tag{78}$$

Since $s, T \in I_q$, then $\int_s^T |D_q\vartheta(\xi)| d_q\xi$ is a finite sum (see (22)). Hence, we can apply Hölder's inequality to get

$$|\vartheta(s)|^2 \le (T-s)\int_s^T |D_q\vartheta(\xi)|^2 d_q\xi. \tag{79}$$

Multiplying (76) by (79), one obtains

$$|\vartheta(s)|^4 \le s(T-s)\left(\int_0^s |D_q\vartheta(\xi)|^2 d_q\xi\right)\left(\int_s^T |D_q\vartheta(\xi)|^2 d_q\xi\right), \tag{80}$$

i.e.,

$$|\vartheta(s)|^2 \le \sqrt{s(T-s)}\left(\int_0^s |D_q\vartheta(\xi)|^2 d_q\xi\right)^{1/2}\left(\int_s^T |D_q\vartheta(\xi)|^2 d_q\xi\right)^{1/2}. \tag{81}$$

Using the inequality $2AB \le A^2 + B^2$, $A, B \in \mathbb{R}$, it holds that

$$|\vartheta(s)|^2 \le \frac{\sqrt{s(T-s)}}{2}\left[\int_0^s |D_q\vartheta(\xi)|^2 d_q\xi + \int_s^T |D_q\vartheta(\xi)|^2 d_q\xi\right], \tag{82}$$

i.e., (recall that $s = q^\sigma T$ and $\vartheta(T) = 0$)

$$|\vartheta(q^\sigma T)|^2 \le \frac{\sqrt{q^\sigma T(T-q^\sigma T)}}{2}\int_0^T |D_q\vartheta(\xi)|^2 d_q\xi, \quad \sigma \in \mathbb{N}_0. \tag{83}$$

Consider now the function

$$w(t) = D_q\vartheta(t/q), \quad 0 < t < T. \tag{84}$$

By (69), one has

$$-D_q w(t) + a(t)D_q\vartheta(t)(t) = f(t)\varphi(\vartheta(t)), \quad 0 < t < T. \tag{85}$$

Multiplying (85) by $\vartheta(t)$ and integrating over $(0, T)$, one obtains

$$-\int_0^T \vartheta(\xi)D_q w(\xi)d_q\xi + \int_0^T a(\xi)\vartheta(\xi)D_q\vartheta(\xi)d_q\xi = \int_0^T f(\xi)\varphi(\vartheta(\xi))\vartheta(\xi)d_q\xi. \tag{86}$$

On the other hand, using the integration by parts rule (see Lemma 10) and the boundary conditions (70), one has

$$-\int_0^T \vartheta(\xi)D_q w(\xi)d_q\xi = \int_0^T w(q\xi)D_q\vartheta(\xi)d_q\xi. \tag{87}$$

Hence, by (86) and the definition of w, one deduces that

$$\int_0^T |D_q\vartheta(\xi)|^2 d_q\xi = \int_0^T f(\xi)\varphi(\vartheta(\xi))\vartheta(\xi)d_q\xi - \int_0^T \boxtimes a(\xi)\vartheta(\xi)D_q\vartheta(\xi)d_q\xi. \tag{88}$$

Next, using (71), one obtains

$$\int_0^T |D_q\vartheta(\xi)|^2 d_q\xi \le L_\varphi\int_0^T |f(\xi)|\vartheta^2(\xi)d_q\xi + \int_0^T |a(\xi)||\vartheta(\xi)||D_q\vartheta(\xi)|d_q\xi. \tag{89}$$

Furthermore, by (83) and property (ii) of Lemma 7, one deduces that

$$\begin{aligned}\int_0^T |D_q\vartheta(\xi)|^2 d_q\xi \le \frac{L_\varphi}{2}&\left(\int_0^T |D_q\vartheta(\xi)|^2 d_q\xi\right)\\ &\cdot\left(\int_0^T |f(\xi)|\sqrt{\xi(T-\xi)}d_q\xi\right)\\ &+\frac{1}{\sqrt{2}}\left(\int_0^T |a(\xi)|[\xi(T-\xi)]^{1/4}|D_q\vartheta(\xi)|d_q\xi\right)\\ &\cdot\left(\int_0^T |D_q\vartheta(\xi)|^2 d_q\xi\right)^{1/2}.\end{aligned} \tag{90}$$

Therefore, by Hölder's inequality, it holds that

$$\begin{aligned}\int_0^T |D_q\vartheta(\xi)|^2 d_q\xi \le \frac{L_\varphi}{2}&\left(\int_0^T |D_q\vartheta(\xi)|^2 d_q\xi\right)\\ &\cdot\left(\int_0^T |f(\xi)|\sqrt{\xi(T-\xi)}d_q\xi\right)\\ &+\frac{1}{\sqrt{2}}\left(\int_0^T |a(\xi)|^2\sqrt{\xi(T-\xi)}d_q\xi\right)^{1/2}\\ &\cdot\left(\int_0^T |D_q\vartheta(\xi)|^2 d_q\xi\right).\end{aligned} \tag{91}$$

Next, we claim that

$$\int_0^T |D_q\vartheta(\xi)|^2 d_q\xi \ne 0. \tag{92}$$

Indeed, suppose that $\int_0^T |D_q\vartheta(\xi)|^2 d_q\xi = 0$. By Definition 5, one obtains

$$(1-q)T\sum_{\tau=0}^{\infty} q^\tau |D_q\vartheta(q^\tau T)||D_q\vartheta(q^\tau T)|^2 = 0, \tag{93}$$

which yields

$$D_q\vartheta(q^\tau T) = 0, \tau \in \mathbb{N}_0. \tag{94}$$

In particular, for $\tau = 1$, one has

$$D_q\vartheta(T) = 0, \tag{95}$$

i.e.,

$$\vartheta(T) - \vartheta(qT) = 0. \tag{96}$$

Since $\vartheta(T) = 0$, one deduces that $\vartheta(qT) = 0$, which contradicts (72). This proves (92). Now, dividing (91) by $\int_0^T |D_q\vartheta(\xi)|^2 \, d_q\xi > 0$, it holds that

$$1 \le \frac{L_\varphi}{2}\int_0^T |f(\xi)|\sqrt{\xi(T-\xi)}\, d_q\xi + \frac{1}{\sqrt{2}}\left(\int_0^T |a(\xi)|^2\sqrt{\xi(T-\xi)}\, d_q\xi\right)^{1/2}, \tag{97}$$

which yields (73).

Using the inequality

$$\xi(T-\xi) \le \frac{T^2}{4}, \quad 0 < \xi < T, \tag{98}$$

one deduces from Theorem 20 the following result.

Corollary 21. *Suppose that $\vartheta \in C^1([0, T])$ is a solution to (69) and (70) satisfying (72). Then*

$$1 \le \frac{L_\varphi T}{4}\int_0^T |f(\xi)|\, d_q\xi + \frac{\sqrt{T}}{2}\left(\int_0^T |a(\xi)|^2 \, d_q\xi\right)^{1/2}. \tag{99}$$

Consider now the second order q-difference equation

$$-D_q\left(D_q\vartheta\right)(t/q) = f(t)\varphi(\vartheta(t)), \quad 0 < t < T, \tag{100}$$

under the boundary conditions (70), where $f \in C([0, T])$ and $\varphi : \mathbb{R} \to \mathbb{R}$ satisfies (71). Notice that (100) is a special case of (69) with $a \equiv 0$. Hence, by Theorem 20 and Corollary 21, one deduces the following results.

Corollary 3.2. *Suppose that $\vartheta \in C^1([0, T])$ is a solution to (100) and (70) satisfying (72). Then*

$$\int_0^T \sqrt{\xi(T-\xi)}|f(\xi)|\, d_q\xi \ge \frac{2}{L_\varphi}. \tag{101}$$

Corollary 22. *Suppose that $\vartheta \in C^1([0, T])$ is a solution to (100) and (70) satisfying (72). Then*

$$\int_0^T |f(\xi)|\, d_q\xi \ge \frac{4}{L_\varphi T}. \tag{102}$$

Remark 23. Inequality (102) with $\varphi(x) = x$ ($L_\varphi = 1$) is the q-analogue of Lyapunov inequality (5) with $m_1 = 0$ *and* $m_2 = T$.

4. Conclusion

Integral inequalities involving the function and its gradient are very useful in the study of existence, uniqueness, and qualitative properties of solutions to ordinary and partial differential equations. Motivated by the importance of q-calculus in applications, integral inequalities involving the function and its q-derivative are obtained. Namely, we derived the q-analogue of some Poincaré-type inequalities and Sobolev-type inequalities. We also established the q-analogue of some Lyapunov-type inequalities. We hope that our results will serve as a useful inspiration for future works in the context of q-calculus.

Data Availability

No data were used in this study.

Conflicts of Interest

The authors declare no conflict of interest.

Authors' Contributions

All authors contributed equally to this work.

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