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Research Article

A Relaxed Self-Adaptive Projection Algorithm for Solving the Multiple-Sets Split Equality Problem

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In this article, we introduce a relaxed self-adaptive projection algorithm for solving the multiple-sets split equality problem. Firstly, we transfer the original problem to the constrained multiple-sets split equality problem and a fixed point equation system is established. Then, we show the equivalence of the constrained multiple-sets split equality problem and the fixed point equation system. Secondly, we present a relaxed self-adaptive projection algorithm for the fixed point equation system. The advantage of the self-adaptive step size is that it could be obtained directly from the iterative procedure. Furthermore, we prove the convergence of the proposed algorithm. Finally, several numerical results are shown to confirm the feasibility and efficiency of the proposed algorithm.

1. Introduction

Let H_1, H_2, and H_3 be real Hilbert spaces. For $i = 1, 2, \cdots, t$, $j = 1, 2, \cdots, r$, C_i and Q_j are nonempty closed convex subsets of Hilbert spaces H_1 and H_2, respectively, and assume that $A : H_1 \longrightarrow H_3, B : H_2 \longrightarrow H_3$ are two bounded linear operators. The multiple-sets split equality problem (MSSEP) is to find x and y satisfying the property

$$x \in C = \bigcap_{i=1}^{t} C_i,\ y \in Q = \bigcap_{j=1}^{r} Q_j \text{ such that } Ax = By. \quad (1)$$

When $B = I$, MSSEP (1) reduces to the multiple-sets split feasibility problem

$$\text{find a point } x \in C = \bigcap_{i=1}^{t} C_i, Ax \in Q = \bigcap_{j=1}^{r} Q_j, \quad (2)$$

which is applied to intensity-modulated radiation therapy [1–11], signal processing [12–21], and image reconstruction [22–38]. Censor et al. [39] proposed the proximity function $p(x)$ to measure the distance of a point to all sets

$$p(x) = \frac{1}{2}\sum_{i=1}^{t} l_i \left\| x - P_{C_i}(x) \right\|^2 + \frac{1}{2}\sum_{j=1}^{r} \lambda_j \left\| Ax - P_{Q_j}(Ax) \right\|^2, \quad (3)$$

where $l_i > 0$ for all i, and $\lambda_j > 0$ for all j with $\sum_{i=1}^{t} l_i + \sum_{j=1}^{r} \lambda_j = 1$. To solve (2), they considered the following constrained MSSEP:

$$\text{find a point } x \in \Omega \text{ such that } x \text{ solves } (2), \quad (4)$$

and then presented the projection method

$$x_{k+1} = P_{\Omega}(x_k - s\nabla p(x)), \quad (5)$$

where $s > 0$ and Ω is an auxiliary simple nonempty closed convex set with $\Omega \cap S \neq \varnothing$, and S denotes the solution set of (2). The convergence of the projection method was obtained under some mild conditions.

When $t = r = 1$, MSSEP (1) reduces to the split equality problem which was introduced by Moudafi [40] as follows:

$$\text{find two points } x \in C, y \in Q \text{ such that } Ax = By, \quad (6)$$

which is applied to the game theory [41] and optimal control and approximation theory [42]. The following alternating CQ algorithm (ACQ) was introduced by Moudafi [40] as follows:

$$\begin{cases} x_{k+1} = P_C(x_k - \gamma_k A^*(Ax_k - By_k)), \\ y_{k+1} = P_Q(y_k + \beta_k B^*(Ax_{k+1} - By_k)), \end{cases} \quad (7)$$

where $\gamma_k, \beta_k \in (\varepsilon, \min\{(1/\lambda_A), (1/\lambda_B)\} - \varepsilon)$ for small enough $\varepsilon > 0$, A^* and B^* denote the adjoint of A and B, respectively. λ_A and λ_B are the spectral radiuses of A^*A and B^*B, respectively. Since the computation of P_C and P_Q onto a closed convex subset might be hard to be implemented, Fukushima [43] suggested a way to compute the projection onto a level set of a convex function by considering a sequence of projections onto half-spaces containing the original level set. Then, Moudafi [44] introduced the following relaxed alternating CQ algorithm (RACQ):

$$\begin{cases} x_{k+1} = P_{C_k}(x_k - \gamma_k A^*(Ax_k - By_k)), \\ y_{k+1} = P_{Q_k}(y_k + \beta_k B^*(Ax_{k+1} - By_k)), \end{cases} \quad (8)$$

where C_k and Q_k are two sequences of closed convex sets.

Recently, Dang et al. [45] gave the following relaxed two-point projection method to solve MSSEP (1):

$$\begin{cases} x_{k+1} = P_{\Omega_1}\left(x_k - \gamma\left(\sum_{i=1}^{t} \alpha_i\left(x_k - P_{C_{i,k}}(x_k)\right) + A^T(Ax_k - By_k)\right)\right), \\ y_{k+1} = P_{\Omega_2}\left(y_k - \gamma\left(\sum_{j=1}^{r} \beta_j\left(y_k - P_{Q_{j,k}}(y_k)\right) - B^T(Ax_{k+1} - By_k)\right)\right), \end{cases} \quad (9)$$

where $\gamma \in (0, \min\{\{1/21/2, 1/(4\|A\|^2)(1/4\|A\|^2), 1/(4\|B\|^2)\}\})$, $C_{i,k}$, $i = 1, 2, \cdots, r$ and $Q_{j,k}$, $j = 1, 2, \cdots, t$ are two sequences of closed convex sets corresponding to C_i and Q_j, respectively. $\Omega_1 \subset H_1$ and $\Omega_2 \subset H_2$ are auxiliary simple sets. $\alpha_i > 0$ for all i, and $\beta_j > 0$ for all j with $\sum_{i=1}^{t}\alpha_i + \sum_{j=1}^{r}\beta_j = 1$. Under some mild conditions, the weak convergence of the algorithm (9) was obtained.

Noting that the determination of the stepsize γ of algorithm (9) depends on the operator (matrix) norms $\|A\|$ and $\|B\|$. This implies that if we implement the relaxed two-point projection method (9), one first need to calculate operator norms of A and B, which is in general not an easy work in practice. To overcome this weakness, Lopez et al. [46] and Zhao and Yang [47] introduced self-adaptive methods of which the advantage of the methods is that the stepsizes do not need prior knowledge of the operator norms. Motivated by them, we introduce a relaxed self-adaptive projection algorithm for solving the multiple-sets split equality problem. First, we transfer the origin problem to the constrained multiple-sets split equality problem and establish the fixed point equation system. We show the equivalence of the constrained multiple-sets split equality problem and the fixed point equation system. Second, based on the fixed point equation system, we present a relaxed self-adaptive projection algorithm for solving the constrained multiple-sets split equality problem, and the convergence of the proposed algorithm is obtained. Finally, several numerical results are shown to confirm the feasibility and efficiency of the proposed algorithm.

The remainder of this paper is organized as follows. Section 2 shows some preliminaries and notations used for subsequent analysis. In Section 3, we transfer the origin problem to the constrained multiple-sets split equality problem and establish the fixed point equation system and propose a relaxed self-adaptive projection algorithm for solving the constrained multiple-sets split equality problem. The convergence of the proposed algorithm is obtained. In Section 4, several numerical results are shown to confirm the effectiveness of our algorithm.

2. Preliminaries

Throughout this paper, we use $\longrightarrow$ and $\rightharpoonup$ to denote the strong convergence and weak convergence, respectively. We write $\omega_w(x_k) = \{x : \exists x_{k_j} \rightharpoonup x\}$ to indicate the weak ω-limit set of $\{x_k\}$. For any $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \le \|x - y\|, \forall y \in C. \quad (10)$$

It is well known that P_C is nonexpansive and firmly nonexpansive. Moreover, P_C has the following well-known properties (see for example [48]).

Lemma 1. *Let $C \subset H$ be nonempty, closed and convex. Then for all $x, y \in H$ and $z \in C$,*

(i) $\langle x - P_C x, z - P_C x\rangle \le 0$

(ii) $\|P_C x - P_C y\|^2 \le \langle P_C x - P_C y, x - y\rangle$;

(iii) $\|P_C x - z\|^2 \le \|x - z\|^2 - \|P_C x - x\|^2$.

Definition 2. Let $f : H \longrightarrow R$ be convex. The subdifferential of f at x is defined as

$$\partial f(x) = \{\xi \in H \mid f(y) \ge f(x) + \langle \xi, y - x\rangle, \forall y \in H\}. \quad (11)$$

An element of $\partial f(x)$ is said to be a subgradient.

Lemma 3. *Suppose $f : H \longrightarrow R$ is a convex function, then it is subdifferentiable everywhere and its subdifferentials are uniformly bounded set of H.*

3. Algorithm and Its Convergence

In this section, we focus on a relaxed self-adaptive projection algorithm and obtain the convergence of the proposed algorithm. Following the idea of Censor et al. [39], we give two additional closed convex sets $\Omega_1 \subset H_1$ and $\Omega_2 \subset H_2$ and consider the constrained multiple-sets split equality problem

$$\text{find } x \in \Omega_1,\ y \in \Omega_2 \text{ such that } (x, y) \text{ solves } (1), \tag{12}$$

where the sets C_i and Q_j can be expressed by

$$\begin{aligned} C_i &= \{x \in H_1 \mid c_i(x) \le 0\}, \\ Q_j &= \left\{y \in H_2 \mid q_j(y_k) \le 0\right\}, \end{aligned} \tag{13}$$

$c_i : H_1 \longrightarrow R$ and $q_j : H_2 \longrightarrow R$ are convex functions for all $i = 1, 2, \cdots, t$ and $j = 1, 2, \cdots, r$, and Γ denotes the solution set of (32). Define

$$C_{i,k} = \left\{x \in H_1 \mid c_i(x_k) + \left\langle \xi_{i,k}, x - x_k \right\rangle \le 0\right\}, \tag{14}$$

where $\xi_{i,k} \in \partial c_i(x_k)$ and

$$Q_{j,k} = \left\{y \in H_2 \mid q_j(y_k) + \left\langle \eta_{j,k}, y - y_k \right\rangle \le 0\right\}, \tag{15}$$

where $\eta_{j,k} \in \partial q_j(y_k)$. It is easily seen that $C_i \subset C_{i,k}$ and $Q_j \subset Q_{j,k}$ for all k. Notice that $C_{i,k}$ and $Q_{j,k}$ are half-spaces and thus the corresponding projections have closed-form expressions. Hence, we focus on the following multiple-sets split equality problem (CMSSEP):

$$\begin{aligned} &\text{find } x \in \Omega_1,\ y \in \Omega_2 \text{ to solve } x \\ &\quad \in C = \bigcap_{i=1}^{t} C_{i,k}, y \in Q = \bigcap_{j=1}^{r} Q_{j,k} \text{such that } Ax = By. \end{aligned} \tag{16}$$

Now, we define the proximity function $p_k(x, y)$:

$$\begin{aligned} p_k(x, y) = \frac{1}{2}\sum_{i=1}^{t} \alpha_i \left\| x - P_{C_{i,k}}(x) \right\|^2 + \frac{1}{2}\sum_{j=1}^{r} \beta_j \left\| y - P_{Q_{j,k}}(y) \right\|^2 \\ + \frac{1}{2}\|Ax - By\|^2, \end{aligned} \tag{17}$$

where $\alpha_i > 0$ for all i, and $\beta_j > 0$ for all j with $\sum_{i=1}^{t} \alpha_i + \sum_{j=1}^{r} \beta_j = 1$.

Using the proximity function $p_k(x, y)$, we can obtain the following technical lemmas.

Lemma 4. *Assume that (16) is consistent (i.e.,(16) has a solution) and denotes its solution set by Γ. If $(x\, y) \in \Gamma$, then it solves the fixed point equation system*

$$\begin{cases} x = P_{\Omega_1}\left(x - \lambda\left(\sum_{i=1}^{t} \alpha_i \left(x - P_{C_{i,k}}(x)\right) + A^T(Ax - By)\right)\right), \\ y = P_{\Omega_2}\left(y - \beta\left(\sum_{j=1}^{r} \beta_j \left(y - P_{Q_{j,k}}(y)\right) - B^T(Ax - By)\right)\right). \end{cases} \tag{18}$$

Proof. To solve the problem (16), we consider the minimization problem

$$\min\left\{p_k(x, y) \mid x \in \Omega_1, y \in \Omega_2\right\}. \tag{19}$$

(19) leads to the following unconstrained optimization problem:

$$\min_{x \in \Omega_1, y \in \Omega_2} \left\{\delta_{\Omega_1}(x) + \delta_{\Omega_2}(y) + p_k(x, y)\right\}, \tag{20}$$

where δ_{Ω_i} is a indicator function of Ω_i for $i = 1, 2$ defined by

$$\delta_{\Omega_i}(x) = \begin{cases} 0, & x \in \Omega_i, \\ +\infty, & \text{otherwise}. \end{cases} \tag{21}$$

Note that $\partial\delta_{\Omega_1}(x) = N_{\Omega_1}(x)$ and $\partial\delta_{\Omega_2}(y) = N_{\Omega_2}(y)$, where N_{Ω_1} and N_{Ω_2} are the normal cone of the convex sets Ω_1 and Ω_1, respectively. From the optimality conditions of (20), it yields

$$\begin{cases} 0 \in \sum_{i=1}^{t} \alpha_i \left(x - P_{C_{i,k}}(x)\right) + A^T(Ax - By) + \partial\delta_{\Omega_1}(x), \\ 0 \in \sum_{j=1}^{r} \beta_j \left(y - P_{Q_{j,k}}(y)\right) - B^T(Ax - By) + \partial\delta_{\Omega_2}(y), \end{cases} \tag{22}$$

which means that, for $\lambda > 0, \beta > 0$,

$$\begin{aligned} x - \lambda\left(\sum_{i=1}^{t} \alpha_i \left(x - P_{C_{i,k}}(x)\right) + A^T(Ax - By)\right) &= x + \lambda\partial\delta_{\Omega_1}(x), \\ y - \beta\left(\sum_{j=1}^{r} \beta_j \left(y - P_{Q_{j,k}}(y)\right) - B^T(Ax - By)\right) &= y + \beta\partial\delta_{\Omega_2}(y), \end{aligned} \tag{23}$$

that is,

$$\begin{aligned} x &= \left(I + \lambda N_{\Omega_1}\right)^{-1}\left(x - \lambda\left(\sum_{i=1}^{t} \alpha_i \left(x - P_{C_{i,k}}(x)\right) + A^T(Ax - By)\right)\right), \\ y &= \left(I + \beta N_{\Omega_2}\right)^{-1}\left(y - \beta\left(\sum_{j=1}^{r} \beta_j \left(y - P_{Q_{j,k}}(y)\right) - B^T(Ax - By)\right)\right). \end{aligned} \tag{24}$$

Since $(I+\lambda N_{\Omega_1})^{-1}=P_{\Omega_1}$ and $(I+\beta N_{\Omega_2})^{-1}=P_{\Omega_2}$, we obtain

$$\begin{cases} x=P_{\Omega_1}\left(x-\lambda\left(\sum_{i=1}^{t}\alpha_i\left(x-P_{C_{i,k}}(x)\right)+A^T(Ax-By)\right)\right),\\ y=P_{\Omega_2}\left(y-\beta\left(\sum_{j=1}^{r}\beta_j\left(y-P_{Q_{j,k}}(y)\right)-B^T(Ax-By)\right)\right). \end{cases} \tag{25}$$

Thus, the desired result can be obtained.

The following lemma reveals that ESEP (16) is equivalent to the fixed point equation system (18).

Lemma 5. *Assume that the problem (16) is consistent.* $(x^*,y^*)\in\Gamma$ *solves ESEP (2) if and only if* (x^*,y^*) *solves the fixed point equation system (18).*

Proof. From Lemma 4, we reveal that (x^*,y^*) can solve (16); it also can solve (18). Next, we will prove that (x^*,y^*) can solve (18), it also can solve (16). Obviously, one has $x^*\in\Omega_1$, and $y^*\in\Omega_2$. It follows from the proposition of projection that

$$\begin{cases} \left\langle x^*-\lambda\left(\sum_{i=1}^{t}\alpha_i\left(x^*-P_{C_{i,k}}(x^*)\right)+A^T(Ax^*-By^*)\right)-x^*,u-x^*\right\rangle\le 0, u\in\Gamma,\\ \left\langle y^*-\beta\left(\sum_{j=1}^{r}\beta_j\left(y^*-P_{Q_{j,k}}(y^*)\right)-B^T(Ax^*-By^*)\right)-y^*,v-y^*\right\rangle\le 0, v\in\Gamma. \end{cases} \tag{26}$$

which means

$$\begin{cases} \left\langle -\lambda\left(\sum_{i=1}^{t}\alpha_i\left(x^*-P_{C_{i,k}}(x^*)\right)+A^T(Ax^*-By^*)\right),u-x^*\right\rangle\le 0, u\in\Gamma,\\ \left\langle -\beta\left(\sum_{j=1}^{r}\beta_j\left(y^*-P_{Q_{j,k}}(y^*)\right)-B^T(Ax^*-By^*)\right),v-y^*\right\rangle\le 0, v\in\Gamma. \end{cases} \tag{27}$$

Hence, from Lemma 3, we add two inequalities to obtain

$$\sum_{i=1}^{t}\alpha_i\left\|x^*-P_{C_{i,k}}(x^*)\right\|^2+\sum_{j=1}^{r}\beta_j\left\|y^*-P_{Q_{j,k}}(y^*)\right\|^2 + \langle Ax^*-By^*,Bv-Au+Ax^*-By^*\rangle\le 0. \tag{28}$$

Furthermore, from $Au=Bv$, we deduce

$$\begin{aligned} &\|x^*-P_{C_{i,k}}(x^*)\|=0, \text{for } i=1,2,\cdots,t,\\ &\left\|y^*-P_{Q_{j,k}}(y^*)\right\|^2=0, \text{for } j=1,2,\cdots,r,\\ &\|Ax^*-By^*\|^2=0. \end{aligned} \tag{29}$$

Thus, (x^*,y^*) solves ESEP (16). This completes the proof.

Based on (18), we can introduce a relaxed self-adaptive projection algorithm to solve (16), with $\sigma_k\in(0,1)$.

Alggorithm 6. Let $x_0\in H_1, y_0\in H_2$ be arbitrary. We calculate the $(k+1)$th iterate via the following formula

$$\begin{cases} u_k=P_{\Omega_1}\left(x_k-\lambda_k\left(\sum_{i=1}^{t}\alpha_i\left(x_k-P_{C_{i,k}}(x_k)\right)+A^T(Ax_k-By_k)\right)\right),\\ x_{k+1}=\gamma_k x_k+(1-\gamma_k)u_k,\\ v_k=P_{\Omega_2}\left(y_k-\lambda_k\left(\sum_{j=1}^{r}\beta_j\left(y_k-P_{Q_{j,k}}(y_k)\right)-B^T(Ax_k-By_k)\right)\right),\\ y_{k+1}=\gamma_k y_k+(1-\gamma_k)v_k, \end{cases} \tag{30}$$

where the stepsize λ_k is chosen by

$$\lambda_k=2\sigma_k\frac{\sum_{i=1}^{t}\alpha_i\|x_k-P_{C_{i,k}}(x_k)\|^2+\sum_{j=1}^{r}\beta_j\|y_k-P_{Q_{j,k}}(y_k)\|^2+\|Ax_k-By_k\|^2}{\|\sum_{i=1}^{t}\alpha_i\left(x_k-P_{C_{i,k}}(x_k)\right)+A^T(Ax_k-By_k)\|^2+\|\sum_{j=1}^{r}\beta_j\left(y_k-P_{Q_{j,k}}(y_k)\right)-B^T(Ax_k-By_k)\|^2}=4\sigma_k\frac{p_k(x_k,y_k)}{\|\nabla p_k(x_k,y_k)\|^2}, \tag{31}$$

with $\sigma_k\in(1,0)$.

Next, we will focus on the convergence analysis of Algorithm 6.

Theorem 7. *Assume* $\lim_{k\to\infty}\gamma_k=0$, $\sum_{k=1}^{\infty}\gamma_k=\infty$ *and* $\sigma_k\in[M_1,M_2]\subset(0,1)$, *then the sequence* (x_k,y_k) *generated by Algorithm 6 converges to a solution of (1).*

Proof. Taking $(x^*,y^*)\in\Gamma$, one has

$$Ax^*=By^*. \tag{32}$$

From (30) and the fact that the projection is nonexpansive, we have

$$\|u_k - x^*\|^2 = \left\| P_{\Omega_1}\left(x_k - \lambda_k \left(\sum_{i=1}^{t} \alpha_i \left(x_k - P_{C_{i,k}}(x_k) \right) + A^T(Ax_k - By_k) \right) \right) - x^* \right\|^2$$
$$\leq \left\| x_k - \lambda_k \left(\sum_{i=1}^{t} \alpha_i \left(x_k - P_{C_{i,k}}(x_k) \right) + A^T(Ax_k - By_k) \right) - x^* \right\|^2$$
$$= \left\| x_k - x^* \|^2 + (\lambda_k)^2 \| \sum_{i=1}^{t} \alpha_i \left(x_k - P_{C_{i,k}}(x_k) \right) + A^T(Ax_k - By_k) \right\|^2$$
$$- 2\lambda_k \left\langle \sum_{i=1}^{t} \alpha_i \left(x_k - P_{C_{i,k}}(x_k) \right) + A^T(Ax_k - By_k), x_k - x^* \right\rangle. \quad (33)$$

Since

$$-2\lambda_k \left\langle \sum_{i=1}^{t} \alpha_i \left(x_k - P_{C_{i,k}}(x_k) \right) + A^T(Ax_k - By_k), x_k - x^* \right\rangle$$
$$= -2\lambda_k \left\langle \sum_{i=1}^{t} \alpha_i \left(x_k - P_{C_{i,k}}(x_k) \right), x_k - x^* \right\rangle - 2\lambda_k \left\langle A^T(Ax_k - By_k), x_k - x^* \right\rangle$$
$$= -2\lambda_k \sum_{i=1}^{t} \alpha_i \left\langle x_k - P_{C_{i,k}}(x_k), x_k - x^* \right\rangle - 2\lambda_k \langle Ax_k - By_k, Ax_k - Ax^* \rangle$$
$$\leq -2\lambda_k \sum_{i=1}^{t} \alpha_i \|x_k - P_{C_{i,k}}(x_k)\|^2 - \lambda_k \|Ax_k - By_k\|^2 - \lambda_k \|Ax_k - Ax^*\|^2 + \lambda_k \|By_k - Ax^*\|^2, \quad (34)$$

together with (33), we deduce

$$\|u_k - x^*\|^2 \leq \|x_k - \lambda_k \left(\sum_{i=1}^{t} \alpha_i \left(x_k - P_{C_{i,k}}(x_k) \right) + A^T(Ax_k - By_k) \right) - x^*\|^2$$
$$= \|x_k - x^*\|^2 + (\lambda_k)^2 \| \sum_{i=1}^{t} \alpha_i \left(x_k - P_{C_{i,k}}(x_k) \right) + A^T(Ax_k - By_k)\|^2$$
$$- 2\lambda_k \sum_{i=1}^{t} \alpha_i \|x_k - P_{C_{i,k}}(x_k)\|^2 - \lambda_k \|Ax_k - By_k\|^2 - \lambda_k \|Ax_k - Ax^*\|^2 + \lambda_k \|By_k - Ax^*\|^2. \quad (35)$$

Similarly, we have

$$\|v_k - y^*\|^2 = \|P_{\Omega_2}\left(y_k - \lambda_k \left(\sum_{j=1}^{r} \beta_j \left(y_k - P_{Q_{j,k}}(y_k) \right) - B^T(Ax_k - By_k) \right) \right) - y^*\|^2$$
$$\leq \|y_k - \lambda_k \left(\sum_{j=1}^{r} \beta_j \left(y_k - P_{Q_{j,k}}(y_k) \right) - B^T(Ax_k - By_k) \right) - y^*\|^2$$
$$\leq \|y_k - y^*\|^2 + (\lambda_k)^2 \| \sum_{j=1}^{r} \beta_j \left(y_k - P_{Q_{j,k}}(y_k) \right) - B^T(Ax_k - By_k))\|^2$$
$$- 2\lambda_k \sum_{i=1}^{t} \beta_j \|y_k - P_{Q_{j,k}}(y_k)\|^2 - \lambda_k \|By_k - By^*\|^2 - \lambda_k \|Ax_k - By_k\|^2 + \lambda_k \|Ax_k - By^*\|^2. \quad (36)$$

From (35) and (36), it follows

$$\|u_k - x^*\|^2 + \|v_k - y^*\|^2 \leq \|x_k - x^*\|^2 + \|y_k - y^*\|^2 - \lambda_k \left(2 \left(\sum_{i=1}^{t} \alpha_i \|x_k - P_{C_{i,k}}(x_k)\|^2 + \sum_{i=1}^{t} \beta_j \|y_k - P_{Q_{j,k}}(y_k)\|^2 + \|Ax_k - By_k\|^2 \right) \right.$$
$$\left. - \lambda_k \left(\| \sum_{i=1}^{t} \alpha_i \left(x_k - P_{C_{i,k}}(x_k) \right) + A^T(Ax_k - By_k)\|^2 + \| \sum_{j=1}^{r} \beta_j \left(y_k - P_{Q_{j,k}}(y_k) \right) - B^T(Ax_k - By_k)\|^2 \right) \right), \quad (37)$$

which together with (31) means

$$\|u_k - x^*\|^2 + \|v_k - y^*\|^2 \leq \|x_k - x^*\|^2 + \|y_k - y^*\|^2. \quad (38)$$

Furthermore, it follows from (31) and (38) that

$$\|x_{k+1} - x^*\|^2 + \|y_{k+1} - y^*\|^2 = \|\gamma_k x_k + (1 - \gamma_k) u_k - x^*\|^2 + \|\gamma_k y_k + (1 - \gamma_k) v_k - y^*\|^2$$
$$\leq \gamma_k \|x_k - x^*\|^2 + (1 - \gamma_k) \|u_k - x^*\|^2 + \gamma_k \|y_k - y^*\|^2 + (1 - \lambda_k) \|v_k - y^*\|^2$$
$$\leq \gamma_k \left(\|x_k - x^*\|^2 + \|y_k - y^*\|^2 \right) + (1 - \gamma_k) \left(\|u_k - x^*\|^2 + \|v_k - y^*\|^2 \right)$$
$$\leq \|x_k - x^*\|^2 + \|y_k - y^*\|^2, \quad (39)$$

By induction, one has

$$\|x_{k+1} - x^*\|^2 + \|y_{k+1} - y^*\|^2 \leq \|x_0 - x^*\|^2 + \|y_0 - y^*\|^2. \quad (40)$$

Hence, $\{x_n\}$ and $\{y_n\}$ are bounded. Following (31), (36), and (39), we have

$$\|x_{k+1} - x^*\|^2 + \|y_{k+1} - y^*\|^2 \leq \gamma_k \left(\|x_k - x^*\|^2 + \|y_k - y^*\|^2 \right) + (1 - \gamma_k) \left(\|u_k - x^*\|^2 + \|v_k - y^*\|^2 \right) \leq \|x_k - x^*\|^2$$
$$+ \|y_k - y^*\|^2 - (1 - \gamma_k) \lambda_k \left(2 \left(\sum_{i=1}^{t} \alpha_i \|x_k - P_{C_{i,k}}(x_k)\|^2 + \sum_{i=1}^{t} \beta_j \|y_k - P_{Q_{j,k}}(y_k)\|^2 + \|Ax_k - By_k\|^2 \right) \right.$$
$$\left. - \lambda_k \left(\| \sum_{i=1}^{t} \alpha_i \left(x_k - P_{C_{i,k}}(x_k) \right) + A^T(Ax_k - By_k)\|^2 + \| \sum_{j=1}^{r} \beta_j \left(y_k - P_{Q_{j,k}}(y_k) \right) - B^T(Ax_k - By_k)\|^2 \right) \right). \quad (41)$$

Without loss of generality, we can assume that there is $\sigma > 0$ such that $4(1-\gamma_k)\sigma_k(1-\sigma_k) > \sigma$ for all k. Setting $s_k = \|x_k - x^*\|^2 + \|y_k - y^*\|^2$, together with (41), we have the following inequality

$$\sigma \frac{(p_k(x_k,y_k))^2}{\|\nabla p_k(x_k, y_k)\|^2} + s_{k+1} - s_k \le 0. \tag{42}$$

Since s_k is eventually decreasing, we obtain s_k as convergent. From (42), we have $\lim_{k\to\infty} p_k(x_k, y_k) = 0$. Furthermore,

$$\lim_{k\to\infty} \|x_k - P_{C_{i,k}}(x_k)\|^2 = 0, fori = 1, 2, \cdots, t, \tag{43}$$

$$\lim_{k\to\infty} \|y_k - P_{Q_{j,k}}(y_k)\|^2 = 0, forj = 1, 2, \cdots, r, \tag{44}$$

$$\lim_{k\to\infty} \|Ax_k - By_k\|^2 = 0. \tag{45}$$

Furthermore,

$$\|x_{k+1} - x_k\| = \|\gamma_k x_k + (1-\gamma_k)u_k - x_k\| = (1-\gamma_k)\|u_k - x_k\| \le (1-\gamma_k)\lambda_k \left(\sum_{i=1}^{t} \alpha_i \|x_k - P_{C_{i,k}}(x_k)\| + \|A^T(Ax_k - By_k)\|\right), \tag{46}$$

which with (41), (45), and the assumption on γ_k means

$$\lim_{k\to\infty} \|x_{k+1} - x_k\|^2 = 0. \tag{47}$$

Note that

$$\|x_{k+1} - u_k\| = \|\gamma_k x_k + (1-\gamma_k)u_k - u_k\| = \gamma_k \|x_k - u_k\|, \tag{48}$$

we have

$$\lim_{k\to\infty} \|x_{k+1} - u_k\|^2 = 0. \tag{49}$$

(47) and (49) imply

$$\lim_{k\to\infty} \|x_k - u_k\|^2 = 0. \tag{50}$$

Similarly, we have

$$\begin{aligned} &\lim_{k\to\infty} \|y_{k+1} - y_k\|^2 = 0, \\ &\lim_{k\to\infty} \|y_{k+1} - v_k\|^2 = 0, \\ &\lim_{k\to\infty} \|y_k - v_k\|^2 = 0. \end{aligned} \tag{51}$$

Thus, $\{x_k\}$ and $\{y_k\}$ are asymptotically regular. Notice that

$$\begin{aligned} &\left\|\sum_{i=1}^{t} \alpha_i \left(x_k - P_{C_{i,k}}(x_k)\right) + A^T(Ax_k - By_k))\right\|^2 \\ &\quad + \left\|\sum_{j=1}^{r} \beta_j \left(y_k - P_{Q_{j,k}}(y_k)\right) - B^T(Ax_k - By_k))\right\|^2 \\ &\le 2\left(\sum_{i=1}^{t} \alpha_i \|\left(x_k, -, P_{C_{i,k}}, (x_k)\right\|^2 + \|A\|^2 \|Ax_k - By_k\|^2 \right. \\ &\quad \left. + \sum_{j=1}^{r} \beta_j \|y_k - P_{Q_{j,k}}(y_k)\|^2 + \|B\|^2 \|Ax_k - By_k\|^2\right) \\ &\le 2 \max\left\{1, \|A\|^2 + \|B\|^2\right\} \left(\sum_{i=1}^{t} \alpha_i \|\left(x_k, -, P_{C_{i,k}}, (x_k)\right\|^2 \right. \\ &\quad \left. + \|Ax_k - By_k\|^2 + \sum_{j=1}^{r} \beta_j \|y_k - P_{Q_{j,k}}(y_k)\|^2\right), \end{aligned} \tag{52}$$

which implies that

$$\lambda_k \ge \sigma_k \frac{1}{\max\left\{1, \|A\|^2 + \|B\|^2\right\}}. \tag{53}$$

Moreover, it follows from (22) that

$$\begin{aligned} \left\|\frac{x_{k+1} - x_k}{\lambda_k}\right\| &= (1-\gamma_k)\frac{1}{\lambda_k}\|u_k - x_k\| \le (1-\gamma_k) \\ &\quad \cdot \left(\sum_{i=1}^{t} \alpha_i \|x_k - \mathrm{P}_{C_{i,k}}(x_k)\| + \|A^T(Ax_k - By_k)\|\right), \end{aligned} \tag{54}$$

which with (43), (45), and the assumption on γ_k yields

$$\lim_{k\to\infty} \left\|\frac{x_{k+1} - x_k}{\lambda_k}\right\| = 0. \tag{55}$$

Similarly, one has

$$\lim_{k\to\infty} \left\|\frac{y_{k+1} - y_k}{\lambda_k}\right\| = 0. \tag{56}$$

Let $\bar{x}$ and $\bar{y}$ be, respectively, weak cluster points of the sequences $\{x_k\}$ and $\{y_k\}$, then there exist two subsequences of $\{x_k\}$ and $\{y_k\}$ (again labeled $\{x_k\}$ and $\{y_k\}$ which converge weakly to $\bar{x}$ and $\bar{y}$). Next, we will show that $(\bar{x}, \bar{y}) \in \Gamma$. It follows from (30) that

$$\frac{x_{k+1}-x_k}{\lambda_k(1-\gamma_k)}-\lambda_k\left(\sum_{i=1}^{t}\alpha_i\left(x_k-P_{C_{i,k}}(x_k)\right)+A^T(Ax_k-By_k)\right)\in N_{\Omega_1}\left(\frac{x_{k+1}-\gamma_k x_k}{1-\gamma_k}\right),$$
$$\frac{y_{k+1}-y_k}{\lambda_k(1-\gamma_k)}-\lambda_k\left(\sum_{j=1}^{r}\beta_j\left(y_k-P_{Q_{j,k}}(y_k)\right)-B^T(Ax_k-By_k)\right)\in N_{\Omega_2}\left(\frac{y_{k+1}-\gamma_k y_k}{1-\gamma_k}\right). \quad (57)$$

From the graphs of the maximal monotone operators, N_C and N_Q are weakly-strongly closed, and by passing to the limit in the last inclusions, we obtain that $\bar{x}\in\Omega_1$ and $\bar{y}\in\Omega_2$.

On the other hand, from Lemma 1 and the definition of $C_{i,k}$, one has

$$c_i(x_k)\le\left\langle\xi^{i,k},x_k-P_{C_{i,k}}(x_k)\right\rangle\le\|\xi^{i,k}\|\|x_k-P_{C_{i,k}}(x_k)\|\le M_1\|x_k-P_{C_{i,k}}(x_k)\|, \quad (58)$$

where M satisfies $\|\xi^{i,k}\|\le M_1$ for all k. The lower semicontinuity of function $c_i(x)$ and (41) assert that

$$c_i(\bar{x})\le\liminf_{k\to\infty}c_i(x_k)\le 0. \quad (59)$$

Thus, $\bar{x}\in C_i$ for $i=1,2,\cdots,t$. Likewise, we can obtain

$$q_j(x_k)\le\left\langle\eta^{j,k},y_k-P_{Q_{j,k}}(y_k)\right\rangle\le\|\eta^{j,k}\|\|y_k-P_{Q_{j,k}}(y_k)\|\le M_2\|y_k-P_{Q_{j,k}}(y_k)\|, \quad (60)$$

where M_2 satisfies $\|\eta^{j,k}\|\le M_2$ for all k. The lower semicontinuity of function $q_j(x)$ and (42) lead to

$$q_i(\bar{y})\le\liminf_{k\to\infty}q_i(y_k)\le 0. \quad (61)$$

Thus, $\bar{y}\in Q_j$ for $j=1,2,\cdots,r$. Moreover, the weak convergence of Ax_k-By_k to $A\bar{x}-B\bar{y}$ and the lower semicontinuity of the squared norm imply

$$\|A\bar{x}-B\bar{y}\|\le\liminf_{k\to\infty}\|Ax_k-By_k\|=0, \quad (62)$$

hence, $(\bar{x},\bar{y})\in\Gamma$. This completes the proof.

4. Numerical Examples

We are in a position to show numerical examples to demonstrate the performance and convergence of Algorithm 6. The whole programs are written in MATLAB 7.0. All the numerical results are carried out on a personal Lenovo computer with Intel®Core™ i7-7500 U CPU 2.70 GHz and RAM 4.00 GB. We denote the vector with all elements 1 by e in what follows.

Example 8. Let

$$A=\begin{pmatrix}3&-1&2\\2&1&0\\3&0&3\end{pmatrix},B=\begin{pmatrix}4&-4&2&1\\3&-1&4&3\\5&1&0&4\end{pmatrix} \quad (63)$$

$C_1=\{x\in R^3\mid x_1+5x_2^2+4x_3\le 0\}$, $C_2=\{x\in R^3\mid 3x_1+10x_3\le 0\}$, $Q_1=\{y\in R^4\mid 2y_1-3y_2-2y_3+4y_4\le 0\}$, and $Q_2=\{y\in R^4\mid 2y_1^2-y_2+4y_3-3y_4\le 0\}$. Find $x\in C=C_1\cap C_2, y\in Q=Q_1\cap Q_2$ such that $Ax=By$.

Example 9. Let

$$A=\begin{pmatrix}0.2620&0.0268&0.2589\\0.5697&0.5004&0.0458\\0.3595&0.8270&0.2464\end{pmatrix},$$
$$B=\begin{pmatrix}0.6607&0.0130&0.0335&0.9213\\0.3294&0.7180&0.4060&0.9840\\0.6594&0.3911&0.7163&0.9834\end{pmatrix} \quad (64)$$

$C_1=\{x\in R^3\mid x_1^4+x_2^2-2x_3^2-1\le 0\}$, $C_2=\{x\in R^3\mid 2x_1^2+x_2^3-3x_3^2-1\le 0\}$, $Q_1=\{y\in R^4\mid 2y_1^3-y_2^2+2y_3^3+6y_4-2\le 0\}$, and $Q_2=\{y\in R^4\mid 2y_1^2+3y_3^2+2y_4^2-2\le 0\}$. Find $x\in C=C_1\cap C_2, y\in Q=Q_1\cap Q_2$ such that $Ax=By$.

Example 10. Let $A=(a_{ij})_{J\times N}$ and $B=(b_{ij})_{J\times M}$. $C_1=\{x_1\in R^N\mid\|x_1\|\le 2\}$, $C_2=\{x_2\in R^N\mid -e\le x_2\le 3e\}$. $Q_1=\{y_1\in R^M\mid -2e\le y_1\le 6e\}$, and $Q_2=\{y_2\in R^M\mid\|y_2\|\le 4\}$, where $\{a_{ij}\},\{b_{ij}\}\in(0,1)$ are all generated randomly; J, N and M are positive integers. Find $x\in C=C_1\cap C_2, y\in Q=Q_1\cap Q_2$ such that $Ax=By$.

In this example, we consider $J=10, N=10$, and $M=20$; $J=20, N=30$, and $M=40$; and $J=40, N=50$, and $M=60$ and three initial values:

(i) Case 1 $x=\text{ones}(N,1), y=\text{ones}(M,1)$;

(ii) Case 2 $x=10*\text{ones}(N,1), y=10*\text{ones}(M,1)$;

(iii) Case 3 $x=-10*\text{ones}(N,1), y=-10*\text{ones}(M,1)$.

We take $\Omega_1=C_{1,n}$, $\Omega_2=Q_{1,n}$ when the algorithm iterates to step n, $\gamma_k=1/20k$, $\sigma_k=(1/4)+(1/2k)$, $\alpha_1=\alpha_2=\beta_1=\beta_2=1/4$ in Algorithm 6. In the following tables and figures, we denote Algorithm 6 and the algorithm in reference [45] by QSPA and RTPPM, respectively. And we set " n ", " s " and " x^*, " and " y^* " to express the number of iteration, CPU time in seconds, and the final solution, respectively. Init. denote the initial points, and $p_k(x,y)\le\varepsilon=10^{-4}$ is used as the stop

Table 1: The numerical results of Example 8.

Init.	QSPR	RTPPM
$x_1 = (0, 0, 0)^T$	$n = 14, s = 0.000671$	$n = 7684, s = 0.198442$
$y_1 = (1, 0, -1, 1)^T$	$x^* = (0.0196, -0.0222, -0.0055)^T$ $y^* = (0.0345, 0.0119, 0.0071, -0.0357)^T$	$x^* = (0.0342, -0.0734, -0.0153)^T$ $y^* = (0.0509, 0.0012, -0.0023, -0.0495)^T$
$x_1 = (0, 1, 1)^T$	$n = 15, s = 0.000636$	$n = 49, s = 0.001585$
$y_1 = (0, 0, 0, 0)^T$	$x^* = (-0.0750, 0.0232, 0.0181)^T$ $y^* = (-0.0279, 0.0205, 0.0014, -0.0109)^T$	$x^* = (-0.2420, -0.0354, 0.0589)^T$ $y^* = (-0.0721, 0.0426, -0.0254, -0.0566)^T$
$x_1 = (1, 1, 1)^T$	$n = 258, s = 0.008365$	$n = 34587, s = 0.816461$
$y_1 = (1, 1, -1, -1)^T$	$x^* = (-0.1554, -0.0125, -0.0255)^T$ $y^* = (-0.0951, 0.0345, 0.0180, -0.0257)^T$	$x^* = (0.0836, -0.1013, -0.0566)^T$ $y^* = (0.0496, -0.0180, 0.0025, -0.0371)^T$

Table 2: The numerical results of Example 9.

Init.	QSPR	RTPPM
$x_1 = (1, 1, 1)^T$	$n = 16, s = 0.001268$	$n = 478, s = 0.029797$
$y_1 = (1, 1, 1, 1)^T$	$x^* = (0.1710, 0.1525, 0.1824)^T$ $y^* = (0.1110, 0.1218, 0.1283, 0.0100)^T$	$x^* = (1.2294, 0.7761, 0.9712)^T$ $y^* = (0.6171, 0.7022, 0.6533, 0.1692)^T$
$x_1 = 10(1, 1, 1)^T$	$n = 37, s = 0.001877$	$n = 1687, s = 0.079442$
$y_1 = 10(1, 1, 1, 1)^T$	$x^* = (0.2973, 0.3380, 0.7617)^T$ $y^* = (0.6560, 0.3316, 0.2638, -0.1878)^T$	$x^* = (1.5257, 0.4196, 1.5156)^T$ $y^* = (0.9622, 0.8416, 0.2119, 0.1513)^T$
$x_1 = 100(1, 1, 1)^T$	$n = 63, s = 0.003101$	$n = 2651, s = 0.110352$
$y_1 = 100(1, 1, 1, 1)^T$	$x^* = (0.5363, -0.1663, 2.0146)^T$ $y^* = (0.9380, 0.0704, -0.1862, 0.0354)^T$	$x^* = (1.5276, 0.3916, 1.5165)^T$ $y^* = (0.9713, 0.8464, 0.1785, 0.1454)^T$

Table 3: The numerical results of Example 10.

				QSPR with λ_n		QSPR with 0.5 λ_n		RTPPM with λ_n	
	J	N	M	n	s	n	s	n	s
Case 1	10	10	20	30	0.003122	48	0.004302	808	0.050933
	20	30	40	37	0.005304	94	0.011647	1994	0.258627
	40	50	60	91	0.013088	188	0.028659	4014	1.286561
Case 2	10	10	20	56	0.011269	125	0.031055	3236	0.201379
	20	30	40	107	0.038957	171	0.039266	1762	0.233102
	40	50	60	295	0.041663	351	0.100628	5084	1.673274
Case 3	10	10	20	67	0.008644	98	0.016412	812	0.051558
	20	30	40	118	0.015576	178	0.028107	1953	0.270206
	40	50	60	233	0.058625	302	0.108850	4176	1.360695

criterion. The numerical results can be seen from Tables 1–3 and Figures 1–4. For Figures 3 and 4, take $J = 20, N = 30$, and $M = 40$ in Example 10.

From Tables 1–3, we can see that the iterative number and CPU time of Algorithm 6 is less algorithm RTPPM. Figures 1–4 indicate that Algorithm 6 is more stable than RTPPM.

Furthermore, for testing the stationary property of iterative number, we carry 500 experiments for the initial point which are presented randomly, such as

$$x_1 = \text{rand}\,(3, 1), y_1 = \text{rand}\,(4, 1), \tag{65}$$

in Example 9, the results can be found in Figure 1.

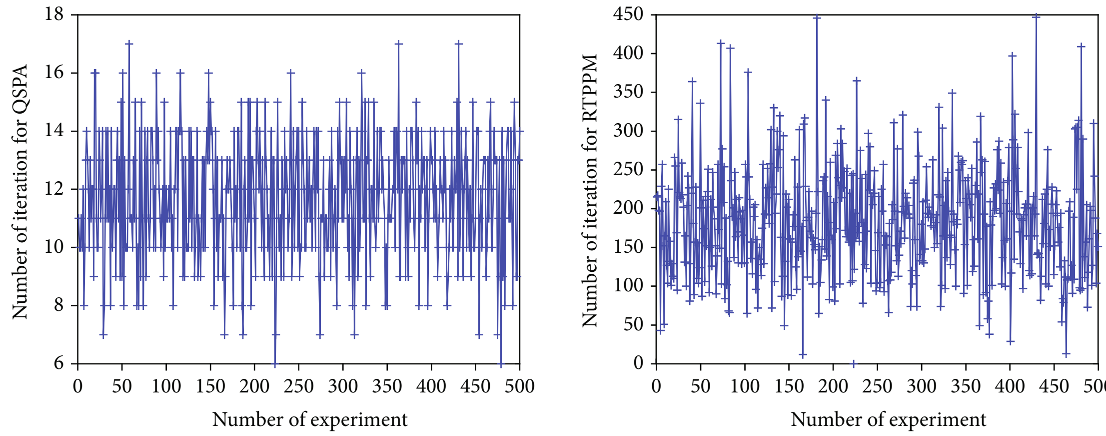

Figure 1: The iteration number of QSPA and RTPPM.

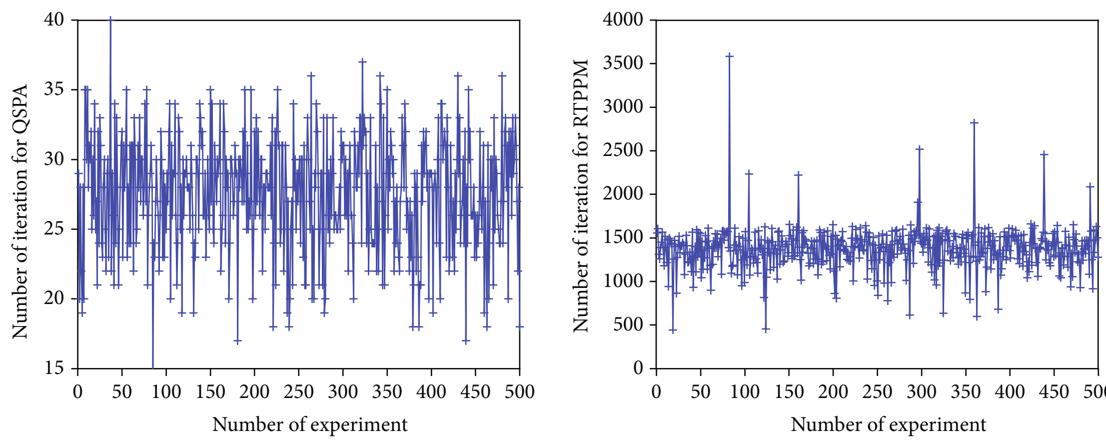

Figure 2: The iteration number of QSPA and RTPPM.

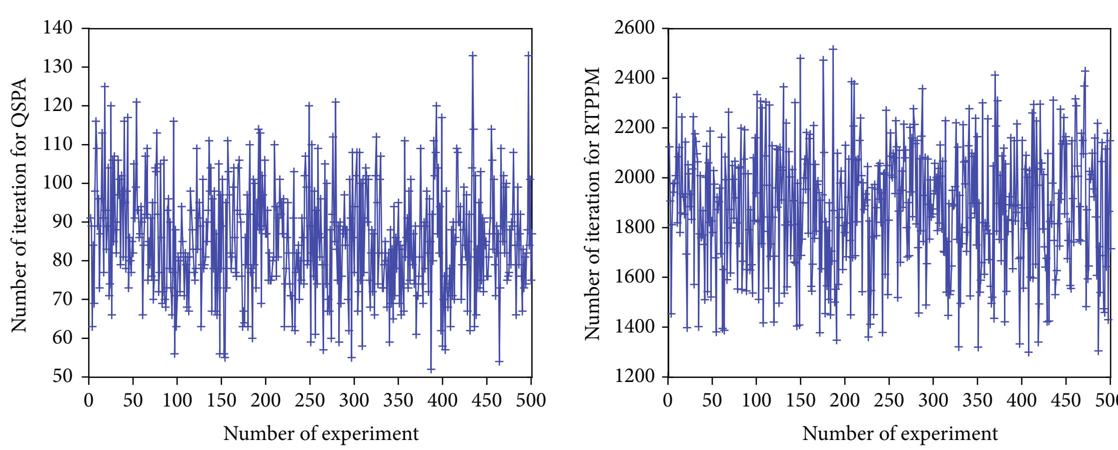

Figure 3: The iteration number of QSPA and RTPPM.

On the other initial point, such as

$$x_1 = \text{rand}\,(3, 1) * 10, y_1 = \text{rand}\,(4, 1) * 10, \tag{66}$$

in Example 9, the results can be found in Figure 2.

Similarly, we carry 500 experiments for the initial point which are presented randomly, such as

$$x_1 = \text{rand}\,(N, 1), y_1 = \text{rand}\,(M, 1), \tag{67}$$

in Example 10, the results can be found in Figure 3.

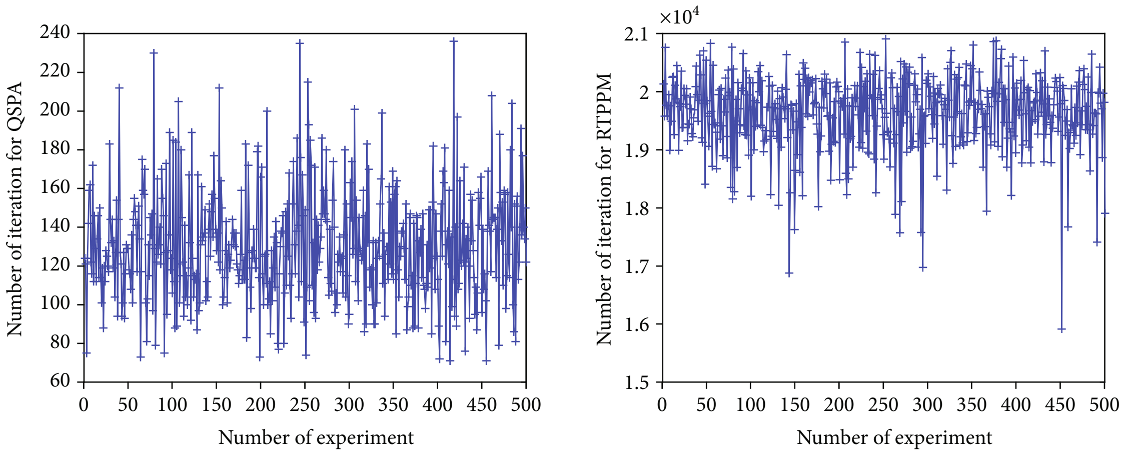

Figure 4: The iteration number of QSPA and RTPPM.

On the other initial point, such as

$$x_1 = \text{rand}\,(N, 1) * 10, y_1 = \text{rand}\,(M, 1) * 10, \tag{68}$$

in Example 10, the results can be found in Figure 4.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

Authors' Contributions

Each author equally contributed to this paper and read and approved the final manuscript.

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