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Research Article

Blow-Up Solutions for a Singular Nonlinear Hadamard Fractional Boundary Value Problem

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We consider singular nonlinear Hadamard fractional boundary value problems. Using properties of Green's function and a fixed point theorem, we show that the problem has positive solutions which blow up. Finally, some examples are provided to explain the applications of the results.

1. Introduction

Fractional-order differential equations appear extensively in a variety of applications in science and engineering; see, for instance, [1–13] and the references therein.

In [14], Hadamard introduced a new definition of fractional derivatives which differs from the Riemann-Liouville and Caputo fractional derivatives in the sense that its kernel integral contains the logarithmic function of an arbitrary exponent. Hadamard fractional derivatives are viewed as a generalization of the operator $\delta = x(d/dx)$. For further details, properties, and generalizations of this type of derivative, we refer the reader to [5, 15–21] and the references therein.

The study of existence, uniqueness, and global asymptotic behavior of a continuous solution of fractional differential equations involving Hadamard fractional derivatives has been investigated by several researchers; see, for example, [22–32].

In [23], Ahmad and Ntouyas studied the following problem:

$$\begin{cases} {}^{\mathscr{H}}\mathscr{D}^{\alpha} u(r) - g(r, u(r)) = 0, r \in (1, e), \\ u(1) = 0, \quad u(e) = \dfrac{1}{\Gamma(\beta)} \displaystyle\int_1^e \left(\ln \frac{e}{s}\right)^{\beta - 1} \frac{u(s)}{s} ds, \beta > 0, \end{cases} \tag{1}$$

where ${}^{\mathscr{H}}\mathscr{D}^{\alpha}$ is the Hadamard fractional derivative order $\alpha \in (1, 2]$ and $g : [1, e] \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function satisfying

$$|g(r, x) - g(r, y)| \leq L|x - y|, \quad \text{for each } r \in [1, e] \text{ and } x, y \in \mathbb{R}, \tag{2}$$

where $L > 0$ denotes a convenient Lipschitz constant.

The authors used the classical Banach fixed point theorem to obtain the existence and uniqueness of a solution for the abovementioned problem.

In [24], the authors studied the existence of solutions for a fractional boundary value problem involving Hadamard-type fractional differential inclusions and integral boundary conditions. Their approach was based on standard fixed point theorems for multivalued maps. In [25], the authors used some classical ideas of fixed point theory to investigate the existence and uniqueness of solutions of a boundary value problem comprising nonlinear Hadamard fractional differential equations and nonlocal nonconserved boundary conditions in terms of the Hadamard integral. In [15], the authors studied a Cauchy problem for a differential equation with a left Caputo-Hadamard fractional derivative. By using Banach's fixed point theorem, they proved the existence and uniqueness of the solution in the space of continuously differentiable functions.

The primary objective of this paper is to address the existence and qualitative properties of a solution for the following problem:

$$\begin{cases} {}^{\mathscr{H}}\mathscr{D}^{\alpha}u(r)-\lambda f(r,u(r))=0, r\in(1,e), \\ u>0, \text{in } (1,e), \\ \lim_{r\to 1^{+}}(\ln(r))^{2-\alpha}u(r)=a>0, u(e)=b>0, \end{cases} \quad (3)$$

where ${}^{\mathscr{H}}\mathscr{D}^{\alpha}$ is the Hadamard fractional derivative of order $\alpha\in(1,2]$, $\lambda\geq 0$, and f satisfies

(H_1) $f:(1,e)\times[0,\infty)\longrightarrow[0,\infty))$ is continuous, such that for each each fixed $r\in(1,e)$, $s\longrightarrow f(r,s)$ is nondecreasing on $[0,\infty)$.

(H_2) For all $c>0,\int_1^e(\ln r)(1-\ln r)^{\alpha-1}f(r,c(\ln r)^{\alpha-2})dr<\infty$.

The following are some examples of functions that satisfy hypotheses (H_1) and (H_2).

(i) $f(r,s)=\sqrt{s}$, which is not a Lipschitz function on $[0,\infty)$

(ii) $f(r,s)=(\ln r)^{\gamma(2-\alpha)}(1-\ln r)^{-\beta}s^{\gamma}$, where $0<\beta<\alpha$ and $\gamma\geq 0$. It is to be noted that $f(r,s)$ is singular at $r=1$

(iii) $f(r,s)=p(r)s^{\gamma}$, where p is any nonnegative continuous function on $[1,e]$ and $\gamma\in[0,2/(2-\alpha))$.

Before stating our main result, we explain some notations.

1.1. Notations

(i) $\mathscr{B}^{+}((1,e)):=\{g|g:(1,e)\longrightarrow 0,\infty)$is a measurable function$\}$. If $\alpha\in(1,2]$, then

(ii) $C_{\alpha,\ln}([1,e]):=\{g:(\ln r)^{2-\alpha}g(r)\in C([1,e])\}$

(iii) $G_{\alpha}(r,s)$ is Green's function of the operator $u\longrightarrow -{}^{\mathscr{H}}\mathscr{D}^{\alpha}u$ on $(1,e)$ with $\lim_{r\to 1^{+}}(\ln(r))^{2-\alpha}u(r)=0$ and $u(e)=0$.

(iv) $u_0(r):=a((\ln r)^{\alpha-2}-(\ln r)^{\alpha-1})+b(\ln r)^{\alpha-1}$ is the unique solution of the problem

$$\begin{cases} {}^{\mathscr{H}}\mathscr{D}^{\alpha}u(r)=0, \quad r\in(1,e), \\ u>0, \text{in } (1,e), \\ \lim_{r\to 1^{+}}(\ln(r))^{2-\alpha}u(r)=a>0, \quad u(e)=b>0 \end{cases} \quad (4)$$

(v) Assuming (H_1) and (H_2), we define

$$\lambda_0:=\inf_{r\in(1,e)}\frac{u_0(r)}{\int_1^e G_{\alpha}(r,s)(f(s,u_0(s))/s)ds} \quad (5)$$

It will be proven that $\lambda_0>0$.

The main result of this paper can be stated as follows.

Theorem 1. *Let $\alpha\in(1,2]$ and assume that hypotheses (H_2) and (H_2) are satisfied. Then, for $\lambda\in[0,\lambda_0)$, problem (3) has a solution $u_{\lambda}\in C_{\alpha,\ln}([1,e])$ satisfying for all $r\in(1,e]$,*

$$\left(1-\frac{\lambda}{\lambda_0}\right)\min(a,b)(\ln r)^{\alpha-2}\leq u_{\lambda}(r)\leq\max(a,b)(\ln r)^{\alpha-2}. \quad (6)$$

Remark 2.

(i) For $\alpha\in(1,2)$, we have $\lim_{r\to 1^{+}}u_{\lambda}(r)=\infty$.

(ii) If $\lambda=0$, then u_0 satisfies (6).

The remainder of this paper is organized as follows. In Section 2, some relevant properties of Hadamard fractional calculus are presented. Additionally, we construct Green's function and establish certain interesting inequalities. Theorem 1 is proven in Section 3. To illustrate our existence results, some examples are provided at the end of Section 3.

2. Preliminaries

We recall some relevant properties concerning Hadamard fractional derivative. For more details, the reader can see Section 2.7 of [19].

Definition 3. The Hadamard fractional integral of order $\gamma>0$ of the function h is defined as

$$\left({}^{\mathscr{H}}\mathscr{I}^{\gamma}h\right)(r):=\frac{1}{\Gamma(\gamma)}\int_1^r\left(\ln\frac{r}{s}\right)^{\gamma-1}\frac{h(s)}{s}ds, \quad 1\leq r\leq e. \quad (7)$$

For $\gamma=0$, we define ${}^{\mathscr{H}}\mathscr{I}^{0}h=h$.

Definition 4. Let $\gamma>0$ and $[\gamma]$ its integer part. The Hadamard fractional derivative of order γ of the function h is defined as

$$\left({}^{\mathscr{H}}\mathscr{D}^{\gamma}h\right)(r):=\delta^{n}\frac{1}{\Gamma(n-\gamma)}\int_1^r\left(\ln\frac{r}{s}\right)^{n-\gamma-1}\frac{h(s)}{s}ds, \quad 1\leq r\leq e, \quad (8)$$

where $n=[\gamma]+1$ and $\delta=r(d/dr)$.

Example 5. (Property 2.24 of [19]).

If $\gamma,\sigma>0$, then

$$\begin{aligned} \left({}^{\mathscr{H}}\mathscr{I}^{\gamma}(\ln s)^{\sigma-1}\right)(r)&=\frac{\Gamma(\sigma)}{\Gamma(\sigma+\gamma)}(\ln r)^{\sigma+\gamma-1}, \\ \left({}^{\mathscr{H}}\mathscr{D}^{\gamma}(\ln s)^{\sigma-1}\right)(r)&=\frac{\Gamma(\sigma)}{\Gamma(\sigma-\gamma)}(\ln r)^{\sigma-\gamma-1}. \end{aligned} \quad (9)$$

In particular, if $\sigma = 1$ and $\gamma \in (0,1)$, then ${}^{\mathscr{H}}\mathscr{D}^{\gamma}1(r) = (1/\Gamma(1-\gamma))(\ln r)^{-\gamma}$.

Lemma 6. *(see [5]).*
Let $\beta > \gamma > 0$ and $h \in C((1,e)) \cap L^1((1,e))$. Then,

(i) ${}^{\mathscr{H}}\mathscr{D}^{\gamma}({}^{\mathscr{H}}\mathscr{I}^{\beta}h) = {}^{\mathscr{H}}\mathscr{I}^{\beta-\gamma}h$ *and* ${}^{\mathscr{H}}\mathscr{D}^{\gamma}({}^{\mathscr{H}}\mathscr{I}^{\gamma}h) = h$

(ii) The equality $({}^{\mathscr{H}}\mathscr{D}^{\gamma}h)(r) = 0$ is valid on $(1,e)$ if, and only if,

$$h(r) = \sum_{j=1}^{m} c_j (\ln r)^{\gamma-j}, \tag{10}$$

where $c_j \in \mathbb{R}, j = 1, \cdots, m$, and m is the smallest integer greater than or equal to γ.

(iii) If ${}^{\mathscr{H}}\mathscr{D}^{\gamma}h \in C((1,e)) \cap L^1((1,e))$, then

$${}^{\mathscr{H}}\mathscr{I}^{\gamma}\left({}^{\mathscr{H}}\mathscr{D}^{\gamma}h\right)(r) = h(r) + \sum_{j=1}^{m} c_j (\ln r)^{\gamma-j}, \tag{11}$$

where $c_j \in \mathbb{R}, j = 1, \cdots, m$, and m is the smallest integer greater than or equal to γ.

Lemma 7. *Let $\alpha \in (1,2]$ and $h \in C([1,e])$.*

The unique solution of the problem

$$\begin{cases} {}^{\mathscr{H}}\mathscr{D}^{\alpha}u(r) + h(r) = 0, & 1 < r < e, \\ \lim_{r \to 1^+} (\ln(r))^{2-\alpha}u(r) = 0, & u(e) = 0, \end{cases} \tag{12}$$

is given by

$$u(r) = \int_1^e G_\alpha(r,s)\frac{h(s)}{s}\,ds, \tag{13}$$

where

$$G_\alpha(r,s) = \frac{1}{\Gamma(\alpha)}\begin{cases} (\ln r - \ln r \ln s)^{\alpha-1} - \left(\ln\left(\frac{r}{s}\right)\right)^{\alpha-1}, & 1 \le s \le r \le e, \\ (\ln r - \ln r \ln s)^{\alpha-1}, & 1 \le r \le s \le e. \end{cases} \tag{14}$$

Proof. By Lemma 6, the solution of problem (12) can be written as

$$u(r) = c_1(\ln r)^{\alpha-1} + c_2(\ln r)^{\alpha-2} - \frac{1}{\Gamma(\alpha)}\int_1^r \left(\ln\frac{r}{s}\right)^{\alpha-1}\frac{h(s)}{s}\,ds. \tag{15}$$

Since $\lim_{r \to 1^+}(\ln(r))^{2-\alpha}u(r) = 0$ and $u(e) = 0$, we obtain $c_2 = 0$ and

$$c_1 = \frac{1}{\Gamma(\alpha)}\int_1^e \left(\ln\frac{e}{s}\right)^{\alpha-1}\frac{h(s)}{s}\,ds. \tag{16}$$

Therefore,

$$\begin{aligned} u(r) &= \int_1^e \frac{(\ln r)^{\alpha-1}}{\Gamma(\alpha)}\left(\ln\frac{e}{s}\right)^{\alpha-1}\frac{h(s)}{s}\,ds \\ &\quad - \int_1^r \frac{1}{\Gamma(\alpha)}\left(\ln\frac{r}{s}\right)^{\alpha-1}\frac{h(s)}{s}\,ds \\ &= \int_1^e G_\alpha(r,s)\frac{h(s)}{s}\,ds, \end{aligned} \tag{17}$$

where $G_\alpha(r,s)$ is given by (14).

In Figure 1, we give the representation of the Green function $G_{3/2}(t,s)$ with the contours and the projections on some coordinate planes. In particular, one can see that $G_{3/2}(t,s)$ is nonnegative.

Lemma 8. *Let $1 < \alpha \le 2$. Then,*

(i) $G_\alpha(r,s) \in C([1,e] \times [1,e])$

(ii) On $(1,e) \times (1,e)$, one has

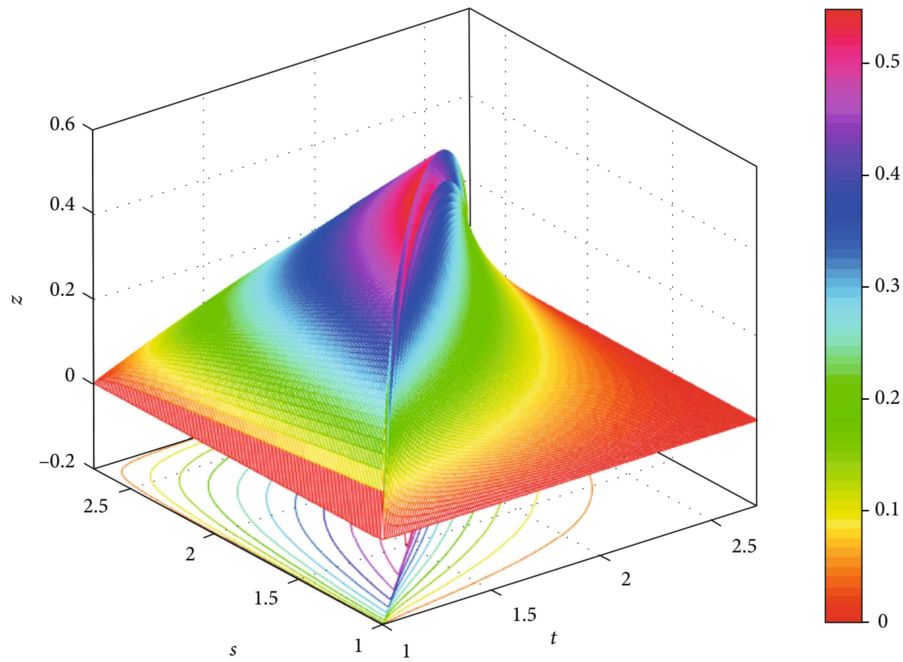
$$(\alpha-1)H_\alpha(r,s) \le \Gamma(\alpha)G_\alpha(r,s) \le H_\alpha(r,s), \tag{18}$$

where $H_\alpha(r,s) := (\ln r - \ln r \ln s)^{\alpha-2}\min(\ln r, \ln s)(1 - \max(\ln r, \ln s))$.

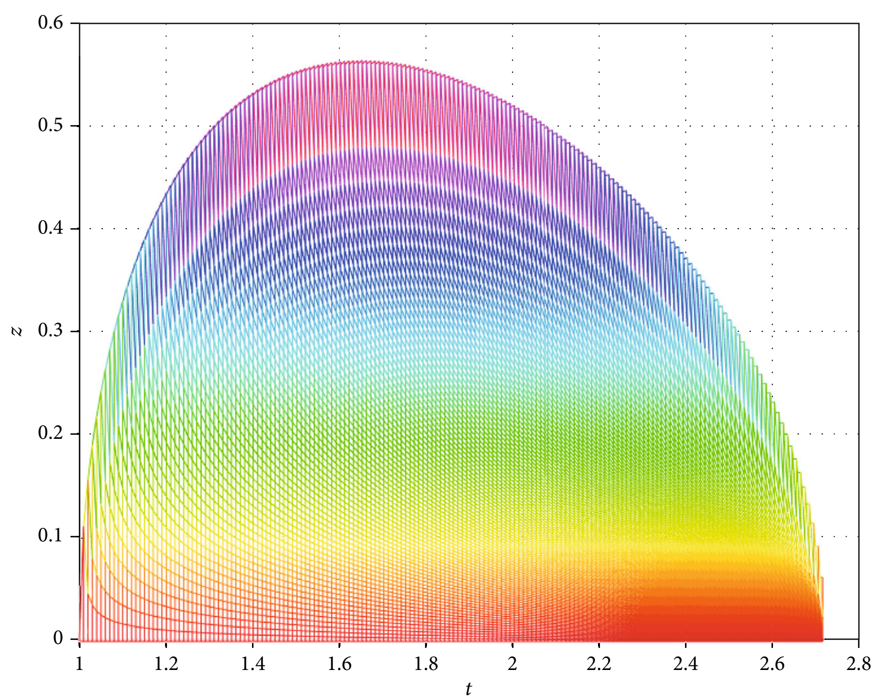
In particular, $G_\alpha(r,s) \ge 0$

(iii) On $(1,e) \times (1,e)$, one has

$$\begin{cases} (\alpha-1)(\ln s - \ln s \ln r)(\ln r - \ln r \ln s)^{\alpha-1} \le \Gamma(\alpha)G_\alpha(r,s), \\ \Gamma(\alpha)G_\alpha(r,s) \le (\ln s - \ln s \ln s)(\ln r - \ln r \ln s)^{\alpha-2} \end{cases} \tag{19}$$

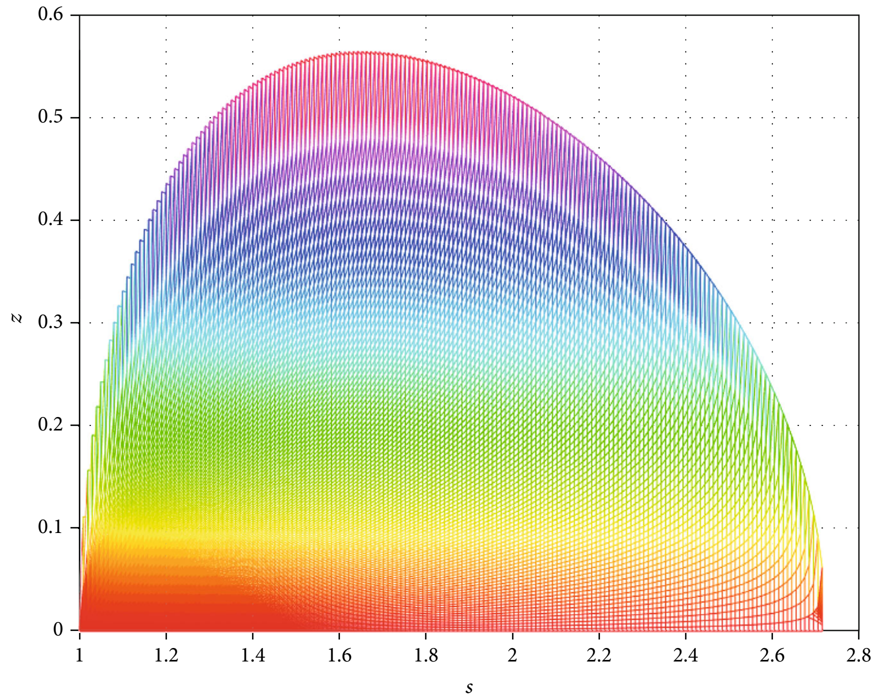


(a) $G_\alpha(t,s)$ and contours



(b) Projection on tz

Figure 1: Continued.



(c) Projection on sz

FIGURE 1: $G_\alpha(t, s)$ for $\alpha = 3/2$.

(iv) For each $r, \xi, s \in (1, e)$, the following holds:

$$\frac{G_\alpha(r,\xi)G_\alpha(\xi,s)}{G_\alpha(r,s)} \le \frac{1}{(\alpha-1)\Gamma(\alpha)}(\ln\xi)^{\alpha-1}(1-\ln\xi)^{\alpha-1}. \quad (20)$$

Proof. It is easy to check that (i) holds.

To prove (ii), for $r, s \in (1, e)$, we have

$$\begin{aligned} G_\alpha(r,s) &= \frac{1}{\Gamma(\alpha)}\left((\ln r - \ln r \ln s)^{\alpha-1} - \left((\ln r - \ln s)^+\right)^{\alpha-1}\right), \\ &= \frac{1}{\Gamma(\alpha)}(\ln r - \ln r\ln s)^{\alpha-1}\left(1 - \left(\frac{(\ln r - \ln s)^+}{\ln r - \ln r \ln s}\right)^{\alpha-1}\right), \end{aligned} \quad (21)$$

where $(\ln r - \ln s)^+ = \max(\ln r - \ln s, 0)$.

Therefore, inequalities in (18) follow from the fact that

$$(\alpha-1)(1-\xi) \le 1 - \xi^{\alpha-1} \le 1 - \xi, \quad \text{for } \xi \in [0,1]. \quad (22)$$

By using (18) and the fact that $(\ln r)(\ln s) \le \min(\ln r, \ln s) \le \ln s$ and $(1-\ln r)(1-\ln s) \le (1 - \max(\ln r, \ln s)) \le (1-\ln s)$, we obtain (19).

Next, we aim at proving (iv). Let $r, \xi, s \in (1, e)$ and put

$$\rho(r,s) := \min(\ln r, \ln s)(1 - \max(\ln r, \ln s)). \quad (23)$$

From (18), we have

$$\frac{G_\alpha(r,\xi)G_\alpha(\xi,s)}{G_\alpha(r,s)} \le \frac{(\ln\xi)^{\alpha-2}(1-\ln\xi)^{\alpha-2}}{(\alpha-1)\Gamma(\alpha)}\frac{\rho(r,\xi)\rho(\xi,s)}{\rho(r,s)}. \quad (24)$$

By symmetry, one can verify that

$$\frac{\rho(r,\xi)\rho(\xi,s)}{\rho(r,s)} \le (\ln\xi)(1-\ln\xi). \quad (25)$$

Hence, the required results follow from (24) and (25).

3. Proof of Theorem 1

We aim at proving Theorem 1. First, we need to establish some preliminary results. For $1 < \alpha \le 2$, we denote by

(i) $\mathcal{J}_{\alpha,\ln} := \{g \in \mathcal{B}^+((1,e)) : \int_1^e (1-\ln\xi)^{\alpha-1}(\ln\xi)^{\alpha-1} g(\xi)d\xi < \infty\}$

(ii) For $g \in \mathcal{B}^+((1,e))$,

$$\mathcal{A}_g := \sup_{r,s\in(1,e)} \int_1^e \frac{G_\alpha(r,\xi)G_\alpha(\xi,s)}{G_\alpha(r,s)}\frac{g(\xi)}{\xi}d\xi \quad (26)$$

(iii) For $g \in \mathcal{B}^+((1,e))$,

$$\mathcal{W}g(r) := \int_1^e G_\alpha(r,s)\frac{g(s)}{s}ds, \quad \text{for } r \in [1,e] \quad (27)$$

(iv) For $a, b > 0$, we recall that $u_0(r) := a((\ln r)^{\alpha-2} - (\ln r)^{\alpha-1}) + b(\ln r)^{\alpha-1}$ is the unique solution of problem (4).

Note that for $r \in (1, e]$,

$$\min (a, b)(\ln r)^{\alpha-2} \leq u_0(r) \leq \max (a, b)(\ln r)^{\alpha-2}. \quad (28)$$

We recall that $C_{\alpha,\ln}([1, e]) := \{g : (1, e] \longrightarrow \mathbb{R}, (\ln \xi)^{2-\alpha} g \in C([1, e])\}$.

Proposition 9. *Let $\alpha \in (1, 2)$ and $g \in \mathscr{B}^+((1, e))$, then*

(i) $\mathscr{W} g \in C_{\alpha,\ln}([1, e]) \Leftrightarrow \int_1^e (1 - \ln s)^{\alpha-1}(\ln s)g(s)ds < \infty$

(ii) Let g be such that the function $s \longrightarrow (1 - \ln s)^{\alpha-1}(\ln s)g(s) \in C((1, e)) \cap L^1((1, e))$, then $\mathscr{W} g \in C_{\alpha,\ln}([1, e])$ and it is the unique solution of the problem

$$\begin{cases} {}^{\mathscr{H}}\mathscr{D}^{\alpha}u(r) = -g(r), & 1 < r < e, \\ \lim_{r\to 1^+} (\ln (r))^{2-\alpha}u(r) = 0, & u(e) = 0. \end{cases} \quad (29)$$

Proof.

(i) The property follows from Lemma 8 (ii).

(ii) From (i), $\mathscr{W} g \in C_{\alpha,\ln}([1, e])$ and by using again Lemma 8 (ii), we have

$$\mathscr{W}|g|(r) \leq \frac{1}{\Gamma(\alpha)} (\ln r)^{\alpha-2} \int_1^e (\ln s)(1 - \ln s)^{\alpha-1} \frac{|g(s)|}{s} ds, \quad (30)$$

which implies by Example 5 (i) that ${}^{\mathscr{H}}\mathscr{I}^{2-\alpha}(\mathscr{W}|g|)$ is bounded on $(1, e)$.

Therefore, we have

$$\begin{aligned} {}^{\mathscr{H}}\mathscr{I}^{2-\alpha}(\mathscr{W} g)(r) &= \frac{1}{\Gamma(2-\alpha)} \int_1^r \left(\ln \frac{r}{s}\right)^{1-\alpha} \frac{\mathscr{W} g(s)}{s} ds \\ &= \frac{1}{\Gamma(2-\alpha)} \int_1^r \left(\ln \frac{r}{s}\right)^{1-\alpha} \frac{1}{s} \\ &\quad \cdot \left(\int_1^e G_\alpha(s, \xi) \frac{g(\xi)}{\xi} d\xi\right) ds \\ &= \int_1^e \mathscr{K}(r, \xi) \frac{g(\xi)}{\xi} d\xi, \end{aligned} \quad (31)$$

where

$$\mathscr{K}(r, \xi) = \frac{1}{\Gamma(2-\alpha)} \int_1^r \left(\ln \frac{r}{s}\right)^{1-\alpha} \frac{G_\alpha(s, \xi)}{s} ds. \quad (32)$$

We claim that

$$\mathscr{K}(r, \xi) = (1 - \ln \xi)^{\alpha-1} \ln r - (\ln r - \ln \xi)^+, \quad (33)$$

where $(\ln r - \ln \xi)^+ = \max (\ln r - \ln \xi, 0)$.
Indeed, from (32) and (14), we have

$$\begin{aligned} \mathscr{K}(r, \xi) = \frac{1}{\Gamma(\alpha)\Gamma(2-\alpha)} \Bigg[&\int_1^r (1 - \ln \xi)^{\alpha-1} \left(\ln \frac{r}{s}\right)^{1-\alpha} \frac{(\ln s)^{\alpha-1}}{s} ds \\ &- \int_1^r \left(\ln \frac{r}{s}\right)^{1-\alpha} ((\ln s - \ln \xi)^+)^{\alpha-1} \frac{1}{s} ds\Bigg]. \end{aligned} \quad (34)$$

Now using the fact that

$$\int_\zeta^\xi (\xi - \kappa)^{1-\alpha}(\kappa - \zeta)^{1-\alpha} d\kappa = \Gamma(\alpha)\Gamma(2-\alpha)(\xi - \zeta), \quad (35)$$

we deduce that

$$\int_1^r (\ln s)^{\alpha-1} \left(\ln \frac{r}{s}\right)^{1-\alpha} \frac{1}{s} ds = \Gamma(\alpha)\Gamma(2-\alpha) \ln r. \quad (36)$$

On the other hand,

(i) if $1 \leq \xi \leq r$, then by using (35), we get

$$\begin{aligned} &\int_1^r ((\ln s - \ln \xi)^+)^{\alpha-1} \left(\ln \frac{r}{s}\right)^{1-\alpha} \frac{1}{s} ds \\ &= \int_\xi^r (\ln s - \ln \xi)^{\alpha-1}(\ln r - \ln s)^{1-\alpha} \frac{1}{s} ds \\ &= \Gamma(\alpha)\Gamma(2-\alpha)(\ln r - \ln \xi) \end{aligned} \quad (37)$$

(ii) if $1 \leq r \leq \xi$, then obviously

$$\int_1^r ((\ln s - \ln \xi)^+)^{\alpha-1} \left(\ln \frac{r}{s}\right)^{1-\alpha} \frac{1}{s} ds = 0 \quad (38)$$

Hence,

$$\begin{aligned} &\int_1^r ((\ln s - \ln \xi)^+)^{\alpha-1} \left(\ln \frac{r}{s}\right)^{1-\alpha} \frac{1}{s} ds \\ &= \Gamma(\alpha)\Gamma(2-\alpha)(\ln r - \ln \xi)^+. \end{aligned} \quad (39)$$

So, (33) follows from (34), (36), and (39).
Next, we claim that

$${}^{\mathscr{H}}\mathscr{D}^{\alpha}(\mathscr{W} g)(r) := \delta^2 \left({}^{\mathscr{H}}\mathscr{I}^{2-\alpha}(\mathscr{W} g)\right)(r) = -g(r), \quad \text{for } r \in (1, e). \quad (40)$$

Indeed, from (33), we have

$$^{\mathscr{H}}\mathscr{I}^{2-\alpha}(\mathscr{W}g)(r)=\int_1^e \mathscr{K}(r,s)\frac{g(s)}{s}ds=\int_1^e\Big((\ln r)(1-\ln s)^{\alpha-1} - (\ln r-\ln s)^+\Big)\frac{g(s)}{s}ds$$
$$=(\ln r)\int_1^r\left((1-\ln s)^{\alpha-1}-1\right)\frac{g(s)}{s}ds + (\ln r)\int_r^e(1-\ln s)^{\alpha-1}\frac{g(s)}{s}ds + \int_1^r(\ln s)\frac{g(s)}{s}ds := \mathbb{J}_1(r)+\mathbb{J}_2(r)+\mathbb{J}_3(r). \tag{41}$$

From the hypothesis, the function $s\longrightarrow(1-\ln s)^{\alpha-1}(g(s)/s)$ is continuous and integrable near e while the function $s\longrightarrow(\ln s)(g(s)/s)$ becomes continuous and integrable near 1. So $\mathbb{J}_2(r)$ and $\mathbb{J}_3(r)$ are differentiable on $(1,e)$.

On the other hand by observing that

$$1-(1-\ln s)^{\alpha-1}=O(\ln s)\text{near}1, \tag{42}$$

we deduce that $\mathbb{J}_1(r)$ is differentiable on $(1,e)$.

Therefore,

$$\delta\left({}^{\mathscr{H}}\mathscr{I}^{2-\alpha}(\mathscr{W}g)\right)(r)=\int_1^r\left((1-\ln s)^{\alpha-1}-1\right)\frac{g(s)}{s}ds + \int_r^e(1-\ln s)^{\alpha-1}\frac{g(s)}{s}ds. \tag{43}$$

Applying for the second time the δ-derivative, we obtain

$$^{\mathscr{H}}\mathscr{D}^{\alpha}(\mathscr{W}g)(r)=\delta^2\left({}^{\mathscr{H}}\mathscr{I}^{2-\alpha}(\mathscr{W}g)\right)(r)=-g(r). \tag{44}$$

By Lemma 8 (ii) and (iii), for each $s\in[1,e]$, we have

$$\lim_{r\to1^+}\left((\ln r)^{2-\alpha}G_\alpha(r,s)\right)=0,$$
$$0\le\left((\ln r)^{2-\alpha}G_\alpha(r,s)\right)\le\frac{1}{\Gamma(\alpha)}(\ln s)(1-\ln s)^{\alpha-1}. \tag{45}$$

This implies by the dominated convergence theorem that

$$\lim_{r\to1^+}(\ln r)^{2-\alpha}\mathscr{W}g(r)=0. \tag{46}$$

Similarly, we have $(\mathscr{W}g)(e)=0$.

Finally, the uniqueness follows from Lemma 6 (ii).

Remark 10. The property of the above proposition remains true for $\alpha=2$.

Lemma 11. *Let $1<\alpha\le2$ and $g\in\mathscr{J}_{\alpha,\ln}$, then*

(i) $\mathscr{A}_g<\infty$

(ii) For all $r\in[1,e]$,

$$\int_1^e G_\alpha(r,s)u_0(s)\frac{g(s)}{s}ds\le\mathscr{A}_g u_0(r) \tag{47}$$

(iii) The family

$$\Lambda_g=\left\{\frac{1}{u_0(r)}\int_1^e G_\alpha(r,s)u_0(s)\frac{h(s)}{s}ds,\ |h|\le g\right\} \tag{48}$$

is relatively compact in $C([1,e])$.

Proof.

(i) As consequence of Lemma 8 (iv) and definition of $\mathscr{J}_{\alpha,\ln}$, we obtain $\mathscr{A}_g<\infty$.

(ii) Observe that for each $r,s\in(1,e)$, we have

$$\lim_{\xi\to1}\frac{G_\alpha(s,\xi)}{G_\alpha(r,\xi)}=\frac{(\ln s)^{\alpha-2}-(\ln s)^{\alpha-1}}{(\ln r)^{\alpha-2}-(\ln r)^{\alpha-1}} \tag{49}$$

Using this fact, Fatou's lemma, and (26), we deduce that

$$\int_1^e G_\alpha(r,s)\left(\frac{(\ln s)^{\alpha-2}-(\ln s)^{\alpha-1}}{(\ln r)^{\alpha-2}-(\ln r)^{\alpha-1}}\right)\frac{g(s)}{s}ds \le\liminf_{\xi\to1}\int_1^e\frac{G_\alpha(r,\xi)G_\alpha(\xi,s)}{G_\alpha(r,s)}\frac{g(s)}{s}ds\le\mathscr{A}_g. \tag{50}$$

That is,

$$\int_1^e G_\alpha(r,s)\left((\ln s)^{\alpha-2}-(\ln s)^{\alpha-1}\right)\frac{g(s)}{s}ds \le\mathscr{A}_g\left((\ln r)^{\alpha-2}-(\ln r)^{\alpha-1}\right),\quad\text{for } r\in[1,e]. \tag{51}$$

Similarly, since $\lim_{\xi\to e}(G_\alpha(s,\xi)/G_\alpha(r,\xi))=(\ln s)^{\alpha-1}/(\ln r)^{\alpha-1}$, we obtain

$$\int_1^e G_\alpha(r,s)(\ln s)^{\alpha-1}\frac{g(s)}{s}ds\le\mathscr{A}_g(\ln r)^{\alpha-1},\quad\text{for } r\in[1,e]. \tag{52}$$

Hence, (47) follows by combining (51) and (52).

(iii) It follows from (ii) and (i) that the family Λ_g is uniformly bounded

By (19) and (28), for $(r,s)\in[1,e]\times[1,e]$, we have

$$\left|\frac{G_\alpha(r,s)}{u_0(r)}u_0(s)\frac{g(s)}{s}\right|\le\frac{1}{\Gamma(\alpha)}\frac{\max(a,b)}{\min(a,b)}(\ln s-\ln s\ln s)^{\alpha-1}g(s). \tag{53}$$

Since the function $(r,s) \longrightarrow G_\alpha(r,s)/u_0(r) \in C([1,e] \times [1,e])$ and $g \in \mathcal{J}_{\alpha,\ln}$, we deduce by (53) that Λ_g is equicontinuous in $[1,e]$ and becomes relatively compact in $C([1,e])$ by Ascoli's theorem.

Proof of Theorem 1. We let

$$g_0(r) := \frac{1}{u_0(r)} f(r, u_0(r)), \quad \text{for } r \in (1,e). \tag{54}$$

By hypotheses (H_1) and (H_2) and (28), we have $g_0 \in \mathcal{J}_{\alpha,\ln}$.

Define

$$\lambda(r) := \frac{u_0(r)}{\int_1^e G_\alpha(r,s)(f(s,u_0(s))/s)ds}, \quad \lambda_0 := \inf_{r\in(1,e)} \lambda(r). \tag{55}$$

Using (54) and (47), we obtain

$$\int_1^e G_\alpha(r,s)\frac{f(s,u_0(s))}{s}ds = \int_1^e G_\alpha(r,s)u_0(s)\frac{g_0(s)}{s}ds \le \mathcal{A}_{g_0} u_0(r). \tag{56}$$

Therefore, $\lambda_0 \ge 1/\mathcal{A}_{g_0} > 0$.

Let $0 < \lambda \le \lambda_0$ and

$$S = \left\{ v \in C([1,e]) : \left(1 - \frac{\lambda}{\lambda_0}\right) \le v \le 1 \right\}. \tag{57}$$

For $v \in S$, define T by

$$Tv(r) = 1 - \frac{\lambda}{u_0(r)} \int_1^e G_\alpha(r,s)\frac{f(s, v(s)u_0(s))}{s}ds. \tag{58}$$

By using (H_1),(H_2), and Lemma 11 (iii), we prove that $T(S)$ is relatively compact in $C([1,e])$.

From (58), (H_1), and (55), we deduce that $T(S) \subseteq S$.

Next, by simple arguments, one can prove that T is a compact operator.

Therefore, it has a fixed point $v_\lambda \in S$ satisfying

$$v_\lambda(r) = 1 - \frac{\lambda}{u_0(r)} \int_1^e G_\alpha(r,s)\frac{f(s, v_\lambda(s)u_0(s))}{s}ds. \tag{59}$$

Let $u_\lambda(r) = v_\lambda(r)u_0(r)$. Then, $u_\lambda \in C_{\alpha,\ln}([1,e])$ and satisfies

$$u_\lambda(r) = u_0(r) - \lambda \int_1^e G_\alpha(r,s)\frac{f(s,u_\lambda(s))}{s}ds. \tag{60}$$

Since $v_\lambda \in S$, it follows from (28) that

$$\left(1 - \frac{\lambda}{\lambda_0}\right) \min(a,b)(\ln r)^{\alpha-2} \le u_\lambda(r) \le \max(a,b)(\ln r)^{\alpha-2}. \tag{61}$$

By using (H_1), (61), and (H_2), we deduce that the function $s \longrightarrow (\ln s)(1-\ln s)^{\alpha-1} f(s,u_\lambda(s)) \in C((1,e)) \cap L^1((1,e))$.

Hence, from (60), Proposition 9 (ii), and (4), we conclude that u_λ is a solution of problem (3).

Example 12. Let $0 < \beta < 1$. Then, for some $\lambda_0 > 0$ and each $\lambda \in [0, \lambda_0)$, problem

$$\begin{cases} {}^{\mathcal{H}}\mathcal{D}^{4/3}u(r) - \lambda(\ln(r))^{-\beta}u^{3/2} = 0, & r \in (1,e), \\ u > 0, \text{ in } (1,e), \\ \lim_{r\to1^+} (\ln(r))^{2/3}u(r) = a > 0, & u(e) = b > 0, \end{cases} \tag{62}$$

has a solution u_λ in $C_{4/3,\ln}([1,e])$ satisfying

$$\left(1 - \frac{\lambda}{\lambda_0}\right) \min(a,b)(\ln r)^{-2/3} \le u_\lambda(r) \le \max(a,b)(\ln r)^{-2/3}. \tag{63}$$

Observe that the nonlinearity considered in this example is singular at $r = 1$.

Example 13. Let $1 < \alpha < 2, \gamma \in [0, 2/(2-\alpha))$ and $p \in C^+([1,e])$.

Then, there exists a constant $\lambda_0 > 0$ such that for $\lambda \in [0, \lambda_0)$, problem

$$\begin{cases} {}^{\mathcal{H}}\mathcal{D}^{\alpha}u(r) - \lambda p(r)u^{\gamma} = 0, & r \in (1,e), \\ u > 0, \text{ in } (1,e), \\ \lim_{r\to1^+} (\ln(r))^{2-\alpha}u(r) = a > 0, & u(e) = b > 0, \end{cases} \tag{64}$$

admit a solution u_λ in $C_{\alpha,\ln}([1,e])$ satisfying

$$\left(1 - \frac{\lambda}{\lambda_0}\right) \min(a,b)(\ln r)^{\alpha-2} \le u_\lambda(r) \le \max(a,b)(\ln r)^{\alpha-2}. \tag{65}$$

In particular, for $= 3/2, \gamma = 2$, and $a = b = 1$, we have from (4) and (55),

$$\lambda(r) := \frac{(\ln r)^{-1/2}}{\int_1^e G_{3/2}(r,s)(\ln s)^{-1}(p(s)/s)ds}. \tag{66}$$

Therefore, by choosing some continuous functions p_i $(i = 1,2,3,4)$ in (66), we obtain the following graph for $\lambda_i(r) := \lambda(r)$ with $p = p_i$ and a numerical value of the constant $\lambda_0 := \inf_{r\in(1,e)} \lambda_i(r)$. In Figure 2, we collect the graph of functions λ_i, and in Table 1, we summarize the numerical value of λ_0.

Example 14. Let $1 < \alpha < 2$ and $\beta \in [0, 2/(2-\alpha))$. Then, Theorem 1 can be applied for $f(r,s) := ((\ln r)^{\alpha-2} + s)^{\beta}$, where $(r,s) \in (1,e) \times [0,\infty)$.

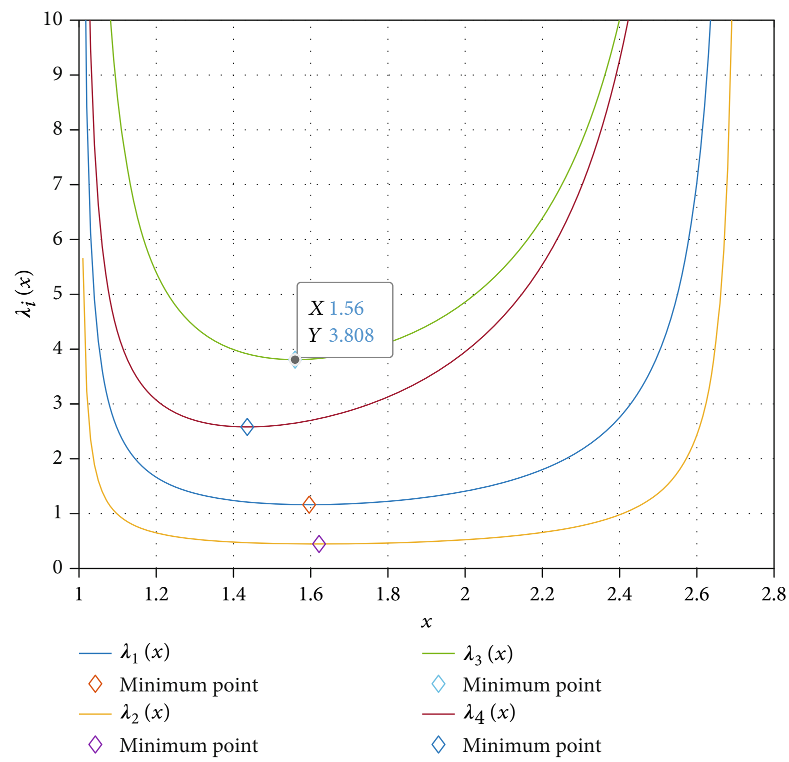


FIGURE 2: Graph of the functions λ_i.

TABLE 1: Value of λ_0.

Function $p(s)$ in (66)	Value of λ_0
$p_1(s) = s^{3/2}$	$\lambda_0 = 1.162$
$p_2(s) = e^s$	$\lambda_0 = 0.446$
$p_3(s) = (\ln s)^{1/2}$	$\lambda_0 = 3.808$
$p_4(s) = \sin s$	$\lambda_0 = 2.581$

4. Conclusion

In this paper, we have considered singular nonlinear Hadamard fractional boundary value problems. By using estimates on Green's function and the Schauder fixed point theorem, we have proven the existence of a positive solution which blows up.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors' Contributions

All authors contributed equally to writing of this paper. All authors read and approved the final manuscript.

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