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Research Article

Inequalities of Hardy Type via Superquadratic Functions with General Kernels and Measures for Several Variables on Time Scales

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We use the properties of superquadratic functions to produce various improvements and popularizations on time scales of the Hardy form inequalities and their converses. Also, we include various examples and interpretations of the disparities in the literature that exist. In particular, our findings can be seen as refinements of some recent results closely linked to the time-scale inequalities of the classical Hardy, Pólya-Knopp, and Hardy-Hilbert. Some continuous inequalities are derived from the main results as special cases. The essential results will be proved by making use of some algebraic inequalities such as the Minkowski inequality, the refined Jensen inequality, and the Bernoulli inequality on time scales.

1. Introduction

In [1], Hardy claimed this fundamental inequality and proved it:

$$\int_0^\infty \left(\frac{1}{\theta}\int_0^\theta g(\eta)d\eta\right)^q d\theta \le \left(\frac{q}{q-1}\right)^q \int_0^\infty g^q(\theta)d\theta, \quad (1)$$

where $1 < q < \infty$, $g \ge 0$, and $(q/(q-1))^q$ are sharp. They have emerged in the literature since the discovery of (1) numerous papers concerned with new arguments, generalizations, and extensions. One of the most common generalizations for (1) is the disparity of Pólya-Knopp's inequality (see [2]), which is

$$\int_0^\infty \exp\left(\frac{1}{\theta}\int_0^\theta \ln g(\eta)d\eta\right)d\theta \le e\int_0^\infty g(\theta)d\theta. \quad (2)$$

In [3], Kaijser et al. signalized that both (1) and (2) are special states of the Hardy-Knopp's inequality:

$$\int_0^\infty \Theta\left(\frac{1}{\theta}\int_0^\theta g(\eta)d\eta\right)\frac{d\theta}{\theta} \le \int_0^\infty \Theta(g(\theta))d\theta\frac{d\theta}{\theta}, \quad (3)$$

where $\Theta \in C((0,\infty),\mathbb{R})$ is a convex function.

In [4], Cizmeija et al. proved that if $\zeta : (0,\alpha) \longrightarrow \mathbb{R} \ge 0$, Θ is a convex on (β,γ) where $-\infty \le \beta \le \gamma \le \infty$, $g : (0,\alpha) \longrightarrow \mathbb{R}$ with $g(\theta) \in (\beta,\gamma)$, $\forall\theta \in (0,\alpha)$ as an integrable function and υ is defined by

$$\upsilon(\eta) \coloneqq \eta\int_\eta^\alpha \frac{\zeta(\theta)}{\theta^2}d\theta, \quad \text{for } \eta \in (0,\alpha), \quad (4)$$

then the integral inequality

$$\int_0^\infty \zeta(\theta)\Theta\left(\frac{1}{\theta}\int_0^\theta g(\eta)d\eta\right)\frac{d\theta}{\theta} \le \int_0^\infty \upsilon(\theta)\Theta(g(\theta))\frac{d\theta}{\theta}, \quad (5)$$

is valid.

In [5], Kaijser et al. applied the inequality of Jensen for convex functions and the theorem of Fubini to establish an invitingly popularization (1). Particularly, it was proved that if $\zeta : (0,\alpha) \longrightarrow \mathbb{R} \ge 0$ and $l : (0,\alpha)\times(0,\alpha) \longrightarrow \mathbb{R} \ge 0, 0<\alpha \le \infty$ such that

$$L(\theta) \coloneqq \int_0^\theta l(\theta,\eta)d\eta > 0, \theta \in (0,\alpha), \quad (6)$$

and $\Theta \in C(I,\mathbb{R}), I \subseteq \mathbb{R}$ is a convex function, $g : (0,\alpha) \longrightarrow \mathbb{R}$ such that $g(\theta) \in I, \forall\theta \in (0,\alpha)$ be integrable function, and υ is defined by

$$\upsilon(\eta) \coloneqq \eta\int_\eta^\alpha \xi(\theta)\frac{l(\theta,\eta)}{L(\theta)}\frac{d\theta}{\theta} < \infty, \quad \eta \in (0,\alpha), \quad (7)$$

then the integral inequality

$$\int_0^\infty \xi(\theta)\Theta(A_l g(\theta))\frac{d\theta}{\theta} \le \int_0^\infty \upsilon(\theta)\Theta(g(\theta))\frac{d\theta}{\theta}, \quad (8)$$

is valid, where $A_l g$ is defined by

$$A_l g(\theta) \coloneqq \frac{1}{L(\theta)}\int_0^\theta l(\theta,\eta)g(\eta)d\eta, \theta \in (0,\alpha). \quad (9)$$

As a popularization of (8), Krulic et al. [6] have demonstrated that if $(\Omega_1, \sum_1, \mu_1)$ and $(\Omega_2, \sum_2, \mu_2)$ are two measure spaces with positive σ finite measures $\zeta : \Omega_1 \longrightarrow \mathbb{R} \ge 0$ and $l : \Omega_1 \times \Omega_2 \longrightarrow \mathbb{R} \ge 0$ such that

$$L(\theta) \coloneqq \int_{\Omega_2} l(\theta,\eta)d\mu_2(\eta) > 0, \quad \theta \in \Omega_1, \quad (10)$$

and Θ is a convex function on an interval $I \subseteq \mathbb{R}$, $g : \Omega_2 \longrightarrow \mathbb{R} \ge\ > 0$ with $g(\Omega_2) \subseteq I$ be measurable function and υ is defined by

$$\upsilon(\eta) \coloneqq \left(\int_{\Omega_1} \xi(\theta)\left(\frac{l(\theta,\eta)}{L(\theta)}\right)^{q/p} d\mu_1(\theta)\right)^{q/p} < \infty, \quad \eta \in \Omega_2, \quad (11)$$

then the integral inequality

$$\left(\int_{\Omega_1} \xi(\theta)\Theta^{q/p}(A_l g(\theta))d\mu_1(\theta)\right)^{1/q} \le \left(\int_{\Omega_2} \upsilon(\eta)\Theta(g(\eta))d\mu_2(\eta)\right)^{1/q}, \quad (12)$$

is valid, where $0 < p \le q < \infty$ and $A_l g : \Omega_1 \longrightarrow \mathbb{R}$ are defined by

$$A_l g(\theta) \coloneqq \frac{1}{L(\theta)}\int_{\Omega_2} l(\theta,\eta)g(\eta)d\mu_2(\eta), \quad \theta \in \Omega_1. \quad (13)$$

Observe that inequality (12) is a generalization of Hardy inequality (1). Namely, let $\Omega_1 = \Omega_2 = \mathbb{R}_+ = (0,\infty)$, $d\mu_1(\theta) = d\theta, d\mu_2(\eta) = d\eta$ and $u(\theta) = 1/\theta$, and if $1 < p = q < \infty$ and $\Theta : [0,\infty) \longrightarrow \mathbb{R}$ are defined by $\Theta(\theta) = \theta^p$, then (1) is followed directly from (12), which can be rewritten with $g(\eta^{p/(p-1)})\eta^{1/(p-1)}$ instead of $g(\eta)$ and

$$l(\theta,\eta) \coloneqq \frac{1}{\theta}\chi_{0<\eta\le\theta<\infty}(\theta,\eta). \quad (14)$$

In the same setting, except with $g(\eta)\eta^{1/p}$ instead of $g(\eta)$ and with

$$l(\theta,\eta) \coloneqq \left(\frac{\theta}{\eta}\right)^{1/q}(\theta+\eta)^{-1}, \quad (15)$$

relation (12) becomes the Hardy-Hilbert integral inequality (see [7]).

$$\int_0^\infty \left(\int_0^\infty \frac{g(\eta)}{\theta+\eta}d\eta\right)^p d\theta \le \left(\frac{\pi}{\sin(\pi/p)}\right)^p \int_0^\infty g^p(\theta)d\theta. \quad (16)$$

In [8], Abramovich et al. considered a superquadratic function Θ instead of a convex function Θ and obtained the following refinement of inequality (12) in the particular case $p = q$, as

$$\begin{aligned}&\int_{\Omega_1} \zeta(\theta)\Theta(A_l g(\theta))d\mu_1(\theta)\\ &\quad + \int_{\Omega_1}\int_{\Omega_2} \zeta(\theta)\frac{l(\theta,\eta)}{L(\theta)}\Theta(|g(\eta) - A_l g(\theta)|)d\mu_1(\theta)d\mu_2(\eta)\\ &\le \int_{\Omega_2} \upsilon(\eta)\Theta(g(\eta))d\mu_2(\eta).\end{aligned} \quad (17)$$

In [9], Aleksandra et al. proved that, if $\lambda \le 1, (\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ are two measure spaces with positive σ-finite measures, $\zeta : \Omega_1 \longrightarrow \mathbb{R} \ge 0, l : \Omega_1 \times \Omega_2 \longrightarrow \mathbb{R} \ge 0$ such that $L : \Omega_1 \longrightarrow \mathbb{R}$ is defined as in (10), $\Theta \in C(I,\mathbb{R})$, $I \subseteq \mathbb{R}$ is a convex function, $g : \Omega_2 \longrightarrow \mathbb{R} \ge 0$ such that $g(\Omega_2) \subseteq I$ be measurable function and is defined by

$$\upsilon(\eta) \coloneqq \left(\int_{\Omega_1} \zeta(\theta)\left(\frac{l(\theta,\eta)}{L(\theta)}\right)^\lambda d\mu_1(\theta)\right)^{1/\lambda} < \infty, \quad \eta \in \Omega_2, \quad (18)$$

then the integral inequality

$$\int_{\Omega_1} \zeta(\theta)\Theta^{\lambda}(A_l g(\theta))d\mu_1(\theta) + \lambda \int_{\Omega_1}\int_{\Omega_2} \zeta(\theta)\frac{l(\theta,\eta)}{L(\theta)} \cdot \Theta^{\lambda-1}(|g(\eta) - A_l g(\theta)|)d\mu_1(\theta)d\mu_2(\eta) \leq \left(\int_{\Omega_2} v(\eta)\Theta(g(\eta))d\mu_2(\eta)\right)^{\lambda}, \quad (19)$$

is valid, where $A_l g : \Omega_1 \longrightarrow \mathbb{R}$ is defined by (13).

In the past few years, several researchers have suggested the study of dynamic time-scale inequalities. In [10], the authors showed a number of Hardy-type inequalities with a general kernel on time scale. Namely, they have determined that if $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ are two time-scale measure spaces, $l : \Omega_1 \times \Omega_2 \longrightarrow \mathbb{R} \geq 0$ such that

$$L(\theta) \coloneqq \int_{\Omega_2} l(\theta,\eta)\Delta\eta < \infty, \theta \in \Omega_1, \quad (20)$$

and $\zeta : \Omega_1 \longrightarrow \mathbb{R}_+ \geq 0$ such that

$$v(\eta) \coloneqq \int_{\Omega_1} \frac{l(\theta,\eta)\zeta(\theta)}{L(\theta)}\Delta\theta < \infty, \quad \eta \in \Omega_2, \quad (21)$$

then the integral inequality

$$\int_{\Omega_1} \zeta(\theta)\Theta\left(\frac{1}{L(\theta)}\int_{\Lambda} l(\theta,\eta)g(\eta)\Delta\eta\right)\Delta\theta \leq \int_{\Omega_2} v(\eta)\Theta(g(\eta))\Delta\eta, \quad (22)$$

is available for all $\Delta\mu_2$-integrable $g : \Omega_2 \longrightarrow \mathbb{R}$ such that $g(\Omega_2) \subset I$ and $\Theta \in C(I, \mathbb{R}), I \subset \mathbb{R}$ are a convex function.

Moreover, Donchev et al. [11] improved the inequality (22) by replacing the function $g(\eta)$ by an m-tuple of functions $\boldsymbol{g}(\eta) = (g_1(\eta), g_2(\eta), \cdots, g_m(\eta))$ such that $g_1(\eta), g_2(\eta), \cdots, g_m(\eta)$ are $\Delta\mu_2$-integrable on Ω_2 in the following way. If $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ are two time-scale measure spaces, $U \subset \mathbb{R}^m$ a convex set and $l : \Omega_1 \times \Omega_2 \longrightarrow \mathbb{R}_+$ such that

$$L(\theta) \coloneqq \int_{\Omega_2} l(\theta,\eta)\Delta\eta < \infty, \quad \theta \in \Omega_1, \quad (23)$$

and $\zeta : \Omega_1 \longrightarrow \mathbb{R}$ such that

$$v(\eta) \coloneqq \int_{\Omega_1} \frac{l(\theta,\eta)\zeta(\theta)}{L(\theta)}\Delta\theta < \infty, \quad \eta \in \Omega_2, \quad (24)$$

then for every a convex function Θ, the integral inequality

$$\int_{\Omega_1} \zeta(\theta)\Theta\left(\frac{1}{L(\theta)}\int_{\Omega_2} l(\theta,\eta)\boldsymbol{g}(\eta)\Delta\eta\right)\Delta\theta \leq \int_{\Omega_2} v(\eta)\Theta(\boldsymbol{g}(\eta))\Delta\eta, \quad (25)$$

is available for all $\Delta\mu_2$-integrable functions $\boldsymbol{g} : \Omega_2 \longrightarrow \mathbb{R}^m$ such that $\boldsymbol{g}(\Omega_2) \subset U \subset \mathbb{R}^m$.

In [12], the authors have specified the time scale version of (17). That is, they proved it if $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ are two time-scale measure spaces with positive σ-finite measures, $\zeta : \Omega_1 \longrightarrow \mathbb{R} \geq 0$ and $l : \Omega_1 \times \Omega_2 \longrightarrow \mathbb{R} \geq 0$ such that $l(\theta, .)$ is a $\Delta\mu_2$-integrable function for $\theta \in \Omega_2$, and $L : \Omega_1 \longrightarrow \mathbb{R}$ is defined as

$$L(\theta) \coloneqq \int_{\Omega_2} l(\theta,\eta)\Delta\mu_2(\eta) > 0, \quad \theta \in \Omega_1, \quad (26)$$

$$v(\eta) \coloneqq \int_{\Omega_1} \zeta(\theta)\frac{l(\theta,\eta)}{L(\theta)}\Delta\mu_1(\theta) < \infty, \quad \eta \in \Omega_2. \quad (27)$$

If $\Theta : [\alpha, \infty) \longrightarrow \mathbb{R} \geq 0, (\alpha \geq 0)$ and a superquadratic function, then

$$\int_{\Omega_1} \zeta(\theta)\Theta(A_l g(\theta))\Delta\mu_1(\theta) + \int_{\Omega_1}\int_{\Omega_2} \zeta(\theta)\frac{l(\theta,\eta)}{L(\theta)}\Theta \cdot (|g(\eta) - A_l g(\theta)|)\Delta\mu_1(\theta)\Delta\mu_2(\eta) \leq \int_{\Omega_2} v(\eta)\Theta(g(\eta))\Delta\mu_2(\eta), \quad (28)$$

is available for all $\Delta\mu_2$-integrable function $g : \Omega_2 \longrightarrow \mathbb{R} \geq 0$, and $A_l g$ is defined by

$$(A_l g)(\theta) \coloneqq \frac{1}{L(\theta)}\int_{\Omega_2} l(\theta,\eta)g(\eta)\Delta\mu_2(\eta), \quad \theta \in \Omega_1. \quad (29)$$

In [13], Saker et al. obtained the following refined Jensen's inequality for superquadratic

$$\Theta\left(\frac{\int_{\Omega_2} l(\theta,\eta)g(\eta)\Delta\mu_2(\eta)}{\int_{\Omega_2} l(\theta,\eta)\Delta\mu_2(\eta)}\right) \leq \int_{\Omega_2} \frac{l(\theta,\eta)}{\int_{\Omega_2} l(\theta,\eta)\Delta\mu_2(\eta)}[\Theta(g(\eta)) - \Theta(|g(\eta) - A_l g(\theta)|)]\Delta\mu_2(\eta), \quad (30)$$

and in the same paper, he employed the above result to derive the following inequality of Hardy type:

$$\int_{\Omega_1} \zeta(\theta)\Theta^{\lambda}(A_l g(\theta))\Delta\mu_1(\theta) + \lambda\int_{\Omega_1}\int_{\Omega_2} \zeta(\theta) \cdot \frac{l(\theta,\eta)}{L(\theta)}\Theta^{\lambda-1}(A_l g(\theta))\Theta(|g(\eta) - A_l g(\theta)|)\Delta\mu_1(\eta) \leq \left(\int_{\Omega_2} v(\eta)\Theta(g(\eta))\Delta\mu_2(\eta)\right)^{\lambda}, \quad (31)$$

where

$$v(\eta) := \left(\int_{\Omega_1} \zeta(\theta) \left(\frac{l(\theta, \eta)}{L(\theta)} \right)^{\lambda} \Delta\mu_1(\theta) \right)^{1/\lambda} < \infty, \quad \mu \in \Omega_2, \tag{32}$$

$\lambda \geq 1$, $\zeta : \Omega_1 \longrightarrow \mathbb{R} \geq 0$, and $l : \Omega_1 \times \Omega_2 \longrightarrow \mathbb{R} \geq 0$ such that $l(\theta, .)$ is a $\Delta\mu_2$-integrable function for $\theta \in \Omega_2$ and $L : \Omega_1 \longrightarrow \mathbb{R}$ is defined by (26), $\Theta : [0, \infty) \longrightarrow \mathbb{R} \geq 0$ is a superquadratic function, and $A_l g$ is defined by (29).

Another development of Hardy-type inequality (28) has been made by Bibi [14] and Fabelurin [15] as follows. If $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ are two time-scale measure spaces, $\zeta : \Omega_1 \longrightarrow \mathbb{R} \geq 0$ and $l : \Omega_1 \times \Omega_2 \longrightarrow \mathbb{R} \geq 0$ such that $l(\theta, .)$ are a $\Delta\mu_2$-integrable function for $\theta \in \Omega_2$, $L : \Omega_1 \longrightarrow \mathbb{R}$ is defined by (26) and $\Theta \in C(K_m, \mathbb{R})$ is a superquadratic function, then

$$\begin{aligned} &\int_{\Omega_1} \xi(\theta)\Theta((A_l \boldsymbol{g})(\theta))\Delta\mu_1(\theta) + \int_{\Omega_1}\int_{\Omega_2} \xi(\theta) \\ &\cdot \frac{l(\theta, \eta)}{L(\theta)} \Theta(|\boldsymbol{g}(\eta) - (A_l \boldsymbol{g})(\theta)|)\Delta\mu_2(\eta) \\ &\leq \int_{\Omega_2} v(\eta)\Theta(\boldsymbol{g}(\eta))\Delta\mu_2(\eta), \end{aligned} \tag{33}$$

is available for all $\Delta\mu_2$-integrable functions $\boldsymbol{g} : \Omega_2 \longrightarrow \mathbb{R}^m$ such that $\boldsymbol{g}(\Omega_2) \subset K_m$, where $A_l \boldsymbol{g} : \Omega_1 \longrightarrow \mathbb{R}$ is defined by

$$(A_l \boldsymbol{g})(\theta) := \frac{1}{L(\theta)} \int_{\Omega_2} l(\theta, \eta)\boldsymbol{g}(\eta)\Delta\mu_2(\eta), \quad \theta \in \Omega_1. \tag{34}$$

For developing of dynamic inequalities on time scale calculus, we refer the reader to the articles [16–26].

Motivated by the above results, our major aim in this paper is to deduce few nouveau general Hardy-type inequalities for multivariate superquadratic functions that involve more general kernels on arbitrary time scales.

The paper is governed as follows: We remember some basic notions, definitions, and results of multivariate superquadratic functions on time scales in Preliminaries. In Inequalities with General Kernel, we obtain the extensions to the general kernel of Hardy-type inequality. In Inequalities with Specific Time Scales, we extend the latest results from Inequalities with General Kernel to several specific time scales. In Inequalities with Specific Time Scales, we discuss several particular cases of Hardy-type inequality by choosing such special kernels. In Inequalities with Specific Kernels, we derive enhanced forms of certain well-known Hardy-Hilbert-type inequalities.

2. Preliminaries

In this section, we will present some fundamental concepts and effects to integrals of time scales and for multivariate superquadratic functions which will be useful to deduce our major results. Let $\mathbb{R}^m$ be the Euclidean space, $\boldsymbol{\theta} := (\theta_1, \theta_2, \cdots, \theta_m) \in \mathbb{R}^m$, $\boldsymbol{\eta} := (\eta_1, \eta_2, \cdots, \eta_m) \in \mathbb{R}^m$, and $\boldsymbol{g}(t) := (g_1(t), g_2(t), \cdots, g_m(t))$ be the function defined on $\boldsymbol{\theta} \subset \mathbb{R}^m$. Throughout this supplement, we utilize the following notations:

$$\begin{aligned} \boldsymbol{\theta}.\boldsymbol{\eta} &:= (\theta_1\eta_1, \theta_2\eta_2, \cdots, \theta_m\eta_m), \\ |\boldsymbol{\theta}| &:= (|\theta_1|, |\theta_2|, \cdots, |\theta_m|) \text{ and} \\ \langle \boldsymbol{\theta}, \boldsymbol{\eta} \rangle &:= \sum_{i=1}^{m} \theta_i \eta_i. \end{aligned} \tag{35}$$

Also, $\boldsymbol{\theta} \leq \boldsymbol{\eta} (\theta < \eta)$ means that $\theta_i \leq \eta_i (\theta_i < \eta_i)$, $\forall 1 \leq i \leq m$, and $\mathbf{0} := (0, 0, \cdots, 0)$ is the null vector. The subsets K_m and K_m^+ in $\mathbb{R}^m$ are defined by

$$\begin{aligned} K_m &:= [0, \infty)^m := \{\boldsymbol{\theta} \in \mathbb{R}^m : \mathbf{0} \leq \boldsymbol{\theta}\}, \\ K_m^+ &:= [0, \infty)^m := \{\boldsymbol{\theta} \in \mathbb{R}^m : \mathbf{0} < \boldsymbol{\theta}\}. \end{aligned} \tag{36}$$

Now, we arraign the definition and few essential properties of superquadratic functions that premised in [27].

Definition 1. A function $\Theta : K_m \longrightarrow \mathbb{R}$ is named a superquadratic function if $\forall \boldsymbol{\theta} \in K_m$, $\exists c(\theta) \in \mathbb{R}^m$ such that

$$\Theta(\boldsymbol{\eta}) - \Theta(\boldsymbol{\theta}) - \Theta(|\boldsymbol{\eta} - \boldsymbol{\theta}|) \geq \langle c(\boldsymbol{\theta}), \boldsymbol{\eta} - \boldsymbol{\theta} \rangle, \forall \boldsymbol{\eta} \in K_m. \tag{37}$$

If $-\Theta$ is a superquadratic, then Θ is a subquadratic, and the reverse inequality of (37) is available.

In the following, we recall a couple of beneficial examples of a superquadratic function.

Example 1. By [2], Example 1, the power function $\Theta : [0, \infty) \longrightarrow \mathbb{R}$, defined by $\Theta(\theta) := \theta^p$, is called a superquadratic if $p \geq 2$ and a subquadratic if $1 < p \leq 2$ (it is also readily seen that if $0 < p \leq 1$ then θ^p is a subquadratic function). Since the sum of superquadratic functions is also superquadratic, then

$$\Theta(\boldsymbol{\theta}) := \sum_{i=1}^{m} \theta_i^p, \tag{38}$$

is a superquadratic on K_m for each $p \geq 2$.

Example 2 ([2], Examples 4, 5, and 6,). By utilizing the same argument as in Example 1, the functions $\Theta_1, \Theta_2, \Theta_3 : K_m \longrightarrow \mathbb{R}$ defined as

$$\Theta_1(\boldsymbol{\theta}) \coloneqq \sum_{i=1}^{m}(\theta_i \cosh \theta_i - \sinh \theta_i),$$
$$\Theta_2(\boldsymbol{\theta}) \coloneqq \ln\left(1 + \sum_{i=1}^{m}\theta_i\right) - \sum_{i=1}^{m}\theta_i, \tag{39}$$
$$\Theta_3(\boldsymbol{\theta}) \coloneqq \begin{cases} \sum_{i=1,i\neq j}^{m} \theta_i^2 \ln \theta_i, & \text{if } \theta_i > 0, \theta_j = 0, \\ 0, & \text{if } \boldsymbol{\theta} = 0, \end{cases}$$

are superquadratic.

The following lemma shows that nonnegative superquadratic functions are indeed convex functions.

Lemma 2. *Suppose that Θ is a superquadratic with $\mathbf{c}(\theta) \coloneqq (c_1(\theta), c_2(\theta), \cdots, c_n(\theta))$ as in Definition 1. Then*

(i) $\Theta(0) \leq 0$ and $c_i(0) \leq 0 \forall 1 \leq i \leq m$

(ii) If $\Theta(0) \coloneqq 0$ and $\nabla\Theta(0) \coloneqq 0$, then $c_i(\theta) \coloneqq \partial_i g(\theta)$, whenever $\partial_i g(\theta)$ exists for some index $1 \leq i \leq m$ at $\boldsymbol{\theta} \in K_m$

(iii) If $\Theta \geq 0$, then Θ is convex and $\Theta(0) \coloneqq 0$ and $\nabla\Theta(0) \coloneqq 0$.

In the following, we recall the inequality of Minkowski and the inequality of Jensen for superquadratic functions on time scales which are utilized in the proof of the essential results. The following definitions and theorems are referred from [28, 29]. Let $\mathbb{T}_i$, $1 \leq i \leq m$ be time scales, and

$$\begin{aligned} \Lambda^m &\coloneqq \mathbb{T}_1 \times \mathbb{T}_2 \times \cdots \times \mathbb{T}_m \\ &\coloneqq \{t = (t_1, t_2, \cdots, t_m) \colon t_i \in \mathbb{T}_i, 1 \leq i \leq m\}, \end{aligned} \tag{40}$$

is called an m-dimensional time scale. Consider E to be Δ-measurable subplot of Λ^m and $g : E \longrightarrow \mathbb{R}$ a Δ-measurable function; then, the corresponding Δ-integral named Lebesgue Δ-integral is denoted by

$$\begin{aligned} &\int_E g(t_1, t_2, \cdots, t_m)\Delta_1 t_1 \cdots \Delta_m t_m, \\ &\int_E g(t)\Delta t, \int_E g d\mu_\Delta \text{or} \int_E g(t) d\mu_\Delta(t), \end{aligned} \tag{41}$$

where μ_Δ is a σ-additive Lebesgue Δ-measure on Λ^m. Also, if $\boldsymbol{g}(t) \coloneqq (g_1(t), g_2(t), \cdots, g_m(t))$ is an m-tuple of functions such that $g_1, g_2, \cdots, g_m$ are Lebesgue Δ-integrable on E, then $\int_E \boldsymbol{g} d\mu_\Delta$ denotes the m-tuple:

$$\left(\int_E g_1 d\mu_\Delta, \cdots, \int_E g_m d\mu_\Delta\right), \tag{42}$$

i.e., Δ-integral acts on each component of g.

Lemma 3. *Assume $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ are two time-scale measure spaces, and suppose that $u \geq 0, v \geq 0$ and $g \geq 0$ on Ω_1, Ω_2 and $\Omega_1 \times \Omega_2$, respectively. If $q \geq 1$, then*

$$\begin{aligned} &\left(\int_{\Omega_1}\left(\int_{\Omega_2} g(\theta,\eta)v(\eta)d\mu_2(\eta)\right)^q u(\theta)d\mu_1(\theta)\right)^{1/q} \\ &\quad \leq \int_{\Omega_2}\left(\int_{\Omega_1} g^q(\theta,\eta)u(\theta)d\mu_1(\theta)\right)v(\eta)d\mu_2(\eta), \end{aligned} \tag{43}$$

is available provided all integrals in (43) exist. If $0 < q < 1$ and

$$\int_{\Omega_1}\left(\int_{\Omega_2} gvd\mu_2\right)^q ud\mu_1 > 0, \quad \int_{\Omega_2} gvd\mu_2 > 0, \tag{44}$$

is available, then (43) is reversed. For $q < 0$, in addition with (44), if

$$\int_{\Omega_1} g^q ud\mu_1 > 0, \tag{45}$$

is available, then the sign of (43) is reversed.

Theorem 4 ([14], Theorem 3.1). *Assume $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ are two finite-dimensional time-scale measure spaces. Let $\Theta \in C(K_m, \mathbb{R}) \geq 0$ be continuous and superquadratic, $l : \Omega_1 \times \Omega_2 \longrightarrow \mathbb{R} \geq 0$ such that $l(\theta, .)$ is $\Delta\mu_2$-integrable for $\theta \in \Omega_2$. Then, the inequality*

$$\Theta\left(\frac{\int_{\Omega_2} l(\theta,\eta)\boldsymbol{g}(\eta)\Delta\mu_2(\eta)}{\int_{\Omega_2} l(\theta,\eta)\Delta\mu_2(\eta)}\right) \leq \frac{\int_{\Omega_2} l(\theta,\eta)\left(\Theta(\boldsymbol{g}(\eta)) - \Theta\left(\left|\boldsymbol{g}(\eta) - 1/\int_{\Omega_2} l(\theta,\eta)\Delta\mu_2(\eta)\int_{\Omega_2} l(\theta,\eta)\boldsymbol{g}(\eta)\Delta\mu_2(\eta)\right|\right)\right)\Delta\mu_2(\eta)}{\int_{\Omega_2} l(\theta,\eta)\Delta\mu_2(\eta)}, \tag{46}$$

holds for all functions $\boldsymbol{g}$ such that $\boldsymbol{g}(E) \subset K_m$. If Θ is a subquadratic, then (46) is reversed.

3. Inequalities with General Kernel

In this section, we get the Hardy inequality for several variables via multivariate superquadratic functions. Before presenting the results, we labeled the following hypothesis.

(A1) $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ are two time-scale measure spaces with positive σ-finite measures

(A2) $l : \Omega_1 \times \Omega_2 \longrightarrow \mathbb{R} \geq 0$ such that

$$L(\theta) := \int_{\Omega_2} l(\theta, \eta) \Delta\mu_2(\eta) < \infty, \quad \theta \in \Omega_1. \tag{47}$$

(A3) $\xi : \Omega_1 \longrightarrow \mathbb{R}$ is $\Delta\mu_1$-integrable, and the function ω is defined by

$$\omega(\eta) := \left(\int_{\Omega_1} \xi(\theta) \left(\frac{l(\theta, \eta)}{L(\theta)} \right)^{\lambda} \Delta\mu_1(\theta) \right)^{1/\lambda} < \infty, \quad \eta \in \Omega_2, \tag{48}$$

where $\lambda \geq 1$.

Theorem 5. *Assume (A1)–(A3) are satisfied. If $\Theta \in C(K_m, \mathbb{R}) \geq 0$ and is superquadratic, then*

$$\begin{aligned} &\int_{\Omega_1} \xi(\theta) \Theta^{\lambda}((A_l\boldsymbol{g})(\theta)) \Delta\mu_1(\theta) + \lambda \int_{\Omega_1} \int_{\Omega_2} \xi(\theta) \frac{l(\theta, \eta)}{L(\theta)} \\ &\quad \cdot \Theta^{\lambda-1}((A_k\boldsymbol{g})(\theta)) \Theta(|\boldsymbol{g}(\eta) - A_1\boldsymbol{g}(\theta)|) \Delta\mu_2(\eta) \\ &\leq \left(\int_{\Omega_2} \omega(\eta) \Theta(\boldsymbol{g}(\eta)) \Delta\mu_2(\eta) \right)^{\lambda}, \end{aligned} \tag{49}$$

is available for $\boldsymbol{g} : \Omega_2 \longrightarrow \mathbb{R}^m$ that is a nonnegative $\Delta\mu_2$-integrable function such that $\boldsymbol{g}(\Omega_2) \subset K_m$ and $A_l\boldsymbol{g} : \Omega_1 \longrightarrow \mathbb{R}$ defined by

$$(A_l\boldsymbol{g})(\theta) := \frac{1}{L(\theta)} \int_{\Omega_2} l(\theta, \eta) \boldsymbol{g}(\eta) \Delta\mu_2(\eta), \quad \theta \in \Omega_1. \tag{50}$$

If Θ is subquadratic and $0 < \lambda < 1$, then (49) is reversed.

Proof. We begin with an explicit identity

$$\Theta((A_l\boldsymbol{g})(\theta)) := \Theta\left(\frac{1}{L(\theta)} \int_{\Omega_2} l(\theta, \eta) \boldsymbol{g}(\eta) \Delta\mu_2(\eta) \right). \tag{51}$$

By applying the refined Jensen inequality (46) on (51), we find

$$\begin{aligned} &\Theta((A_l\boldsymbol{g})(\theta)) + \frac{1}{L(\theta)} \int_{\Omega_2} l(\theta, \eta) \Theta(|\boldsymbol{g}(\eta) - A_l\boldsymbol{g}(\theta)|) \Delta\mu_2(\eta) \\ &\quad \leq \frac{1}{L(\theta)} \int_{\Omega_2} l(\theta, \eta) \Theta(\boldsymbol{g}(\eta)) \Delta\mu_2(\eta). \end{aligned} \tag{52}$$

Then, since $\lambda \geq 1$ and $\Theta \geq 0$, we get

$$\begin{aligned} &\left(\Theta((A_l\boldsymbol{g})(\theta)) + \frac{1}{L(\theta)} \int_{\Omega_2} l(\theta, \eta) \Theta(|\boldsymbol{g}(\eta) - (A_l\boldsymbol{g})(\theta)|) \Delta\mu_2(\eta) \right)^{\lambda} \\ &\quad \leq \left(\frac{1}{L(\theta)} \int_{\Omega_2} l(\theta, \eta) \Theta(\boldsymbol{g}(\eta)) \Delta\mu_2(\eta) \right)^{\lambda}. \end{aligned} \tag{53}$$

Furthermost, by utilizing the famous inequality of Bernoulli, it ensues that the L. H. S. of (53) became

$$\begin{aligned} &\left(\Theta((A_l\boldsymbol{g})(\theta)) + \frac{1}{L(\theta)} \int_{\Omega_2} l(\theta, \eta) \Theta(|\boldsymbol{g}(\eta) - (A_l\boldsymbol{g})(\theta)|) \Delta\mu_2(\eta) \right)^{\lambda} \\ &\quad \leq \Theta^{\lambda}((A_l\boldsymbol{g})(\theta)) + \lambda \frac{\Theta^{\lambda-1}((A_l\boldsymbol{g})(\theta))}{L(\theta)} \int_{\Omega_2} l(\theta, \eta) \Theta \\ &\qquad \cdot (|\boldsymbol{g}(\eta) - (A_l\boldsymbol{g})(\theta)|) \Delta\mu_2(\eta), \end{aligned} \tag{54}$$

that is, we get

$$\begin{aligned} &\Theta^{\lambda}((A_l\boldsymbol{g})(\theta)) + \lambda \frac{\Theta^{\lambda-1}((A_l\boldsymbol{g})(\theta))}{L(\theta)} \int_{\Omega_2} l(\theta, \eta) \Theta \\ &\quad \cdot (|\boldsymbol{g}(\eta) - (A_l\boldsymbol{g})(\theta)|) \Delta\mu_2(\eta) \\ &\quad \leq \left(\frac{1}{L(\theta)} \int_{\Omega_2} l(\theta, \eta) \Theta(\boldsymbol{g}(\eta)) \Delta\mu_2(\eta) \right)^{\lambda}. \end{aligned} \tag{55}$$

Multiplying (55) by $\xi(\theta)$ and integrating it over Ω_1 with respect to $\Delta\mu_1(\theta)$, we have

$$\begin{aligned} &\int_{\Omega_1} \xi(\theta) \Theta^{\lambda}((A_l\boldsymbol{g})(\theta)) \Delta\mu_1(\theta) + \lambda \int_{\Omega_1} \xi(\theta) \\ &\quad \cdot \left(\frac{1}{L(\theta)} \int_{\Omega_1} l(\theta, \eta) \Theta(|\boldsymbol{g}(\eta) - (A_l\boldsymbol{g})(\theta)|) \Delta\mu_2(\eta) \right) \Delta\mu_1(\theta) \\ &\leq \int_{\Omega_1} \xi(\theta) \left(\frac{1}{L(\theta)} \int_{\Omega_1} l(\theta, \eta) \Theta(\boldsymbol{g}(\eta)) \Delta\mu_2(\eta) \right)^{\lambda} \Delta\mu_1(\theta). \end{aligned} \tag{56}$$

Applying the inequality of Minkowski on the R. H. S. of (56), we get

$$\int_{\Omega_1}\xi(\theta)\Theta\left(\frac{1}{L(\theta)}\int_{\Omega 2}l(\theta,\eta)\Theta(\boldsymbol{g}(\eta))\Delta\mu_2(\eta)\right)^{\lambda}\Delta\mu_1(\theta)$$
$$\leq\left(\int_{\Omega_2}\Theta(\boldsymbol{g}(\eta))\left(\int_{\Omega_2}\xi(\theta)\left(\frac{l(\theta,\eta)}{L(\theta)}\right)^{\lambda}\Delta\mu_1(\theta)\right)^{1/\lambda}\Delta\mu_2(\eta)\right)^{\lambda}. \tag{57}$$

Finally, substituting (57) into (56) and utilizing the definition (48) of the weight function ω, we get

$$\int_{\Omega_1}\xi(\theta)\Theta^{\lambda}((A_l\boldsymbol{g})(\theta))\Delta\mu_1(\theta)+\lambda\int_{\Omega_1}\int_{\Omega_2}\xi(\theta)\frac{l(\theta,\eta)}{L(\theta)}\Theta^{\lambda-1}$$
$$\cdot((A_l\boldsymbol{g})(\theta))\Theta(|\boldsymbol{g}(\eta)-(A_l\boldsymbol{g})(\theta)|)\Delta\mu_1(\theta)\Delta\mu_2(\eta)$$
$$\leq\left(\int_{\Omega_2}\omega(\eta)\Theta(\boldsymbol{g}(\eta))\Delta\mu_2(\eta)\right)^{\lambda}, \tag{58}$$

which is (49). If Θ is subquadratic and $0<\lambda<1$, the corresponding results can be obtained similarly.

Remark 6. If $\lambda=1$ and $m=1$ in Theorem 5, then (49) reduces to (28) premised in Introduction.

Remark 7. For the Lebesgue scale measures $\Delta\mu_1(\theta)=\Delta\theta$, $\Delta\mu_2(\eta)$ and $m=1$, Theorem 5 coincides with Theorem 2.1.1 in [30].

Remark 8. As a special case of Theorem 5 when $\mathbb{T}=\mathbb{R}$ and $m=1$, we have the inequality (19).

Corollary 9. *Given that ξ and $(A_l\boldsymbol{g})(\theta)$ are as in Theorem 5 and $\omega\geq 0$, then, since $\Theta\geq 0$ and superquadratic, the second term on the L. H. S. of (49) is nonnegative and the integral inequality*

$$\int_{\Omega_1}\xi(\theta)\Theta^{\lambda}((A_l\boldsymbol{g})(\theta))\Delta\mu_1(\theta)\leq\left(\int_{\Omega 2}\omega(\eta)\Theta(\boldsymbol{g}(\eta))\Delta\mu_2(\eta)\right)^{\lambda}, \tag{59}$$

is valid.

Remark 10. By taking $\lambda=1$ in Corollary 9, inequality (59) reduces to (25).

Remark 11. For the Lebesgue scale measures $\Delta\mu_1(\theta)=\Delta\theta$, $\Delta\mu_2(\eta)=\Delta\eta$ and $m=1$, Corollary 9 coincides with Corollary 2.1.2 in [30].

Remark 12. Rewrite (49) with $\lambda=qp^{-1}\geq 1$ such that $0<p\leq q<\infty$ or $-\infty<p\leq q<0$; then

$$\int_{\Omega_1}\xi(\theta)\Theta^{q/p}((A_l\boldsymbol{g})(\theta))\Delta\mu_1(\theta)+\frac{q}{p}\int_{\Omega_1}\int_{\Omega_2}\xi(\theta)\frac{l(\theta,\eta)}{L(\theta)}\Theta^{q/p-1}$$
$$\cdot((A_l\boldsymbol{g})(\theta))\Theta(|\boldsymbol{g}(\eta)-(A_l\boldsymbol{g})(\theta)|)\Delta\mu_1(\theta)\Delta\mu_2(\eta)$$
$$\leq\left(\int_{\Omega_2}\omega(\eta)\Theta(\boldsymbol{g}(\eta))\Delta\mu_2(\eta)\right)^{q/p}. \tag{60}$$

Remark 13. For $m=1$, inequality (60) coincides with inequality (3.13) in ([28], Remark 3.5).

Remark 14. In Remark 12, since $\Theta\geq 0$, then the second term on the L. H. S. of (60) is nonnegative. Hence, (60) reduces to

$$\int_{\Omega_1}\xi(\theta)\Theta^{q/p}((A_l\boldsymbol{g})(\theta))\Delta\mu_1(\theta)\leq\left(\int_{\Omega_2}\omega(\eta)\Theta(\boldsymbol{g}(\eta))\Delta\mu_2(\eta)\right)^{q/p}, \tag{61}$$

which is a refinement of the Hardy-type inequality in ([27], Remark 2.1.4) and [6].

In the following, we labeled some specific superquadratic functions starting with power functions.

Theorem 15. *Assume (A1)–(A3) are satisfied. If $\boldsymbol{g}_i:\Omega_2\longrightarrow\mathbb{R}(1\leq i\leq m)$ are $\Delta\mu_2$-integrable functions such that $\boldsymbol{g}_i(\Omega_2)\subset[0,\infty)$, then the inequality*

$$\int_{\Omega_1}\xi(\theta)\left(\sum_{i=1}^{m}(A_l\boldsymbol{g}_i)^p(\theta)\right)^{r}\Delta\mu_1(\theta)+\lambda\int_{\Omega_1}\int_{\Omega_2}\xi(\theta)\frac{l(\theta,\eta)}{L(\theta)}$$
$$\cdot\left(\sum_{i=1}^{m}(A_k\boldsymbol{g}_i)^p(\theta)\right)^{\lambda-1}\left(\sum_{i=1}^{m}\left|\boldsymbol{g}_i(\eta)-(A_l\boldsymbol{g}_i)(\theta)^p\right|\right)$$
$$\cdot\Delta\mu_1(\theta)\Delta\mu_2(\eta)\leq\left(\int_{\Omega_2}\omega(\eta)\left(\sum_{i=1}^{m}(\boldsymbol{g}_i(\eta))^p\right)\Delta\mu_2(\eta)\right)^{\lambda}, \tag{62}$$

is valid, where $p\geq 2$ and

$$(A_lg_i)(\theta):=\frac{1}{L(\theta)}\int_{\Omega_2}l(\theta,\eta)g_i(\eta)\Delta\mu_2(\eta),\theta\in\Omega_1. \tag{63}$$

If $0<\lambda<1$ and $1<p\leq 2$, then (62) is reversed.

Proof. We get the result from Theorem 5 by putting

$$\Theta(\theta):=\sum_{i=1}^{m}\theta_i^p, \tag{64}$$

in (49).

Remark 16. For $m = 1$, Theorem 15 reduces to Corollary 3.1 in [13]. In particular, for $p = 1$ and $\lambda = 1$, Theorem 15 reduces to Remark 3.11 in [13].

Remark 17. For the Lebesgue scale measures $\Delta\mu_1(\theta) = \Delta\theta$, $\Delta\mu_2(\eta) = \Delta\eta$ and $m = 1$. Theorem 15 coincides with Corollary 2.1.5 in [30].

Theorem 18. *Assume (A1)–(A3) are satisfied. If $g_i : \Omega_2 \longrightarrow \mathbb{R}(1 \le i \le m)$ are $\Delta\mu_2$-integrable functions such that $g_i(\Omega_2) \subset [0, \infty)$, then the inequality*

$$\begin{aligned}\int_{\Omega_1} \xi(\theta)\left(\sum_{i=1}^{m}(\exp (A_i g_i)(\theta) - (A_i g_i)(\theta) - 1)\right)^{\lambda} \Delta\mu_1(\theta) + I \\ \le \left(\int_{\Omega_2} \omega(\eta)\left(\sum_{i=1}^{m}(g_i(\eta) - \log g_i(\eta) - 1)\right)\Delta\mu_2(\eta)\right)^{\lambda},\end{aligned} \tag{65}$$

is valid, where

$$\begin{aligned} I \coloneqq \lambda\int_{\Omega_1}\int_{\Omega_2} \xi(\theta)\frac{l(\theta,\eta)}{L(\theta)}\Bigg(\sum_{i=1}^{m}(\exp |\log g_i(\eta) \\ - (A_l g_i)(\theta)| - |\log g_i(\eta) - (A_l g_i)(\theta)| - 1)\Bigg) \\ \times \left(\sum_{i=1}^{m}(\exp (A_l g_i)(\theta) - (A_l g_i)(\theta) - 1)\right)^{\lambda-1} \Delta\mu_1(\theta)\Delta\mu_2(\eta),\end{aligned} \tag{66}$$

and

$$(A_l g_i)(\theta) \coloneqq \frac{1}{L(\theta)}\int_{\Omega_2} l(\theta,\eta) \log g_i(\eta)\Delta\mu_2(\eta), \theta \in \Omega_1. \tag{67}$$

If $0 < \lambda < 1$, then (65) is reversed.

Proof. We get the result from Theorem 5 by putting

$$\Theta(\theta) \coloneqq \sum_{i=1}^{m}(\exp (\theta_i) - \theta_i - 1), \tag{68}$$

in (49) and with $\log g(\eta)$ instead of $g(\eta)$.

Remark 19. By taking $m = 1$ in Theorem 18, inequality (65) reduces to inequality 3.16 in [28], Corollary 3.2.

Remark 20. For $m = 1$ and $\lambda = 1$, the relation (65) that is regarded as a generalization and a refinement of the Pólya-Knopp's inequality which coincided with Remark 3.12 in [13].

Theorem 21. *Assume (A1)–(A3) are satisfied. If $g_i : \Omega_2 \longrightarrow \mathbb{R}(1 \le i \le m)$ are $\Delta\mu_2$-integrable functions such that $g_i(\Omega_2) \subset [0, \infty)$, then the inequality*

$$\begin{aligned}\int_{\Omega_1} \sum_{i=1}^{m}[(A_i g_i)(\theta) \cosh (A_i g_i)(\theta) - \sinh (A_i g_i)(\theta)]^{\lambda} \\ \cdot \xi(\theta)\Delta\mu_1(\theta) + \lambda\int_{\Omega_1}\int_{\Omega 2}\xi(\theta)\frac{l(\theta,\eta)}{L(\theta)} \\ \cdot \left(\sum_{i=1}^{m}[(A_i g_i)(\theta) \cosh (A_i g_i)(\theta) - \sinh (A_i g_i)(\theta)]\right)^{\lambda-1} \\ \times \sum_{i=1}^{m}[|g_i(\eta) - (A_i g_i)(\theta)| \cosh (|g_i(\eta) - (A_i g_i)(\theta)|) \\ - \sinh (|g_i(\eta) - (A_i g_i)(\theta)|)]\Delta\mu_1(\theta)\Delta\mu_2(\eta) \\ \le \left(\int_{\Omega_2} \omega(\eta)\sum_{i=1}^{m}([g_i(\eta) - \cosh (g_i(\eta)) - \sinh (g_i(\eta))]\Delta\mu_2(\eta))\right)^{\lambda},\end{aligned} \tag{69}$$

is valid, where $A_l g_i$ is defined as in (63). If $0 < \lambda < 1$, then (69) is reversed.

Proof. We get the result from Theorem 5 by putting

$$\Theta(\theta) \coloneqq \sum_{i=1}^{m}(\theta_i \cosh \theta_i - \sinh \theta_i), \tag{70}$$

in (49).

Remark 22. For $\lambda = 1$, Theorem 21 reduces to Theorem 2.5 in [14]. In particular, for $m = 1$ and $\lambda = 1$, Theorem 21 coincides with Corollary 2.6 in [14].

Theorem 23. *Assume (A1)–(A3) are satisfied. If$g_i : \Omega_2 \longrightarrow \mathbb{R}(1 \le i \le m)$ are $\Delta\mu_2$-integrable functions such that $g_i(\Omega_2) \subset [0, \infty)$, then the inequality.*

$$\begin{aligned}\int_{\Omega_1} \xi(\theta)\left(\ln\left(1 + \sum_{i=1}^{m}(A_i g_i)(\theta)\right) - \sum_{i=1}^{m}(A_i g_i)(\theta)\right)^{\lambda} \Delta\mu_1(\theta) \\ + \lambda\int_{\Omega_1}\int_{\Omega_1}\xi(\theta)\frac{l(\theta,\eta)}{L(\theta)}\Bigg(\ln\left(1 + \sum_{i=1}^{m}(A_i g_i)(\theta)\right) \\ - \sum_{i=1}^{m}(A_i g_i)(\theta)\Bigg)^{\lambda-1} \times \Bigg(\ln\left(1 + \sum_{i=1}^{m}|g_i(\eta)A_i g_i(\theta)|\right) \\ - \sum_{i=1}^{m}|g_i(\eta)A_i g_i(\theta)|\Bigg)^{\lambda} \Delta\mu_1(\theta)\Delta\mu_2(\eta) \\ \le \left(\int_{\Omega_2} \omega(\eta)\left(\ln\left(1 + \sum_{i=1}^{m} g_i(\eta)\right) - \sum_{i=1}^{m} g_i(\eta)\right)\Delta\mu_2(\eta)\right)^{\lambda},\end{aligned} \tag{71}$$

is valid, where $A_l g_i$ is defined as in (63). If $0 < \lambda < 1$, then (71) is reversed.

Proof. We get the result from Theorem 5 by putting

$$\Theta(\theta) := \sum_{i=1}^{m} \theta_i^2 \ln \theta_i, \tag{72}$$

in (49) with the assumption $0 \ln 0 = 0$.

Remark 24. For $\lambda = 1$, Theorem 23 reduces to Theorem 2.7 in [14]. In particular, for $m = 1$ and $\lambda = 1$, Theorem 23 coincides with Corollary 2.8 in [14].

Theorem 25. *Assume (A1)–(A3) are satisfied. If $g_i : \Omega_2 \longrightarrow \mathbb{R}(1 \le i \le m)$ are $\Delta\mu_2$-integrable functions such that $g_i(\Omega_2) \subset [0, \infty)$, then the inequality*

$$\begin{aligned}
&\int_{\Omega_1} \xi(\theta)\left(\ln\left(1 + \sum_{i=1}^{m}(A_i g_i)(\theta)\right) - \sum_{i=1}^{m}(A_i g_i)(\theta)\right)^{\lambda} \Delta\mu_1(\theta) \\
&\quad + \lambda \int_{\Omega_1}\int_{\Omega_1} \xi(\theta)\frac{l(\theta,\eta)}{L(\theta)}\left(\ln\left(1 + \sum_{i=1}^{m}(A_i g_i)(\theta)\right)\right. \\
&\quad \left. - \sum_{i=1}^{m}(A_i g_i)(\theta)\right)^{\lambda-1} \times \left(\ln\left(1 + \sum_{i=1}^{m}\left|g_i(\eta) A_i g_i(\theta)\right|\right)\right. \\
&\quad \left. - \sum_{i=1}^{m}\left|g_i(\eta) A_i g_i(\theta)\right|\right)^{\lambda} \Delta\mu_1(\theta)\Delta\mu_2(\eta) \\
&\le \left(\int_{\Omega_2} \omega(\eta)\left(\ln\left(1 + \sum_{i=1}^{m} g_i(\eta)\right) - \sum_{i=1}^{m} g_i(\eta)\right)\Delta\mu_2(\eta)\right)^{\lambda},
\end{aligned} \tag{73}$$

is valid, where $A_l g_i$ is defined as in (63). If $0 < \lambda < 1$, then (73) is reversed.

Proof. We get the result from Theorem 5 by taking

$$\Theta(\theta) := \ln\left(1 + \sum_{i=1}^{m}\theta_i\right) - \sum_{i=1}^{m}\theta_i, \tag{74}$$

in (49).

Remark 26. For $\lambda = 1$, Theorem 25 reduces to Theorem 2.9 in [14]. In particular, for $m = 1$ and $\lambda = 1$, Theorem 25 coincides with Corollary 2.10 in [14].

Now, to wrap up this section, we consider yet another implementation of Theorem 5 rigged with finite measure spaces.

Corollary 27. *Let the supposition of Theorem 5 be satisfied and denote $\int_{\Omega_1}\Delta\mu_1(\theta) = |\Omega_1|$ and $\int_{\Omega_2}\Delta\mu_2(\theta) = |\Omega_2|$ such that $|\Omega_1|, |\Omega_2| < \infty$: setting $l(\theta,\eta)$ and $\xi(\theta) = 1$. Then, $L(\theta) = \int_{\Omega_2}\Delta\mu_2(\theta) = |\Omega_2|$ and*

$$\begin{aligned}
\omega(\eta) &:= \left(\int_{\Omega_1}\left(\frac{1}{|\Omega_2|}\right)^{\lambda}\Delta\mu_1(\theta)\right)^{1/\lambda} \\
&= \left(\frac{1}{|\Omega_2|^{\lambda}}\int_{\Omega_1}\Delta\mu_1(\theta)\right)^{1/\lambda} = \frac{|\Omega_1|^{1/\lambda}}{|\Omega_2|}.
\end{aligned} \tag{75}$$

Hence, the following inequality

$$\begin{aligned}
&\int_{\Omega_1}\Theta\left(\frac{1}{|\Omega_2|}\int_{\Omega_2}\boldsymbol{g}(\eta)\Delta\mu_2(\eta)\right)^{\lambda}\Delta\mu_1(\theta) \\
&\quad + \frac{\lambda}{|\Omega_2|}\int_{\Omega_1}\int_{\Omega_2}\Theta\left(\frac{1}{|\Omega_2|}\int_{\Omega_2}\boldsymbol{g}(\eta)\Delta\mu_2(\eta)\right)^{\lambda-1} \\
&\quad \times \Theta\left(\left|\boldsymbol{g}(\eta) - \frac{1}{|\Omega_2|}\int_{\Omega_2}\boldsymbol{g}(\eta)\Delta\mu_2(\eta)\right|\right)\Delta\mu_1(\theta)\Delta\mu_2(\eta) \\
&\le \frac{|\Omega_1|}{|\Omega_2|}\left(\int_{\Omega_2}\Theta(\boldsymbol{g}(\eta))\Delta\mu_2(\eta)\right)^{\lambda},
\end{aligned} \tag{76}$$

is valid. If Θ is subquadratic and $0 < \lambda < 1$, then (76) is reversed.

Remark 28. By taking $m = 1$ in Corollary 27, inequality (76) reduces to inequality 3.19 in [28], Corollary 3.2.

Remark 29. For the Lebesgue scale measures $\Delta\mu_1(\theta) = \Delta\theta$, $\Delta\mu_2(\eta) = \Delta\eta$ and $m = 1$, Corollary 27 coincides with Corollary 2.1.6 in [30].

Remark 30. For $\mathbb{T} = \mathbb{R}$, $m = 1$, and $\lambda = 1$, Corollary 27 reduces to Corollary 3.3 in [8].

4. Inequalities with Specific Time Scales

In this section, by selecting few different time scales, we get some consequential inequalities. More precisely, assume $0 \le \alpha < \beta \le \infty$ are points in $\mathbb{T}$ and $S_1 := \{(\theta,\eta) \in \mathbb{T} : 0 \le \alpha < \eta \le \theta < \beta\}$. Applying Theorem 5 to $\Omega_2 = \Omega_2 = [\alpha,\beta)_{\mathbb{T}}$, $\Delta\mu_1(\theta) = \Delta\theta$, and $\Delta\mu_2(\eta) = \Delta\eta$, we get the following conclusion.

Theorem 31. *Assume $0 \le \alpha < \beta \le \infty$ and $l : [\alpha,\beta)_{\mathbb{T}} \times [\alpha,\beta)_{\mathbb{T}} \longrightarrow \mathbb{R} \ge 0$ such as $L(\theta) := \int_{\alpha}^{\theta} k(\theta,\eta)\Delta\eta < \infty, \theta \in [\alpha,\beta)_{\mathbb{T}}$*

Suppose that $\xi(\theta)$: $[\alpha,\beta)_{\mathbb{T}} \longrightarrow \mathbb{R}$ and

$$\omega(\eta) := \left(\int_{\eta}^{\beta}\xi(\theta)\left(\frac{l(\theta,\eta)}{L(\theta)}\right)^{\lambda}\Delta\theta\right)^{1/\lambda} < \infty, \eta \in [\alpha,\beta)_{\mathbb{T}}, \tag{77}$$

where $\lambda \geq 1$. If $\Theta \in C(K_m, \mathbb{R}) \geq 0$ and is superquadratic, then

$$\begin{aligned}&\int_\alpha^\beta \xi(\theta)\Theta^\lambda((A_l\boldsymbol{g})(\theta))\Delta\theta + \lambda\int_\alpha^\beta\int_\alpha^\theta \xi(\theta)\frac{l(\theta,\eta)}{L(\theta)}\Theta^{\lambda-1}\\&\cdot((A_l\boldsymbol{g})(\theta))\Theta(|\boldsymbol{g}(\eta)-(A_l\boldsymbol{g})(\theta)|)\Delta\theta\Delta\eta\\&\leq\left(\int_\alpha^\beta \omega(\eta)\Theta(\boldsymbol{g}(\eta))\Delta\eta\right)^\lambda,\end{aligned}\tag{78}$$

is available for all nonnegative integrable functions $\boldsymbol{g}: [\alpha,\beta)_\mathbb{T} \longrightarrow \mathbb{R}^m$ and for $A_l\boldsymbol{g}: [\alpha,\beta)_\mathbb{T} \longrightarrow \mathbb{R}$ defined as

$$(A_l\boldsymbol{g})(\theta) \coloneqq \frac{1}{L(\theta)}\int_\alpha^\theta l(\theta,\eta)\boldsymbol{g}(\eta)\Delta\eta,\ \theta\in[\alpha,\beta)_\mathbb{T}. \tag{79}$$

If $0<\lambda<1$ and Θ are subquadratic, then (78) is reversed.

Remark 32. By taking $m=1$ and replacing $\xi(\theta)$, $\omega(\eta)$, and $l(\theta,\eta)$, respectively, $\xi(\theta)/(\theta-\alpha)$, $\omega(\eta)/(\eta-\alpha)$, and $l_{\chi_{S_1}}(\theta,\eta)$ where χ_{S_1} denotes the characteristic function over S_1 in Theorem 31, inequality (78) reduces to inequality 4.1 in [28], Theorem 4.1.

On the other hand, for $0\leq\alpha<\beta\leq\infty$, consider the set

$$S_2 \coloneqq \{(\theta,\eta)\in\mathbb{T} : \beta<\theta\leq\eta<\infty\}. \tag{80}$$

Then, putting $\Omega_1=\Omega_2=[\beta,\infty)_\mathbb{T}$ where $\mathbb{T}$ is a time scale, $\Delta\mu_1(\theta)=\Delta\theta$ and $\Delta\mu_2(\eta)=\Delta\eta$. We obtain a dual form of Theorem 31 as follows.

Theorem 33. *Suppose that $0\leq\beta<\infty\tilde{\xi}(\theta): [\beta,\infty)_\mathbb{T}\longrightarrow\mathbb{R} \geq 0$ and $\tilde{l}: [\beta,\infty)_\mathbb{T}\times[\beta,\infty)_\mathbb{T}\times[\beta,\infty)_\mathbb{T}\longrightarrow\mathbb{R}\geq 0$ such that*

$$\begin{aligned}&\tilde{L}(\theta)\coloneqq\int_\theta^\infty \tilde{l}(\theta,\eta)\Delta\eta<\infty,\quad \theta\in[\beta,\infty)_\mathbb{T},\\&\tilde{\omega}(\eta)\coloneqq\left(\int_\beta^\eta\tilde{\xi}(\theta)\left(\frac{\tilde{l}(\theta,\eta)}{L(\theta)}\right)^\lambda\Delta\theta\right)^{1/\lambda}<\infty,\eta\in[\beta,\infty)_\mathbb{T},\end{aligned}\tag{81}$$

where $\lambda\geq1$. If $\Theta\in C(K_m\mathbb{R})\geq0$ and superquadratic, then

$$\begin{aligned}&\int_\beta^\infty\tilde{\xi}(\theta)\Theta^\lambda((A_l\boldsymbol{g})(\theta))\Delta\theta+\lambda\int_\beta^\infty\int_\theta^\infty\tilde{\xi}(\theta)\frac{\tilde{l}(\theta,\eta)}{L(\theta)}\Theta^{\lambda-1}\\&\cdot\left(\left(\tilde{A}_l\boldsymbol{g}\right)(\theta)\right)\Theta(|\boldsymbol{g}(\eta)-(A_l\boldsymbol{g})(\theta)|)\Delta\theta\Delta\eta\\&\leq\left(\int_\beta^\infty\tilde{\omega}(\eta)\Theta(\boldsymbol{g}(\eta))\Delta\eta\right),\end{aligned}\tag{82}$$

is available for all nonnegative $\Delta\eta$ -integrable functions $\boldsymbol{g}: [\beta,\infty)_\mathbb{T}\longrightarrow\mathbb{R}^m$ and for the operator $\tilde{A}_l g: [\beta,\infty)_\mathbb{T}\longrightarrow\mathbb{R}$ defined by

$$\left(\tilde{A}_l\boldsymbol{g}\right)(\theta)\coloneqq\frac{1}{\tilde{L}(\theta)}\int_\theta^\infty\tilde{l}(\theta,\eta)\boldsymbol{g}(\eta)\Delta\eta,\quad\theta\in[\beta,\infty)_\mathbb{T}. \tag{83}$$

If Θ is subquadratic and $0<\lambda<1$, then (82) is reversed.

Remark 34. By taking $m=1$ and replacing $\tilde{\xi}(\theta)$, $\tilde{\omega}(\eta)$, and $\tilde{l}(\theta,\eta)$, respectively, by $\tilde{\xi}(\theta)/(\theta-\alpha)$, $\tilde{\omega}(\eta)/(\eta-\alpha)$, and $\tilde{l}_{\chi_{S_2}}(\theta,\eta)$ where χ_{S_2} denotes the characteristic function over S_2 in Theorem 33; inequality (82) reduces to inequality 4.7 in [28], Theorem 4.2.

5. Inequalities with Specific Kernels

In this section, we find some consequential inequalities of the Hardy type by selecting specific kernels and weight functions.

Corollary 35. *Suppose that the assumptions of Theorem 31 are satisfied only with*

$$l(\theta,\eta)\coloneqq0,\quad \text{if } \alpha\leq\eta\leq\sigma(\theta)\leq\beta. \tag{84}$$

Define

$$L(\theta)\coloneqq\int_\alpha^{\sigma(\theta)}l(\theta,\eta)\Delta\eta>0,\quad\theta\in[\alpha,\beta)_\mathbb{T}. \tag{85}$$

If $\Theta\in C(K_m,\mathbb{R})\geq0$ and is superquadratic, then (78) is available for all nonnegative $\Delta\eta$-integrable functions $g: [\alpha,\beta)_\mathbb{T}\longrightarrow\mathbb{R}^m$ defined as

$$(A_l\boldsymbol{g})(\theta)\coloneqq\frac{1}{L(\theta)}\int_\alpha^\sigma l(\theta,\eta)\,\boldsymbol{g}(\eta)\Delta\eta,\quad\theta\in[\alpha,\beta)_\mathbb{T}. \tag{86}$$

If Θ is subquadratic and $0<\lambda<1$, then (78) is reversed.

Corollary 36. *Assume that the assumptions of Theorem 31 is satisfied only with*

$$l(\theta,\eta)\coloneqq0, \text{if } \alpha\leq\sigma(\theta)\leq\eta\leq\beta. \tag{87}$$

Define

$$L(\theta)\coloneqq\int_{\sigma(\theta)}^\beta l(\theta,\eta)\Delta\eta>0,\quad\theta\in[\alpha,\beta)_\mathbb{T}. \tag{88}$$

If $\Theta\in C(K_m,\mathbb{R})\geq0$ and is superquadratic, then (78) is available for all nonnegative integrable functions $\boldsymbol{g}: [\alpha,\beta)_\mathbb{T}\longrightarrow\mathbb{R}^m$

$$(A_l\boldsymbol{g})(\theta)\coloneqq\frac{1}{L(\theta)}\int_{\sigma(\theta)}^\beta l(\theta,\eta)g(\eta)\Delta\eta,\quad\theta\in[\alpha,\beta)_\mathbb{T}. \tag{89}$$

If Θ is subquadratic and $0<\lambda<1$, then (78) is reversed.

Corollary 37. *Assume that the assumptions of Theorem 31 is satisfied only with* $l : [\alpha, \beta)_{\mathbb{T}} \times [\alpha, \beta)_{\mathbb{T}} \longrightarrow \mathbb{R}$ *defined as*

$$l(\theta, \eta) := \begin{cases} 1, & \text{if } 0 \leq \alpha \leq \eta < \sigma(\theta) \leq \beta, \\ 0, & \text{otherwise}, \end{cases} \tag{90}$$

and $\xi(\theta)$*:* $[\alpha, \beta)_{\mathbb{T}} \longrightarrow \mathbb{R}$*; then* $L(\theta) := \int_\alpha^{\sigma(\theta)} l(\theta, \eta)\Delta\eta = \sigma(\theta) - \alpha, \theta \in [\alpha, \beta)_{\mathbb{T}}$*, and* $A_l \boldsymbol{g}(\theta)$ *in this case is the classical Hardy and denoted by*

$$(H\boldsymbol{g})(\theta) := \frac{1}{\sigma(\theta) - \alpha} \int_\alpha^{\sigma(\theta)} \boldsymbol{g}(\eta)\Delta\eta, \quad \theta \in [\alpha, \beta)_{\mathbb{T}}. \tag{91}$$

If we let

$$\omega(\eta) := \left(\int_\eta^\beta \xi(\theta) \left(\frac{1}{\sigma(\theta) - \alpha} \right)^\lambda \Delta\theta \right) < \infty, \quad \eta \in [\alpha, \beta)_{\mathbb{T}}, \tag{92}$$

where $\lambda \geq 1$*, then (78) became*

$$\begin{aligned} &\int_\alpha^\beta \xi(\theta)\Theta^\lambda \left(\frac{1}{\sigma(\theta) - \alpha} \int_\alpha^{\sigma(\theta)} \boldsymbol{g}(\eta)\Delta\eta \right) \Delta\theta \\ &\quad + \lambda \int_\alpha^\beta \int_\eta^\beta \Theta^{\lambda-1} \left(\frac{1}{\sigma(\theta) - \alpha} \int_\alpha^{\sigma(\theta)} \boldsymbol{g}(\eta)\Delta\eta \right) \Theta \\ &\quad \cdot \left(\left| \boldsymbol{g}(\eta) - \frac{1}{\sigma(\theta) - \alpha} \int_\alpha^{\sigma(\theta)} \boldsymbol{g}(\eta)\Delta\eta \right| \right) \frac{\xi(\theta)}{\sigma(\theta) - \alpha} \Delta\theta\Delta\eta \\ &\leq \left(\int_\alpha^\beta \omega(\eta)\Theta(\boldsymbol{g}(\eta))\Delta\eta \right). \end{aligned} \tag{93}$$

If Θ *is subquadratic and* $0 < \lambda < 1$*, then (93) is reversed.*

Remark 38. For $m = 1$ and replacing $\xi(\theta), \omega(\eta)$ by $\xi(\theta)/(\theta - \alpha)$ and $\omega(\eta)/(\eta - \alpha)$ in (93), Corollary 37 coincides with Example 4.1 in [13].

Remark 39. By taking $\mathbb{T} = \mathbb{R}, \alpha = 0$, and replacing $\xi(\theta), \omega(\eta)$ by $\xi(\theta)/\theta$ and$\omega(\eta)/\eta$ in (93), we have

$$\begin{aligned} &\int_0^\beta \xi(\theta)\Theta^\lambda \left(\theta^{-1} \int_0^\theta \boldsymbol{g}(\eta)d\eta \right) \frac{d\theta}{\theta} \\ &\quad + \lambda \int_0^\beta \int_\eta^\beta \Theta^{\lambda-1} \left(\frac{1}{\theta} \int_0^\theta \boldsymbol{g}(\eta)d\eta \right) \Theta \\ &\quad \cdot \left(\left| \boldsymbol{g}(\eta) - \frac{1}{\theta} \int_0^\theta \boldsymbol{g}(\eta)d\eta \right| \right) \frac{\xi(\theta)}{\theta^2} d\theta d\eta \\ &\leq \left(\int_0^\beta \omega(\eta)\Theta(\boldsymbol{g}(\eta)) \frac{d\eta}{\eta} \right)^\lambda, \end{aligned} \tag{94}$$

where

$$\omega(\eta) := \eta \left(\int_\eta^\beta \xi(\theta) \left(\frac{1}{\theta} \right)^\lambda \frac{d\theta}{\theta} \right)^{1/\lambda}, \eta \in [0, \beta). \tag{95}$$

If Θ is subquadratic and $0 < \lambda < 1$, then (94) is reversed, which is a refinement of 4.6 in [28], Remark 4.2.

Corollary 40. *In Corollary 37, if* $\alpha = 0$ *and* $\xi(\theta) = 1/\theta$*, then (93) reduces to*

$$\begin{aligned} &\int_0^\beta \Theta^\lambda \left(\frac{1}{\sigma(\theta)} \int_0^\theta \boldsymbol{g}(\eta)d\eta \right) \frac{d\theta}{\theta} \\ &\quad + \lambda \int_0^\beta \int_\eta^\beta \Theta^{\lambda-1} \left(\frac{1}{\sigma(\theta)} \int_0^{\sigma(\theta)} \boldsymbol{g}(\eta)\Delta\eta \right) \Theta \\ &\quad \cdot \left(\left| \boldsymbol{g}(\eta) - \frac{1}{\sigma(\theta)} \int_0^{\sigma(\theta)} \boldsymbol{g}(\eta)\Delta\eta \right| \right) \frac{1}{\theta\sigma(\theta)} \Delta\theta\Delta\eta \\ &\leq \left(\int_0^\beta \omega(\eta)\Theta(\boldsymbol{g}(\eta))\Delta\eta \right)^\lambda, \end{aligned} \tag{96}$$

where

$$\omega(\eta) := \left(\int_\eta^\beta \left(\frac{1}{\sigma(\theta)} \right)^\lambda \frac{\Delta\theta}{\theta} \right)^{1/\lambda} < \infty, \eta \in [\alpha, \beta)_{\mathbb{T}}. \tag{97}$$

Furthermore, if $\beta = \infty$*, then (96) becomes*

$$\begin{aligned} &\int_0^\infty \Theta^\lambda \left(\frac{1}{\sigma(\theta)} \int_0^{\sigma(\theta)} \boldsymbol{g}(\eta)\Delta\eta \right) \frac{\Delta\theta}{\theta} \\ &\quad + \lambda \int_0^\infty \int_\eta^\infty \Theta^{\lambda-1} \left(\frac{1}{\sigma(\theta)} \int_0^{\sigma(\theta)} \boldsymbol{g}(\eta)\Delta\eta \right) \Theta \\ &\quad \cdot \left(\left| \boldsymbol{g}(\eta) - \frac{1}{\sigma(\theta)} \int_0^{\sigma(\theta)} \boldsymbol{g}(\eta)\Delta\eta \right| \right) \frac{1}{\theta\sigma(\theta)} \Delta\theta\Delta\eta \\ &\leq \left(\int_0^\infty \omega(\eta)\Theta(\boldsymbol{g}(\eta))\Delta\eta \right)^\lambda, \end{aligned} \tag{98}$$

where

$$\omega(\eta) := \left(\int_\eta^\infty \left(\frac{1}{\sigma(\theta)} \right)^\lambda \frac{\Delta\theta}{\theta} \right)^{1/\lambda} < \infty, \eta \in [\alpha, \infty)_{\mathbb{T}}. \tag{99}$$

Remark 41. For $\lambda = 1$, inequality (96) reduces to

$$\begin{aligned} &\int_0^\beta \Theta \left(\frac{1}{\sigma(\theta)} \int_0^\theta \boldsymbol{g}(\eta)\Delta\eta \right) \frac{\Delta\theta}{\theta} + \int_0^\beta \int_\eta^\beta \Theta \\ &\quad \cdot \left(\left| \boldsymbol{g}(\eta) - \frac{1}{\sigma(\theta)} \int_0^{\sigma(\theta)} \boldsymbol{g}(\eta)\Delta\eta \right| \right) \frac{1}{\theta\sigma(\theta)} \Delta\theta\Delta\eta \\ &\leq \int_0^\beta \omega(\eta)\Theta(\boldsymbol{g}(\eta))\Delta\eta, \end{aligned} \tag{100}$$

where

$$\omega(\eta) := \int_{\eta}^{\beta} \left(\frac{\Delta\theta}{\theta\sigma(\theta)} \right) = \left(\frac{1}{\eta} - \frac{1}{\beta} \right), \eta \in [\alpha, \beta)_{\mathbb{T}}, \quad (101)$$

while inequality (98) reduces to

$$\int_0^{\infty} \Theta\left(\frac{1}{\sigma(\theta)} \int_0^{\sigma(\theta)} \boldsymbol{g}(\eta)\Delta\eta \right) \frac{\Delta\theta}{\theta} + \int_0^{\infty}\int_{\eta}^{\infty} \Theta \cdot \left(\left| \boldsymbol{g}(\eta) - \frac{1}{\sigma(\theta)} \int_0^{\sigma(\theta)} \boldsymbol{g}(\eta)\Delta\eta \right| \right) \frac{1}{\theta\sigma(\theta)} \Delta\theta\Delta\eta \le \int_0^{\infty} \Theta(\boldsymbol{g}(\eta)) \frac{\Delta\eta}{\eta}. \quad (102)$$

Example 3. Considering Theorem 33 with $l : [\beta, \infty)_{\mathbb{T}} \times [\beta, \infty)_{\mathbb{T}} \longrightarrow \mathbb{R}$ defined by

$$l(\theta, \eta) := \begin{cases} {}^{1/\eta\sigma(\eta)} \text{if } \eta \ge \theta \\ 0, \text{otherwise,} \end{cases} \quad (103)$$

and $\xi(\theta)$: $[\beta, \infty)_{\mathbb{T}} \longrightarrow \mathbb{R} \ge 0$, then

$$L(\theta) := \int_{\theta}^{\infty} l(\theta, \eta)\Delta\eta = \int_{\theta}^{\infty} \frac{1}{\eta\sigma(\eta)} = -\int_{\theta}^{\infty} \left(\frac{1}{\eta} \right)^{\Delta} \Delta\eta = \frac{1}{\theta}, \theta \in [\beta, \infty)_{\mathbb{T}}. \quad (104)$$

The operator $A_l g(\theta)$ is defined as

$$(A_l \boldsymbol{g})(\theta) := \theta \int_{\theta}^{\infty} \frac{1}{\eta\sigma(\eta)} \boldsymbol{g}(\eta)\Delta\eta, \theta \in [\beta, \infty)_{\mathbb{T}}, \quad (105)$$

and if we let

$$\omega(\eta) := \left(\int_{\eta}^{\beta} \theta^{-1} \left(\frac{\theta}{\eta\sigma(\eta)} \right)^{\lambda} \Delta\theta \right)^{1/\lambda} < \infty, \eta \in [\beta, \infty)_{\mathbb{T}}, \quad (106)$$

where $\lambda \ge 1$, then (82) became

$$\int_0^{\infty} \Theta^{\lambda} \left(\theta \int_{\theta}^{\infty} \frac{1}{\eta\sigma(\eta)} \boldsymbol{g}(\eta)\Delta\eta \right) \frac{\Delta\theta}{\theta} + \lambda \int_{\beta}^{\infty}\int_{\theta}^{\infty} \Theta^{\lambda-1} \left(\theta \int_{\theta}^{\infty} \frac{1}{\eta\sigma(\eta)} \boldsymbol{g}(\eta)\Delta\eta \right) \Theta \cdot \left(\left| \boldsymbol{g}(\eta) - \theta \int_0^{\infty} \frac{1}{\eta\sigma(\eta)} \boldsymbol{g}(\eta)\Delta\eta \right| \right) \frac{1}{\eta\sigma(\eta)} \Delta\theta\Delta\eta \le \left(\int_{\beta}^{\infty} \omega(\eta)\Theta(\boldsymbol{g}(\eta))\Delta\eta \right)^{\lambda}. \quad (107)$$

If Θ is subquadratic and $0 < \lambda < 1$, then (107) is reversed.

Remark 42. For $\lambda = 1$, inequality (107) reduces to

$$\int_{\beta}^{\infty} \Theta\left(\theta \int_{\beta}^{\infty} \frac{1}{\eta\sigma(\eta)} \boldsymbol{g}(\eta)\Delta\eta \right) \frac{\Delta\theta}{\theta} + \lambda \int_{\beta}^{\infty}\int_{\theta}^{\infty} \Theta \cdot \left(\left| \boldsymbol{g}(\eta) - \theta \int_0^{\infty} \frac{1}{\eta\sigma(\eta)} \boldsymbol{g}(\eta)\Delta\eta \right| \right) \frac{1}{\eta\sigma(\eta)} \Delta\theta\Delta\eta \le \int_{\beta}^{\infty} \omega(\eta)\Theta(\boldsymbol{g}(\eta))\Delta\eta, \quad (108)$$

where

$$\omega(\eta) := \frac{1}{\eta\sigma(\eta)} \int_{\eta}^{\beta} \Delta\theta = \frac{1}{\eta\sigma(\eta)} (\beta - \eta). \quad (109)$$

6. Some Particular Cases

In this section, we obtain a popularization and a refinement of the classical inequality of the Hardy-Hilbert type (16) for numerous variables on time scales. It is clarified in the result below.

Theorem 43. *Assume that the assumptions of Theorem 31 are satisfied only with $\Omega_1 = \Omega_2 = [0, \infty)_{\mathbb{T}}$, $p > 1$, $\lambda > 0$ and replace $\Delta\mu_1(\theta)$ and $\Delta\mu_2(\eta)$ by the Lebesgue scale measure $\Delta\theta$ and $\Delta\eta$. Furthermore, define*

$$L_1(\theta) := \int_0^{\infty} \frac{(\theta/\eta)^{-1/p}}{\theta + \eta} \Delta\eta \text{ and } L_2(\eta) := \left(\int_0^{\infty} \left(\frac{\theta/\eta^{1-(1/p)}}{\theta + \eta} \right)^{\lambda} \Delta\theta \right)^{1/\lambda}. \quad (110)$$

If $\lambda \ge 1$ and $p \ge 2$, then

$$\int_0^{\infty} (L_1(\theta))^{\lambda(1-p)} \left(\int_0^{\infty} \frac{\boldsymbol{g}(\eta)}{\theta + \eta} \Delta\eta \right)^{\lambda p} \Delta\theta + \lambda \int_0^{\infty}\int_0^{\infty} \eta^{-1/p} L_1^{(\lambda-1)(1-p)}(\theta) \left(\int_0^{\infty} \frac{\boldsymbol{g}(\eta)}{\theta + \eta} \Delta\eta \right)^{p(\lambda-1)} \times \left| \boldsymbol{g}(\eta)\eta 1/p - \frac{1}{L_1(\theta)} \int_0^{\infty} \frac{\boldsymbol{g}(\eta)}{\theta + \eta} \Delta\eta \right|^p \frac{\theta^{\frac{l}{p}-1}}{\theta + \eta} \Delta\theta\Delta\eta \le \left(\int_0^{\infty} L_2(\eta)\boldsymbol{g}(\eta)\Delta\eta \right)^{\lambda}, \quad (111)$$

is available for all nonnegative integrable $\Delta\eta$-integrable functions $\boldsymbol{g} : [\alpha, \beta)_{\mathbb{T}} \longrightarrow \mathbb{R}^m$. If $0 < \lambda < 1$, then (111) is reversed.

Proof. Utilizing $\xi(\theta) := (L_1(\theta)/\theta)^{\lambda}$ and

$$l(\theta, \eta) := \begin{cases} \dfrac{\left(\frac{\eta}{\theta}\right)^{-1/p}}{\theta + \eta}, \text{if } \theta \ne 0, \eta \ne 0, \theta + \eta \ne 0 \\ 0, \text{otherwise,} \end{cases} \quad (112)$$

in Theorem 15, we obtain

$$L(\theta) := \int_0^\infty \frac{\left(\frac{\eta}{\theta}\right)^{-1/p}}{\theta+\eta} \Delta\eta = L_1(\theta),$$
$$\omega(\eta) := \left(\int_0^\infty \xi(\theta)\left(\frac{l(\theta,\eta)}{L(\theta)}\right)^\lambda \Delta\theta\right) := \left(\int_0^\infty \left(\frac{L_1(\theta)}{\theta}\right)^\lambda \left(\frac{l(\theta,\eta)}{L(\theta)}\right)^\lambda \Delta\theta\right)^{\frac{1}{\lambda}}$$
$$:= \left(\int_0^\infty \left(\frac{l(\theta,\eta)}{\theta}\right)^\lambda \Delta\theta\right)^{\frac{1}{\lambda}} := \left(\left(\eta^{-1}\int_0^\infty \frac{\left(\frac{\eta}{\theta}\right)^{1-1/p}}{\theta+\eta}\right)^\lambda \Delta\theta\right)^{\frac{1}{\lambda}}$$
$$:= \frac{1}{\eta}\left(\left(\int_0^\infty \frac{\left(\frac{\eta}{\theta}\right)^{1-1/p}}{\theta+\eta}\right)^\lambda \Delta\theta\right)^{\frac{1}{\lambda}} \frac{L_2(\eta)}{\eta}, \tag{113}$$

and the operator $(A_l\boldsymbol{g})(\theta)$ in this case is defined as

$$(A_l\boldsymbol{g})(\theta) := \frac{1}{L_1(\theta)}\int_0^\infty \frac{\left(\frac{\eta}{\theta}\right)^{-1/p}}{\theta+\eta}\boldsymbol{g}(\eta)\Delta\eta. \tag{114}$$

Utilizing $(A_l\boldsymbol{g})(\theta)$ in (62), we obtain

$$\int_0^\infty \left(\frac{L_1(\theta)}{\theta}\right)^\lambda \left(\frac{1}{L_1(\theta)}\int_0^\infty \frac{\left(\frac{\eta}{\theta}\right)^{-1/p}}{\theta+\eta}\boldsymbol{g}(\eta)\Delta\eta\right)^{\lambda p}\Delta\theta$$
$$+\lambda\int_0^\infty\int_0^\infty \left(\frac{L_1(\theta)}{\theta}\right)^\lambda \left(\frac{\left(\frac{\eta}{\theta}\right)^{-1/p}}{(\theta+\eta)L_1(\theta)}\right)\left(\frac{1}{L_1(\theta)}\int_0^\infty \frac{\left(\frac{\eta}{\theta}\right)^{-1/p}}{\theta+\eta}\boldsymbol{g}(\eta)\Delta\eta\right)^{p(\lambda-1)}$$
$$\times\left|\boldsymbol{g}(\eta)-\frac{1}{L_1(\theta)}\int_0^\infty \frac{\left(\frac{\eta}{\theta}\right)^{-1/p}}{\theta+\eta}\boldsymbol{g}(\eta)\Delta\eta\right|^p\Delta\theta\Delta\eta$$
$$\le\left(\int_0^\infty \frac{K_2(\eta)}{\eta}\boldsymbol{g}^p(\eta)\Delta\eta\right)^\lambda. \tag{115}$$

Hence,

$$\int_0^\infty (L_1(\theta))^{\lambda(1-p)}\left(\int_0^\infty \frac{\boldsymbol{g}(\eta)\eta^{-1/p}}{\theta+\eta}\Delta\eta\right)^{\lambda p}\Delta\theta$$
$$+\lambda\int_0^\infty\int_0^\infty L_1^{(\lambda-1)(1-p)}(\theta)\left(\int_0^\infty \frac{\boldsymbol{g}(\eta)\eta^{-\frac{1}{p}}}{\theta+\eta}\Delta\eta\right)^{p(\lambda-1)}$$
$$\times\left|\boldsymbol{g}(\eta)-\frac{1}{L_1(\theta)}\left(\frac{1}{\theta}\right)\int_0^\infty \frac{\boldsymbol{g}(\eta)\eta^{-p^{-1}}}{\theta+\eta}\Delta\eta\right|^p\left(\frac{\eta^{-p^{-1}}}{\theta+\eta}\right)\left(\theta^{-1}\right)^{1-p^{-1}}\Delta\theta\Delta\eta$$
$$\le\left(\int_0^\infty \frac{K_2(\eta)}{\eta}\boldsymbol{g}^p(\eta)\Delta\eta\right)^\lambda. \tag{116}$$

Finally, replacing $g(\eta)$ by $g(\eta)\eta^{1/p}$ in (116), we get (111). The cases $0<\lambda<1$ and $1<p\le 2$ are proved in the same way.

Remark 44. For $m=1$, Theorem 43 reduces to Theorem 5.1 in [13]. In particular, for $\lambda=1$, Theorem 43 is a refinement of Theorem 5.5 in [10].

Remark 45. By taking $\mathbb{T}=\mathbb{R}$, $\lambda=1$, and $p\ge 2$, in Theorem 43 and utilizing the known fact that

$$\int_0^\infty \frac{\left(\frac{\eta}{\theta}\right)^{-1/p}}{\theta+\eta}d\eta = \int_0^\infty \frac{\left(\frac{\eta}{\theta}\right)^{1-1/p}}{\theta+\eta}d\theta = \frac{\pi}{\sin\left(\frac{\pi}{p}\right)}, \tag{117}$$

then (111) becomes

$$\int_0^\infty\left(\int_0^\infty \frac{\boldsymbol{g}(\eta)}{\theta+\eta}d\eta\right)^p d\theta + \left(\frac{\pi}{\sin(\pi p^{-1})}\right)^{p-1}\int_0^\infty\int_0^\infty \eta^{-p^{-1}}$$
$$\cdot\left|\boldsymbol{g}(\eta)\eta^{p^{-1}}-\frac{\sin(\pi p^{-1})}{\pi}\theta^{p^{-1}}\int_0^\infty \frac{\boldsymbol{g}(\eta)}{\theta+\eta}d\eta\right|^p\frac{\theta^{p^{-1}-1}}{\theta+\eta}d\theta d\eta$$
$$\le\left(\frac{\pi}{\sin(\pi p^{-1})}\right)^p\int_0^\infty \boldsymbol{g}^p(\eta)d\eta, \tag{118}$$

which is a refinement of (16). For $m=1$, (118) has been established in [3], Corollary 3.2.

In the following theorem, we introduce a generalized form of (111) on time scales.

Theorem 46. *Suppose that $\lambda>0, p>1$ and $s,\delta\in\mathbb{R}$. Furthermore, assume*

$$\left[\left(\int_0^\infty \frac{\theta^\delta(\eta/\theta)^{(s-2/p)+1}}{(\theta+\eta)^s}\right)^\lambda\Delta\theta\right]^{1/\lambda} \text{ and } L_1(\theta) := \int_0^\infty \frac{\left(\eta\theta^{-1}\right)^{s-2/p}}{(\theta+\eta)^s}\Delta\eta, \tag{119}$$

where $\lambda\ge 1$ and $p\ge 2$; then

$$\int_0^\infty (L_1(\theta))^{\lambda(1-p)}\theta^{\lambda(\delta-s+1)}\left(\int_0^\infty \frac{\boldsymbol{g}(\eta)}{(\theta+\eta)^s}\Delta\eta\right)^{p\lambda}$$
$$+\lambda\int_0^\infty\int_0^\infty \eta_p^{2-s}L_1^{(\lambda-1)(1-p)}(\theta)\left(\int_0^\infty \frac{\boldsymbol{g}(\eta)}{(\theta+\eta)^s}\Delta\eta\right)^{p(\lambda-1)}$$
$$\times\left|\boldsymbol{g}(\eta)\eta_p^{2-s}-\frac{1}{L_1(\theta)}\theta_p^{2-s}\int_0^\infty \frac{\boldsymbol{g}(\eta)}{(\theta+\eta)^s}\Delta\eta\right|^p$$
$$\cdot\frac{\theta^{p\lambda+(s-2)(1+p\lambda-p)/p}}{(\theta+\eta)^s}\Delta\theta\Delta\eta\le\left(\int_0^\infty L_2(\eta)\boldsymbol{g}^p(\eta)\Delta\eta\right)^\lambda, \tag{120}$$

is available for all nonnegative integrable functions $g:[\alpha,\beta)_{\mathbb{T}}\longrightarrow\mathbb{R}^m$. If $0<\lambda<1$ and $1<p\le 2$, then (120) is reversed.

Proof. Rewrite (62) in Theorem 15 with $\Omega_1=\Omega_2=[0,\infty)_{\mathbb{T}}$, $\Delta\mu_1(\theta)=\Delta\theta$, and $\Delta\mu_2(\eta)=\Delta\eta$. Let us define $\xi(\theta):=\left(L_1(\theta)\theta^{\delta-1}\right)^\lambda$ and

$$l(\theta, \eta) := \begin{cases} \dfrac{\binom{\eta}{\theta}^{s-2/p}}{(\theta+\eta)^s}, \text{if } \theta \neq 0, \eta \neq 0, \theta + \eta \neq 0 \\ 0, \text{otherwise.} \end{cases} \tag{121}$$

We have

$$\begin{aligned} L(\theta) &:= \int_0^\infty \frac{\binom{\eta}{\theta}^{s-2/p}}{(\theta+\eta)^s} \Delta\eta = L_1(\theta), \\ \omega(\eta) &:= \int_0^\infty \xi(\theta) \left(\frac{l(\theta,\eta)}{L(\theta)} \Delta\theta \right)^{\lambda-1} := \left(\int_0^\infty \frac{L_1(\theta)\theta^\delta}{\theta} \frac{l(\theta,\eta)}{L_1(\theta)} \Delta\theta \right)^{\lambda-1} \\ &:= \left(\int_0^\infty \frac{L_1(\theta)\theta^\delta}{\theta} \frac{l(\theta,\eta)}{L_1(\theta)} \Delta\theta \right)^{\lambda-1} := \left(\frac{1}{\eta} \int_0^\infty \frac{\theta^\delta \binom{\eta}{\theta}_P^{s-2}}{\theta(\theta+\eta)^s} g(\eta) \Delta\theta \right)^{\lambda-1} \\ &:= \frac{1}{\eta} \left(\left(\int_0^\infty \frac{\theta^\delta \binom{\eta}{\theta}^{\frac{s-2}{P}+1}}{(\theta+\eta)^s} \right)^\lambda \Delta\theta \right)^{\lambda-1} := \frac{L_2(\eta)}{\eta}, \end{aligned} \tag{122}$$

and the operator $(A_l g)(\theta)$ in this case is defined as

$$(A_l \boldsymbol{g})(\theta) := \frac{1}{L_1(\theta)} \int_\theta^\infty \frac{\binom{\eta}{\theta}^{s-2/p}}{(\theta+\eta)^s} \boldsymbol{g}(\eta) \Delta\eta. \tag{123}$$

Now, substituting L, ω and $(A_l g)(\theta)$ in (62), we get

$$\begin{aligned} &\int_0^\infty \left(L_1(\theta)\theta^{\delta-1} \right)^\lambda \left(\frac{1}{L_1(\theta)^s} \int_0^\infty \frac{\binom{\eta}{\theta}^{s-2/p}}{(\theta+\eta)^s} \boldsymbol{g}(\eta) \Delta\theta \right)^{p\lambda} \Delta\theta \\ &\quad + \lambda \int_0^\infty \int_0^\infty \left(L_1(\theta)\theta^{\delta-1} \right)^\lambda \left(\frac{\binom{\eta}{\theta}^{s-2/p}}{(\theta+\eta)^s L_1(\theta)} \right) \left(\frac{1}{L_1(\theta)} \int_0^\infty \frac{\binom{\eta}{\theta}^{s-2/p}}{(\theta+\eta)^s} \boldsymbol{g}(\eta) \Delta\theta \right)^{p(\lambda-1)} \\ &\quad \times \left| \boldsymbol{g}(\eta) - \frac{1}{L_1(\theta)} \int_0^\infty \frac{\binom{\eta}{\theta}^{s-2/p}}{(\theta+\eta)^s} \boldsymbol{g}(\eta) \Delta\eta \right|^p \Delta\theta\Delta\eta \\ &\leq \left(\int_0^\infty \frac{L_2(\eta)}{\eta} \boldsymbol{g}^p(\eta) \Delta\eta \right)^\lambda. \end{aligned} \tag{124}$$

Hence,

$$\begin{aligned} &\int_0^\infty (L_1(\theta))^{\lambda(1-p)} \theta^{\delta\lambda} \left(\frac{1}{\theta} \right)^{\lambda(s-1)} \left(\int_0^\infty \frac{\boldsymbol{g}(\eta)^{s-2/p}}{(\theta+\eta)^s} \Delta\eta \right)^{p\lambda} \\ &\quad + \lambda \int_0^\infty \int_0^\infty L_1{}^{(\lambda-1)(1-p)}(\theta) \left(\int_0^\infty \frac{\boldsymbol{g}(\eta)^{s-2/p}}{(\theta+\eta)^s} \Delta\eta \right)^{p(\lambda-1)} \\ &\quad \times \left| \boldsymbol{g}(\eta) - \frac{1}{L_1(\theta)} \theta_p^{s-2} \int_0^\infty \frac{(\eta)^{s-2/p} \boldsymbol{g}(\eta)}{(\theta+\eta)^s} \Delta\eta \right|^p \left(\frac{(\eta)_p^{s-2}}{(\theta+\eta)^s} \right) \\ &\quad \cdot \left(\frac{1}{\theta} \right) \frac{\theta^{p\lambda+(s-2)(1+p\lambda-p)/p}}{(\theta+\eta)^s} \Delta\theta\Delta\eta \leq \left(\int_0^\infty L_2(\eta) \boldsymbol{g}^p(\eta) \Delta\eta \right)^\lambda. \end{aligned} \tag{125}$$

Finally, considering (125) with $g(\eta)\eta^{(2-s/p)}$ instead of $g(\eta)$, we obtain (120). The cases $0 < \lambda < 1$ and $1 < p \leq 2$ are proved in the same way.

Remark 47. For $m = 1$, Theorem 46 coincides with Theorem 5.2 in [13].

Remark 48. Clearly, for $p > 1, \delta = 0$, and $s = 1$, Theorem 46 reduces to Theorem 43.

7. Conclusion and Future Work

The study of dynamic inequalities on time scales has a lot of scope. This research article is devoted to some general Hardy-type dynamic inequalities and their converses on time scales. Inequalities are considered in rather general forms and contain several special integral inequalities. In particular, our findings can be seen as refinements of some recent results closely linked to the time-scale inequalities of the classical Hardy, Pólya-Knopp, and Hardy-Hilbert. We use some algebraic inequalities such as the Minkowski inequality, the refined Jensen inequality and the Bernoulli inequality on time scales to prove the essential results in this paper. The performance of the superquadratic method for functions is reliable and effective to obtain new dynamic inequalities on time scales. This method has more advantages: it is direct and concise. Thus, the proposed method can be extended to some forms for Hardy's and related dynamic inequalities in mathematical and physical sciences. Our computed outcomes can be very useful as a starting point to get some continuous inequalities, especially from the obtained dynamic inequalities. In the future, we will get some discrete inequalities from the main results. Also, we will suppose that $g(t) := (g_1(t), \cdots, g(t))$ is an m-tuple of functions and $t = (t_1, t_2, \cdots, t_n)$ is n-tuple of variables to get the general forms of Hardy's and related inequalities on time scales. Similarly, in the future, we can present such inequalities by using Riemann-Liouville-type fractional integrals and fractional derivatives on time scales. It will also be very interesting to present such inequalities on quantum calculus.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally. All the authors read and approved the final manuscript.

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