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Research Article

On the Uniqueness of Meromorphic Functions on Annuli in terms of Deficiencies

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The purpose of this article is to study the uniqueness of meromorphic functions on annuli. Under a certain condition about deficiencies, we prove some new uniqueness theorems of meromorphic functions on the annulus $\mathbb{A} = \{z\colon (1/R_0) < |z| < R_0\}$, where $1 < R_0 \leq +\infty$.

1. Introduction and Main Results

In this article, we assume that the readers are familiar with the classical notations and definitions of Nevanlinna theory (refer to [1, 2]). The main purpose of this article is to study the uniqueness of meromorphic functions on annuli. For the necessary concepts and notations of the Nevanlinna theory of meromorphic functions on annuli, such as $T_0(r, f)$, $m_0(r, f)$, and $N_0(r, f)$, refer to the excellent summarizations [3–11].

Let a be a value in the extended complex plane $\overline{\mathbb{C}}$, and let f and g be two meromorphic functions on the annulus $\mathbb{A} = \{z\colon (1/R_0) < |z| < R_0\}$, where $1 < R_0 \leq +\infty$. Then, we say that f and g share a IM (ignoring multiplicities) when $f - a$ and $g - a$ have the same zeros, and furthermore, we say that f and g share a CM (counting multiplicities) when $f - a$ and $g - a$ have the same zeros with the same multiplicities. As mentioned in [3, 4], the reduced counting function $\overline{N}_0(r, 1/(f - a))$ is defined by

$$\int_{1/r}^{1} \frac{\overline{n}_1(t, (1/(f - a)))}{t} \mathrm{d}t + \int_{1}^{r} \frac{\overline{n}_2(t, (1/(f - a)))}{t} \mathrm{d}t, \quad (1)$$

where $\overline{n}_1(t, 1/(f - a))$ and $\overline{n}_2(t, 1/(f - a))$ are the functions counting (counting only once) the zeros of $f - a$ in $\{z\colon t < |z| \leq 1\}$ and $\{z\colon 1 < |z| \leq t\}$, respectively. Similarly, we denote by $\overline{N}_0^E(r, a)$ $(\overline{N}_0^D(r, a))$ the reduced counting function of common zeros (different zeros) of $f - a$ and $g - a$ on $\mathbb{A}$, where $\overline{N}_0^D(r, a) = \overline{N}_0(r, 1/(f - a)) + \overline{N}_0(r, 1/(g - a)) - 2\overline{N}_0^E(r, a)$. It is obvious that f and g share a IM on $\mathbb{A}$ if $\overline{N}_0^D(r, a) = 0$. Following the definitions in [2], we say that f and g share a "IM" if $\overline{N}_0^D(r, a) = o(T_0(r, f)) + o(T_0(r, g))$, and we say that f and g share a "CM" if $\overline{N}_0(r, 1/(f - a)) + \overline{N}_0(r, 1/(g - a)) - 2\overline{N}_0^C(r, a) = o(T_0(r, f)) + o(T_0(r, g))$, in which $\overline{N}_0^C(r, a)$ denotes the reduced counting function of common zeros with the same multiplicities of $f - a$ and $g - a$.

For a nonconstant meromorphic function f on the annulus $\mathbb{A}$, it is named as a transcendental meromorphic function on $\mathbb{A}$ provided that

$$\limsup_{r \longrightarrow \infty} \frac{T_0(r, f)}{\log r} = \infty, \quad 1 < r < R_0 = +\infty,$$
$$\text{or } \limsup_{r \longrightarrow R_0} \frac{T_0(r, f)}{-\log(R_0 - r)} = \infty, \quad 1 < r < R_0 < +\infty, \quad (2)$$

respectively. In fact, the transcendental meromorphic functions are also known as admissible meromorphic functions. If f is a transcendental meromorphic function on $\mathbb{A}$, then we have $S(r, f) = o(T_0(r, f))$ for all $1 < r < R_0$ except for a set Δ_r such that $\int_{\Delta_r} r^{\lambda-1} \mathrm{d}r < +\infty$ or a set Δ_r' such that $\int_{\Delta_r'} \mathrm{d}r/((R_0 - r)^{\lambda+1}) < +\infty$, respectively.

There existed many famous results about the uniqueness theory of meromorphic functions sharing values. In 1926, Nevanlinna [12] proved the celebrated five-value theorem.

Theorem 1. *Let f and g be two nonconstant meromorphic functions in $\mathbb{C}$, and let $a_i\,(i=1,2,3,4,5)$ be five distinct values in $\overline{\mathbb{C}}$. If f and g share the values a_i IM for $i=1,2,3,4,5$ in $\mathbb{C}$, then $f\equiv g$.*

Since that time, a series of results emerged in large numbers, which discussed and generalized the five-value theorem (Theroem 1). For the main results about the generalizations of Theroem 1 in simply connected regions, we can refer to [7, 13–16]. For instance, Zheng [15, 16] and Fang [13] obtained the generalization of the five-value theorem in an angular domain and in the unit disc, respectively. For the generalizations on multiply-connected regions, we can refer to [5, 17, 18].

Recently, Cao et al. [17, 18] proved the following five-value theorem on annuli (the case of $R_0=+\infty$ was derived from [5] by Kondratyuk and Laine).

Theorem 2. *Let f and g be two transcendental meromorphic functions on the annulus $\mathbb{A}=\{z\colon (1/R_0)<|z|<R_0\}$, where $1<R_0\le+\infty$. Let a_i $(i=1,2,3,4,5)$ be five distinct values in $\overline{\mathbb{C}}$. If f and g share the values a_i IM for $i=1,2,3,4,5$ on $\mathbb{A}$, then $f\equiv g$.*

From the very point of sharing small functions, we studied above theorems in [19] and provided the following uniqueness theorem of meromorphic functions sharing four small functions on annuli.

Theorem 3. *Let f and g be two transcendental meromorphic functions on the annulus $\mathbb{A}=\{z\colon (1/R_0)<|z|<R_0\}$, where $1<R_0\le+\infty$. Let $a_i\equiv a_i(z)\,(i=1,2,3,4)$ be four distinct small functions with respect to f and g on $\mathbb{A}$. If f and g share $a_i\,(i=1,2,3,4)$ IM and*

$$\sum_{i=1}^{4}\widetilde{N}_0(r,a_i)\ne S(r,f), \tag{3}$$

then $f\equiv g$, where $\widetilde{N}_0(r,a_i)$ is the reduced counting function which counts the multiple common zeros of $f-a_i$ and $g-a_i$ on $\mathbb{A}$.

In this article, we mainly investigate whether Theroem 2 holds if f and g dissatisfy the condition of sharing values. From the very point of deficiencies, we deal with this question and propose the following uniqueness theorem without conditions of sharing values. This theorem generalizes Theroem 2.

Theorem 4. *Let f and g be two transcendental meromorphic functions on the annulus $\mathbb{A}=\{z\colon (1/R_0)<|z|<R_0\}$, where $1<R_0\le+\infty$, and let $a_i\,(i=1,2,3,4,5)$ be five distinct complex numbers in $\overline{\mathbb{C}}$. Then, we have $f\equiv g$ provided that*

$$\sum_{i=1}^{5}\delta_0^D(a_i)>\frac{14}{3}, \tag{4}$$

where the deficiencies $\delta_0^D(a_i)$ are defined as

$$1-\limsup_{r\longrightarrow\infty}\frac{\overline{N}_0^D(r,a_i)}{T_0(r,f)+T_0(r,g)}, \tag{5}$$

when $R_0=+\infty$, or

$$1-\limsup_{r\longrightarrow R_0}\frac{\overline{N}_0^D(r,a_i)}{T_0(r,f)+T_0(r,g)}, \tag{6}$$

when $R_0<+\infty$, respectively.

In special, if f and g share $a_i\,(i=1,2,3,4,5)$ "IM," then it is obvious that $\sum_{i=1}^{5}\delta_0^D(a_i)=5$, which satisfies the condition

$$\sum_{i=1}^{5}\delta_0^D(a_i)>\frac{14}{3}, \tag{7}$$

of Theroem 4. In view of the discussion above, we deduce a corollary as follows. This corollary partly improves Theroem 2 in the sense that IM is replaced with "IM."

Corollary 1. *Let f and g be two transcendental meromorphic functions on the annulus $\mathbb{A}=\{z\colon (1/R_0)<|z|<R_0\}$, where $1<R_0\le+\infty$, and let $a_i\,(i=1,2,3,4,5)$ be five distinct complex numbers in $\overline{\mathbb{C}}$. If f and g share $a_i\,(i=1,2,3,4,5)$ "IM," then $f\equiv g$.*

2. Some Lemmas

In this section, we will give some necessary lemmas, where the third lemma is motivated by the ideas of [20–22].

Lemma 1 (see [4], Theroem 1). *Let f be a nonconstant meromorphic function on the annulus $\mathbb{A}=\{z\colon (1/R_0)<|z|<R_0\}$, where $R_0\le+\infty$, and let $\lambda\ge0$. Then,*

(i) If $R_0=+\infty$, then $m_0(r,f'/f)=O(\log(rT_0(r,f)))$ for $R\in(1,\infty)$ except for a set Δ_r such that $\int_{\Delta_r}r^{\lambda-1}dr<+\infty$.

(ii) If $R_0<+\infty$, then $m_0(r,f'/f)=O(\log(T_0(r,f)/R_0-r))$ for $r\in(1,R_0)$ except for a set Δ_r' such that $\int_{\Delta_r'}dr/((R_0-r)^{\lambda+1})<+\infty$.

Lemma 2 (see [18], Theorem 2.3). *Let f be a nonconstant meromorphic function on the annulus $\mathbb{A}=\{z\colon (1/R_0)<|z|<R_0\}$, where $1<R_0\le+\infty$. Let $a_1,a_2,\ldots,a_q$ be q distinct complex numbers in the extended complex plane $\overline{\mathbb{C}}$. Then,*

$$(q-2)T_0(r,f)<\sum_{j=1}^{q}\overline{N}_0\left(r,\frac{1}{f-a_j}\right)+S(r,f). \tag{8}$$

Inspired by the ideas of [20–22], we propose the following lemma and give the proof.

Lemma 3. *Let f and g be two transcendental meromorphic functions on $\mathbb{A}=\{z\colon (1/R_0)<|z|<R_0\}$, where $1<R_0\le+\infty$, and let $a_i\,(i=1,2,3,4,5)$ be five distinct complex numbers in $\mathbb{C}\cup\{\infty\}$. If $f\equiv g$, then*

$$\overline{N}_0^E(r,a_i) \le \sum_{j=1,j\ne i}^{5} \overline{N}_0^D(r,a_j) + S(r,f) + S(r,g), \quad (9)$$

where $\overline{N}_0^E(r,a_i)$ ($\overline{N}_0^D(r,a_i)$) is the reduced counting function of the common (different) zeros of $f-a_i$ and $g-a_i$ on $\mathbb{A}$ ($i=1,2,3,4,5$).

Proof. Without loss of generality, we suppose that $a_1 = 0$, $a_2 = 1$, $a_3 = \infty$, $a_4 = a$, and $a_5 = b$, in which a, b are two distinct complex numbers such that $a, b \ne 0, 1, \infty$. Otherwise, a Möbius transformation as

$$\frac{f-a_1}{f-a_3}\frac{a_2-a_3}{a_2-a_1}, \quad (10)$$

will be done. Then, set

$$h = h_1 - h_2, \quad (11)$$

where

$$h_1 = \frac{(f-g)f'g'}{f(f-1)g(g-1)(g-a)}, \qquad h_2 = \frac{(f-g)f'g'}{g(g-1)f(f-1)(f-a)}. \quad (12)$$

Noting that h_1 can be expressed by

$$\frac{f'}{f-1}\left[\frac{g'}{ag} + \frac{g'}{a(a-1)(g-a)} - \frac{g'}{(a-1)(g-1)}\right] + \frac{1}{a-1}\left(\frac{f'}{f} - \frac{f'}{f-1}\right)\left(\frac{g'}{g-a} - \frac{g'}{g-1}\right). \quad (13)$$

By Lemma 1, we have $m_0(r,h_1) = S(r,f) + S(r,g)$. Similarly, we get $m_0(r,h_2) = S(r,f) + S(r,g)$, and thus,

$$m_0(r,h) = S(r,f) + S(r,g), \quad (14)$$

holds.

Next, we will estimate the counting function $N_0(r,h)$. A simple computation yields

$$h = (f-g)^2 \frac{f'g'}{f(f-1)(f-a)g(g-1)(g-a)}. \quad (15)$$

Then, it is easy to see that the poles of h only come from the zeros of $f, g, f-1, g-1, f-a$, and $g-a$ and the poles of f and g on $\mathbb{A}$. Now, let z_0 be a common zero of f and g on $\mathbb{A}$ with multiplicity p and q, respectively. Without loss of generality, assume that $p \ge q$. Then, it follows that z_0 is a zero of $(f-g)^2$ with multiplicity at least $2q$ and that z_0 is a pole of

$$\frac{f'g'}{f(f-1)(f-a)g(g-1)(g-a)}, \quad (16)$$

with multiplicity 2. We consequently know that z_0 is not a pole of h, and hence the poles of h cannot occur at the common zeros of f and g. By similar methods, we can conclude that the poles of h cannot occur at the common zeros of $f-1$ and $g-1$, the common zeros of $f-a$ and $g-a$, and the common poles of f and g, so the poles of h only come from the different zeros of $f, g, f-1, g-1, f-a$, and $g-a$ and the different poles of f and g on $\mathbb{A}$. In order to analyze these different zeros and different poles, we distinguish the following distinct cases.

Case 1: let z_1 be a zero of f which is not a zero of g. Then, by using the equation (15), we find that z_1 is a pole of h with multiplicity at most 1 if $g(z_1) \ne 1, \infty, a$; otherwise, z_1 is a pole of h with multiplicity at most 2.

Case 2: let z_2 be a zero of $f-1$, which is not a zero of $g-1$. It is clear that z_2 is a pole of h with multiplicity at most 1 if $g(z_2) \ne 0, \infty, a$; otherwise, z_2 is a pole of h with multiplicity at most 2.

Case 3: let z_3 be a pole of f, which is not a pole of g. It is clear that z_2 is a pole of h with multiplicity at most 1 if $g(z_3) \ne 0, 1, a$; otherwise, z_3 is a pole of h with multiplicity at most 2.

Case 4: let z_4 be a zero of $f-a$, which is not a zero of $g-a$. It is clear that z_4 is a pole of h with multiplicity at most 1 if $g(z_4) \ne 0, 1, \infty$; otherwise, z_4 is a pole of h with multiplicity at most 2.

Case 5: let z_5 be a zero of g, which is not a zero of f. It is clear that z_5 is a pole of h with multiplicity at most 1 if $f(z_5) \ne 1, \infty, a$; otherwise, z_5 is a pole of h with multiplicity at most 2.

Case 6: let z_6 be a zero of $g-1$, which is not a zero of $f-1$. It is clear that z_6 is a pole of h with multiplicity at most 1 if $f(z_6) \ne 0, \infty, a$; otherwise, z_6 is a pole of h with multiplicity at most 2.

Case 7: let z_7 be a pole of g, which is not a pole of f. It is clear that z_7 is a pole of h with multiplicity at most 1 if $f(z_7) \ne 0, 1, a$; otherwise, z_7 is a pole of h with multiplicity at most 2.

Case 8: let z_8 be a zero of $g-a$, which is not a zero of $f-a$. It is clear that z_8 is a pole of h with multiplicity at most 1 if $f(z_8) \ne 0, 1, \infty$; otherwise, z_8 is a pole of h with multiplicity at most 2.

In view of these cases, we obtain

$$N_0(r,h) \le \overline{N}_0^D(r,0) + \overline{N}_0^D(r,1) + \overline{N}_0^D(r,\infty) + \overline{N}_0^D(r,a), \quad (17)$$

which means

$$N_0(r,h) \le \sum_{i=1}^{4} \overline{N}_0^D(r,a_i). \quad (18)$$

Combining (14) with (18), we get

$$T_0(r,h) \le \sum_{i=1}^{4} \overline{N}_0^D(r,a_i) + S(r,f) + S(r,g). \quad (19)$$

If $h \equiv 0$, then $h_1 \equiv h_2$. This implies that $f \equiv g$, which is impossible, so $h \equiv 0$ holds. Furthermore, it follows from (15) that the common zeros of $f - b$ and $g - b$ must be the zeros of h. This implies that

$$\overline{N}_0^E(r,b) \le N_0\left(r,\frac{1}{h}\right) \le T_0(r,h), \quad (20)$$

which further implies that

$$\overline{N}_0^E(r,a_5) \le \sum_{i=1}^{4} \overline{N}_0^D(r,a_i) + S(r,f) + S(r,g), \quad (21)$$

combined with (19). Similarly, we can derive other inequations as

$$\begin{aligned}\overline{N}_0^E(r,a_i) \le &\sum_{j=1,j\ne i}^{5} \overline{N}_0^D(r,a_j) + S(r,f)\\ &+ S(r,g),\ (i = 1,2,3,4).\end{aligned} \quad (22)$$

Therefore, we have proved Lemma 3. □

3. The Proof of Theorem 4

On the contrary, we suppose that $f \equiv g$, and then it follows from Lemma 3 that

$$\overline{N}_0^E(r,a_i) \le \sum_{j=1,j\ne i}^{5} \overline{N}_0^D(r,a_j) + S(r,f) + S(r,g), \quad (23)$$

for $i = 1,2,3,4,5$. Thus, noting that

$$\overline{N}_0\left(r,\frac{1}{f-a_i}\right) + \overline{N}_0\left(r,\frac{1}{g-a_i}\right) = 2\overline{N}_0^E(r,a_i) + \overline{N}_0^D(r,a_i), \quad (24)$$

we know

$$\begin{aligned}&\overline{N}_0\left(r,\frac{1}{f-a_i}\right) + \overline{N}_0\left(r,\frac{1}{g-a_i}\right)\\ &\quad\le \overline{N}_0^D(r,a_i) + \sum_{j=1,j\ne i}^{5} 2\overline{N}_0^D(r,a_j) + S(r,f) + S(r,g).\end{aligned} \quad (25)$$

This yields that

$$\begin{aligned}&\overline{N}_0\left(r,\frac{1}{f-a_i}\right) + \overline{N}_0\left(r,\frac{1}{g-a_i}\right)\\ &\quad\le \sum_{j=1}^{5} 2\overline{N}_0^D(r,a_j) - \overline{N}_0^D(r,a_i) + S(r,f) + S(r,g),\end{aligned} \quad (26)$$

for $i = 1,2,3,4,5$, which further yields that

$$\begin{aligned}&\sum_{i=1}^{5} \overline{N}_0\left(r,\frac{1}{f-a_i}\right) + \sum_{i=1}^{5} \overline{N}_0\left(r,\frac{1}{g-a_i}\right)\\ &\quad\le 9\sum_{i=1}^{5} \overline{N}_0^D(r,a_i) + S(r,f) + S(r,g).\end{aligned} \quad (27)$$

On the other hand, by utilizing Lemma 2, we find

$$\begin{aligned}3T_0(r,f) &< \sum_{i=1}^{5} \overline{N}_0\left(r,\frac{1}{f-a_i}\right) + S(r,f),\\ 3T_0(r,g) &< \sum_{i=1}^{5} \overline{N}_0\left(r,\frac{1}{g-a_i}\right) + S(r,f).\end{aligned} \quad (28)$$

Therefore, it follows from (27) that

$$3T_0(r,f) + 3T_0(r,g) \le 9\sum_{i=1}^{5} \overline{N}_0^D(r,a_i) + S(r,f) + S(r,g), \quad (29)$$

which means

$$\frac{1}{3} \le \sum_{i=1}^{5} \frac{\overline{N}_0^D(r,a_i)}{T_0(r,f) + T_0(r,g)} + \frac{S(r,f) + S(r,g)}{T_0(r,f) + T_0(r,g)}. \quad (30)$$

Consequently, from (30), we have

$$\sum_{i=1}^{5} \delta_0^D(a_j) \le \frac{14}{3}, \quad (31)$$

which is a contradiction. Hence, the proof is completed.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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References

[1] L. Yang, *Value Distribution Theory*, Science Press, Beijing, China, 1993.
[2] H. X. Yi and C. C. Yang, *Uniqueness Theory of Meromorphic Functions*, Kluwer Academic Publishers Group, Dordrecht, Netherlands, 2003.
[3] A. Y. Khrystiyanyn and A. A. Kondratyuk, “On the Nevanlinna theory for meromorphic functions on annuli. I,” *Matematychni Studii*, vol. 23, no. 1, pp. 19–30, 2005.
[4] A. Y. Khrystiyanyn and A. A. Kondratyuk, “On the Nevanlinna theory for meromorphic functions on annuli. II,” *Matematychni Studii*, vol. 24, no. 2, pp. 57–68, 2005.

[5] A. A. Kondratyuk and I. Laine, *Meromorphic Functions in Multiply Connected Domains, Fourier Series Methods in Complex Analysis*, Vol. 9-111, University of Joensuu, Department of Mathematics, Report series, Joensuu, Finland, 2006.

[6] R. Korhonen, “Nevanlinna theory in an annulus, value distribution theory and related topics,” *Advances in Complex Analysis and Its Applications*, vol. 3, pp. 167–179, 2004.

[7] N. Wu and Q. Ge, “On uniqueness of meromorphic functions sharing five small functions on annuli,” *Bulletin of the Iranian Mathematical Society*, vol. 41, no. 3, pp. 713–722, 2015.

[8] H. Y. Xu and Z. J. Wu, “The shared set and uniqueness of meromorphic functions on annuli,” *Abstract and Applied Analysis*, vol. 2013, Article ID 758318, 10 pages, 2013.

[9] H. Y. Xu and Z. J. Wu, “The inequality and its application of algebroid functions on annulus concerning some polynomials,” *Publicationes Mathematicae Debrecen*, vol. 94, no. 3-4, pp. 269–287, 2019.

[10] H. Y. Xu and Z. J. Wu, “The fundamental inequality for algebroid functions on annuli concerning small algebroid functions,” *Journal of Mathematical Inequalities*, vol. 13, no. 4, 2019.

[11] H. Y. Xu and Z. X. Xuan, “The uniqueness of analytic functions on annuli sharing some values,” *Abstract and Applied Analysis*, vol. 2012, Article ID 896596, 13 pages, 2012.

[12] R. Nevanlinna, “Einige Eindeutigkeitssätze in der Theorie der Meromorphen Funktionen,” *Acta Mathematica*, vol. 48, no. 3-4, pp. 367–391, 1926.

[13] M. Fang, “Uniqueness of admissible meromorphic functions in the unit disc,” *Science in China Series A: Mathematics*, vol. 42, no. 4, pp. 367–381, 1999.

[14] W. Lin, S. Mori, and K. Tohge, “Uniqueness theorems in an angular domain,” *Tohoku Mathematical Journal*, vol. 58, no. 4, pp. 509–527, 2006.

[15] J. H. Zheng, “On uniqueness of meromorphic functions with shared values in one angular domains,” *Complex Variables, Theory and Application: An International Journal*, vol. 48, no. 9, pp. 777–785, 2003.

[16] J. H. Zheng, “On uniqueness of meromorphic functions with shared values in some angular domains,” *Canadian Mathematical Bulletin*, vol. 47, no. 1, pp. 152–160, 2004.

[17] T. B. Cao and H. X. Yi, “Uniqueness theorems of meromorphic functions sharing sets IM on annuli,” *Acta Mathematica Sinica*, vol. 54, no. 4, pp. 623–632, 2011, in Chinese.

[18] T. B. Cao, H. X. Yi, and H. Y. Xu, “On the multiple values and uniqueness of meromorphic functions on annuli,” *Computers & Mathematics with Applications*, vol. 58, no. 7, pp. 1457–1465, 2009.

[19] D. W. Meng, S. Y. Liu, and N. Lu, “Uniqueness of meromorphic functions sharing four small functions on annuli,” *Mathematica Slovaca*, vol. 69, no. 4, pp. 815–824, 2019.

[20] K. Ishizaki, “Meromorphic functions sharing small functions,” *Archiv der Mathematik*, vol. 77, no. 3, pp. 273–277, 2001.

[21] W. H. Yao, “Meromorphic functions sharing four small functions IM,” *Indian Journal of Pure and Applied Mathematics*, vol. 34, no. 7, pp. 1025–1033, 2003.

[22] H. X. Yi, “Uniqueness theorems for meromorphic functions concerning small functions,” *Indian Journal of Pure and Applied Mathematics*, vol. 32, no. 6, pp. 903–914, 2001.