Hindawi
Journal of Function Spaces
Volume 2020, Article ID 6517068, 17 pages
https://doi.org/10.1155/2020/6517068

Research Article

Inequalities Involving Conformable Approach for Exponentially Convex Functions and Their Applications

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Received 11 July 2019; Accepted 3 October 2019; Published 27 January 2020

Academic Editor: Gisele Mophou

In the article, we present several Hermite–Hadamard-type inequalities for the exponentially convex functions via conformable integrals. As applications, we give new inequalities for certain bivariate means.

1. Introduction

Recently, the fractional calculus has attracted the attention of several researchers [1–14]. The affect and motivation of the fractional calculus in both theoretical and applied science and engineering arose out substantially. Numerous studies are related with the discrete versions of the fractional calculus which benefit from countless applications in the theory of time scales, physics, different fields of engineering, chemistry, and so forth.

In the past decades, the subject of fractional integrals has attracted the attention for mathematicians working on inequality theory and convexity. Fractional integral operators are sometimes the gateway to physical problems that cannot be expressed by classical integral, sometimes for the solution of problems expressed in fractional order. In recent years, a lot of new operator definitions have been given, and the properties and structures of these operators have been examined. Some of these operators are very close to classical operators in terms of their characteristics and definitions. In general, they have nonlocality property and defined with singular kernels.

The derivatives in this calculus seemed complicated and lost some of the basic properties that usual derivatives have such as the product rule and the chain rule. However, the semigroup properties of these operators behave well in some cases. Recently, the authors in [15] defined a new well-behaved simple derivative called "conformable fractional derivative" which depends just on the basic limit definition of the derivative. It will define the derivative of higher order (i.e., order $\alpha > 1$) and also define the integral of order $0 < \alpha \leq 1$ only. It will also prove the product rule and the mean value theorem and solve some (conformable) differential equations where the fractional exponential function $e^{\xi^{\alpha}/\alpha}$ plays an important rule.

Almost every mathematician knows the importance of convex sets and convex functions in many fields of mathematics, for example, in nonlinear programming and optimization theory. By using the concept of convexity, several integral inequalities have been introduced such as Jensen, Hermite–Hadamard, and Slater inequalities. But, the well-known one of them is the celebrated Hermite–Hadamard inequality.

Let $K \subseteq \mathbb{R}$ be an interval and $\chi\colon K \longrightarrow \mathbb{R}$ be a convex function. Then, the double inequality

$$\chi\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b \chi(x)\mathrm{d}x \leq \frac{\chi(a)+\chi(b)}{2}, \tag{1}$$

holds for all $a, b \in K$ with $a \neq b$.

It is easy to see that if χ is concave on K, then one has the reverse of inequality (1). Many upper and lower bounds for the mean value of a convex function can be derived by use of inequality (1). Recently, the generalizations, improvements, refinements, extensions, and variants for Hermite–Hadamard inequality (1) can be found in the literature [16–19].

It is well known that convex functions have wide applications in pure and applied mathematics [20–35] and many other natural sciences [21–55]. The development in the history of convex function is the minimum of the differentiable convex functions that can be characterized by variational inequalities.

Motivated and inspired by the previous study, a comprehensive investigation of exponentially convex functions for conformable integral in the present paper is new. The class of exponentially convex functions was introduced by Antczak [56] and Dragomir and Gomm [57]. Inspired by these facts, Awan et al. [58] defined a new class of convexity, namely, the exponentially convex function. The growth of research on big data analysis, time space management, and deep learning has recently increased interest in information theory involving exponentially convex functions. The smoothness of exponentially convex functions is exploited for statistical learning, sequential prediction, and stochastic optimization.

Now, we recall and introduce some definitions for various convex functions.

Definition 1. A nonempty set $K \subseteq \mathbb{R}$ is said to be convex if $\lambda x + (1-\lambda)y \in K$ for all $x, y \in K$ and $\lambda \in [0,1]$.

Definition 2. Let $\chi\colon K \longrightarrow \mathbb{R}$ be a real-valued function. Then, χ is said to be convex (concave) if the inequality holds

$$\chi[(1-\xi)x + \xi y] \leq (1-\xi)\chi(x) + \xi\chi(y), \tag{2}$$

which holds for all $x, y \in K$ and $\xi \in [0,1]$. χ is said to be concave if $-\chi$ is convex.

Definition 3. A positive real-valued function $\chi\colon K \subseteq \mathbb{R} \longrightarrow (0,\infty)$ is said to be exponentially convex if the inequality

$$e^{\chi[(1-\xi)x+\xi y]} \leq (1-\xi)e^{\chi(x)} + \xi e^{\chi(y)}, \tag{3}$$

holds for all $x, y \in K$ and $\xi \in [0,1]$.

It is well known that $x \in K$ is the minimum of the differentiable exponentially convex functions χ if and only if it satisfies

$$\langle \chi'(x)e^{\chi(x)}, y - x\rangle \geq 0, \tag{4}$$

for all $y \in K$. Inequality (4) is known as the exponentially variational inequality.

The Riemann–Liouville fractional integral, conformable derivative, and conformable integral are very important in the theory of fractional calculus which are defined as follows.

Definition 4. Let $\alpha > 0$ and $\chi \in L_1[\mu,\nu]$. Then, the Riemann–Liouville integral $J^{\alpha}_{\mu+}\chi$ and $J^{\alpha}_{\nu-}\chi$ of order α is defined by

$$\begin{aligned} J^{\alpha}_{\mu+}\chi(\xi) &= \frac{1}{\Gamma(\alpha)}\int_{\mu}^{\xi}(\xi - x)^{\alpha-1}\chi(x)\mathrm{d}x, \quad \xi > \mu,\\ J^{\alpha}_{\nu-}\chi(\xi) &= \frac{1}{\Gamma(\alpha)}\int_{\xi}^{\nu}(x - \xi)^{\alpha-1}\chi(x)\mathrm{d}x, \quad \xi < \nu, \end{aligned} \tag{5}$$

respectively, where $\Gamma(\alpha) = \int_0^{\infty} e^{-\xi}\xi^{\alpha-1}\mathrm{d}\xi$ is the Euler gamma function [59–61].

Definition 5. Let $\alpha \in (0,1)$ and $\chi\colon [0,\infty) \longrightarrow \mathbb{R}$ be a real-valued function. Then, the α-order conformable derivative of f at $\xi > 0$ is defined by

$$D_{\alpha}(\chi)(\xi) = \lim_{\varepsilon \longrightarrow 0}\frac{\chi\left(\xi + \varepsilon\xi^{1-\alpha}\right) - \chi(\xi)}{\varepsilon}. \tag{6}$$

If $\lim_{\xi \longrightarrow 0^+}\chi^{(\alpha)}(\xi)$ exists, then we define

$$D_{\alpha}(\chi)(0) = \lim_{\xi \longrightarrow 0^+} D_{\alpha}(\chi)(\xi). \tag{7}$$

Additionally, if χ is differentiable, then

$$D_{\alpha}(\chi)(\xi) = \xi^{1-\alpha}\chi'(\xi), \tag{8}$$

where

$$\chi'(\xi) = \lim_{\varepsilon \longrightarrow 0}\frac{\chi(\xi + \varepsilon) - \chi(\xi)}{\varepsilon}. \tag{9}$$

We write $\chi^{\alpha}(\xi)$ for $D_{\alpha}(\chi)(\xi)$ to denote the α-order conformable derivative of χ at ξ. Additionally, we say that χ is α-differentiable if the α-order conformable derivative of χ exists.

Definition 6. Let $\alpha \in (0,1]$ and $0 \leq \mu < \nu$. Then, the real-valued function $\chi\colon [\mu,\nu] \longrightarrow \mathbb{R}$ is said to be α integrable on $[\mu,\nu]$ if the integral

$$\int_a^b \chi(x)\mathrm{d}_{\alpha}x = \int_a^b \chi(x)x^{\alpha-1}\mathrm{d}x, \tag{10}$$

exists and is finite.

Remark 1. Let $\alpha \in (0,1]$. Then, it is well known that

$$I^{\mu}_{\alpha}(\chi)(\xi) = I^{\mu}_{1}\left(t^{\alpha-1}\chi\right) = \int_{\mu}^{\xi}\frac{\chi(x)}{x^{1-\alpha}}\mathrm{d}x, \tag{11}$$

where the integral is the classical Riemann improper integral.

Anderson [62] gave a variant of the Hermite–Hadamard inequality which is as follows.

Theorem 1. *Let $\alpha \in (0,1]$ and $\chi\colon I \longrightarrow \mathbb{R}$ be an α-differentiable function such that $D_\alpha(\chi)$ is monotone. Then, the double inequality*

$$\chi\left(\frac{\mu+\nu}{2}\right) \leq \frac{\alpha}{\nu^\alpha-\mu^\alpha}\int_\mu^\nu \chi(x)\mathrm{d}x \leq \frac{\chi(\mu)+\chi(\nu)}{2}, \tag{12}$$

holds for all $\mu,\nu \in I$ with $\mu \neq \nu$.

Note that, if $\alpha = 1$, then inequality (12) leads to the classical Hermite–Hadamard inequality (1).

The main purpose of the article is to present new Hermite–Hadamard-type inequalities for exponentially convex functions via the conformable integrals and find new inequalities for certain bivariate means.

2. Main Results

First of all, we provide an integral identity which can be proved easily by the readers.

Lemma 1. *Let $\alpha \in (0,1]$ and $\chi\colon [\mu,\nu] \longrightarrow (0,\infty)$ be an α-fractional exponentially differentiable function on (μ,ν). The identity*

$$\begin{aligned}
&\frac{(\nu^\alpha-x^\alpha)e^{\chi(\nu)}+(x^\alpha-\nu^\alpha)e^{\chi(\mu)}}{\nu^\alpha-a^\alpha}-\frac{\alpha}{\nu^\alpha-\mu^\alpha}\int_\mu^\nu e^{\chi(x)}\mathrm{d}_\alpha x\\
&=\frac{x-\mu}{\nu^\alpha-\mu^\alpha}\left[\int_0^1\left[((1-\xi)\mu+\xi x)^{2\alpha-1}-x^\alpha((1-\xi)\mu+\xi x)^{\alpha-1}\right]\times D_\alpha\left(e^\chi\right)((1-\xi)\mu+\xi x)\chi^{(\alpha)}((1-\xi)\mu+\xi x)\xi^{1-\alpha}\mathrm{d}\xi\right]\\
&+\frac{\nu-x}{\nu^\alpha-\mu^\alpha}\left[\int_0^1\left[((1-\xi)\nu+\xi x)^{2\alpha-1}-x^\alpha((1-\xi)\nu+\xi x)^{\alpha-1}\right]\times D_\alpha\left(e^\chi\right)((1-\xi)\nu+\xi x)\chi^{(\alpha)}((1-\xi)\nu+\xi x)\xi^{1-\alpha}\mathrm{d}\xi\right],
\end{aligned} \tag{13}$$

holds for any $x \in [\mu,\nu]$ if $D_\alpha(e^\chi) \in L_\alpha([\mu,\nu])$.

Remark 2. Let $\alpha = 1$. Then, identity (13) reduces to

$$\begin{aligned}
&\frac{(\nu-x)e^{\chi(\nu)}+(x-\nu)e^{\chi(\mu)}}{\nu-\mu}-\frac{1}{\nu-\mu}\int_\mu^\nu e^{\chi(x)}\mathrm{d}x\\
&=\frac{(x-\mu)^2}{\mu-\nu}\left[\int_0^1(\xi-1)e^{\chi((1-\xi)\mu+\xi x)}\chi'((1-\xi)\mu+\xi x)\mathrm{d}\xi\right]\\
&+\frac{(\nu-x)^2}{\nu-\mu}\left[\int_0^1(\xi-1)e^{\chi((1-\xi)\nu+\xi x)}\chi'((1-\xi)\nu+\xi x)\mathrm{d}\xi\right].
\end{aligned} \tag{14}$$

Theorem 2. *Let $\mu,\nu>0$ with $\mu<\nu$, $\alpha \in (0,1)$, and $\chi\colon [\mu,\nu] \longrightarrow \mathbb{R}$ be an α–differentiable exponentially function on (μ,ν). Then, the inequality*

$$\begin{aligned}
&\left|\frac{(\nu^\alpha-x^\alpha)e^{\chi(\nu)}+(x^\alpha-\nu^\alpha)e^{\chi(\mu)}}{\nu^\alpha-\mu^\alpha}-\frac{\alpha}{\nu^\alpha-\mu^\alpha}\int_\mu^\nu e^{\chi(x)}\mathrm{d}_\alpha x\right|\\
&\leq \frac{x-\mu}{\nu^\alpha-\mu^\alpha}\theta_1(\alpha)+\frac{\nu-x}{\nu^\alpha-\mu^\alpha}\theta_2(\alpha),
\end{aligned} \tag{15}$$

holds, where

$$\begin{aligned}
\theta_1(\alpha)&=\frac{(x^\alpha-\mu^\alpha)}{12}\left\{3\left|e^{\chi(\mu)}\chi'(\mu)\right|+\left|e^{\chi(x)}\chi'(x)\right|+\Delta(\mu,x)\right\},\\
\theta_2(\alpha)&=\frac{1}{60}\left[\left\{12\nu^\alpha+3\left[x\nu^{\alpha-1}+x^{\alpha-1}\nu\right]-18x^\alpha\right\}\left|e^{\chi(\nu)}\chi'(\nu)\right|\right.\\
&\quad+\left\{2\nu^\alpha+3\left[x\nu^{\alpha-1}+x^{\alpha-1}\nu\right]-8x^\alpha\right\}\left|e^{\chi(x)}\chi'(x)\right|\\
&\quad\left.+\left\{3\nu^\alpha+2\left[x\nu^{\alpha-1}+x^{\alpha-1}\nu\right]-7x^\alpha\right\}\Delta(\nu,x)\right],\\
\Delta(\mu,x)&=\left\{\left|e^{\chi(\mu)}\chi'(x)\right|+\left|e^{\chi(x)}\chi'(\mu)\right|\right\},\\
\Delta(\nu,x)&=\left\{\left|e^{\chi(\nu)}\chi'(x)\right|+\left|e^{\chi(x)}\chi'(\nu)\right|\right\}.
\end{aligned} \tag{16}$$

Proof. It follows from Lemma 1 and the convexity of the functions $x^{\alpha-1}$, $-x^\alpha$, and $|(e^\chi)'|$ that

$$\begin{aligned}
&\left|\frac{(\nu^\alpha - x^\alpha)e^{\chi(\nu)} + (x^\alpha - \nu^\alpha)e^{\chi(\mu)}}{\nu^\alpha - \mu^\alpha} - \frac{\alpha}{\nu^\alpha - \mu^\alpha}\int_\mu^\nu e^{\chi(x)}\mathrm{d}_\alpha x\right| \\
&\quad = \frac{x-\mu}{\nu^\alpha - \mu^\alpha}\int_0^1 \left[x^\alpha - ((1-t)\mu + xt)^\alpha\right]\left|e^{\chi((1-\xi)\mu + x\xi)}\chi'((1-\xi)\mu + x\xi)\right|\mathrm{d}\xi \\
&\qquad + \frac{\nu - x}{\nu^\alpha - \mu^\alpha}\int_0^1 \int_0^1 \left[((1-\xi)\nu + x\xi)^\alpha - x^\alpha\right]\left|e^{\chi((1-\xi)\nu + x\xi)}\chi'((1-\xi)\nu + x\xi)\right|\mathrm{d}\xi \\
&\quad = \frac{x-\mu}{\nu^\alpha - \mu^\alpha}\int_0^1 \left[x^\alpha - ((1-\xi)\mu + xt)^\alpha\right]\left|e^{\chi((1-\xi)\mu + x\xi)}\chi'((1-\xi)\mu + x\xi)\right|\mathrm{d}\xi \\
&\qquad + \frac{\nu - x}{\nu^\alpha - \mu^\alpha}\int_0^1 \left[((1-\xi)\nu + xt)^{\alpha+1-1} - x^\alpha\right]\left|e^{\chi((1-\xi)\nu + x\xi)}e^{\chi((1-\xi)\nu + x\xi)}\chi'((1-\xi)\nu + x\xi)\right|\mathrm{d}\xi \\
&\quad \leq \frac{x-\mu}{\nu^\alpha - \mu^\alpha}\int_0^1 \left[(x^\alpha - ((1-\xi)\mu^\alpha + \xi x^\alpha))\right]\left|e^{\chi((1-\xi)\mu + x\xi)}\chi'((1-\xi)\mu + \xi x)\right|\mathrm{d}\xi \\
&\qquad + \frac{b - x}{\nu^\alpha - \mu^\alpha}\int_0^1 \left[((1-\xi)\nu + \xi x)^{\alpha-1}((1-\xi)\nu + \xi x) - x^\alpha\right]\left|e^{\chi((1-\xi)\nu + x\xi)}\chi'((1-\xi)\nu + \xi x)\right|\mathrm{d}\xi \\
&\quad \leq \frac{x-\mu}{\nu^\alpha - \mu^\alpha}\int_0^1 \left[(x^\alpha - ((1-\xi)\mu^\alpha + \xi x^\alpha))\right]\left|e^{\chi((1-\xi)\mu + x\xi)}\chi'((1-\xi)\mu + \xi x)\right|\mathrm{d}\xi \\
&\qquad + \frac{\nu - x}{\nu^\alpha - \mu^\alpha}\int_0^1 \left[\left((1-\xi)\nu^{\alpha-1} + \xi x^{\alpha-1}\right)((1-\xi)\nu + \xi x) - x^\alpha\right]\left|e^{\chi((1-\xi)\nu + x\xi)}\chi'((1-\xi)\nu + \xi x)\right|\mathrm{d}\xi \\
&\quad \leq \frac{x-\mu}{\nu^\alpha - \mu^\alpha}\int_0^1 \left[(x^\alpha - ((1-\xi)\mu^\alpha + \xi x^\alpha))\right]\left[(1-\xi)\left|e^{\chi(\mu)} + \xi\left|e^{\chi(x)}\right|\right]\left[(1-\xi)\left|\chi'(\mu)\right| + \xi\left|\chi'(x)\right|\right]\mathrm{d}\xi \\
&\qquad + \frac{\nu - x}{\nu^\alpha - \mu^\alpha}\int_0^1 \left[\left((1-\xi)\nu^{\alpha-1} + \xi x^{\alpha-1}\right)((1-\xi)\nu + \xi x) - x^\alpha\right] \\
&\qquad \times\left[(1-\xi)\left|e^{\chi(\nu)}\right| + \xi\left|e^{\chi(x)}\right|\right]\left[(1-\xi)\left|\chi'(\nu)\right| + \xi\left|\chi'(x)\right|\right]\mathrm{d}\xi \\
&\quad \leq \frac{x-\mu}{\nu^\alpha - \mu^\alpha}\int_0^1 \left[(x^\alpha - ((1-\xi)\mu^\alpha + \xi x^\alpha))\right] \\
&\qquad \times\left[(1-\xi)^2\left|e^{\chi(\mu)}\chi'(\mu)\right| + \xi^2\left|e^{f(x)}\chi'(x)\right| + \xi(1-\xi)\left\{\left|e^{\chi(\mu)}\chi'(x)\right| + \left|e^{\chi(x)}\chi'(\mu)\right|\right\}\right]\mathrm{d}\xi \\
&\qquad + \frac{\nu - x}{\nu^\alpha - \mu^\alpha}\int_0^1 \left[\left((1-\xi)\nu^{\alpha-1} + \xi x^{\alpha-1}\right)((1-\xi)\nu + \xi x) - x^\alpha\right] \\
&\qquad \times\left[(1-\xi)^2\left|e^{\chi(\nu)}\chi'(\nu)\right| + \xi^2\left|e^{\chi(x)}\chi'(x)\right| + \xi(1-\xi)\left\{\left|e^{\chi(\nu)}\chi'(x)\right| + \left|e^{\chi(x)}\chi'(\nu)\right|\right\}\right]\mathrm{d}\xi \\
&\quad \leq \frac{x-\mu}{\nu^\alpha - \mu^\alpha}\int_0^1 \left[(x^\alpha - ((1-\xi)\mu^\alpha + \xi x^\alpha))\right] \\
&\qquad \times\left[(1-\xi)^2\left|e^{\chi(\mu)}\chi'(\mu)\right| + \xi^2\left|e^{\chi(x)}\chi'(x)\right| + \xi(1-\xi)\Delta(\mu, x)\right]\mathrm{d}\xi \\
&\qquad + \frac{\nu - x}{\nu^\alpha - \mu^\alpha}\int_0^1 \left[\left((1-\xi)\nu^{\alpha-1} + \xi x^{\alpha-1}\right)((1-\xi)\nu + \xi x) - x^\alpha\right] \\
&\qquad \times\left[(1-\xi)^2\left|e^{\chi(\nu)}\chi'(\nu)\right| + \xi^2\left|e^{\chi(x)}\chi'(x)\right| + \xi(1-\xi)\Delta(\nu, x)\right]\mathrm{d}\xi \\
&\quad = \frac{x-\mu}{\nu^\alpha - \mu^\alpha}\theta_1(\alpha) + \frac{\nu - x}{\nu^\alpha - \mu^\alpha}\theta_2(\alpha).
\end{aligned} \tag{17}$$

□

Remark 3. Let $\alpha = 1$. Then, inequality (15) leads to

$$\left|\frac{(\nu - x)e^{\chi(\nu)} + (x - \nu)e^{\chi(\mu)}}{\nu - \mu} - \frac{1}{\nu - \mu}\int_{\mu}^{\nu} e^{\chi(x)}\mathrm{d}x\right| \leq \frac{(x-\mu)^2}{\nu - \mu}\left[\frac{3|e^{\chi(\mu)}\chi'(\mu)| + |e^{\chi(x)}\chi'(x)| + \Delta(x,\mu)}{12}\right] + \frac{(\nu - x)^2}{\nu - \mu}\left[\frac{3|e^{\chi(\nu)}\chi'(\nu) + |e^{\chi(x)}\chi'(x)| + \Delta(x,\nu)}{12}\right]. \tag{18}$$

Theorem 3. *Let $\mu, \mu > 0$ with $\mu < \nu$, $\alpha \in (0,1)$, $p, q > 1$ such that $(1/p) + (1/q) = 1$, and χ: $[\mu,\nu] \longrightarrow \mathbb{R}$ be an α-differentiable function on (μ,ν). Then, the inequality*

$$\left|\frac{(\nu^\alpha - x^\alpha)e^{\chi(\nu)} + (x^\alpha - \nu^\alpha)e^{\chi(\mu)}}{\nu^\alpha - \mu^\alpha} - \frac{\alpha}{\nu^\alpha - \mu^\alpha}\int_{\mu}^{\nu} e^{\chi(x)}\mathrm{d}_\alpha x\right| \leq \frac{x-\mu}{\nu^\alpha - \mu^\alpha}\left[(\varphi_1(\alpha,p))^{1/p}\left(\frac{2\left\{\left|e^{\chi}(\mu)\chi'(\mu)\right|^q + \left|e^{\chi}(x)\chi'(x)\right|^q\right\} + \Delta_1(x,\mu)}{6}\right)^{1/q}\right] + \frac{\nu - x}{\nu^\alpha - \mu^\alpha}\left[(\varphi_2(\alpha,p))^{1/p}\left(\frac{2\left\{\left|e^{\chi}(\nu)\chi'(\nu)\right|^q + \left|e^{\chi}(x)\chi'(x)\right|^q\right\} + \Delta_2(x,\nu)}{6}\right)^{1/q}\right] \tag{19}$$

holds if $D_\alpha(e^\chi) \in L_\alpha([\mu,\nu])$ and $|(e^\chi)'|^q$ is convex on $[\mu,\nu]$, where

$$\varphi_1(\alpha,p) = \int_0^1 [x^\alpha - ((1-\xi)\mu^\alpha + \xi x^\alpha)]^p \mathrm{d}\xi, \tag{20}$$

$$\varphi_2(\alpha,p) = \int_0^1 \left[((1-t)\nu + \xi x)^{\alpha-1}((1-t)\nu + \xi x) - x^\alpha\right]^p \mathrm{d}\xi, \tag{21}$$

$$\Delta_1(x,\mu) = \left|e^{\chi(\mu)}\chi'(x)\right|^q + \left|e^{\chi(x)}\chi'(\mu)\right|^q, \quad \Delta_2(x,\nu) = \left|e^{\chi(\nu)}\chi'(x)\right|^q + \left|e^{\chi(x)}\chi'(\nu)\right|^q. \tag{22}$$

Proof. From (13) and (20)–(22), we clearly see that

$$\left|\frac{(\nu^\alpha - x^\alpha)e^{\chi(\nu)} + (x^\alpha - \nu^\alpha)e^{\chi(\mu)}}{\nu^\alpha - \mu^\alpha} - \frac{\alpha}{\nu^\alpha - \mu^\alpha}\int_{\mu}^{\nu} e^{\chi(x)}\mathrm{d}_\alpha x\right| \leq \frac{x-\mu}{\nu^\alpha - \mu^\alpha}\int_0^1 [(x^\alpha - ((1-\xi)\mu^\alpha + \xi x^\alpha))]\left|e^{\chi((1-\xi)\mu + x\xi)}\chi'((1-\xi)\mu + \xi x)\right|\mathrm{d}\xi + \frac{\nu - x}{\nu^\alpha - \mu^\alpha}\int_0^1 \left[((1-\xi)\nu + \xi x)^{\alpha-1}((1-\xi)\nu + \xi x) - x^\alpha\right]\left|e^{\chi((1-\xi)\nu + xt)}\chi'((1-\xi)\nu + \xi x)\right|\mathrm{d}\xi. \tag{23}$$

Making use of Hölder's inequality, one has

$$
\begin{aligned}
&\int_0^1 [(x^\alpha - ((1-\xi)\mu^\alpha + \xi x^\alpha))] \left| e^{\chi((1-\xi)\mu + x\xi)} \chi'((1-\xi)\mu + \xi x) \right| \mathrm{d}\xi \\
&\le \left(\int_0^1 (x^\alpha - ((1-\xi)\mu^\alpha + \xi x^\alpha))^p d\xi \right)^{1/p} \left(\left| e^{\chi((1-\xi)\mu + x\xi)} \chi'((1-\xi)\mu + \xi x) \right|^q \mathrm{d}\xi \right)^{1/q} \\
&\le (\varphi_1(\alpha, p))^{1/p} \left(\int_0^1 \left((1-\xi)\left| e^\chi(\mu) \right|^q + \xi \left| e^\chi(x) \right|^q \right) \left((1-\xi)\left| \chi'(\mu) \right|^q + \xi \left| \chi'(x) \right|^q \right) \mathrm{d}\xi \right)^{1/q} \\
&= (\varphi_1(\alpha, p))^{1/p} \left(\int_0^1 \left((1-\xi)^2 \left| e^\chi(\mu)\chi'(\mu) \right|^q + \xi^2 \left| e^\chi(x)\chi'(x) \right|^q \right) + \xi(1-\xi)\Delta_1(x,\mu) \mathrm{d}\xi \right)^{1/q} \\
&= (\varphi_1(\alpha, p))^{1/p} \left(\frac{2\left\{ \left| e^\chi(\mu)\chi'(\mu) \right|^q + \left| e^\chi(x)\chi'(x) \right|^q \right\} + \Delta_1(x,\mu)}{6} \right)^{1/q}.
\end{aligned} \tag{24}
$$

Similarly, we have

$$
\begin{aligned}
&\int_0^1 \left[((1-\xi)\nu + \xi x)^{\alpha-1} ((1-\xi)\nu + \xi x) - x^\alpha \right] \left| e^{\chi((1-\xi)\nu + x\xi)} \chi'((1-\xi)\nu + \xi x) \right| \mathrm{d}\xi \\
&\le \left(\int_0^1 \left[((1-\xi)\nu + \xi x)^{\alpha-1} ((1-\xi)\nu + \xi x) - x^\alpha \right] \mathrm{d}\xi \right)^{1/p} \left(\int_0^1 \left| e^{\chi((1-\xi)\nu + x\xi)} \chi'((1-\xi)\nu + \xi x) \right|^q \mathrm{d}\xi \right)^{1/q} \\
&\le (\varphi_2(\alpha, p))^{1/p} \left(\frac{2\left\{ \left| e^\chi(\nu)\chi'(\nu) \right|^q + \left| e^\chi(x)\chi'(x) \right|^q \right\} + \Delta_2(x,\nu)}{6} \right)^{1/q}.
\end{aligned} \tag{25}
$$

Therefore, inequality (19) follows from (24) and (25). □

Remark 4. Let $\alpha = 1$. Then, inequality (19) becomes

$$
\begin{aligned}
&\left| \frac{(\nu - x)e^{\chi(\nu)} + (x-\nu)e^{\chi(\mu)}}{\nu - \mu} - \frac{1}{\nu - \mu} \int_\mu^\nu e^{\chi(x)} \mathrm{d}x \right| \\
&\le \left(\frac{1}{p+1} \right)^{1/p} \left[\frac{(x-\mu)^2}{\nu - \mu} \left(\frac{2\left\{ \left| e^\chi(\mu)\chi'(\mu) \right|^q + \left| e^\chi(x)\chi'(x) \right|^q \right\} + \Delta_1(x,\mu)}{6} \right)^{1/q} \right. \\
&\left. + \frac{(\nu - x)^2}{\nu - \mu} \left(\frac{2\left\{ \left| e^\chi(\nu)\chi'(\nu) \right|^q + \left| e^\chi(x)\chi'(x) \right|^q \right\} + \Delta_2(x,\nu)}{6} \right)^{1/q} \right].
\end{aligned} \tag{26}
$$

Theorem 4. *Let* $\mu, \nu > 0$ *with* $\mu < \nu$, $\alpha \in (0,1)$, $p, q > 1$ *such that* $(1/p) + (1/q) = 1$, *and* $\chi: [\mu, \nu] \longrightarrow \mathbb{R}$ *be an* α*-differentiable function on* (μ, ν). *Then, the inequality*

$$\begin{aligned}&\left|\frac{(\nu^\alpha - x^\alpha)e^{\chi(\nu)} + (x^\alpha - \nu^\alpha)e^{\chi(\mu)}}{\nu^\alpha - \mu^\alpha} - \frac{\alpha}{\nu^\alpha - \mu^\alpha}\int_\mu^\nu e^{\chi(x)}\mathrm{d}_\alpha x\right| \\ &\quad\leq \frac{x-\mu}{\nu^\alpha - \mu^\alpha}\left[(\psi_1(\alpha))^{1-(1/q)}\left\{\psi_2(\alpha)\left|e^{\chi(\mu)}\chi'(\mu)\right|^q + \psi_3(\alpha)\left|e^{\chi(x)}\chi'(x)\right|^q + \psi_4(\alpha)\Delta_1(x,\mu)\right\}^{1/q}\right] \\ &\qquad + \frac{\nu - x}{\nu^\alpha - \mu^\alpha}\left[(\vartheta_1(\alpha))^{1-(1/q)}\left\{\vartheta_2(\alpha)\left|e^{\chi(\nu)}\chi'(\nu)\right|^q + \vartheta_3(\alpha)\left|e^{\chi(x)}\chi'(x)\right|^q + \vartheta_4(\alpha)\Delta_2(x,\nu)\right\}^{1/q}\right],\end{aligned} \tag{27}$$

holds if $D_\alpha(e^\chi) \in L_\alpha([\mu,\nu])$ *and* $|(e^\chi)'|^q$ *is convex on* $[\mu,\nu]$, *where*

$$\begin{aligned}\psi_1(\alpha) &= \frac{x^\alpha - \mu^\alpha}{2}, \\ \psi_2(\alpha) &= \frac{x^\alpha - \mu^\alpha}{4}, \\ \psi_3(\alpha) = \psi_4(\alpha) &= \frac{x^\alpha - \mu^\alpha}{12}, \\ \vartheta_1(\alpha) &= \frac{\nu^{\alpha-1}(x + 2\nu) + x^{\alpha-1}(\nu - 4x)}{6}, \\ \vartheta_2(\alpha) &= \frac{12\nu^\alpha + 3\{x^{\alpha-1}\nu + x\nu^{\alpha-1}\} - 18x^\alpha}{60}, \\ \vartheta_3(\alpha) &= \frac{2\nu^\alpha + 3\{x^{\alpha-1}\nu + x\nu^{\alpha-1}\} - 8x^\alpha}{60}, \\ \vartheta_4(\alpha) &= \frac{3\nu^\alpha + 2\{x^{\alpha-1}\nu + x\nu^{\alpha-1}\} - 7x^\alpha}{60},\end{aligned} \tag{28}$$

and $\Delta_1(x,\mu)$ *and* $\Delta_2(x,\nu)$ *are given in Theorem 3.*

Proof. It follows from Lemma 1 that

$$\begin{aligned}&\left|\frac{(\nu^\alpha - x^\alpha)e^{\chi(\nu)} + (x^\alpha - \nu^\alpha)e^{\chi(\mu)}}{\nu^\alpha - \mu^\alpha} - \frac{\alpha}{\nu^\alpha - \mu^\alpha}\int_\mu^\nu e^{\chi(x)}\mathrm{d}_\alpha x\right| \\ &\quad\leq \frac{x-\mu}{\nu^\alpha - \mu^\alpha}\int_0^1 [(x^\alpha - ((1-\xi)\mu^\alpha + \xi x^\alpha))]\left|e^{\chi((1-\xi)\mu + x\xi)}\chi'((1-\xi)\mu + \xi x)\right|\mathrm{d}\xi \\ &\qquad + \frac{\nu - x}{\nu^\alpha - \mu^\alpha}\int_0^1 \left[\left((1-\xi)\nu^{\alpha-1} + \xi x^{\alpha-1}\right)((1-\xi)\nu + \xi x) - x^\alpha\right]\left|e^{\chi((1-\xi)\nu + x\xi)}\chi'((1-\xi)\nu + \xi x)\right|\mathrm{d}\xi.\end{aligned} \tag{29}$$

Making use of power mean inequality, we get

$$\begin{aligned}&\left(\int_0^1 [(x^\alpha - ((1-\xi)\mu^\alpha + \xi x^\alpha))]\left|e^{\chi((1-\xi)\mu + x\xi)}\chi'((1-\xi)\mu + \xi x)\right|\mathrm{d}\xi\right) \\ &\quad\leq \left(\int_0^1 [(x^\alpha - ((1-\xi)\mu^\alpha + \xi x^\alpha))\mathrm{d}\xi]\right)^{1-(1/q)} \\ &\qquad \times \left(\int_0^1 \left[(x^\alpha - ((1-\xi)\mu^\alpha + \xi x^\alpha))\left|e^{\chi((1-\xi)\mu + x\xi)}\chi'((1-\xi)\mu + \xi x)\right|^q\mathrm{d}\xi\right]\right)^{1/q}.\end{aligned} \tag{30}$$

Similarly, one has

$$\begin{aligned}
&\int_0^1 \left[\left((1-\xi)\nu^{\alpha-1}+\xi x^{\alpha-1}\right)((1-\xi)\nu+\xi x)-x^{\alpha}\right]\left|1e^{\chi((1-\xi)\nu+x\xi)}\chi'((1-\xi)\nu+\xi x)\right|\mathrm{d}\xi\\
&\leq\left(\int_0^1\left[\left((1-\xi)\nu^{\alpha-1}+\xi x^{\alpha-1}\right)((1-\xi)\nu+\xi x)-x^{\alpha}\right]\mathrm{d}\xi\right)^{1/q}\\
&\quad\times\left(\int_0^1\left[\left((1-\xi)\nu^{\alpha-1}+\xi x^{\alpha-1}\right)((1-\xi)b+\xi x)-x^{\alpha}\right]\left|e^{\chi((1-\xi)\nu+x\xi)}\chi'((1-\xi)\nu+\xi x)\right|^q\mathrm{d}\xi\right)^{1/q}.
\end{aligned}\tag{31}$$

From the convexity of $|(e^{\chi})'|^q$, we get

$$\begin{aligned}
&\int_0^1\left[\left(x^{\alpha}-((1-\xi)\mu^{\alpha}+\xi x^{\alpha})\right)\left|e^{\chi((1-\xi)\mu+x\xi)}\chi'((1-\xi)\mu+\xi x)\right|^q\right]\mathrm{d}\xi\\
&\leq\int_0^1\left[\left(x^{\alpha}-((1-\xi)\mu^{\alpha}+\xi x^{\alpha})\right)\left[(1-\xi)^2\left|e^{\chi(\mu)}\chi'(\mu)\right|^q+\xi^2\left|e^{\chi(x)}\chi'(x)\right|^q+\xi(1-\xi)\Delta_1(x,\mu)\right]\right]\mathrm{d}\xi\\
&=\psi_2(\alpha)\left|e^{\chi(\mu)}\chi'(\mu)\right|^q+\psi_3(\alpha)\left|e^{\chi(x)}\chi'(x)\right|^q+\psi_4(\alpha)\Delta_1(x,\mu),
\end{aligned}\tag{32}$$

$$\begin{aligned}
&\int_0^1\left[\left((1-\xi)\nu^{\alpha-1}+\xi x^{\alpha-1}\right)((1-\xi)\nu+tx)-x^{\alpha}\right]\left|e^{\chi((1-\xi)\nu+x\xi)}\chi'((1-\xi)\nu+\xi x)\right|^q\mathrm{d}\xi\\
&\leq\int_0^1\left[\left((1-\xi)\nu^{\alpha-1}+\xi x^{\alpha-1}\right)((1-\xi)\nu+\xi x)-x^{\alpha}\right]\\
&\quad\times\left[(1-\xi)^2\left|e^{\chi(\nu)}\chi'(\nu)\right|^q+\xi^2\left|e^{\chi(x)}\chi'(x)\right|^q+\xi(1-\xi)\Delta_2(x,\nu)\right]\mathrm{d}\xi\\
&=\vartheta_2(\alpha)\left|e^{\chi(\mu)}\chi'(\mu)\right|^q+\vartheta_3(\alpha)\left|e^{\chi(x)}\chi'(x)\right|^q+\vartheta_4(\alpha)\Delta_2(x,\nu),
\end{aligned}\tag{33}$$

where we have used the facts that

$$\begin{aligned}
\psi_1(\alpha)&=\int_0^1\left[\left(x^{\alpha}-((1-\xi)\mu^{\alpha}+\xi x^{\alpha})\right)\right]\mathrm{d}\xi,\\
\vartheta_1(\alpha)&=\int_0^1\left[\left((1-\xi)\nu^{\alpha-1}+\xi x^{\alpha-1}\right)((1-\xi)\nu+\xi x)-x^{\alpha}\right]\mathrm{d}\xi.
\end{aligned}\tag{34}$$

Therefore, Theorem 4 follows from (30)–(33). □

Remark 5. Let $\alpha=1$. Then, inequality (27) leads to

$$\begin{aligned}
&\left|\frac{(\nu-x)e^{\chi(\nu)}+(x-\nu)e^{\chi(\mu)}}{\nu-\mu}-\frac{1}{\nu-\mu}\int_{\mu}^{\nu}e^{\chi(x)}\mathrm{d}x\right|\\
&\leq\left(\frac{1}{2}\right)^{1-(1/q)}\left[\frac{(x-\mu)^2}{\nu-\mu}\left\{\frac{3\left|e^{\chi(\mu)}\chi'(\mu)\right|^q+\left|e^{\chi(x)}\chi'(x)\right|^q+\Delta_1(x,\mu)}{12}\right\}^{1/q}\right.\\
&\quad\left.+\frac{(\nu-x)^2}{\nu-\mu}\left\{\frac{3\left|e^{\chi(\nu)}\chi'(\nu)\right|^q+\left|e^{\chi(x)}\chi'(x)\right|^q+\Delta_2(x,\nu)}{12}\right\}^{1/q}\right].
\end{aligned}\tag{35}$$

Theorem 5. *Let $\mu, \nu > 0$ with $\mu < \nu$, $\alpha \in (0,1)$, $p, q > 1$ such that $(1/p) + (1/q) = 1$, and $\chi \colon [\mu, \nu] \longrightarrow \mathbb{R}$ be an α-differentiable function on (μ, ν). Then, the inequality*

$$\begin{aligned}
&\left|\frac{(\nu^\alpha - x^\alpha)e^{\chi(\nu)} + (x^\alpha - \nu^\alpha)e^{\chi(\mu)}}{\nu^\alpha - \mu^\alpha} - \frac{\alpha}{\nu^\alpha - \mu^\alpha}\int_\mu^u e^{\chi(x)}\mathrm{d}_\alpha x\right| \\
&\leq \frac{x-\mu}{\nu^\alpha - \mu^\alpha}\left[\left(\aleph_1(\alpha)\right)^{1-(1/q)}\left\{\aleph_2(\alpha)\left|e^{\chi(\mu)}\chi'(\mu)\right|^q + \aleph_3(\alpha)\left|e^{\chi(x)}\chi'(x)\right|^q + \aleph_4(\alpha)\Delta_1(x,\mu)\right\}\right]^{1/q} \\
&\quad + \frac{\nu - x}{\nu^\alpha - \mu^\alpha}\left[\left(\Im_1(\alpha)\right)^{1-(1/q)}\left\{\Im_2(\alpha)\left|e^{\chi(\nu)}\chi'(\nu)\right|^q + \Im_3(\alpha)\left|e^{\chi(x)}\chi'(x)\right|^q + \Im_4(\alpha)\Delta_2(x,\nu)\right\}^{1/q}\right],
\end{aligned} \tag{36}$$

holds if $D_\alpha(e^\chi) \in L_\alpha([\mu,\nu])$ and $|(e^\chi)'|^q$ is convex on $[\mu,\nu]$, where

$$\begin{aligned}
\aleph_1(\alpha) &= x^\alpha - \frac{x^{\alpha+1} - \mu^{\alpha+1}}{(\alpha+1)(x-\mu)}, \\
\aleph_2(\alpha) &= \frac{1}{3(\alpha+1)(\alpha+2)(\alpha+3)(x-\mu)^3} \times \Big(x^\alpha(\alpha+1)(\alpha+2)(\alpha+3)(x-\mu)^3 + 3\mu^{\alpha+1}(\alpha+2)(\alpha+3)(x-\mu)^2 \\
&\quad + 6\mu^{\alpha+2}(\alpha+3)(x-\mu) - 6\left\{x^{\alpha+3} - \mu^{\alpha+3}\right\}\Big), \\
\aleph_3(\alpha) &= \frac{1}{3(\alpha+1)(\alpha+2)(\alpha+3)(x-\mu)^3} \times \Big(x^\alpha(\alpha+1)(\alpha+2)(\alpha+3)(x-\mu)^3 - 3x^{\alpha+1}(\alpha+2)(\alpha+3)(x-\mu)^2 \\
&\quad + 6x^{\alpha+2}(\alpha+3)(x-\mu) - 6\left\{x^{\alpha+3} - \mu^{\alpha+3}\right\}\Big), \\
\aleph_4(\alpha) &= \frac{1}{6(\alpha+1)(\alpha+2)(\alpha+3)(x-\mu)^3} \times \Big(x^\alpha(\alpha+1)(\alpha+2)(\alpha+3)(x-\mu)^3 - 6\left\{x^{\alpha+2} - \nu^{\alpha+2}\right\} + 12\left\{x^{\alpha+3} - \nu^{\alpha+3}\right\}\Big), \\
\Im_1(\alpha) &= \left(\frac{x^{\alpha+1} - \nu^{\alpha+1}}{(\alpha+1)(\nu - x)}\right) - x^\alpha, \\
\Im_2(\alpha) &= \frac{1}{3(\alpha+1)(\alpha+2)(\alpha+3)(x-\nu)^3} \times \Big(6\left\{x^{\alpha+3} - \nu^{\alpha+3}\right\} - 6\nu^{\alpha+2}(\alpha+3)(x-\nu) - 3\nu^{\alpha+1}(\alpha+2)(\alpha+3)(x-\nu) \\
&\quad - x^\alpha(\alpha+1)(\alpha+2)(\alpha+3)(x-\nu)^3\Big), \\
\Im_3(\alpha) &= \frac{1}{3(\alpha+1)(\alpha+2)(\alpha+3)(x-\nu)^3} \times \Big(3x^{\alpha+1}(\alpha+2)(\alpha+3)(x-\nu)^2 - 6(\alpha+3)(x-\nu)\left\{x^{\alpha+2} - \nu^{\alpha+2}\right\} + 6\left\{x^{\alpha+3} - \nu^{\alpha+3}\right\} \\
&\quad - x^\alpha(\alpha+1)(\alpha+2)(\alpha+3)(x-\nu)^3\Big), \\
\Im_4(\alpha) &= \frac{1}{6(\alpha+1)(\alpha+2)(\alpha+3)(x-\nu)^3} \times \Big(6\left\{x^{\alpha+2} + \nu^{\alpha+2}\right\}(\alpha+3)(x-\nu) - 12\left\{x^{\alpha+3} - \nu^{\alpha+3}\right\} - x^\alpha(\alpha+1)(\alpha+2)(\alpha+3)(x-\nu)^3\Big).
\end{aligned} \tag{37}$$

Proof. From Definition 3 and Lemma 1, we get

$$
\begin{aligned}
&\left|\frac{(\nu^\alpha - x^\alpha)e^{\chi(\nu)} + (x^\alpha - \nu^\alpha)e^{\chi(\mu)}}{\nu^\alpha - \mu^\alpha} - \frac{\alpha}{\nu^\alpha - \mu^\alpha}\int_\mu^\nu e^{\chi(x)}\mathrm{d}_\alpha x\right| \\
&= \left|\frac{x-\mu}{\nu^\alpha - \mu^\alpha}\int_0^1 \left[((1-\xi)\mu + \xi x)^{2\alpha-1} - x^\alpha((1-\xi)\mu + \xi x)^{\alpha-1}\right] D_\alpha\left(e^\chi\right)((1-\xi)\mu + \xi x)\chi^{(\alpha)}((1-\xi)\mu + \xi x)\mathrm{d}\xi\right| \\
&\quad + \left|\frac{\nu - x}{\nu^\alpha - a^\alpha}\int_0^1 \left[((1-\xi)\nu + \xi x)^{2\alpha-1} - x^\alpha((1-\xi)\nu + \xi x)^{\alpha-1}\right] D_\alpha\left(e^\chi\right)((1-\xi)\nu + \xi x)\chi^{(\alpha)}((1-\xi)\nu + \xi x)\mathrm{d}\xi\right| \\
&\le \frac{x-\mu}{\nu^\alpha - \mu^\alpha}\int_0^1 \left[x^\alpha - ((1-\xi)\mu + \xi x)^\alpha\right]\left|e^\chi((1-\xi)\mu + \xi x)\chi'((1-\xi)\mu + \xi x)\right|\mathrm{d}\xi \\
&\quad + \frac{\nu - x}{\nu^\alpha - \mu^\alpha}\int_0^1 \left[((1-\xi)\nu + \xi x)^\alpha - x^\alpha\right]\left|e^\chi((1-\xi)\nu + \xi x)\chi'((1-\xi)\nu + \xi x)\right|\mathrm{d}\xi.
\end{aligned}
\tag{38}
$$

Making use of power mean inequality, we get

$$
\begin{aligned}
&\int_0^1 \left[x^\alpha - ((1-\xi)\mu + \xi x)^\alpha\right]\left|e^\chi((1-\xi)\mu + \xi x)\chi'((1-\xi)\mu + \xi x)\right|\mathrm{d}\xi \\
&\le \left(\int_0^1 \left(x^\alpha - ((1-\xi)\mu + \xi x)^\alpha\right)\right)^{1-(1/q)} \\
&\quad \times \left(\int_0^1 \left[x^\alpha - ((1-\xi)\mu + \xi x)^\alpha\right]\left|e^\chi((1-\xi)\mu + \xi x)\chi'((1-\xi)\mu + \xi x)\right|^q\mathrm{d}\xi\right)^{1/q}.
\end{aligned}
\tag{39}
$$

Similarly, we have

$$
\begin{aligned}
&\int_0^1 \left[((1-\xi)\nu + \xi x)^\alpha - x^\alpha\right]\left|e^\chi((1-\xi)\mu + \xi x)\chi'((1-\xi)\nu + \xi x)\right|\mathrm{d}\xi \\
&\le \left(\int_0^1 \left(((1-\xi)\nu + \xi x)^\alpha - x^\alpha\right)\right)^{1-(1/q)} \\
&\quad \times \left(\int_0^1 \left[((1-\xi)\nu + \xi x)^\alpha - x^\alpha\right]\left|e^\chi((1-\xi)\nu + \xi x)\chi'((1-\xi)\nu + \xi x)\right|^q\mathrm{d}\xi\right)^{1/q}.
\end{aligned}
\tag{40}
$$

It follows from the convexity of $|(e^{\chi})'|^q$ that

$$\begin{aligned}
&\int_0^1 [x^\alpha - ((1-\xi)\mu + \xi x)^\alpha]\left|e^{\chi}((1-\xi)\mu + \xi x)\chi'((1-\xi)\mu + \xi x)\right|^q \mathrm{d}\xi \\
&\le \int_0^1 [x^\alpha - ((1-\xi)\mu + \xi x)^\alpha]\Big((1-\xi)^2\left|e^{\chi(\mu)}\chi'(\mu)\right|^q + \xi^2\left|e^{\chi(x)}\chi'(x)\right|^q + \xi(1-\xi)\Delta_1(x,\mu)\Big)\mathrm{d}\xi \\
&= \aleph_2(\alpha)\left|e^{\chi(\mu)}\chi'(\mu)\right|^q + \aleph_3(\alpha)\left|e^{\chi(x)}\chi'(x)\right|^q + \aleph_4(\alpha)\Delta_1(x,\mu),
\end{aligned} \tag{41}$$

$$\begin{aligned}
&\int_0^1 [((1-\xi)\nu + \xi x)^\alpha - x^\alpha]\left|e^{\chi}((1-\xi)\nu + \xi x)\chi'((1-\xi)\nu + \xi x)\right|^q \mathrm{d}\xi \\
&\le \int_0^1 [((1-\xi)\nu + \xi x)^\alpha - x^\alpha]\Big((1-\xi)^2\left|e^{\chi(\nu)}\chi'(\nu)\right|^q + \xi^2\left|e^{\chi(x)}\chi'(x)\right|^q + \xi(1-\xi)\Delta_2(x,\nu)\Big)\mathrm{d}\xi \\
&= \mathfrak{I}_2(\alpha)\left|e^{\chi(\nu)}\chi'(\nu)\right|^q + \mathfrak{I}_3(\alpha)\left|e^{\chi(x)}\chi'(x)\right|^q + \mathfrak{I}_4(\alpha)\Delta_2(x,\nu),
\end{aligned} \tag{42}$$

where we have used the facts that

$$\begin{aligned}
\aleph_1(\alpha) &= \int_0^1 [x^\alpha - ((1-\xi)\mu + \xi x)^\alpha]\mathrm{d}\xi = x^\alpha - \frac{x^{\alpha+1} - \mu^{\alpha+1}}{(\alpha+1)(x-\mu)}, \\
\mathfrak{I}_1(\alpha) &= \int_0^1 [((1-\xi)\nu + \xi x)^\alpha - x^\alpha]\mathrm{d}\xi = \left(\frac{x^{\alpha+1} - \nu^{\alpha+1}}{(\alpha+1)(\nu - x)}\right) - x^\alpha.
\end{aligned} \tag{43}$$

Therefore, inequality (36) follows from (38)–(42). □

Theorem 6. *Let $\mu, \nu > 0$ with $\mu < \nu$, $\alpha \in (0,1)$, $p, q > 1$ such that $(1/p) + (1/q) = 1$, and $\chi\colon [\mu,\nu] \longrightarrow \mathbb{R}$ be an α–differentiable function on (μ,ν). Then, the inequality*

$$\begin{aligned}
&\left|\frac{(\nu^\alpha - x^\alpha)e^{\chi(\nu)} + (x^\alpha - \nu^\alpha)e^{\chi(\mu)}}{\nu^\alpha - \mu^\alpha} - \frac{\alpha}{\nu^\alpha - \mu^\alpha}\int_\mu^\nu e^{\chi(x)}\mathrm{d}_\alpha x\right| \\
&\le \frac{x-\mu}{\nu^\alpha - \mu^\alpha}\left[\psi_1(\alpha)e^{\chi(Q_1(\alpha)/\psi_1(\alpha))}\chi'\left(\frac{Q_1(\alpha)}{\psi_1(\alpha)}\right)\right] + \frac{\nu - x}{\nu^\alpha - a^\alpha}\left[\vartheta_1(\alpha)e^{\chi(Q_2(\alpha)/\vartheta_1(\alpha))}\chi'\left(\frac{Q_2(\alpha)}{\vartheta_1(\alpha)}\right)\right],
\end{aligned} \tag{44}$$

holds if $D_\alpha(e^{\chi}) \in L_\alpha([\mu,\nu])$ and $|(e^{\chi})'|^q$ is concave on $[\mu,\nu]$, where

$$\begin{aligned}
\psi_1(\alpha) &= \frac{x^\alpha - \mu^\alpha}{2}, \\
\vartheta_1(\alpha) &= \frac{\nu^{\alpha-1}(x + 2\nu) + x^{\alpha-1}(\nu - 4x)}{6}, \\
Q_1(\alpha) &= \frac{(x^\alpha - \mu^\alpha)(2\mu + x)}{6}, \\
Q_2(\alpha) &= \frac{3(\nu^{\alpha+1} + x^{\alpha+1}) + (\nu + x)\{\nu^{\alpha-1}x + x^{\alpha-1}\nu\} - 6x^\alpha(\nu + 2x)}{12}.
\end{aligned} \tag{45}$$

Proof. We clearly see that $|(e^{\chi})'|$ is concave due to $|(e^{\chi})'|^{q}$ is concave for $q>1$. From Definition 3, Lemma 1, the concavity of $|(e^{\chi})'|$, and Jensen's integral inequality, we get

$$\begin{aligned}&\left|\frac{(\nu^{\alpha}-x^{\alpha})e^{\chi(\nu)}+(x^{\alpha}-\nu^{\alpha})e^{\chi(\mu)}}{\nu^{\alpha}-\mu^{\alpha}}-\frac{\alpha}{\nu^{\alpha}-\mu^{\alpha}}\int_{\mu}^{\nu}e^{\chi(x)}\mathrm{d}_{\alpha}x\right|\\&\quad\leq\frac{x-\mu}{\nu^{\alpha}-\mu^{\alpha}}\int_{0}^{1}\left[x^{\alpha}-((1-\xi)\mu^{\alpha}+\xi x^{\alpha})\right]\left|e^{\chi((1-\xi)\mu+\xi x)}\chi'((1-\xi)\mu+\xi x)\right|\mathrm{d}\xi\\&\qquad+\frac{\nu-x}{\nu^{\alpha}-\mu^{\alpha}}\int_{0}^{1}\left[\left((1-\xi)\nu^{\alpha-1}+\xi x^{\alpha-1}\right)((1-\xi)\nu+\xi x)-x^{\alpha}\right]\left|e^{\chi((1-\xi)\nu+\xi x)}\chi'((1-\xi)\nu+\xi x)\right|\mathrm{d}\xi,\end{aligned}\tag{46}$$

$$\begin{aligned}&\int_{0}^{1}[[x^{\alpha}-((1-\xi)\mu+\xi x)^{\alpha}]]\left|e^{\chi((1-\xi)\mu+\xi x)}\chi'((1-\xi)\mu+\xi x)\right|\mathrm{d}\xi\\&\quad\leq\left(\int_{0}^{1}[x^{\alpha}-((1-\xi)\mu^{\alpha}+\xi x^{\alpha})]\mathrm{d}\xi\right)\left|e^{\chi\left(\int_{0}^{1}[x^{\alpha}-((1-\xi)\mu^{\alpha}+\xi x^{\alpha})]((1-\xi)\mu+\xi x)\mathrm{d}\xi/\int_{0}^{1}[x^{\alpha}-((1-\xi)\mu^{\alpha}+\xi x^{\alpha})]\mathrm{d}\xi\right)}\right.\\&\qquad\left.\times\chi'\left(\frac{\int_{0}^{1}[x^{\alpha}-((1-\xi)\mu^{\alpha}+\xi x^{\alpha})]((1-\xi)\mu+\xi x)\mathrm{d}\xi}{\int_{0}^{1}[x^{\alpha}-((1-\xi)\mu^{\alpha}+\xi x^{\alpha})]\mathrm{d}\xi}\right)\right|\\&\quad=\psi_{1}(\alpha)e^{\chi\left(Q_{1}(\alpha)/\psi_{1}(\alpha)\right)}\chi'\left(\frac{Q_{1}(\alpha)}{\psi_{1}(\alpha)}\right),\end{aligned}\tag{47}$$

$$\begin{aligned}&\int_{0}^{1}\left[\left((1-\xi)\nu^{\alpha-1}+\xi x^{\alpha-1}\right)((1-\xi)\nu+\xi x)-x^{\alpha}\right]\left|e^{\chi((1-\xi)\nu+\xi x)}\chi'((1-\xi)\nu+\xi x)\right|\mathrm{d}\xi\\&\quad\leq\left(\int_{0}^{1}\left[\left((1-\xi)\nu^{\alpha-1}+\xi x^{\alpha-1}\right)((1-\xi)\nu+\xi x)-x^{\alpha}\right]\mathrm{d}\xi\right)\\&\qquad\times\left|e^{\chi}\left(\frac{\int_{0}^{1}\left[\left((1-\xi)\nu^{\alpha-1}+\xi x^{\alpha-1}\right)((1-\xi)\nu+\xi x)-x^{\alpha}\right]((1-\xi)\nu+x\xi)\mathrm{d}\xi}{\int_{0}^{1}\left[\left((1-\xi)\nu^{\alpha-1}+\xi x^{\alpha-1}\right)((1-\xi)\nu+\xi x)-x^{\alpha}\right]\mathrm{d}\xi}\right)\right.\\&\qquad\left.\times\chi'\left(\frac{\int_{0}^{1}\left[\left((1-\xi)\nu^{\alpha-1}+\xi x^{\alpha-1}\right)((1-\xi)\nu+\xi x)-x^{\alpha}\right]((1-\xi)\nu+x\xi)\mathrm{d}\xi}{\int_{0}^{1}\left[\left((1-\xi)\nu^{\alpha-1}+\xi x^{\alpha-1}\right)^{\alpha}((1-\xi)\nu+\xi x)-x^{\alpha}\right]\mathrm{d}\xi}\right)\right|\\&\quad=\vartheta_{1}(\alpha)e^{\chi\left(Q_{2}(\alpha)/\vartheta_{1}(\alpha)\right)}\chi'\left(\frac{Q_{2}(\alpha)}{\vartheta_{1}(\alpha)}\right),\end{aligned}\tag{48}$$

where we have used the identities

$$\begin{aligned}
\psi_1(\alpha) &= \int_0^1 [x^\alpha - ((1-\xi)\mu^\alpha + \xi x^\alpha)]\mathrm{d}\xi = \frac{x^\alpha - \mu^\alpha}{2},\\
\vartheta_1(\alpha) &= \int_0^1 \left[\left((1-\xi)\nu^{\alpha-1} + \xi x^{\alpha-1}\right)((1-\xi)\nu + \xi x) - x^\alpha\right]\mathrm{d}\xi\\
&= \frac{\nu^{\alpha-1}(x+2\nu) + x^{\alpha-1}(\nu - 4x)}{6},\\
Q_1(\alpha) &= \int_0^1 [x^\alpha - ((1-\xi)\mu^\alpha + \xi x^\alpha)]((1-\xi)\mu + \xi x)\mathrm{d}\xi\\
&= \frac{(x^\alpha - \mu^\alpha)(2\mu + x)}{6},\\
Q_2(\alpha) &= \int_0^1 \left[\left((1-\xi)\nu^{\alpha-1} + t\xi x^{\alpha-1}\right)((1-\xi)\nu + \xi x) - x^\alpha\right]((1-\xi)\nu + x\xi)\mathrm{d}\xi\\
&= \frac{3\left(\nu^{\alpha+1} + x^{\alpha+1}\right) + (\nu + x)\{\nu^{\alpha-1}x + x^{\alpha-1}\nu\} - 6x^\alpha(\nu + 2x)}{12}.
\end{aligned} \tag{49}$$

Therefore, inequality (44) follows from (46)–(48). □

3. Applications to Special Means

It is very important to give applications in terms of efficiency and usefulness of the obtained results. At the same time, the accuracy of the findings will be confirmed by the applications.

A bivariate function $\Omega\colon (0,\infty)\times(0,\infty) \longrightarrow (0,\infty)$ is said to be a mean if $\min\{a,b\} \le \Omega(a,b) \le \max\{a,b\}$ for all $a,b \in (0,\infty)$. Recently, the bivariate means have been the subject of intensive research [63–75]; in particular, many remarkable inequalities and properties for the bivariate means and their related special functions can be found in the literature [76–85].

In this section, we use the results obtained in Section 2 to derive some inequalities for the weighted arithmetic mean

$$A(\mu,\nu;w_1,w_2) = \frac{w_1\mu + w_2\nu}{w_1 + w_2}, \quad \mu,\nu > 0 \tag{50}$$

and (α,s)-th generalized logarithmic mean

$$L_{(\alpha,s)}(\mu,\nu) = \left(\frac{\alpha\left(\nu^{\alpha+s} - \mu^{\alpha+s}\right)}{(\alpha+s)\left(\nu^\alpha - \mu^\alpha\right)}\right)^{1/s} \tag{51}$$

for $\mu,\nu > 0$ with $\mu \neq \nu$, $s \in \mathbb{R}\backslash\{0\}$ and $\alpha \in [-1,0)\cup(0,1]$.

Let $\chi(x) = x\log x$. Then, Theorems 2–4 and (50) and (51) lead to Theorems 7–9 immediately.

Theorem 7. *Let $\mu,\nu > 0$ with $\mu < \nu$. Then, the inequality*

$$\begin{aligned}
&\left|(\nu^\alpha - \mu^\alpha)A(\mu^s,\nu^s;(x^\alpha - \mu^\alpha),(\nu^\alpha - x^\alpha)) - L^s_{\alpha,s}(\mu,\nu)\right|\\
&\le \frac{s(x-\mu)}{\nu^\alpha - \mu^\alpha}\left\{3|\mu|^{s-1} + |x|^{s-1} + \left|\mu^s x^{-1}\right| + \left|x^s\mu^{-1}\right|\right\}\\
&\quad + \frac{s(\nu - x)}{\nu^\alpha - \mu^\alpha}\left\{\left\{12\nu^\alpha + 3\left[x\nu^{\alpha-1} + \nu x^{\alpha-1}\right] - 18x^\alpha\right\}|\nu|^{s-1} + \left\{2\nu^\alpha + 3\left[x\nu^{\alpha-1} + \nu x^{\alpha-1}\right] - 8x^\alpha\right\}|x|^{s-1} + \left\{3\nu^\alpha + 2\left[x\nu^{\alpha-1} + \nu x^{\alpha-1}\right]\right.\right.\\
&\quad \left.\left. - 7x^\alpha\right\}\left[\left|\nu^s x^{-1}\right| + \left|\nu^{-1}x^s\right|\right]\right\},
\end{aligned} \tag{52}$$

holds for all $\alpha \in (0,1]$, $s>1$, and $x \in [\mu,\nu]$.

Theorem 8. *Let $\mu,\nu>0$ with $\mu<\nu$. Then, the inequality*

$$\begin{aligned}
&\left|(\nu^\alpha-\mu^\alpha)A\left(\mu^s,\nu^s;(x^\alpha-\mu^\alpha),(\nu^\alpha-x^\alpha)\right)-L^s_{\alpha,s}(\mu,\nu)\right|\\
&\leq \frac{s(x-\mu)}{\nu^\alpha-\mu^\alpha}\left[(\varphi_1(\alpha,p))^{1/p}\left\{\frac{2\left|\mu^{q(s-1)}\right|+|x|^{q(s-1)}+\left|\mu^{sq}x^{q(s-1)}\right|+\left|\mu^{s(q-1)}x^{qs}\right|}{6}\right\}^{1/q}\right]\\
&\quad+\frac{s(\nu-x)}{\nu^\alpha-\mu^\alpha}\left[(\varphi_2(\alpha,p))^{1/p}\left\{\frac{2\left|\nu^{q(s-1)}\right|+|x|^{q(s-1)}+\left|\nu^{sq}x^{q(s-1)}\right|+\left|\nu^{s(q-1)}x^{qs}\right|}{6}\right\}^{1/q}\right],
\end{aligned} \tag{53}$$

holds for all $\alpha \in (0,1]$, $s>1$ and $x \in [\mu,\nu]$.

Theorem 9. *Let $\mu,\nu>0$ with $\mu<\nu$. Then, the inequality*

$$\begin{aligned}
&\left|(\nu^\alpha-\mu^\alpha)A\left(\mu^s,\nu^s;(x^\alpha-\mu^\alpha),(\nu^\alpha-x^\alpha)\right)-L^s_{\alpha,s}(\mu,\nu)\right|\\
&\leq \frac{s(x-\mu)}{\nu^\alpha-\mu^\alpha}\left[(\psi_1(\alpha,p))^{1-(1/q)}\left\{\psi_2(\alpha)\left|\mu^{q(s-1)}\right|+\psi_3(\alpha)|x|^{q(s-1)}+\psi_4(\alpha)\left\{\left|\mu^{sq}x^{q(s-1)}\right|+\left|\mu^{s(q-1)}x^{qs}\right|\right\}\right\}^{1/q}\right]\\
&\quad+\frac{s(\nu-x)}{\nu^\alpha-\mu^\alpha}\left[(\vartheta_1(\alpha,p))^{1-(1/q)}\left\{\vartheta_2(\alpha)\left|\nu^{q(s-1)}\right|+\vartheta_3(\alpha)|x|^{q(s-1)}+\vartheta_4(\alpha)\left\{\left|\nu^{sq}x^{q(s-1)}\right|+\left|\nu^{s(q-1)}x^{qs}\right|\right\}\right\}^{1/q}\right]
\end{aligned} \tag{54}$$

holds for all $\alpha \in (0,1]$, $s>1$, and $x \in [\mu,\nu]$, where $\psi_i(\alpha)$ and $\vartheta_i(\alpha)$ for $i=1,2,3$ are defined in Theorem 4.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

This work was supported by the Natural Science Foundation of China (Grant nos. 11971142, 61673169, 11871202, 11701176, 11626101, and 11601485) and the Natural Science Foundation of Zhejiang Province (Grant no. LY19A010012).

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