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Research Article

Existence Results for a Class of the Quasilinear Elliptic Equations with the Logarithmic Nonlinearity

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In this paper, the nonlinear quasilinear elliptic problem with the logarithmic nonlinearity $-\text{div}\,(|\nabla u|^{p-2}\nabla u) = a(x)\varphi_p(u)\log|u| + h(x)\psi_p(u)$ in $\Omega \subset R^n$ was studied. By means of a double perturbation argument and Nehari manifold, the authors obtain the existence results.

1. Introduction

In this paper, we consider the existence of solution to the following problem

$$-\text{div}\,\left(|\nabla u|^{p-2}\nabla u\right) = a(x)\varphi_p(u)\log|u| + h(x)\psi_p(u), \text{ in } \Omega, \tag{1}$$

where $\Omega \subset R^n$, $\varphi_p(z) = |z|^{p-2}z$, $\psi_p(z) = |z|^{p-1}z$, $p > 2$, and $n \geq 1$. We always suppose that $a(x)$ is a sign-changing function; $h(x) \geq 0$ is a $\in C^1$ function.

Equations of the above form are mathematical models occurring in studies of the p-Laplace equation, generalized reaction-diffusion theory [1], non-Newtonian fluid theory [2, 3], non-Newtonian filtration theory [4, 5], and the turbulent flow of a gas in porous medium [6]. In the non-Newtonian fluid theory, the pair p is a characteristic quantity of the medium. Media with $p > 2$ are called dilatant fluids, and those with $p < 2$ are called pseudoplastics. If $p = 2$, they are Newtonian fluids. When $p \neq 2$, the problem becomes more complicated since certain nice properties inherent to the case $p = 2$ seem to be lose or at least difficult to verify. The main differences between $p = 2$ and $p \neq 2$ can be founded in [7, 8].

In recent years, logarithmic nonlinearity is widely used in pseudo-parabolic equations which describe the mathematical and physical phenomena. Equations of the type (1) have been studied by many authors when $p = 2$ (see, for example, [9–12] and the reference therein). To do so, the authors always use the nice properties of Δ, such that, maximum principle and comparison principle and so on. Meanwhile, existence and structure of solutions for such equations with $p > 1$ in bounded domains have also attracted much interest (see [13, 14]).

In the following discussion, we consider two different situations. Firstly, we consider the existence of positive solution for problem (1) with Neumann boundary conditions. In this case, suppose that $\Omega = B_R = B_R(0) \subset R^n$, $a(x) > 0$, $h(x) \geq 0$ are also radial functions, $a(x) = a(|x|)$, $h(x) = h(|x|)$ in B_R. Our strategy in the study of problem (1) is to adopt a double perturbation argument. First, following [15, 16] (see also [17]), for each $0 < \varepsilon < 1$, we consider a family of approximate problems

$$\begin{cases} -\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = a(x)\varphi_p(u)\log\left(\dfrac{u^2+\varepsilon u+\varepsilon}{u+\varepsilon}\right) + h(x)\psi_p(u) & \text{in } B_R, \\ u > 0, & \text{in } B_R, \\ \partial_\nu u = 0, & \text{on } \partial B_R. \end{cases} \tag{2}$$

Then, it is natural to look for a family of solutions of (2) and then to pass the limit as $\varepsilon \to 0$ to obtain a solution to (1).

For each $0 < r < R$, define $A_{rR} := B_R \setminus \bar{B}_r$. Consider the second family of problems

$$\begin{cases} -\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = a(x)\varphi_p(u)\log\left(\dfrac{u^2+\varepsilon u+\varepsilon}{u+\varepsilon}\right) + h(x)\psi_p(u), & \text{in } A_{rR}, \\ u > 0, & \text{in } A_{rR}, \\ u = \theta, & \text{on } \partial \mathrm{B}_r, \\ \partial_\nu u = 0, & \text{on } \partial \mathrm{B}_R. \end{cases} \tag{3}$$

Here, $\theta > 0$ is an appropriate constant. When $r \to 0^+$, we get a solution to (2). The role of problem (3) is that we cannot use Poincare inequality to solve (2) directly by variational methods.

Secondly, we consider the multiple solutions for problem (1) with Dirichlet boundary conditions. In this case, we consider $a(x)$ is a sign-changing function, $h(x) = 0$. The method is based on Nehari manifold and logarithmic Sobolev inequality.

By modification of the methods given in [18–22], we obtain the following results.

Theorem 1. *Let $a(x) > 0, h(x) \geq 0$ be any radial C^1 function. Then, problem (1) has a positive radial solution $u \in C^1(\bar{B}_R \setminus \{0\}) \cap C(\bar{B}_R)$.*

Remark 2. Theorem 1 is valid even if we change the logarithm by a more general singular function. In fact, suppose $g : (0, 1) \to R$ is a smooth function such that

$$\begin{aligned} &\lim_{s\to 0^+} g(s) = -\infty, \\ &\lim_{s\to 0^+} \frac{g(s)}{s^m} = 1, \end{aligned} \tag{4}$$

for some $m \in (0, 1)$. Then, we can perturb g by a family g_ε of smooth functions decreasing in ε, such that $g_\varepsilon(0) = 0$ and $g_\varepsilon(s) \to g(s)$ pointwise in $s \in (0,\infty)$ as $\varepsilon \to 0$. This perturbation can be done in such a way that $g_{\varepsilon 0} \geq 0$ for some $\varepsilon_n > 0$, and then, all the results in Section 2 hold with little modification.

Theorem 3. *Let $h(x) = 0, a(x) \in C(\bar{\Omega})$ and changes sign in $\bar{\Omega}$, satisfying*

$$\max_{\bar{\Omega}} |a(x)| \leq \frac{1}{\mu}, \tag{5}$$

where $\mu = (nL_p/pe)\exp\left((mp^2|\Omega|_n)/ne\right)$, $|\Omega|_n$ is the volume of Ω in R^n. Then, (1) possesses at least two nontrivial solutions.

The paper is organized as follows. In Section 2, we construct a sub- and a supersolution for 3 and finish the proof of Theorem 1. In Section 3, we prove Theorem 3 by the method of Nehari manifold and logarithmic Sobolev inequality.

2. Proof of Theorem 1

2.1. Sub- and Supersolution for 3

Lemma 4. *Suppose that $\theta > 1$. Then, the function $u \equiv 1$ is a subsolution for 3 which does not depend on $0 < \varepsilon \leq 1$ and θ.*

Proof. We just need to see that, since $a(x) > 0$, $h(x) \geq 0$ in B_R, the following inequality holds independently of $0 < \varepsilon \leq 1$ and $\theta > 1$:

$$a(x)\log\left(\frac{1+\varepsilon 1+\varepsilon}{1+\varepsilon}\right) + h(x) \geq \log 1 = 0, \tag{6}$$

We proceed to find a supersolution for 3. Denote by X_r, the following subspace of $H^1(A_{rR})$:

$$X_r := \left\{u \in H^1\left(\mathrm{A}_{rR}\right) | u = 0 \text{ on } \partial \mathrm{B}_r\right\}. \tag{7}$$

For $v \in X_r$, we define the expression:

$$|v|_r := \left(\int_{A_{rR}} |\nabla v|^2 dx\right)^{1/2}. \tag{8}$$

Remark. The expression $|\cdot|_r$ defines a norm on X_r, and $(X_r, |\cdot|_r)$ is a reflexive Banach space. Furthermore, by ([23], (7.44)), the Poincare inequality holds on X_r, that is, there exists $\eta > 0$ such that

$$\int_{A_{rR}} v^p dx \leq \eta \int_{A_{rR}} |\nabla v|^p dx. \tag{9}$$

Next, we work with the radial formulation for $E_{\varepsilon,r}$ in the specific case that $\varepsilon = 1$,

$$\begin{cases} -\left(s^{n-1} o_p\left(u'\right)\right)' = s^{n-1} a(s) o_p(u)\log\left(\dfrac{u^2+u+1}{u+1}\right) + s^{n-1} h(s)\psi_p(u), & \text{in } r < s < R, \\ u > 0, & \text{in } r < s < R, \\ u(r) = \theta, & u'(R) = 0, \end{cases} \tag{10}$$

where $\phi_p(s) = |s|^{p-2}s$. Notice that

$$\log\left(\frac{u^2+u+1}{u+1}\right) \geq 0 \text{ for } u \geq 0. \tag{11}$$

For simplicity, denote

$$f(s,z) = a(s)o_p(z)\log\left(\frac{z^2+z+1}{z+1}\right) + h(s)\psi_p(z). \tag{12}$$

Then, if v solves

$$\begin{cases} -\left(s^{n-1}o_p\left(v'\right)\right)' = s^{n-1}f(s, v+\theta), & \text{in } r < s < R, \\ v > 0, & \text{in } r < s < R, \\ v(r) = \theta, & v'(R) = 0, \end{cases} \tag{13}$$

we will have that $v+\theta$ is a solution of Eq. (10). In order to prove existence of such v, we find a minimum of the functional in the sequel. Let $S \subset X_r$ denote the set of symmetric functions with respect to the origin. We define $\Phi : S \longrightarrow R$ by

$$\Phi(v) = \frac{1}{p}\int_r^R s^{n-1}\left|v'\right|^p ds + \int_r^R s^{n-1}F(s, v(s))s^{n-1}ds, \tag{14}$$

where $F(s, v(s)) = \int_0^t f(s, (z+\theta)^+)dz$ and $z^+ := \max\{z, 0\}$.

Lemma 5. *The functional Φ is C^1, weakly lower semicontinuous and coercive so that there exist $v \in X_r$ such that*

$$\Phi(v) = \min_{u\in X_r}\Phi(u) \text{ and } \Phi'(v) \equiv 0. \tag{15}$$

The proof is standard by (9). Also, since v is a weak solution of (13), we have

$$v(s) = \int_r^s o_p^{-1}\left[t^{1-n}\int_t^R z^{n-1}f(z, v(z)+\theta)dz\right]dt, \tag{16}$$

in which

$$o_p^{-1}(u) = \begin{cases} u^{1/(p-1)}, & \text{if } u \geq 0, \\ -(-u)^{1/(p-1)}, & \text{if } u < 0. \end{cases} \tag{17}$$

Then, we define

$$u_r := v + 0. \tag{18}$$

Lemma 6. *Suppose that $\theta > 1$. Then, the function $\bar{u} \equiv u_r$ is a supersolution for (3) which does not depend on $0 < \varepsilon \leq 1$.*

Lemma 7. *There exists a constant $M > 0$ such that $|u_r|_\infty \leq M$ and the constant M does not depend on $r \in (0, R)$. Moreover, for each $\rho \in (0, R)$, there exist a constant C_ρ and $r_\rho \in (0, R)$ such that we have the following estimates:*

$$|u_r|c^0[\rho, R], |u_r|c^1[\rho, R], \left|o_p\left(u_r'\right)\right|c^1[\rho, R] \leq C\rho. \tag{19}$$

Proof or Lemmas 6 and 7 can be found in [18], we omit them here.

2.2. Existence of Solution for 3. In this section, we use the sub- and supersolution from Section 2.1 (u and u_r, respectively) to obtain a solution for the problem 3. Define the function

$$g_\varepsilon(s,u) := s^{n-1}a(s)o_p(u)\log\left(\frac{u^2+\varepsilon u+\varepsilon}{u+\varepsilon}\right) + s^{n-1}h(s)\psi_p(u) + bu, \quad s \in [r, R], u \geq 0, \tag{20}$$

where we choose b in such a way that the function $u \longrightarrow g_\varepsilon(s, u)$ is increasing in u for all $s \in [r, R]$. Now, starting with $u_0 = u$, we define a sequence u_n such that each u_n satisfies

$$\begin{cases} -\left(s^{n-1}o_p\left(u_{n+1}'\right)\right)' + bu_{n+1} = g_\varepsilon(s, u_n), & \text{in } r < s < R, \\ u_{n+1} > 0, & \text{in } r < s < R, \\ u_{n+1}(r) = \theta, & u_{n+1}'(R) = 0. \end{cases} \tag{21}$$

Let us now recall Lemma 2.1 in [24],

Lemma 8 (weak comparison principle). *Let Ω be a bounded domain in $R^N (N \geq 2)$ with smooth boundary $\partial\Omega$ and $\theta : (0, +\infty) \longrightarrow (0, +\infty)$ is continuous and nondecreasing, let $u_1, u_2 \in W^{1,p}(\Omega)$ satisfy*

$$\int_\Omega |\nabla u_1|^{p-2}\nabla u_1 \nabla v dx + \int_\Omega \theta(u_1)v dx \leq \int_\Omega |\nabla u_2|^{p-2}\nabla u_2 \nabla v dx + \int_\Omega \theta(u_2)v dx. \tag{22}$$

For all nonnegative $v \in W_0^{1,m}(\Omega)$. Then, the inequality

$$u_1 \leq u_2, \text{ on } \partial\Omega, \tag{23}$$

implies that

$$u_1 \leq u_2, \text{ in } \Omega. \tag{24}$$

Lemma 9. *The sequence $\{u_n\}$ is nondecreasing and satisfies $u_0(s) \leq u_n(s) \leq u_{n+1}(s) \leq u_r(s)$ for all $s \in [r, R]$ and all $n \in N$.*

Proof. We just need to see that $u_0 \le u_1 \le u_r$ and the general case follows by induction in an analogous way. We have

$$\begin{cases} -\left(s^{n-1} o_p\left(u_0'\right)\right)' + b(u_0) \le -\left(s^{n-1} o_p\left(u_1'\right)\right)' + b(u_1), & \text{in } r < s < R, \\ (u_0 - u_1)(r) \le 0, \quad (u_0 - u_1)'(R) = 0. \end{cases} \tag{25}$$

So, we can apply Lemma 8 and obtain that $u_0 \le u_1$ in $[r, R]$. On the other hand,

$$\begin{cases} -\left(s^{n-1} o_p(u_1)'\right)' + b(u_1) \le -\left(s^{n-1} o_p(u_r)'\right)' + b(u_r), & \text{in } r < s < R, \\ (u_1 - u_r)(r) \le 0, \quad (u_1 - u_r)'(R) = 0. \end{cases} \tag{26}$$

Again, Lemma 8 implies $u_1 \le u_r$ in $[r, R]$.
By Lemma 9, we define the pointwise limit

$$u_r^\varepsilon(s) := \lim_{n \to \infty} u_n(s), \quad s \in [r, R], \tag{27}$$

and we see that

$$1 \le u_r^\varepsilon \le u_r(s), \quad s \in [r, R]. \tag{28}$$

The function u_r^ε is in fact a solution of 3.

Lemma 10. *The function u_r^ε is a solution of 3, and it belongs to $C^1[r, R]$.*

Proof. For each $n \in N$, we have

$$u_n(s) = \theta + \int_r^s o_p^{-1}\left[t^{1-n}\int_t^R z^{n-1}\left(a(z) o_p(u_n) \log\left(\frac{u_n^2 + \varepsilon u_{n+\varepsilon}}{u_n + \varepsilon}\right) + h(z) v_p(u_n)\right) dz\right] dt. \tag{29}$$

Since we have

$$1 \le u_n \le M, \quad \text{for all n} \in \mathrm{N}, \tag{30}$$

we obtain, as in Lemma 7, that $|\phi_p(u'_p)| C^1[\rho, R]$ is bounded. Then, for a subsequence that we still denote by u_n, we have the convergence

$$u_n \to u_r^\varepsilon \text{ in } C^1[\rho, R]. \tag{31}$$

2.3. Obtaining a Solution for E_ε. In this section, we pass the limit as $r \to 0^+$ and then obtain a solution for 2.

Lemma 11. *For a fixed $0 < \varepsilon \le 1$, the problem (2) has a solution u^ε which is obtained as the limit of u_r^ε as $r \to 0^+$.*

Proof. For simplicity, we omit the dependence on $\varepsilon > 0$ for u_r^ε. We know that

$$u_r(s) = \theta + \int_r^s o_p^{-1}\left[t^{1-n}\int_t^R z^{n-1}\left(a(z) o_p(u_r) \log\left(\frac{u_r^2 + \varepsilon u_r + \varepsilon}{u_r + \varepsilon}\right) + h(z) v_p(u_r)\right) dz\right] dt. \tag{32}$$

Also, we have

$$\begin{aligned} 1 \le u_r \le M, \quad & \text{in} [r, R], \\ 1 \le u_r \le M, \quad & \text{in} [r, R]. \end{aligned} \tag{33}$$

As in Lemma 7, we can prove, for each $\rho \in (0, R)$, there exist a constant $C_\rho > 0$ and $r_\rho \in (0, R)$ such that we have the following estimates:

$$|u_r| c^0[\rho, R], |u_r| c^1[\rho, R], \left|o_p\left(u_r'\right)\right| c^1[\rho, R] \le C\rho. \tag{34}$$

Then, from the compact imbedding $C^1[\rho, R] \to C^0[\rho, R]$, we see that there exist a sequence r_n and u^ε defined on $(0, R]$ such that, if we define $w_n := u_{r_n}$, then

$$\begin{aligned} w_n \to u^\varepsilon \quad & \text{in } \mathrm{C}^1_{loc}(0, R), \\ w_n \to u^\varepsilon \quad & \text{in } \mathrm{C}^1(\rho, R). \end{aligned} \tag{35}$$

2.4. Concluding the Proof of Theorem 1. Now, we would like to pass the limit in the family u^ε obtained in Section 2.3 and get a solution to (1). In order to do that, we need some estimates like the ones in Lemma 7 independently of ε.

First, we observe that the following estimate holds in $(0, R]$

$$1 \le u^\varepsilon \le M. \tag{36}$$

Notice that the family $(u^\varepsilon)_{0<\varepsilon\le 1}$ satisfies $\varepsilon > 0$ for u_r^ε. We know that

$$\begin{aligned} u^\varepsilon(s) = {} & u^\varepsilon\left(\frac{R}{2}\right) + \int_{R/2}^s o_p^{-1}\left[t^{1-n}\int_t^R z^{n-1}\right. \\ & \cdot \left(a(z) o_p(u^\varepsilon) \log\left(\frac{u^{\varepsilon 2} + \varepsilon u^\varepsilon + \varepsilon}{u^\varepsilon + \varepsilon}\right)\right. \\ & \left.\left. + h(z) v_p(u^\varepsilon)\right) dz\right] dt, \end{aligned} \tag{37}$$

if $s \in [R/2, R]$,and

$$\begin{aligned} u^\varepsilon(s) = {} & u^\varepsilon\left(\frac{R}{2}\right) - \int_s^{R/2} o_p^{-1}\left[t^{1-n}\int_t^R z^{n-1}\right. \\ & \cdot \left(a(z) o_p(u^\varepsilon) \log\left(\frac{u^{\varepsilon 2} + \varepsilon u^\varepsilon + \varepsilon}{u^\varepsilon + \varepsilon}\right)\right. \\ & \left.\left. + h(z) v_p(u^\varepsilon)\right) dz\right] dt, \end{aligned} \tag{38}$$

if $s \in (0, R/2]$.

From Eqs. (36)–(38) we see, as in Lemma 7 that, for each for each $\rho \in (0, R)$, there exist a constant $C_\rho > 0$ and $\varepsilon_\rho \in (0, R)$ such that we have the following estimates:

$$|u^\varepsilon|c^0[\rho, R], |u^\varepsilon|c^1[\rho, R], \left|o_p\left(u^{\varepsilon'}\right)\right|c^1[\rho, R] \le C_p \quad \text{for all } \varepsilon \in \left(0, \varepsilon_\rho\right). \tag{39}$$

Now, arguing as in Section 2.4, we can find a function $\boldsymbol{u}$ which satisfies

$$\begin{cases} -\left(s^{n-1}o_p\left(u'\right)\right)' = s^{n-1}\log u + s^{n-1}h(s)u^q, & \text{in } r < s < R, \\ u > 0, & \text{in } r < s < R, \\ u'(R) = 0. & \end{cases} \tag{40}$$

that is, u is a radial solution for the problem (1).

We see that $u \in C^1(0, R) \cap C(0, R]$. Now, extend continuously u to the whole interval $(0, R]$. Indeed, let r_i be a sequence in $(0, R/2)$ with $r_i \to 0$ as $i \to \infty$. From Eq. (13) (after we have passed the limit in ε)

$$u(r_i) = u\left(\frac{R}{2}\right) - \int_{r_i}^{R/2} o_p^{-1}\left[t^{1-n}\int_t^R z^{n-1}\left(a(z)o_p(u)\log u + h(z)v_p(u)\right)dz\right]dt. \tag{41}$$

Then, if $r_j > r_i$, we get

$$\left|u(r_j) - u(r_i)\right| = \left|\int_{r_i}^{rj} o_p^{-1}\left[t^{1-n}\int_t^R z^{n-1}\left(a(z)o_p(u)\log u + h(z)o_p(u)\right)dz\right]dt\right|. \tag{42}$$

From Eq. (36), we obtain that there exists a constant $C > 0$ such that

$$\left|u\left(r_j\right) - u\left(r_i\right)\right| \le C\left|r_j - r_i\right|, \tag{43}$$

so $u(r_i)$ is a Cauchy sequence in R. Let L be the limit of such sequence. By a similar argument, we conclude that if s_i is another sequence in $(0, R/2)$ converging to 0, then we necessarily have $u(s_i) \to L$. So, we have proved that

$$\lim_{r\to 0} u(r) = L, \tag{44}$$

finishing the proof of Theorem 1.

3. Proof of Theorem 3

3.1. Preliminaries. In this section, we consider the multiple solutions for problem (1) with Dirichlet boundary conditions. In this case, we consider $a(x)$ is a sign-changing function, $h(x) = 0$. Moreover, it is necessary to note that the presence of the logarithmic nonlinearity leads to some difficulties in deploying the potential well method. In order to handle this situation, we need the following logarithmic Sobolev inequality which was introduced by [25].

Proposition 12. *Let $p > 1$, $\mu > 0$, and $u \in W^{1,p}(\Omega) \setminus \{0\}$. Then, we have*

$$p\int_{R^n} |u(x)|^p \log\left(\frac{|u(x)|}{\|u\|_{L^p(R^n)}}\right)dx + \frac{n}{p}\log \cdot \left(\frac{p\mu e}{nL_p}\right)\int_{R^n} |u(x)|^p dx \le \mu\int_{R^n} |\nabla u(x)|^p dx, \tag{45}$$

where

$$L_p = \frac{p}{n}\left(\frac{p-1}{e}\right)^{p-1}\pi^{-(p/2)}\left[\frac{\Gamma((n/2)+1)}{\Gamma(n(p-1/p)+1)}\right]^{p/n}. \tag{46}$$

Remark. If $u \in W_0^{1,p}(\Omega)$ then, by defining $u(x) = 0$ for $x \in R^n \setminus \{\Omega\}$, we derive

$$p\int_\Omega |u(x)|^p \log\left(\frac{|u(x)|}{\|u\|_p}\right)dx + \frac{n}{p}\log \cdot \left(\frac{p\mu e}{nL_p}\right)\int_\Omega |u(x)|^p dx \le \mu\int_\Omega |\nabla u(x)|^p dx, \tag{47}$$

for any real number $\mu > 0$.

We start by considering the energy functional J by

$$J(u) = \frac{1}{p}\|\nabla u\|_p^p - \frac{1}{p}\int_\Omega a(x)|u|^p \log|u|dx + \frac{1}{p^2}\int_\Omega a(x)|u|^p dx, \tag{48}$$

in which $\|u\|_p = \|u\|_{L^p}(\Omega)$.

Lemma 13. *For $u \in H_0^1(\Omega)$ and $\int_\Omega a(x)|u|^p dx = 0$, let $M = \max_\Omega |a(x)|$, then it holds*

$$J(u) \ge \left(\frac{1}{p} - \frac{M\mu}{p}\right)\|\nabla u\|_p^p, \tag{49}$$

in which $\mu = (nLp/pe)\exp((mp^2|\Omega|n)/ne)$.

Proof. Using the fact $\int_\Omega a(x)|u|^p dx = 0$, we have

$$J(u) = \frac{1}{p}\|\nabla u\|_p^p - \frac{1}{p}\int_\Omega a(x)|u|^p \log\left(\frac{|u(x)|}{\|u\|_p}\right)dx. \tag{50}$$

Let $\bar{u}(x) = u(x)/\|u\|_p$, then

$$\begin{aligned}\int_\Omega a(x)|u|^p \log\left(\frac{|u(x)|}{\|u\|_p}\right)dx \\ = \int_{\Omega_1} a(x)|u|^p \log|\bar{u}|dx + \int_{\Omega_2} a(x)|u|^p \log|\bar{u}|dx,\end{aligned} \tag{51}$$

where

$$\Omega_1 = \{x \in \Omega, |\bar{u}(x)| < 1\}, \text{and } \Omega_2 = \{x \in \Omega, |\bar{u}(x)| \geq 1\}. \tag{52}$$

By direct calculations, we know

$$\int_{\Omega_1} a(x)|u|^p \log|\bar{u}|dx \leq \frac{M|\Omega|_n}{2e}\|u\|_p^p. \tag{53}$$

Also, by logarithmic Sobolev inequality (47) and (51), we have

$$\begin{aligned}\int_{\Omega_2} a(x)|u|^p \log|\bar{u}|dx &\leq M\left(\int_\Omega |u|^p \log|\bar{u}|dx + \frac{|\Omega|_n}{2e}\|u\|_p^p\right) \\ &\leq M\left[\frac{\mu}{p}\int_\Omega |\nabla u(x)|^p dx - \left(\frac{n}{p^2}\log\left(\frac{p\mu e}{nL_p}\right) - \frac{M|\Omega|_n}{2e}\right)\|u\|_p^p\right].\end{aligned} \tag{54}$$

Then, combining (50), (51), (53), and (54), we have

$$J(u) \geq \left(\frac{1}{p} - \frac{M\mu}{p}\right)\|\nabla u\|_p^p + \left(\frac{n}{p^3}\log\left(\frac{p\mu e}{nL_p}\right) - \frac{|\Omega|_n}{pe}\right)M\|u\|_p^p. \tag{55}$$

Taking $\mu = (nL_p/pe)\exp((mp^2|\Omega|_n)/ne)$ in (55), then

$$\frac{n}{p^3}\log\left(\frac{p\mu e}{nL_p}\right) - \frac{|\Omega|_n}{pe} = 0, \tag{56}$$

we know (49).

Lemma 14. *[19] Let $\{u_m\}$ be a sequence in $W_0^{1,p}(\Omega)$. If $u_m \rightharpoonup u_0$ and $u_m \nrightarrow u_0$ in $W_0^{1,p}(\Omega)$, then*

$$J(u_0) < \underline{\lim}_{m\to\infty} J(u_m). \tag{57}$$

If $u_m \to u_0$ in $W_0^{1,p}(\Omega)$, then

$$J(u_0) = \lim_{m\to\infty} J(u_m). \tag{58}$$

3.2. Multiple Solutions. Inspired by [19], we seek the weak solutions of (E) by Nehari manifold. First, a simple calculation shows that $J(u) \in C^1(W_0^{1,p}(\Omega), R)$, and its derivative is given by

$$\left\langle J'(u), v\right\rangle = \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v dx - \int_\Omega a(x)\varphi_p(u)v \log|u|dx, \tag{59}$$

for all $u, v \in W_0^{1,p}(\Omega)$.

From (49), $J(u)$ is not bounded on $W_0^{1,p}(\Omega)$, but we can prove that $J(u)$ is bounded from below on Nehari manifold

$$N = \left\{u \in W^{1,p}(\Omega) \setminus \{0\}: \left\langle J'(u), u\right\rangle = 0\right\}, \tag{60}$$

where $\langle , \rangle$ denotes the usual duality.

It is clear that all nontrivial critical points of J must lie on N, and as we will see below, local minimizers on N are usually critical points of J. Also, we can see that

$$u \in N \Leftrightarrow \int_\Omega |\nabla u|^p dx - \int_\Omega a(x)|u|^p \log|u|dx = 0. \tag{61}$$

Let $u \in W^{1,p}(R^n) \setminus \{0\}$ and consider the real function $j : \lambda \to J(\lambda u)$ for $\lambda > 0$ defined by

$j(\lambda) \coloneqq J(\lambda u) = \lambda^{\ p}/p\|\nabla u\|_p^p - \lambda^p/p\int_\Omega a(x)|u|^p \log|u|dx - (\lambda^p/p)\log\lambda\int_\Omega a(x)|u|^p dx + (\lambda^p/p^2)\int_\Omega a(x)|u|^p dx$.Such maps are known as fibering maps which were introduced by Drabek and Pohozaev [26].

Then, by direct calculations, we have

$$j'(\lambda) = \frac{\lambda^{p-1}}{p-1}\|\nabla u\|_p^p - \frac{\lambda^{p-1}}{p-1}\int_\Omega a(x)|u|^p \log|\lambda u|dx, \tag{62}$$

$$\begin{aligned}j''(\lambda) = \frac{\lambda^{p-2}}{p-2}\|\nabla u\|_p^p - \frac{\lambda^{p-2}}{p-2}\int_\Omega a(x)|u|^p \log|\lambda u|dx \\ - \frac{\lambda^{p-2}}{p-1}\int_\Omega a(x)|u|^p dx.\end{aligned} \tag{63}$$

Lemma 15. *Let $u \in W^{1,p}(\Omega) \setminus \{0\}$ and $\lambda > 0$. Then, $\lambda u \in N$ if and only if $j'(\lambda) = 0$.*

Proof. First, by direct calculations, we know

$$\begin{aligned}\lambda u \in N \Leftrightarrow \frac{\lambda^{p-1}}{p-1}\left(\|\nabla u\|_p^p - \int_\Omega a(x)|u|^p \log|\lambda u|dx\right) \\ = 0 \Leftrightarrow \lambda j'(\lambda) = 0.\end{aligned} \tag{64}$$

Since $\lambda > 0$, then $\lambda u \in N$ if and only if $j'(\lambda) = 0$.

Then, if $u \in N$, we have $j'(1) = 0$ and $j''(1) = -(1/(p-1))\int_\Omega a(x)|u|^p dx$.

Thus, we can divide N into three subsets N^+, N^-, and N^0, where

$$N^+ = \left\{ u \in N : \int_\Omega a(x)|u|^p dx > 0 \right\},$$
$$N^- = \left\{ u \in N : \int_\Omega a(x)|u|^p dx > 0 \right\}, \tag{65}$$
$$N^0 = \left\{ u \in N : \int_\Omega a(x)|u|^p dx > 0 \right\}.$$

Lemma 16. *If u_0 is a local minimizer for J on N and $u_0 \notin N^0$. Then, $J'(u_0) = 0$.*

Proof. If u_0 is a local minimizer for J on N, by Lagrange multipliers, there exists $\kappa \in R$ such that

$$J'(u_0) = \kappa \chi'(u_0), \tag{66}$$

where $\chi(u) = \|\nabla u\|_p^p - \int_\Omega a(x)|u|^p \log |u| dx$.

Since $u_0 \in N$, then

$$\left\langle J'(u_0), u_0 \right\rangle = 0, \text{and } \kappa \left\langle \chi'(u_0), u_0 \right\rangle = 0. \tag{67}$$

On the other hand, from $u_0 \notin N^0$, we can see

$$\left\langle \chi'(u_0), u_0 \right\rangle = j''_{u_0}(1) = -\frac{\lambda^{p-2}}{p-1} \int_\Omega a(x)|u|^p dx \neq 0. \tag{68}$$

Then, $\kappa = 0$ and $J'(u_0) = 0$.

Proposition 17. *Both N^+ and N^- are nonempty.*

Proof. From (62), $j'(\lambda)$ has a unique turning point at

$$\lambda(u) = \exp\left(\frac{\|\nabla u\|_p^p - \int_\Omega a(x)|u|^p \log |u| dx}{\int_\Omega a(x)|u|^p dx} \right). \tag{69}$$

Since $a(x)$ is sign-changing, then we can take u_1 such that

$$\int_\Omega a(x)|u_1|^p dx < 0, \text{and then } \lambda(u_1)u_1 \in N^+. \tag{70}$$

Also, we can take u_2 such that

$$\int_\Omega a(x)|u_2|^p dx < 0, \text{and then } \lambda(u_2)u_2 \in N^-. \tag{71}$$

Then, both N^+ and N^- are nonempty.

Just like [19], by Lemmas 13–16, we can get the following results.

Lemma 18. *[19] N^+ is bounded; J is bounded below on N^+.*

Lemma 19. *[19] Every minimizing sequence for J on N^- is bounded, $0 \notin \overline{N^-}$, $\inf_{u\in N} - J(u) > 0$.*

Proposition 20. *J has a minimizer on N^+.*

Proof. Let $\{u_m\} \subseteq N^+$ be a minimizing sequence, i.e., $\lim_{m\to\infty} J(u_m) = \inf_{u\in N^+} J(u) < 0$.

By Lemma 18, N^+ is bounded; we may assume that

$$u_m \rightharpoonup u_0, \text{in } W_0^{1,p}(\Omega), \text{and so } u_m \to u_0 \text{ in } L^p(\Omega). \tag{72}$$

Since $\{u_m\} \subseteq N^+$, we can get

$$\begin{aligned} &\int_\Omega a(x)|u_m|^p dx < 0, \int_\Omega a(x)|u_0|^p dx \\ &\quad = \lim_{m\to\infty} \int_\Omega a(x)|u_m|^p dx = \lim_{m\to\infty} p^2 \int_\Omega a(x)|u_m|^p dx < 0, \end{aligned}$$

$$\|\nabla u_m\|_p^p - \int_\Omega a(x)|u_m|^p \log |u_m| dx = 0. \tag{73}$$

Suppose $u_m \nrightarrow u_0$ in $W_0^{1,p}(\Omega)$, then

$$\begin{aligned} &\|\nabla u_0\|_p^p - \int_\Omega a(x)|u_0|^p \log |u_0| dx \\ &\quad < \lim_{m\to\infty} \left(\|\nabla u_m\|_p^p - \int_\Omega a(x)|u_m|^p \log |u_m| dx \right) = 0. \end{aligned} \tag{74}$$

Then, there exists

$$\lambda(u_0) = \exp\left(\frac{\|\nabla u_0\|_p^p - \int_\Omega a(x)|u_0|^p \log |u_0| dx}{\int_\Omega a(x)|u_0|^p dx} \right) > 1, \tag{75}$$

such that $\lambda(u_0)u_0 \in N^+$, and then, J attains minimum at $\lambda(u_0)u_0$.

Hence

$$J(\lambda(u_0)u_0) < J(u_0) \leq \lim_{m\to\infty} J(u_m) = \inf_{u\in N^+} J(u), \tag{76}$$

which is impossible. Hence, $u_m \to u_0$ in $W_0^{1,p}(\Omega)$, $u_0 \in N^+$, and $J(u_0) = \inf_{u\in N^+} J(u) < 0$, this means that u_0 is a minimizer for J on N^+.

Proposition 21. *There exists a minimizer of J on N^-.*

Proof. Let $\{u_m\} \subseteq N^-$ be a minimizing sequence. By Lemma 19, $\{u_m\}$ is bounded; we may assume that

$$u_m \rightharpoonup u_0, \text{in } W_0^{1,p}(\Omega), \text{and so } u_m \to u_0 \text{ in } L^p(\Omega). \tag{77}$$

Since $J(u_m) = 1/p^2 \int_\Omega a(x)|u_m|^p dx$, by Lemma 19, we can get

$$\int_\Omega a(x)|u_0|^p dx = \lim_{m\to\infty} \int_\Omega a(x)|u_m|^p dx > 0. \tag{78}$$

Suppose $u_m \nrightarrow u_0$ in $W_0^{1,p}(\Omega)$, then

$$\|\nabla u_0\|_p^p - \int_\Omega a(x)|u_0|^p \log|u_0| dx < \lim_{m\to\infty}\left(\|\nabla u_m\|_p^p - \int_\Omega a(x)|u_m|^p \log|u_m| dx\right) = 0. \quad (79)$$

Then, there exists

$$\lambda(u_0) = \exp\left(\frac{\|\nabla u_0\|_p^p - \int_\Omega a(x)|u_0|^p \log|u_0| dx}{\int_\Omega a(x)|u_0|^p dx}\right) < 1, \quad (80)$$

such that $\lambda(u_0)u_0 \in N^-$, $\lambda(u_0)u_m \rightharpoonup \lambda(u_0)u_0$, but $\lambda(u_0)u_m \nrightarrow \lambda(u_0)u_0$ in $W_0^{1,p}(\Omega)$.

Hence

$$J(\lambda(u_0)u_0) < \lim_{m\to\infty} J(\lambda(u_0)u_m). \quad (81)$$

Since the map $\lambda \to J(\lambda u_m)$ attains its maximum at $t = 1$,

$$\lim_{m\to\infty} J(\lambda(u_0)u_m) \le \lim_{m\to\infty} J(u_m) = \inf_{u\in N^-} J(u). \quad (82)$$

This means $J(\lambda(u_0)u_0) < \inf_{u\in N^-} J(u)$ is impossible.

Hence, $u_m \to u_0$ in $W_0^{1,p}(\Omega)$, and this means that u_0 is a minimizer for J on N^-.

Proof of Theorem 3. Propositions 20 and 21 show that the energy functional J has two minimizers u_1 on N^+ and u_2 on N^-. Next, by Lemma 16, J has two critical points u_1 and u_2 on $W_0^{1,p}(\Omega)$, which means that the problem (1) has at least two nontrivial solutions under the condition $h(x) = 0$.

Data Availability

All the data in our manuscript are available.

Conflicts of Interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

Authors' Contributions

All authors carried out the proof and conceived of the study. All authors read and approved the final manuscript.

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