Research Article

Employing Locally Finitely $\mathcal{T}$ -Transitive Binary Relations to Prove Coincidence Theorems for Nonlinear Contractions

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In this article, we prove some relation-theoretic results on coincidence and common fixed point for a nonlinear contraction employing a locally finitely $\mathcal{T}$ -transitive binary relation, where $\mathcal{T}$ stands for a self-mapping on the underlying metric space. Our newly proved results deduce sharpened versions of certain relevant results of the existing literature. Finally, we adopt some examples to substantiate the genuineness of our proved results herein.

1. Introduction

The study of order-theoretic metrical fixed point theory was initiated in 1986 by Turinici [1]. After two decades, Ran Reurings [2] extended the classical Banach contraction principle employing partial ordering, which is indeed a natural variant of Turinici's results (cf. [1]). Thereafter, Neito and Rodríguez-López [3] sharpened the fixed point theorem of Ran Reurings [2]. In this continuation, several researchers generalized and extended the fixed point theorem of Neito and Rodríguez-López [3] (see references [4–14]). In 2015, Alam and Imdad [15] proved relatively more natural results of Neito and Rodríguez-López [3] using an amorphous (arbitrary) binary relation. Most recently, Alam et al. [16] obtained a variant of fixed point theorem of Alam and Imdad [15] under nonlinear contractions. As noticed in [16], results proved under nonlinear contractions cannot be extended to the amorphous binary relation. Often, proving the fixed point result under nonlinear contractions, the underlying binary relation requires transitivity. In order to weaken the transitivity condition, Alam et al. [16] utilized the locally finitely $\mathcal{T}$ -transitive binary relation.

The aim of this paper is to extend the results of Alam et al. [16] to a pair of self-mappings defined on a relational

metric space. The locally finitely $\mathcal{T}$ -transitive binary relation is employed, which covers previously known results for nonlinear contractions. Finally, we furnish some illustrative examples to demonstrate the worth of our newly proved results.

2. Preliminaries

In this sequel, firstly we recall some known relevant definitions.

Definition 1 (see [17, 18]). Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be self-mappings on a nonempty set $\mathcal{M}$, then

- (i) If $\mathcal{T}_2(u) = \mathcal{T}_1(u)$ for some u in $\mathcal{M}$, then u is called a coincidence point of $\mathcal{T}_1$ and $\mathcal{T}_2$
- (ii) If $u \in \mathcal{M}$ is a coincidence point of $\mathcal{T}_1$ and $\mathcal{T}_2$ and $\bar{u} \in \mathcal{M}$ such that $\bar{u} = \mathcal{T}_2(u) = \mathcal{T}_1(u)$, then $\bar{u}$ is called a point of coincidence of $\mathcal{T}_1$ and $\mathcal{T}_2$
- (iii) If $u \in \mathcal{M}$ is a coincidence point of $\mathcal{T}_1$ and $\mathcal{T}_2$ such that $u = \mathcal{T}_2 u = \mathcal{T}_1 u$, then u is called a common fixed point of $\mathcal{T}_1$ and $\mathcal{T}_2$
- (iv) If $\mathcal{T}_2(\mathcal{T}_1 u) = \mathcal{T}_1(\mathcal{T}_2 u)$ for all u in $\mathcal{M}$, then $\mathcal{T}_1$ and $\mathcal{T}_2$ are called commuting

- (v) If $\mathcal{T}_1$ and $\mathcal{T}_2$ commute at their coincidence points, i.e., for any $u \in \mathcal{M}$, $\mathcal{T}_2 u = \mathcal{T}_1 u \implies \mathcal{T}_2(\mathcal{T}_1 u) = \mathcal{T}_1(\mathcal{T}_2 u)$, then $\mathcal{T}_1$ and $\mathcal{T}_2$ are called weakly compatible.

Definition 2 (see [19–21]). Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be self-mappings on a nonempty set $\mathcal{M}$, then

- (i) If $d(\mathcal{T}_2(\mathcal{T}_1 u), \mathcal{T}_1(\mathcal{T}_2 u)) \leq d(\mathcal{T}_2 u, \mathcal{T}_1 u)$ for all $u \in \mathcal{M}$, then $\mathcal{T}_1$ and $\mathcal{T}_2$ are called weakly commuting.
- (ii) If $\lim_{n \rightarrow \infty} d(\mathcal{T}_2(\mathcal{T}_1 u_n), \mathcal{T}_1(\mathcal{T}_2 u_n)) = 0$, with $\{u_n\}$ being a sequence in $\mathcal{M}$ such that $\lim_{n \rightarrow \infty} \mathcal{T}_2 u_n = \lim_{n \rightarrow \infty} \mathcal{T}_1 u_n$, then $\mathcal{T}_1$ and $\mathcal{T}_2$ are called compatible.
- (iii) If for all sequences $\{u_n\} \subset \mathcal{M}$, with $\mathcal{T}_2 u_n \xrightarrow{d} \mathcal{T}_2 u \implies \mathcal{T}_1 u_n \xrightarrow{d} \mathcal{T}_1 u$, then $\mathcal{T}_1$ is said to be $\mathcal{T}_2$ -continuous at u in $\mathcal{M}$. Moreover, $\mathcal{T}_1$ is called $\mathcal{T}_2$ -continuous if it is $\mathcal{T}_2$ -continuous at each point of $\mathcal{M}$.

3. Relation-Theoretic Notions and Some Auxiliary Results

For a nonempty set $\mathcal{M}$, a subset $\mathcal{R}$ of $\mathcal{M}^2$ is called a binary relation on $\mathcal{M}$. $\mathcal{R}$ is defined as $u\mathcal{R}v$ or $(u, v) \in \mathcal{R}$. Usually, the universal relation and empty relation (void relation), respectively, on $\mathcal{M}$ are denoted by $\mathcal{M}^2$ and $\emptyset$.

Definition 3 (see [22, 23]). A binary relation $\mathcal{R}$ on a nonempty set $\mathcal{M}$ is called

- (i) Reflexive if $(u, u) \in \mathcal{R}$, $\forall u \in \mathcal{M}$
- (ii) Symmetric if whenever $(u, v) \in \mathcal{R}$ and then $(v, u) \in \mathcal{R}$
- (iii) Antisymmetric if whenever $(u, v) \in \mathcal{R}$ and $(v, u) \in \mathcal{R}$ and then $u = v$
- (iv) Transitive if whenever $(u, v) \in \mathcal{R}$ and $(v, z) \in \mathcal{R}$ and then $(u, z) \in \mathcal{R}$
- (v) Complete or connected or dichotomous if $[u, v] \in \mathcal{R}$, $\forall u, v \in \mathcal{M}$
- (vi) Weakly complete or weakly connected or trichotomous if $[u, v] \in \mathcal{R}$ or $u = v$, $\forall u, v \in \mathcal{M}$.

Definition 4 (see [22–27]). Let $\mathcal{R}$ be a binary relation defined on a nonempty set $\mathcal{M}$. Then, $\mathcal{R}$ is said to be

- (i) Amorphous if $\mathcal{R}$ has no specific properties at all
- (ii) Near order if $\mathcal{R}$ is antisymmetric and transitive
- (iii) Pseudo-order if $\mathcal{R}$ is reflexive and antisymmetric
- (iv) Quasiorder or preorder if $\mathcal{R}$ is reflexive and transitive
- (v) Strict order or *sharp order* if $\mathcal{R}$ is irreflexive and transitive
- (vi) Partial order if $\mathcal{R}$ is reflexive, antisymmetric, and transitive.

- (vii) Tolerance if $\mathcal{R}$ is reflexive and symmetric
- (viii) Equivalence if $\mathcal{R}$ is reflexive, symmetric, and transitive

Throughout this paper, $\mathcal{R}$ stands for a nonempty binary relation, but for the sake of simplicity, we often write “binary relation” instead of “nonempty binary relation.” Also, $\mathbb{N}$ stands for the set of natural numbers, while $\mathbb{N}_0$ for the set of whole numbers (i.e., $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$).

Remark 1. Notice that the “universal relation $\mathcal{M}^2$ ” on a nonempty set $\mathcal{M}$ remains a “complete equivalence relation.”

Definition 5 (see [30]). For a binary relation $\mathcal{R}$ defined on a nonempty set $\mathcal{M}$, $U \subseteq \mathcal{M}$ such that the restriction of $\mathcal{R}$ to U is denoted by $\mathcal{R}|_U$ and is defined to be the set $\mathcal{R} \cap U^2$ (i.e., $\mathcal{R}|_U := \mathcal{R} \cap U^2$). In fact, $\mathcal{R}|_U$ is a relation on U induced by $\mathcal{R}$.

Definition 6 (see [15]). A binary relation $\mathcal{R}$ defined on a nonempty set $\mathcal{M}$ is called “ $\mathcal{R}$ -comparative,” if either $(u, v) \in \mathcal{R}$ or $(v, u) \in \mathcal{R}$ for $u, v \in \mathcal{M}$. We denote it by “ $[u, v] \in \mathcal{R}$.”

Definition 7 (see [28]). Given $N \in \mathbb{N}_0$, $N \geq 2$, a binary relation $\mathcal{R}$ defined on a nonempty set $\mathcal{M}$ is called N -transitive if for any $u_0, u_1, \dots, u_N \in \mathcal{M}$,

$$(u_{i-1}, u_i) \in \mathcal{R} \text{ for each } i (1 \leq i \leq N) \implies (u_0, u_N) \in \mathcal{R}. \quad (1)$$

Notice that “2-transitivity” coincides with “transitivity.” Following Turinici [29], $\mathcal{R}$ is called “finitely transitive” if it is “ N -transitive” for some $N \geq 2$.

Definition 8 (see [29]). Given $N \in \mathbb{N}_0$, $N \geq 2$, a binary relation $\mathcal{R}$ defined on a nonempty set $\mathcal{M}$ is called “locally finitely transitive” if for each denumerable (effectively) subset E of $\mathcal{M}$, there exists $N = N(E) \geq 2$ such that $\mathcal{R}|_E$ is “ N -transitive.”

Definition 9 (see [31]). Let $\mathcal{M}$ be a nonempty set equipped with a binary relation $\mathcal{R}$ and $\mathcal{T}$ a self-mapping on $\mathcal{M}$. We say that $\mathcal{R}$ is “ $\mathcal{T}$ -transitive” if for any $u, v, z \in \mathcal{M}$,

$$(\mathcal{T}u, \mathcal{T}v), (\mathcal{T}v, \mathcal{T}z) \in \mathcal{R} \implies (\mathcal{T}u, \mathcal{T}z) \in \mathcal{R}. \quad (2)$$

Henceforth, the notions of “ $\mathcal{T}$ -transitivity” and “locally finitely transitivity” are not only weaker as compared to “transitivity” but also independent of one another.

Definition 10 (see [31]). Let $\mathcal{M}$ be a nonempty set equipped with a binary relation $\mathcal{R}$ and $\mathcal{T}$ a self-mapping on $\mathcal{M}$. Then $\mathcal{R}$ is called “locally $\mathcal{T}$ -transitive” if for each denumerable (effectively) subset E of $\mathcal{T}(\mathcal{M})$, such that $\mathcal{R}|_E$ is “transitive.”

Definition 11 (see [16]). Let $\mathcal{M}$ be a nonempty set equipped with a binary relation $\mathcal{R}$ and $\mathcal{T}$ a self-mapping on $\mathcal{M}$. Then for a given $N \in \mathbb{N}_0$, $N \geq 2$, $\mathcal{R}$ is called “locally finitely

$\mathcal{T}$ -transitive” if for each denumerable (effectively) subset E of $\mathcal{T}(\mathcal{M})$, there exists $N = N(E)$ such that $\mathcal{R}|_E$ is “ N -transitive.”

In view of Proposition 1 [32], the following are predictable.

Proposition 1. Let $\mathcal{M}$ be a nonempty set equipped with a binary relation $\mathcal{R}$ and $\mathcal{T}$ a self-mapping on $\mathcal{M}$. Then,

- (i) $\mathcal{R}$ is $\mathcal{T}$ -transitive $\iff \mathcal{R}|_{\mathcal{T}(\mathcal{M})}$ is transitive
- (ii) $\mathcal{R}$ is locally finitely $\mathcal{T}$ -transitive $\iff \mathcal{R}|_{\mathcal{T}(\mathcal{M})}$ is locally finitely transitive
- (iii) $\mathcal{R}$ is transitive $\implies \mathcal{R}$ is finitely transitive $\implies \mathcal{R}$ is locally finitely transitive $\implies \mathcal{R}$ is locally finitely $\mathcal{T}$ -transitive
- (iv) $\mathcal{R}$ is transitive $\implies \mathcal{R}$ is $\mathcal{T}$ -transitive $\implies \mathcal{R}$ is locally finitely $\mathcal{T}$ -transitive.

Proposition 2. Let $\mathcal{M}$ be a nonempty set, $\mathcal{R}$ a binary relation defined on $\mathcal{M}$, and $(\mathcal{T}_1, \mathcal{T}_2)$ a pair of self-mappings on $\mathcal{M}$ with $\mathcal{T}_1(\mathcal{M}) \subseteq \mathcal{T}_2(\mathcal{M})$. Then,

- (i) $\mathcal{R}$ is $\mathcal{T}_2$ -transitive $\implies \mathcal{R}$ is $\mathcal{T}_1$ -transitive
- (ii) $\mathcal{R}$ is locally finitely $\mathcal{T}_2$ -transitive $\implies \mathcal{R}$ is locally finitely $\mathcal{T}_1$ -transitive
- (iii) $\mathcal{R}$ is transitive $\implies \mathcal{R}$ is finitely transitive $\implies \mathcal{R}$ is locally finitely transitive $\implies \mathcal{R}$ is locally finitely $\mathcal{T}_2$ -transitive
- (iv) $\mathcal{R}$ is transitive $\implies \mathcal{R}$ is finitely $\mathcal{T}_2$ -transitive $\implies \mathcal{R}$ is locally finitely $\mathcal{T}_2$ -transitive
- (v) $\mathcal{R}$ is transitive $\implies \mathcal{R}$ is $\mathcal{T}_2$ -transitive $\implies \mathcal{R}$ is locally $\mathcal{T}_2$ -transitive $\implies \mathcal{R}$ is locally finitely $\mathcal{T}_2$ -transitive
- (vi) $\mathcal{R}$ is transitive $\implies \mathcal{R}$ is $\mathcal{T}_2$ -transitive $\implies \mathcal{R}$ is locally finitely $\mathcal{T}_2$ -transitive.

Definition 12 (see [22]). Let $\mathcal{R}$ be a binary relation defined on a nonempty set $\mathcal{M}$. Then,

- (1) The dual relation of $\mathcal{R}$, denoted by $\mathcal{R}^{-1}$, is defined by $\mathcal{R}^{-1} = \{(u, v) \in \mathcal{M}^2 : (v, u) \in \mathcal{R}\}$.
- (2) “The symmetric closure of $\mathcal{R}$,” denoted by $\mathcal{R}^s$, is defined to be the set $\mathcal{R} \cup \mathcal{R}^{-1}$ (i.e., $\mathcal{R}^s = \mathcal{R} \cup \mathcal{R}^{-1}$). In fact, $\mathcal{R}^s$ is the smallest symmetric relation on $\mathcal{M}$ containing $\mathcal{R}$.

Proposition 3 (see [15]). For a binary relation $\mathcal{R}$ defined on a nonempty set $\mathcal{M}$,

$$(u, v) \in \mathcal{R}^s \iff [u, v] \in \mathcal{R}. \quad (3)$$

Definition 13 (see [15]). Let $\mathcal{M}$ be a nonempty set equipped with a binary relation $\mathcal{R}$. A sequence $\{u_n\} \subset \mathcal{M}$ is called “ $\mathcal{R}$ -preserving” if

$$(u_n, u_{n+1}) \in \mathcal{R} \quad \forall n \in \mathbb{N}_0. \quad (4)$$

Notice that the term “ $\mathcal{R}$ -nondecreasing” is utilized by Shahzad et al. [33] and Roldán-López-de-Hierro and Shahzad [34] instead of “ $\mathcal{R}$ -preserving.”

Definition 14 (see [15]). Let $\mathcal{M}$ be a nonempty set equipped with a binary relation $\mathcal{R}$ and $\mathcal{T}_1$ a self-mapping on $\mathcal{M}$. Then, $\mathcal{R}$ is called $\mathcal{T}_1$ -closed if (for $u, v \in \mathcal{M}$)

$$(u, v) \in \mathcal{R} \implies (\mathcal{T}_1 u, \mathcal{T}_1 v) \in \mathcal{R}. \quad (5)$$

Definition 15 (see [35]). Let $\mathcal{M}$ be a nonempty set equipped with a binary relation $\mathcal{R}$ and $(\mathcal{T}_1, \mathcal{T}_2)$ a pair of self-mappings on $\mathcal{M}$. Then, $\mathcal{R}$ is said to be $(\mathcal{T}_1, \mathcal{T}_2)$ -closed if for all $u, v \in \mathcal{M}$,

$$(\mathcal{T}_2 u, \mathcal{T}_2 v) \in \mathcal{R} \implies (\mathcal{T}_1 u, \mathcal{T}_1 v) \in \mathcal{R}. \quad (6)$$

Definition 15 reduces to Definition 14, if we take $\mathcal{T}_2 = I$ (the identity mapping on $\mathcal{M}$).

Proposition 4 (see [35]). Let $\mathcal{M}$ be a nonempty set equipped with a binary relation $\mathcal{R}$ and $(\mathcal{T}_1, \mathcal{T}_2)$ a pair of self-mappings on $\mathcal{M}$. If $\mathcal{R}$ is “ $(\mathcal{T}_1, \mathcal{T}_2)$ -closed,” then so is $\mathcal{R}^s$.

Now, we recall the metrical notions via a binary relation, namely, “completeness,” “closedness,” “continuity,” “ $\mathcal{T}_2$ -continuity,” and “compatibility.”

Definition 16 (see [35]). Let $(\mathcal{M}, d)$ be a metric space equipped with a binary relation $\mathcal{R}$. We say that $(\mathcal{M}, d)$ is “ $\mathcal{R}$ -complete” if every $\mathcal{R}$ -preserving Cauchy sequence in $\mathcal{M}$ converges in $\mathcal{M}$.

Remark 2. Complete metric space implies $\mathcal{R}$ -complete, for any binary relation $\mathcal{R}$. Converse implication is true with respect to the universal relation.

Definition 17 (see [35]). Let $(\mathcal{M}, d)$ be a metric space equipped with a binary relation $\mathcal{R}$. A subset U of $\mathcal{M}$ is called “ $\mathcal{R}$ -closed” if every $\mathcal{R}$ -preserving sequence in U converges to a point of U .

Remark 3. Every closed subset of a metric space is $\mathcal{R}$ -closed for any binary relation $\mathcal{R}$. Indeed, via the universal relation, the notion of $\mathcal{R}$ -closedness coincides with usual closedness.

Proposition 5 (see [35]). Every $\mathcal{R}$ -complete subspace of a metric space is $\mathcal{R}$ -closed.

Proposition 6 (see [35]). Every $\mathcal{R}$ -closed subspace of an $\mathcal{R}$ -complete metric space is $\mathcal{R}$ -complete.

Definition 18 (see [35]). Let $(\mathcal{M}, d)$ be a metric space equipped with a binary relation $\mathcal{R}$ and $u \in \mathcal{M}$. A mapping $\mathcal{T}_1: \mathcal{M} \longrightarrow \mathcal{M}$ is called “ $\mathcal{R}$ -continuous” at u if for any $\mathcal{R}$ -preserving sequence $\{u_n\}$ such that $u_n \xrightarrow{d} u$, we have $\mathcal{T}_1 u_n \xrightarrow{d} \mathcal{T}_1 u$. Furthermore, $\mathcal{T}_1$ is called $\mathcal{R}$ -continuous if it is $\mathcal{R}$ -continuous at each point of $\mathcal{M}$.

Remark 4. Notice that “continuity” implies “ $\mathcal{R}$ -continuity” for any binary relation $\mathcal{R}$. In fact, under the universal relation, the notion of “ $\mathcal{R}$ -continuity” leads to “continuity.”

Definition 19 (see [35]). Let $(\mathcal{M}, d)$ be a metric space equipped with a binary relation $\mathcal{R}$ and $\mathcal{T}_2$ a self-mapping on $\mathcal{M}$ and $u \in \mathcal{M}$. A mapping $\mathcal{T}_1: \mathcal{M} \rightarrow \mathcal{M}$ is called “ $(\mathcal{T}_2, \mathcal{R})$ -continuous” at u if for any sequence $\{u_n\}$ such that $\{\mathcal{T}_2 u_n\}_d$ is $\mathcal{R}$ -preserving and $\mathcal{T}_2 u_n \xrightarrow{d} \mathcal{T}_2 u$, we have $\mathcal{T}_1 u_n \xrightarrow{d} \mathcal{T}_1 u$. Moreover, $\mathcal{T}_1$ is called “ $(\mathcal{T}_2, \mathcal{R})$ -continuous” if it is “ $(\mathcal{T}_2, \mathcal{R})$ -continuous” at each point of $\mathcal{M}$.

On setting $\mathcal{T}_2 = I$ (the identity mapping on $\mathcal{M}$), Definition 19 leads to Definition 18.

Remark 5. Notice that $\mathcal{T}_2$ -continuity implies $(\mathcal{T}_2, \mathcal{R})$ -continuity, for any binary relation $\mathcal{R}$. In fact, via the universal relation, the notion of $(\mathcal{T}_2, \mathcal{R})$ -continuity leads to $\mathcal{T}_2$ -continuity.

Definition 20 (see [35]). Let $(\mathcal{M}, d)$ be a metric space equipped with a binary relation $\mathcal{R}$ and $(\mathcal{T}_1, \mathcal{T}_2)$ be a pair of self-mappings defined on $\mathcal{M}$. Then, $\mathcal{T}_1$ and $\mathcal{T}_2$ are said to be “ $\mathcal{R}$ -compatible” if for any sequence $\{u_n\} \subset \mathcal{M}$ such that $\{\mathcal{T}_1 u_n\}$ and $\{\mathcal{T}_2 u_n\}$ are $\mathcal{R}$ -preserving and $\lim_{n \rightarrow \infty} \mathcal{T}_2 u_n = \lim_{n \rightarrow \infty} \mathcal{T}_1 u_n$, we have $\lim_{n \rightarrow \infty} d(\mathcal{T}_2(\mathcal{T}_1 u_n), \mathcal{T}_1(\mathcal{T}_2 u_n)) = 0$.

Remark 6. Given a metric space $(\mathcal{M}, d)$ and a binary relation $\mathcal{R}$ on $\mathcal{M}$, “commutativity” $\implies$ “weak commutativity” $\implies$ “compatibility” $\implies$ “ $\mathcal{R}$ -compatibility” $\implies$ “weak compatibility.” Via the universal relation, “ $\mathcal{R}$ -compatibility” reduces to “compatibility.”

Definition 21 (see [15]). Let $(\mathcal{M}, d)$ be a metric space. A binary relation $\mathcal{R}$ on $\mathcal{M}$ is called “ d -self-closed” if for any $\mathcal{R}$ -preserving sequence $\{u_n\}$ such that $u_n \xrightarrow{d} u$, there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ with $[u_{n_k}, u] \in \mathcal{R}, \forall k \in \mathbb{N}_0$.

Definition 22 (see [35]). Let $(\mathcal{M}, d)$ be a metric space and $\mathcal{T}_2$ a self-mapping on $\mathcal{M}$. A binary relation $\mathcal{R}$ on $\mathcal{M}$ is called “ $(\mathcal{T}_2, d)$ -self-closed” if for any $\mathcal{R}$ -preserving sequence $\{u_n\}$ such that $u_n \xrightarrow{d} u$, there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ with $[\mathcal{T}_2 u_{n_k}, \mathcal{T}_2 u] \in \mathcal{R}, \forall k \in \mathbb{N}_0$.

On taking $\mathcal{T}_2 = I$ (the identity mapping on $\mathcal{M}$), Definition 22 reduces to Definition 21.

Definition 23 (see [24]). Let $\mathcal{M}$ be a nonempty set equipped with a binary relation $\mathcal{R}$. A subset U of $\mathcal{M}$ is called “ $\mathcal{R}$ -directed” if for each pair $u, v \in U$, there exists $z \in \mathcal{M}$ such that $(u, z) \in \mathcal{R}$ and $(v, z) \in \mathcal{R}$.

Definition 24 (see [30]). Let $\mathcal{M}$ be a nonempty set equipped with a binary relation $\mathcal{R}$. Given $u, v \in \mathcal{M}$, a path of length k (where k is a natural number) in $\mathcal{R}$ from u to v is a finite sequence $\{z_0, z_1, z_2, \dots, z_k\} \subset \mathcal{M}$ satisfying the following conditions:

- (i) $z_0 = u, z_k = v$
- (ii) $(z_i, z_{i+1}) \in \mathcal{R}$ for each i ($0 \leq i \leq k-1$).

Notice that a path of length k involves $k+1$ elements of $\mathcal{M}$, although they are not necessarily distinct.

Definition 25 (see [35]). Let $\mathcal{M}$ be a nonempty set equipped with a binary relation $\mathcal{R}$. A subset U of $\mathcal{M}$ is called “ $\mathcal{R}$ -connected” if for each pair $u, v \in U$, there exists a path in $\mathcal{R}$ from u to v .

Definition 26 (see [32]). Let $\mathcal{M}$ be a nonempty set equipped with a binary relation $\mathcal{R}$ and $(\mathcal{T}_1, \mathcal{T}_2)$ a pair of self-mappings on $\mathcal{M}$. Then, $\mathcal{R}$ is called “ $(\mathcal{T}_1, \mathcal{T}_2)$ -compatible” if for all $u, v \in \mathcal{M}$,

$$(\mathcal{T}_2 u, \mathcal{T}_2 v) \in \mathcal{R}, \mathcal{T}_2(u) = \mathcal{T}_2(v) \implies \mathcal{T}_1(u) = \mathcal{T}_1(v). \quad (7)$$

Given a nonempty set $\mathcal{M}$, a binary relation $\mathcal{R}$ on $\mathcal{M}$, and a pair of self-mappings $(\mathcal{T}_1, \mathcal{T}_2)$ on $\mathcal{M}$, we employ the following notations:

- (i) “ $C(\mathcal{T}_1, \mathcal{T}_2) := \{u \in \mathcal{M} : \mathcal{T}_2 u = \mathcal{T}_1 u\}$,” i.e., the set of all coincidence points of $\mathcal{T}_1$ and $\mathcal{T}_2$
- (ii) “ $\overline{C}(\mathcal{T}_1, \mathcal{T}_2) := \{\bar{u} \in \mathcal{M} : \bar{u} = \mathcal{T}_2 u = \mathcal{T}_1 u, u \in \mathcal{M}\}$,” i.e., the set of all points of coincidence of $\mathcal{T}_1$ and $\mathcal{T}_2$
- (iii) “ $\mathcal{M}(\mathcal{T}_1, \mathcal{R}) := \{u \in \mathcal{M} : (u, \mathcal{T}_1 u) \in \mathcal{R}\}$ ”
- (iv) “ $\mathcal{M}(\mathcal{T}_1, \mathcal{T}_2, \mathcal{R}) := \{u \in \mathcal{M} : (\mathcal{T}_2 u, \mathcal{T}_1 u) \in \mathcal{R}\}$.”

In order to prove our main results, we utilize the following class of control functions:

$$\Phi = \left\{ \phi: [0, \infty) \rightarrow [0, \infty): \phi(t) < t \text{ for each } t > 0 \text{ and } \limsup_{r \rightarrow t} \phi(r) < t \text{ for each } t > 0 \right\}. \quad (8)$$

In view of the symmetry of d , we can have the following.

Proposition 7. If $(\mathcal{M}, d)$ is a metric space, $\mathcal{R}$ is a binary relation on $\mathcal{M}$, $(\mathcal{T}_1, \mathcal{T}_2)$ is a pair of self-mappings on $\mathcal{M}$, and $\phi \in \Phi$, then the following contractive conditions are equivalent:

- (i) $d(\mathcal{T}_1 u, \mathcal{T}_1 v) \leq \phi(d(\mathcal{T}_2 u, \mathcal{T}_2 v))u, v \in \mathcal{M}$ with $(\mathcal{T}_2 u, \mathcal{T}_2 v) \in \mathcal{R}$
- (ii) $d(\mathcal{T}_1 u, \mathcal{T}_1 v) \leq \phi(d(\mathcal{T}_2 u, \mathcal{T}_2 v))u, v \in \mathcal{M}$ with $[\mathcal{T}_2 u, \mathcal{T}_2 v] \in \mathcal{R}$.

Combining Theorems 4 and 5 of Alam et al. [16], we record the following for the future references.

Theorem 1 (see [16]). Let $(\mathcal{M}, d)$ be a metric space equipped with a binary relation $\mathcal{R}$ and $\mathcal{T}$ a self-mapping on $\mathcal{M}$. Assume that the following conditions hold:

- (a) $(\mathcal{M}, d)$ is $\mathcal{R}$ -complete.
- (b) $\mathcal{R}$ is locally finitely $\mathcal{T}$ -transitive and $\mathcal{T}$ -closed.
- (c) Either $\mathcal{T}$ is $\mathcal{R}$ -continuous or $\mathcal{R}$ is d -self-closed.
- (d) $\mathcal{M}(\mathcal{T}, \mathcal{R})$ is nonempty.
- (e) There exists $\phi \in \Phi$ such that

$$d(\mathcal{T}u, \mathcal{T}v) \leq \phi(d(u, v)), \quad \forall u, v \in \mathcal{M} \text{ with } (u, v) \in \mathcal{R}. \quad (9)$$

Then, $\mathcal{T}$ has a fixed point.

- (f) Furthermore, if $(u)\mathcal{T}(\mathcal{M})$ is $\mathcal{R}^s$ -connected, then $\mathcal{T}$ has a unique fixed point.

In this sequel, we recall some results, which are needed in our foregoing discussion.

Lemma 1 (see [37]). Let $\phi \in \Phi$. If $\{t_n\} \subset (0, \infty)$ is a sequence such that $t_{n+1} \leq \phi(t_n) \forall n \in \mathbb{N}_0$, then $\lim_{n \rightarrow \infty} t_n = 0$.

Lemma 2 (see [38]). Let $\mathcal{M}$ be a nonempty set and $\mathcal{T}_2$ a self-mapping on $\mathcal{M}$. Then, there exists a subset $U \subseteq \mathcal{M}$ such that $\mathcal{T}_2(U) = \mathcal{T}_2(\mathcal{M})$ and $\mathcal{T}_2: U \rightarrow \mathcal{M}$ is one-to-one.

Lemma 3 (see [37]). Let $\mathcal{M}$ be a nonempty set and $(\mathcal{T}_1, \mathcal{T}_2)$ a pair of self-mappings on $\mathcal{M}$. If $\mathcal{T}_1$ and $\mathcal{T}_2$ are weakly compatible, then every point of coincidence of $\mathcal{T}_1$ and $\mathcal{T}_2$ is also a coincidence point of $\mathcal{T}_1$ and $\mathcal{T}_2$.

Lemma 4 (see [39]). Let $(\mathcal{M}, d)$ be a metric space and $\{u_n\}$ a sequence in $\mathcal{M}$. If $\{u_n\}$ is not a Cauchy sequence, then there exist $\epsilon > 0$ and two subsequences $\{u_{n_k}\}$ and $\{u_{m_k}\}$ of $\{u_n\}$ such that

- (i) $k \leq m_k < n_k, \quad \forall k \in \mathbb{N}$
- (ii) $d(u_{m_k}, u_{n_k}) \geq \epsilon$
- (iii) $d(u_{m_k}, u_{p_k}) < \epsilon, \quad \forall p_k \in \{m_k + 1, m_k + 2, \dots, n_k - 2, n_k - 1\}$.

In addition to this, if $\{u_n\}$ also verifies $\lim_{n \rightarrow \infty} d(u_n, u_{n+1}) = 0$, then

$$\lim_{k \rightarrow \infty} d(u_{m_k}, u_{n_k+p}) = \epsilon, \quad \forall p \in \mathbb{N}_0. \quad (10)$$

Lemma 5 (see [29]). Let $\mathcal{M}$ be a nonempty set, $\mathcal{R}$ a binary relation on $\mathcal{M}$, and $\{z_n\}$ an $\mathcal{R}$ -preserving sequence in $\mathcal{M}$. If $\mathcal{R}$ is N -transitive on $Z := \{z_n: n \in \mathbb{N}_0\}$ for some natural number $N \geq 2$, then

$$(z_n, z_{n+1+r(N-1)}) \in \mathcal{R}, \quad \forall n, r \in \mathbb{N}_0. \quad (11)$$

4. Main Results

Firstly, we establish a result on the existence of the coincidence point, which runs as follows.

Theorem 2. Let $(\mathcal{M}, d)$ be a metric space equipped with a binary relation $\mathcal{R}$ and an $\mathcal{R}$ -complete subspace $\mathcal{N}$ of $\mathcal{M}$. Let $(\mathcal{T}_1, \mathcal{T}_2)$ be a pair of self-mappings on $\mathcal{M}$. Suppose that the following conditions hold:

- (a) $\mathcal{T}_1(\mathcal{M}) \subseteq \mathcal{T}_2(\mathcal{M}) \cap \mathcal{N}$
- (b) $\mathcal{R}$ is locally finitely $\mathcal{T}_1$ -transitive and $(\mathcal{T}_1, \mathcal{T}_2)$ -closed
- (c) $\mathcal{M}(\mathcal{T}_1, \mathcal{T}_2, \mathcal{R})$ is nonempty
- (d) There exists $\phi \in \Phi$ such that $d(\mathcal{T}_1 u, \mathcal{T}_1 v) \leq \phi(d(\mathcal{T}_2 u, \mathcal{T}_2 v))$, $\forall u, v \in \mathcal{M}$ with $(\mathcal{T}_2 u, \mathcal{T}_2 v) \in \mathcal{R}$
- (e) $(e_1)\mathcal{T}_1$ and $\mathcal{T}_2$ are $\mathcal{R}$ -compatible
- (e₂) $\mathcal{T}_2$ is $\mathcal{R}$ -continuous
- (e₃) Either $\mathcal{T}_1$ is $\mathcal{R}$ -continuous or $\mathcal{R}$ is $(\mathcal{T}_1, \mathcal{T}_2)$ -compatible and $(\mathcal{T}_2, d)$ -self-closed

Or alternately,

- (e') $(e'_1)\mathcal{N} \subseteq \mathcal{T}_2(\mathcal{M})$
- (e'₂) Either $\mathcal{T}_1$ is $(\mathcal{T}_2, \mathcal{R})$ -continuous or $\mathcal{T}_1$ and $\mathcal{T}_2$ are continuous or $\mathcal{R}$ and $\mathcal{R}|_{\mathcal{N}}$ are $(\mathcal{T}_1, \mathcal{T}_2)$ -compatible and d -self-closed, respectively.

Then, $\mathcal{T}_1$ and $\mathcal{T}_2$ have a coincidence point.

Proof. In lieu of hypothesis (a), it is equivalent to say that $\mathcal{T}_1(\mathcal{M}) \subseteq \mathcal{T}_2(\mathcal{M})$ and $\mathcal{T}_1(\mathcal{M}) \subseteq \mathcal{N}$. With a view of hypothesis (c), let $u_0 \in \mathcal{M}$ such that $(\mathcal{T}_2 u_0, \mathcal{T}_1 u_0) \in \mathcal{R}$. If $\mathcal{T}_2(u_0) = \mathcal{T}_1(u_0)$, then u_0 is a coincidence point of $\mathcal{T}_1$ and $\mathcal{T}_2$, and hence, we are done. Otherwise, as $\mathcal{T}_1(\mathcal{M}) \subseteq \mathcal{T}_2(\mathcal{M})$, there exists some $u_1 \in \mathcal{M}$ such that $\mathcal{T}_2(u_1) = \mathcal{T}_1(u_0)$. Again, since $\mathcal{T}_1(u_1) \in \mathcal{M}$, there exists some $u_2 \in \mathcal{M}$ ($\mathcal{T}_1(\mathcal{M}) \subseteq \mathcal{T}_2(\mathcal{M})$) such that $\mathcal{T}_2(u_2) = \mathcal{T}_1(u_1)$. In the similar fashion, we can obtain a sequence $\{u_n\} \subseteq \mathcal{M}$ of joint iterations of $\mathcal{T}_1$ and $\mathcal{T}_2$ with the initial point u_0 , i.e.,

$$\mathcal{T}_2(u_{n+1}) = \mathcal{T}_1(u_n), \quad \text{for all } n \in \mathbb{N}_0. \quad (12)$$

Now, we assert that $\{\mathcal{T}_2 u_n\}$ and $\{\mathcal{T}_1 u_n\}$ are $\mathcal{R}$ -preserving sequences, i.e.,

$$(\mathcal{T}_1 u_n, \mathcal{T}_1 u_{n+1}) \in \mathcal{R}, \quad \text{for all } n \in \mathbb{N}_0, \quad (13)$$

$$(\mathcal{T}_2 u_n, \mathcal{T}_2 u_{n+1}) \in \mathcal{R}, \quad \text{for all } n \in \mathbb{N}_0. \quad (14)$$

We prove these assertions by mathematical induction, but for the sake of simplicity, we prove only (14). Equation (13) follows with the help of (12) and (14). On utilizing the fact that $u_0 \in \mathcal{M}(\mathcal{T}_1, \mathcal{T}_2, \mathcal{R})$ and (12) (for $n = 0$), we have

$$(\mathcal{T}_2 u_0, \mathcal{T}_2 u_1) \in \mathcal{R}, \quad (15)$$

which implies that (14) is true for $n = 0$. Assume that (14) is true for $n = r > 0$, i.e.,

$$(\mathcal{T}_2 u_r, \mathcal{T}_2 u_{r+1}) \in \mathcal{R}. \quad (16)$$

In view of $(\mathcal{T}_1, \mathcal{T}_2)$ -closedness of $\mathcal{R}$, we have

$$(\mathcal{T}_1 u_r, \mathcal{T}_1 u_{r+1}) \in \mathcal{R}, \quad (17)$$

which on employing (12) gives rise to

$$(\mathcal{T}_2 u_{r+1}, \mathcal{T}_2 u_{r+2}) \in \mathcal{R}. \quad (18)$$

Hence, (14) is true for $n = r + 1$. Thus, by the induction process, (14) is true for $n \in \mathbb{N}_0$. In view of (12) and (14), we obtain (13).

Therefore, $\{\mathcal{T}_2 u_n\}$ and $\{\mathcal{T}_1 u_n\}$ are $\mathcal{R}$ -preserving. Applying the contractive condition (d) to (14) with $\mathcal{T}_2(u_n)$, not equal to $\mathcal{T}_2(u_{n+1})$ for all $n \in \mathbb{N}_0$, we obtain

$$\begin{aligned} d(\mathcal{T}_2 u_{n+1}, \mathcal{T}_2 u_{n+2}) &= d(\mathcal{T}_1 u_n, \mathcal{T}_1 u_{n+1}) \\ &\leq \phi(d(\mathcal{T}_2 u_n, \mathcal{T}_2 u_{n+1})), \quad \text{for all } n \in \mathbb{N}_0. \end{aligned} \quad (19)$$

Owing to Lemma 1, we obtain

$$\lim_{n \rightarrow \infty} d(\mathcal{T}_2 u_n, \mathcal{T}_2 u_{n+1}) = 0. \quad (20)$$

Now, we show that $\{\mathcal{T}_2 u_n\}$ is a Cauchy sequence. Suppose on the contrary that $\{\mathcal{T}_2 u_n\}$ is not Cauchy. Therefore, owing to Lemma 4, there exist $\epsilon > 0$ and two subsequences $\{\mathcal{T}_2 u_{n_k}\}$ and $\{\mathcal{T}_2 u_{m_k}\}$ of the sequence $\{\mathcal{T}_2 u_n\}$ such that $k \leq m_k < n_k$ with $d(\mathcal{T}_2 u_{n_k}, \mathcal{T}_2 u_{m_k}) \geq \epsilon$ and $d(\mathcal{T}_2 u_{n_k}, \mathcal{T}_2 u_{p_k}) < \epsilon$, where $p_k \in \{m_{k+1}, m_{k+2}, \dots, n_{k-2}, n_{k-1}\}$. Furthermore, in view of (20), Lemma 4 assures us that

$$\lim_{n \rightarrow \infty} d(\mathcal{T}_2 u_{m_k}, \mathcal{T}_2 u_{n_k+p}) = \epsilon, \quad \forall p \in \mathbb{N}_0. \quad (21)$$

In view of (12), $\{\mathcal{T}_2 u_n\} \subset \mathcal{T}_1(\mathcal{M}) \subseteq \mathcal{N}$ and hence the range $\mathcal{T}_1: = \{\mathcal{T}_2 u_n: n \in \mathbb{N}_0\}$ (of the sequence $\{\mathcal{T}_2 u_n\}$) is a denumerable subset of $\mathcal{T}_1(\mathcal{M})$. Hence by locally finite $\mathcal{T}_1$ -transitivity of $\mathcal{R}$, there exists a natural number $N = N(E) \geq 2$ such that $\mathcal{R}|_{\mathcal{T}_1}$ is N -transitive.

Since $m_k < n_k$ and $N - 1 > 0$, by the division rule, we have

$$\begin{aligned} &\begin{cases} n_k - m_k = (N - 1)(\mu_k - 1) + (N - \eta_k) \\ \mu_k - 1 \geq 0, 0 \leq N - \eta_k < N - 1 \end{cases} \\ \iff &\begin{cases} n_k + \eta_k = m_k + 1 + (N - 1)\mu_k \\ \mu_k \geq 1, 1 < \eta_k \leq N, \end{cases} \end{aligned} \quad (22)$$

where μ_k and η_k are suitable natural numbers such that η_k lies in the interval $(1, N]$. Hence, without loss of generality, we can choose subsequences $\{\mathcal{T}_2 u_{n_k}\}$ and $\{\mathcal{T}_2 u_{m_k}\}$ of $\{\mathcal{T}_2 u_n\}$ (satisfying (21)) such that η_k remains constant, say η , independent of k . We write

$$m'_k = n_k + \eta = m_k + 1 + (N - 1)\mu_k, \quad (23)$$

where η ($1 < \eta \leq N$) is constant. Owing to (21) and (23), we obtain

$$\lim_{k \rightarrow \infty} d(\mathcal{T}_2 u_{m_k}, \mathcal{T}_2 u_{m'_k}) = \lim_{k \rightarrow \infty} d(\mathcal{T}_2 u_{m_k}, \mathcal{T}_2 u_{n_k+\eta}) = \epsilon. \quad (24)$$

Using triangular inequality, we have

$$\begin{aligned} d(\mathcal{T}_2 u_{m_k+1}, \mathcal{T}_2 u_{m'_k+1}) &\leq d(\mathcal{T}_2 u_{m_k+1}, \mathcal{T}_2 u_{m_k}) \\ &\quad + d(\mathcal{T}_2 u_{m_k}, \mathcal{T}_2 u_{m'_k}) + d(\mathcal{T}_2 u_{m'_k}, \mathcal{T}_2 u_{m'_k+1}), \end{aligned} \quad (25)$$

which, on letting $k \rightarrow \infty$ and using (20) and (24), yields that

$$\lim_{k \rightarrow \infty} d(\mathcal{T}_2 u_{m_k+1}, \mathcal{T}_2 u_{m'_k+1}) = \epsilon. \quad (26)$$

In view of (23) and Lemma 5, we have $(\mathcal{T}_2 u_{m_k}, \mathcal{T}_2 u_{m'_k}) \in \mathcal{R}$. We denote $\delta_k = d(\mathcal{T}_2 u_{m_k}, \mathcal{T}_2 u_{m'_k})$. Now, owing to (12) and hypothesis (d), we have

$$\begin{aligned} d(\mathcal{T}_2 u_{m_k+1}, \mathcal{T}_2 u_{m'_k+1}) &= d(\mathcal{T}_1 u_{m_k}, \mathcal{T}_1 u_{m'_k}) \\ &\leq \phi(d(\mathcal{T}_2 u_{m_k}, \mathcal{T}_2 u_{m'_k})) \\ &= \phi(\delta_k), \end{aligned} \quad (27)$$

so that

$$d(\mathcal{T}_2 u_{m_k+1}, \mathcal{T}_2 u_{m'_k+1}) \leq \phi(\delta_k). \quad (28)$$

Utilizing the fact that $\delta_k \rightarrow \epsilon$ as $k \rightarrow \infty$ (in view of (26)) and by the definition of Φ , we have

$$\limsup_{k \rightarrow \infty} \phi(\delta_k) = \limsup_{t \rightarrow \epsilon} \phi(t) < \epsilon. \quad (29)$$

On making limit superior as $k \rightarrow \infty$ in (28), using (26) and (29), we obtain

$$\epsilon = \limsup_{k \rightarrow \infty} d(\mathcal{T}_2 u_{m_k+1}, \mathcal{T}_2 u_{m'_k+1}) \leq \limsup_{k \rightarrow \infty} \phi(\delta_k) < \epsilon, \quad (30)$$

which shows that our supposition was wrong so that $\{\mathcal{T}_2 u_n\}$ is a Cauchy sequence. Owing to (12), $\{\mathcal{T}_2 u_n\} \subset \mathcal{T}_1(\mathcal{M}) \subseteq \mathcal{N}$ so that $\{\mathcal{T}_2 u_n\}$ is an $\mathcal{R}$ -preserving Cauchy sequence in $\mathcal{N}$. Owing to the $\mathcal{R}$ -completeness of $\mathcal{N}$, there exists $v \in \mathcal{N}$ such that

$$\lim_{n \rightarrow \infty} \mathcal{T}_2(u_n) = v. \quad (31)$$

In lieu of (12) and (31), we obtain

$$\lim_{n \rightarrow \infty} \mathcal{T}_1(u_n) = v. \quad (32)$$

Now, to establish the proof with respect to hypotheses (e) and (e'), firstly suppose the hypothesis (e) holds. $\{\mathcal{T}_2(u_n)\}$ is an $\mathcal{R}$ -preserving convergent sequence. Therefore, applying (e₂) (i.e., $\mathcal{R}$ -continuity of $\mathcal{T}_2$), (14), and (31), we have

$$\lim_{n \rightarrow \infty} \mathcal{T}_2(\mathcal{T}_2 u_n) = \mathcal{T}_2\left(\lim_{n \rightarrow \infty} \mathcal{T}_2 u_n\right) = \mathcal{T}_2(v). \quad (33)$$

Again, in view of hypothesis (e₂) (i.e., $\mathcal{R}$ -continuity of $\mathcal{T}_2$), (13), and (32), we have

$$\lim_{n \rightarrow \infty} \mathcal{T}_2(\mathcal{T}_1 u_n) = \mathcal{T}_2\left(\lim_{n \rightarrow \infty} \mathcal{T}_1 u_n\right) = \mathcal{T}_2(v). \quad (34)$$

Owing to (13) and (14) (i.e., $\{\mathcal{T}_1 u_n\}$ and $\{\mathcal{T}_2 u_n\}$ are $\mathcal{R}$ -preserving, respectively) and $\lim_{n \rightarrow \infty} \mathcal{T}_1(u_n) = \lim_{n \rightarrow \infty} \mathcal{T}_2(u_n) = v$ (due to (31) and (32)), using

assumption (e_1) (i.e., $\mathcal{R}$ -compatibility of $\mathcal{T}_1$ and $\mathcal{T}_2$), we obtain

$$\lim_{n \rightarrow \infty} d(\mathcal{T}_2 \mathcal{T}_1 u_n, \mathcal{T}_1 \mathcal{T}_2 u_n) = 0. \quad (35)$$

Now, we claim that v is a coincidence point of $\mathcal{T}_1$ and $\mathcal{T}_2$. To do this, we utilize hypothesis (e_3) . Assume that $\mathcal{T}_1$ is $\mathcal{R}$ -continuous. Due to (14), (31), and $\mathcal{R}$ -continuity of $\mathcal{T}_1$, we get

$$\lim_{n \rightarrow \infty} \mathcal{T}_1(\mathcal{T}_2 u_n) = \mathcal{T}_1 \left(\lim_{n \rightarrow \infty} \mathcal{T}_2 u_n \right) = \mathcal{T}_1 v. \quad (36)$$

Utilizing (34)–(36) and continuity of d , we get

$$\begin{aligned} d(\mathcal{T}_2 v, \mathcal{T}_1 v) &= d \left(\lim_{n \rightarrow \infty} \mathcal{T}_2 \mathcal{T}_1 u_n, \lim_{n \rightarrow \infty} \mathcal{T}_1 \mathcal{T}_2 u_n \right) \\ &= \lim_{n \rightarrow \infty} d(\mathcal{T}_2 \mathcal{T}_1 u_n, \mathcal{T}_1 \mathcal{T}_2 u_n) \\ &= 0, \end{aligned} \quad (37)$$

so that

$$\mathcal{T}_2(v) = \mathcal{T}_1(v). \quad (38)$$

Therefore, v is a coincidence point of $\mathcal{T}_1$ and $\mathcal{T}_2$. Alternately, suppose that $\mathcal{R}$ is $(\mathcal{T}_2, d)$ -self-closed. Owing to (14) and (31) (i.e., $\{\mathcal{T}_2 u_n\}$ is $\mathcal{R}$ -preserving and $\mathcal{T}_2 u_n \xrightarrow{d} v$, respectively) and $(\mathcal{T}_2, d)$ -self-closedness of $\mathcal{R}$, there exists a subsequence $\{\mathcal{T}_2 u_{n_k}\}$ of $\{\mathcal{T}_2 u_n\}$ such that

$$[\mathcal{T}_2 \mathcal{T}_2 u_{n_k}, \mathcal{T}_2 v] \in \mathcal{R}, \quad \forall k \in \mathbb{N}_0. \quad (39)$$

Since $\mathcal{T}_2 u_{n_k} \xrightarrow{d} v$, equations (31)–(36) also survive for $\{u_{n_k}\}$ instead of $\{u_n\}$. Employing (39), assumption (d), definition of Φ , and Proposition 7, we obtain

$$d(\mathcal{T}_1 \mathcal{T}_2 u_{n_k}, \mathcal{T}_1 v) \leq \phi(d(\mathcal{T}_2 \mathcal{T}_2 u_{n_k}, \mathcal{T}_2 v)), \quad \forall k \in \mathbb{N}_0, \quad (40)$$

and we claim that

$$d(\mathcal{T}_1 \mathcal{T}_2 u_{n_k}, \mathcal{T}_1 v) \leq d(\mathcal{T}_2 \mathcal{T}_2 u_{n_k}, \mathcal{T}_2 v), \quad \forall k \in \mathbb{N}. \quad (41)$$

In order to establish (41), we take a partition $\{\mathbb{N}^0, \mathbb{N}^+\}$ of $\mathbb{N}$, i.e., $\mathbb{N}^0 \cup \mathbb{N}^+ = \mathbb{N}$ and $\mathbb{N}^0 \cap \mathbb{N}^+ = \emptyset$, verifying that

- (i) $d(\mathcal{T}_2 \mathcal{T}_2 u_{n_k}, \mathcal{T}_2 v) = 0, \quad \forall k \in \mathbb{N}^0$
- (ii) $d(\mathcal{T}_2 \mathcal{T}_2 u_{n_k}, \mathcal{T}_2 v) > 0, \quad \forall k \in \mathbb{N}^+$

For case (i), using (39) and the $(\mathcal{T}_1, \mathcal{T}_2)$ -compatibility of $\mathcal{R}$, we get $d(\mathcal{T}_1 \mathcal{T}_2 u_{n_k}, \mathcal{T}_1 v) = 0 \forall k \in \mathbb{N}^0$, and hence, (41) holds for all $k \in \mathbb{N}^0$. For case (ii), owing to definition of Φ , we have $d(\mathcal{T}_1 \mathcal{T}_2 u_{n_k}, \mathcal{T}_1 v) \leq \phi(d(\mathcal{T}_2 \mathcal{T}_2 u_{n_k}, \mathcal{T}_2 v)) < d(\mathcal{T}_2 \mathcal{T}_2 u_{n_k}, \mathcal{T}_2 v)$, $\forall k \in \mathbb{N}^+$, and hence, (41) holds for all $k \in \mathbb{N}^+$. Thus in all cases, (41) holds for all $k \in \mathbb{N}$.

Applying (33)–(35), (41), continuity of d , and triangular inequality, we get

$$\begin{aligned} d(\mathcal{T}_1 v, \mathcal{T}_2 v) &\leq d(\mathcal{T}_1 v, \mathcal{T}_1 \mathcal{T}_2 u_{n_k}) + d(\mathcal{T}_1 \mathcal{T}_2 u_{n_k}, \mathcal{T}_2 \mathcal{T}_1 u_{n_k}) \\ &\quad + d(\mathcal{T}_2 \mathcal{T}_1 u_{n_k}, \mathcal{T}_2 v) \text{ or} \\ d(\mathcal{T}_1 v, \mathcal{T}_2 v) &< d(\mathcal{T}_2 v, \mathcal{T}_2 \mathcal{T}_2 u_{n_k}) + d(\mathcal{T}_1 \mathcal{T}_2 u_{n_k}, \mathcal{T}_2 \mathcal{T}_1 u_{n_k}) \\ &\quad + d(\mathcal{T}_2 \mathcal{T}_1 u_{n_k}, \mathcal{T}_2 v) \longrightarrow 0 \text{ as } k \longrightarrow \infty, \end{aligned} \quad (42)$$

so that

$$\mathcal{T}_2(v) = \mathcal{T}_1(v). \quad (43)$$

Thus, v is a coincidence point of $\mathcal{T}_1$ and $\mathcal{T}_2$. Hence, we are through.

Secondly, suppose that hypothesis (e') holds. Due to hypothesis (e'_1) (i.e., $\mathcal{N} \subseteq \mathcal{T}_2(\mathcal{M})$), one can obtain some $u \in \mathcal{M}$ such that $v = \mathcal{T}_2 u$. Hence, (31) and (32), respectively, reduce to

$$\lim_{n \rightarrow \infty} \mathcal{T}_2(u_n) = \mathcal{T}_2(u), \quad (44)$$

$$\lim_{n \rightarrow \infty} \mathcal{T}_1(u_n) = \mathcal{T}_2(u). \quad (45)$$

Now, we show that u is a coincidence point of $\mathcal{T}_1$ and $\mathcal{T}_2$. To do this, owing to hypothesis (e'_2) , assuming that $\mathcal{T}_1$ is $(\mathcal{T}_2, \mathcal{R})$ -continuous, and then employing (14) and (44), we get

$$\lim_{n \rightarrow \infty} \mathcal{T}_1(u_n) = \mathcal{T}_1(u). \quad (46)$$

Due to (45) and (46), we get

$$\mathcal{T}_2(u) = \mathcal{T}_1(u). \quad (47)$$

Hence, conclusion follows.

Next, suppose that $\mathcal{T}_1$ and $\mathcal{T}_2$ are continuous. In view of Lemma 2, there exists a subset $U \subseteq \mathcal{M}$ with $\mathcal{T}_2(U) = \mathcal{T}_2(\mathcal{M})$ and $\mathcal{T}_2: U \longrightarrow \mathcal{M}$ is one-to-one. We define $H: \mathcal{T}_2(U) \longrightarrow \mathcal{T}_2(\mathcal{M})$ by

$$H(\mathcal{T}_2 a) = \mathcal{T}_1(a) \forall \mathcal{T}_2(a) \in \mathcal{T}_2(U) \text{ where } a \in U. \quad (48)$$

Since $\mathcal{T}_1(\mathcal{M}) \subseteq \mathcal{T}_2(\mathcal{M})$ and $\mathcal{T}_2: U \longrightarrow \mathcal{M}$ is one-to-one, H is well defined. Since $\mathcal{T}_1$ and $\mathcal{T}_2$ are continuous, so is H . Utilizing $\mathcal{T}_2(\mathcal{M}) = \mathcal{T}_2(U)$ and assumptions (a) and (e'_1) , respectively, yields that $\mathcal{T}_1(\mathcal{M}) \subseteq \mathcal{T}_2(U) \cap \mathcal{N}$ and $\mathcal{N} \subseteq \mathcal{T}_2(U)$, which confirms that, without loss of generality, we can find a sequence such that $\{u_n\}_{n=1}^\infty \subset U$ verifying (12), enabling us to choose $u \in U$. Employing (44), (45) and (48) and continuity of H , we get

$$\begin{aligned} \mathcal{T}_1(u) &= H(\mathcal{T}_2 u) = H \left(\lim_{n \rightarrow \infty} \mathcal{T}_2 u_n \right) = \lim_{n \rightarrow \infty} H(\mathcal{T}_2 u_n) \\ &= \lim_{n \rightarrow \infty} \mathcal{T}_1(u_n) = \mathcal{T}_2(u). \end{aligned} \quad (49)$$

Hence, u is a coincidence point of $\mathcal{T}_1$ and $\mathcal{T}_2$, and hence again, we are through.

Lastly, suppose that $\mathcal{R}|_{\mathcal{N}}$ is d -self-closed. Since $\{\mathcal{T}_2 u_n\}$ is $\mathcal{R}|_{\mathcal{N}}$ -preserving (owing to (14)) and $\mathcal{T}_2 u_n \xrightarrow{d} \mathcal{T}_2 u \in \mathcal{N}$ (in view of (44)), owing to d -self-closedness of $\mathcal{R}|_{\mathcal{N}}$, there exists a subsequence $\{\mathcal{T}_2 u_{n_k}\}$ of $\{\mathcal{T}_2 u_n\}$ such that

$$[\mathcal{T}_2 u_{n_k}, \mathcal{T}_2 u] \in \mathcal{R}|_{\mathcal{N}}, \quad \forall k \in \mathbb{N}_0. \quad (50)$$

Applying (12), (50), assumption (d), and Proposition 7, we obtain

$$\begin{aligned} d(\mathcal{T}_2 u_{n_k+1}, \mathcal{T}_1 u) &= d(\mathcal{T}_1 u_{n_k}, \mathcal{T}_1 u) \\ &\leq \phi(d(\mathcal{T}_2 u_{n_k}, \mathcal{T}_2 u)), \quad \forall k \in \mathbb{N}_0. \end{aligned} \quad (51)$$

We claim that

$$d(\mathcal{T}_2 u_{n_k+1}, \mathcal{T}_1 u) \leq d(\mathcal{T}_2 u_{n_k}, \mathcal{T}_2 u), \quad \forall k \in \mathbb{N}. \quad (52)$$

In order to establish (52), we take a partition $\{\mathbb{N}^0, \mathbb{N}^+\}$ of $\mathbb{N}$, i.e., $\mathbb{N}^0 \cup \mathbb{N}^+ = \mathbb{N}$ and $\mathbb{N}^0 \cap \mathbb{N}^+ = \emptyset$, verifying that

- (i) $d(\mathcal{T}_2 u_{n_k}, \mathcal{T}_2 u) = 0, \quad \forall k \in \mathbb{N}^0$
- (ii) $d(\mathcal{T}_2 u_{n_k}, \mathcal{T}_2 u) > 0, \quad \forall k \in \mathbb{N}^+$

For case (i), utilizing (50) and $(\mathcal{T}_1, \mathcal{T}_2)$ -compatibility of $\mathcal{R}$, we get $d(\mathcal{T}_1 u_{n_k}, \mathcal{T}_1 u) = 0 \forall k \in \mathbb{N}^0$, and hence, (52) holds for all $k \in \mathbb{N}^0$. For case (ii), owing to the definition of Φ , we have $d(\mathcal{T}_2 u_{n_k+1}, \mathcal{T}_1 u) = d(\mathcal{T}_1 u_{n_k}, \mathcal{T}_1 u) \leq \phi(d(\mathcal{T}_2 u_{n_k}, \mathcal{T}_2 u)) < d(\mathcal{T}_2 u_{n_k}, \mathcal{T}_2 u), \forall k \in \mathbb{N}^+$, and hence, (52) holds for all $k \in \mathbb{N}^+$. Thus, (52) holds for all $k \in \mathbb{N}$.

Applying (44), (52), and continuity of d , we get

$$\begin{aligned} d(\mathcal{T}_2 u, \mathcal{T}_1 u) &= d\left(\lim_{k \rightarrow \infty} \mathcal{T}_2 u_{n_k+1}, \mathcal{T}_1 u\right) \\ &= \lim_{k \rightarrow \infty} d(\mathcal{T}_2 u_{n_k+1}, \mathcal{T}_1 u) \\ &\leq \lim_{k \rightarrow \infty} d(\mathcal{T}_2 u_{n_k}, \mathcal{T}_2 u) \\ &= 0, \end{aligned} \quad (53)$$

so that

$$\mathcal{T}_2(u) = \mathcal{T}_1(u). \quad (54)$$

Hence, u is a coincidence point $\mathcal{T}_1$ and $\mathcal{T}_2$. Thus in all cases, $\mathcal{T}_1$ and $\mathcal{T}_2$ have a coincidence point on $\mathcal{T}_1$ and $\mathcal{T}_2$, which completes the proof.

In lieu of Propositions 1 and 2, we obtain the following consequence of Theorem 2. $\square$

Corollary 1. *Theorem 2 remains true if “locally finitely $\mathcal{T}_1$ -transitive” (employed in hypothesis (b)) is replaced by one of the following conditions (besides retaining the rest of the hypotheses):*

- (i) $\mathcal{R}$ is transitive
- (ii) $\mathcal{R}$ is finitely transitive
- (iii) $\mathcal{R}$ is $\mathcal{T}_1$ -transitive
- (iv) $\mathcal{R}$ is $\mathcal{T}_2$ -transitive
- (v) $\mathcal{R}$ is finitely $\mathcal{T}_1$ -transitive
- (vi) $\mathcal{R}$ is finitely $\mathcal{T}_2$ -transitive
- (vii) $\mathcal{R}$ is locally $\mathcal{T}_1$ -transitive
- (viii) $\mathcal{R}$ is locally $\mathcal{T}_2$ -transitive.

Corollary 2. *Let $\mathcal{M}$ be a nonempty set, $\mathcal{R}$ a binary relation, and d a metric such that $(\mathcal{M}, d)$ is an $\mathcal{R}$ -complete metric space. Let $(\mathcal{T}_1, \mathcal{T}_2)$ be a pair of self-mappings on $\mathcal{M}$. Assume that the following conditions hold:*

- (a) $\mathcal{T}_1(\mathcal{M}) \subseteq \mathcal{T}_2(\mathcal{M})$
- (b) $\mathcal{R}$ is $(\mathcal{T}_1, \mathcal{T}_2)$ -closed and locally finitely $\mathcal{T}_1$ -transitive
- (c) $\mathcal{M}(\mathcal{T}_1, \mathcal{T}_2, \mathcal{R})$ is nonempty
- (d) There exists $\phi \in \Phi$ such that $d(\mathcal{T}_1 u, \mathcal{T}_1 v) \leq \phi(d(\mathcal{T}_2 u, \mathcal{T}_2 v)) \forall u, v \in \mathcal{M}$ with $(\mathcal{T}_2 u, \mathcal{T}_2 v) \in \mathcal{R}$
- (e) $(e_1)\mathcal{T}_1$ and $\mathcal{T}_2$ are $\mathcal{R}$ -compatible
- (e₂) $\mathcal{T}_2$ is $\mathcal{R}$ -continuous
- (e₃) Either $\mathcal{T}_1$ is $\mathcal{R}$ -continuous or $\mathcal{R}$ is $(\mathcal{T}_1, \mathcal{T}_2)$ -compatible and $(\mathcal{T}_2, d)$ -self-closed

Or alternately,

- (e') (e'₁) There exists an $\mathcal{R}$ -closed subspace $\mathcal{N}$ of $\mathcal{M}$ such that $\mathcal{T}_1(\mathcal{M}) \subseteq \mathcal{N} \subseteq \mathcal{T}_2(\mathcal{M})$
- (e'₂) Either $\mathcal{T}_1$ is $(\mathcal{T}_2, \mathcal{R})$ -continuous or $\mathcal{T}_1$ and $\mathcal{T}_2$ are continuous or $\mathcal{R}$ and $\mathcal{R}|_{\mathcal{N}}$ are $(\mathcal{T}_1, \mathcal{T}_2)$ -compatible and d -self-closed, respectively.

Then, $\mathcal{T}_1$ and $\mathcal{T}_2$ have a coincidence point.

Proof. The proof of the part (e) and the alternative part (e') remains as follows by setting $\mathcal{N} = \mathcal{M}$ in Theorem 2 and utilizing Proposition 6, respectively. $\square$

Corollary 3. *Conclusions of Theorem 2 (even Corollary 1) remain true, if we replace the assumption $(\mathcal{T}_1, \mathcal{T}_2)$ -compatibility of $\mathcal{R}$ (utilized in assumptions (e₃) and (e'₂)) by one of the following conditions while retaining the rest of the hypotheses:*

- (i) $\phi(0) = 0$
- (ii) $\mathcal{T}_2$ is one-to-one.

Proof. Assume that (i) holds. Let $u, v \in \mathcal{M}$ such that $(\mathcal{T}_2 u, \mathcal{T}_2 v) \in \mathcal{R}$ with $\mathcal{T}_2 u = \mathcal{T}_2 v$. Utilizing the contractive condition (d), we get $d(\mathcal{T}_1 u, \mathcal{T}_1 v) \leq \phi(0) = 0$, which implies that $\mathcal{T}_1 u = \mathcal{T}_1 v$. It follows that $\mathcal{R}$ is $(\mathcal{T}_1, \mathcal{T}_2)$ -compatible. Next, assume that (ii) holds. Take $u, v \in \mathcal{M}$ such that $(\mathcal{T}_2 u, \mathcal{T}_2 v) \in \mathcal{R}$ and $\mathcal{T}_2 u = \mathcal{T}_2 v$. Since $\mathcal{T}_2$ is one-to-one, we get $u = v$, which implies that $\mathcal{T}_1 u = \mathcal{T}_1 v$. Hence, $\mathcal{R}$ is $(\mathcal{T}_1, \mathcal{T}_2)$ -compatible.

In lieu of Remarks 2–6, we obtain the more natural form of Theorem 2 in the following consequence. $\square$

Corollary 4. *If the natural metrical notions such as completeness, closedness, compatibility (or commutativity or weak commutativity), continuity, and $\mathcal{T}_2$ -continuity are used instead of their respective $\mathcal{R}$ -analogues, then Theorem 2 (and Corollary 2) remains holding.*

Now, we present a uniqueness result corresponding to Theorem 2.

Theorem 3. *Assume in addition to hypotheses of Theorem 2, the following conditions hold:*

- (f₁): $\mathcal{T}_1(\mathcal{M})$ is $\mathcal{R}|_{\mathcal{T}_2(\mathcal{M})}$ -connected
- (f₂): $\mathcal{R}$ is $(\mathcal{T}_1, \mathcal{T}_2)$ -compatible.

Then, $\mathcal{T}_1$ and $\mathcal{T}_2$ have a unique point of coincidence.

Proof. In lieu of Theorem 2, we have $\overline{C}(\mathcal{T}_1, \mathcal{T}_2) \neq \emptyset$. Let $\bar{u}, \bar{v} \in \overline{C}(\mathcal{T}_1, \mathcal{T}_2)$, then there exist $u, v \in \mathcal{M}$ such that

$$\begin{aligned}\bar{u} &= \mathcal{T}_2(u) = \mathcal{T}_1(u), \\ \bar{v} &= \mathcal{T}_2(v) = \mathcal{T}_1(v).\end{aligned}\quad (55)$$

Now, we assert that $\bar{u} = \bar{v}$. Since $\mathcal{T}_1(u), \mathcal{T}_1(v) \in \mathcal{T}_1(\mathcal{M}) \subseteq \mathcal{T}_2(\mathcal{M})$, by assumption (f_1) , there exists a path (say $\{\mathcal{T}_2 w_0, \mathcal{T}_2 w_1, \mathcal{T}_2 w_2, \dots, \mathcal{T}_2 w_k\}$) of some finite length k in $\mathcal{R}|_{\mathcal{T}_2(\mathcal{M})}$ from $\mathcal{T}_1 u$ to $\mathcal{T}_1 v$ (where $w_0, w_1, w_2, \dots, w_k \in \mathcal{M}$). Owing to (55), without loss of generality, we may set $w_0 = u$ and $w_k = v$, and we have

$$[\mathcal{T}_2 w_i, \mathcal{T}_2 w_{i+1}] \in \mathcal{R}|_{\mathcal{T}_2(\mathcal{M})} \text{ for each } i \ (0 \leq i \leq k-1). \quad (56)$$

Now, we construct the constant sequences $w_n^0 = u$ and $w_n^k = v$. Utilizing (55), we have $\mathcal{T}_2 w_{n+1}^0 = \mathcal{T}_1 w_n^0 = \bar{u}$ and $\mathcal{T}_2 w_{n+1}^k = \mathcal{T}_1 w_n^k = \bar{v} \forall n \in \mathbb{N}_0$. Put $w_0^1 = w_1, w_0^2 = w_2, \dots, w_0^{k-1} = w_{k-1}$. Since $\mathcal{T}_1(\mathcal{M}) \subseteq \mathcal{T}_2(\mathcal{M})$, on the patterns similar to that of Theorem 2, we can obtain sequences $\{w_n^0\}, \{w_n^1\}, \dots, \{w_n^{k-1}\}$ in $\mathcal{M}$ way such that $\mathcal{T}_2(w_{n+1}^1) = \mathcal{T}_1(w_n^1)$, $\mathcal{T}_2(w_{n+1}^2) = \mathcal{T}_1(w_n^2), \dots, \mathcal{T}_2(w_{n+1}^{k-1}) = \mathcal{T}_1(w_n^{k-1}) \forall n \in \mathbb{N}_0$. Hence, we obtain

$$\mathcal{T}_2(w_{n+1}^i) = \mathcal{T}_1(w_n^i) \forall n \in \mathbb{N}_0 \text{ and for each } i \ (0 \leq i \leq k). \quad (57)$$

Now, we assert that

$$[\mathcal{T}_2 w_n^i, \mathcal{T}_2 w_n^{i+1}] \in \mathcal{R} \forall n \in \mathbb{N}_0 \text{ and for each } i \ (0 \leq i \leq k-1). \quad (58)$$

We prove the assertion by using the mathematical induction. It follows from (56) that (58) holds for $n = 0$. Assume that (58) holds for $n = r > 0$, i.e.,

$$[\mathcal{T}_2 w_r^i, \mathcal{T}_2 w_r^{i+1}] \in \mathcal{R} \text{ for each } i \ (0 \leq i \leq k-1). \quad (59)$$

Since $\mathcal{R}$ is $(\mathcal{T}_1, \mathcal{T}_2)$ -closed, using Proposition 4, we obtain

$$[\mathcal{T}_1 w_r^i, \mathcal{T}_1 w_r^{i+1}] \in \mathcal{R} \text{ for each } i \ (0 \leq i \leq k-1). \quad (60)$$

Employing (57) gives rise to

$$[\mathcal{T}_2 w_{r+1}^i, \mathcal{T}_2 w_{r+1}^{i+1}] \in \mathcal{R} \text{ for each } i \ (0 \leq i \leq k-1). \quad (61)$$

Hence, (58) holds for $n = r + 1$. Thus, by induction, (58) holds for all $n \in \mathbb{N}_0$. Now for all $n \in \mathbb{N}_0$ and for each $i \ (0 \leq i \leq k-1)$, define $\delta_n^i = d(\mathcal{T}_2 w_n^i, \mathcal{T}_2 w_n^{i+1})$. Then, we claim that

$$\lim_{n \rightarrow \infty} \delta_n^i = 0 \text{ for each } i \ (0 \leq i \leq k-1). \quad (62)$$

We fix i and distinguish two cases. Firstly, assume that $\delta_{n_0}^i := d(\mathcal{T}_2 w_{n_0}^i, \mathcal{T}_2 w_{n_0}^{i+1}) = 0$ for some $n_0 \in \mathbb{N}_0$, i.e.,

$$\mathcal{T}_2(w_{n_0}^i) = \mathcal{T}_2(w_{n_0}^{i+1}). \quad (63)$$

Now, owing to (57) and (63) and utilizing the hypothesis (f_2) , we obtain $d(\mathcal{T}_1 w_{n_0}^i, \mathcal{T}_1 w_{n_0}^{i+1}) = 0$. Consequently owing to (57), $\delta_{n_0+1}^i = d(\mathcal{T}_2 w_{n_0+1}^i, \mathcal{T}_2 w_{n_0+1}^{i+1}) = d(\mathcal{T}_1 w_{n_0}^i, \mathcal{T}_1 w_{n_0}^{i+1}) = 0$. Thus by induction, we get $\delta_n^i = 0 \forall n \geq n_0$, yielding thereby $\lim_{n \rightarrow \infty} \delta_n^i = 0$, so that $\delta_{n_0+1}^i$ must be “0,” and hence by mathematical induction on n , we get $\delta_n^i = 0 \forall n \geq n_0$ and $\lim_{n \rightarrow \infty} \delta_n^i = 0$. Secondly, assume that $\delta_n^i > 0 \forall n \in \mathbb{N}_0$. Due to (57), (58), and hypothesis (d), we have

$$\begin{aligned}\delta_{n+1}^i &= d(\mathcal{T}_2 w_{n+1}^i, \mathcal{T}_2 w_{n+1}^{i+1}) = d(\mathcal{T}_1 w_n^i, \mathcal{T}_1 w_n^{i+1}) \\ &\leq \phi(d(\mathcal{T}_2 w_n^i, \mathcal{T}_2 w_n^{i+1})),\end{aligned}\quad (64)$$

so that

$$\delta_{n+1}^i \leq \phi(\delta_n^i). \quad (65)$$

In view of Lemma 1 and (65), we have

$$\lim_{n \rightarrow \infty} \delta_n^i = 0. \quad (66)$$

Thus in all cases, (62) is proved for each $i \ (0 \leq i \leq k-1)$. Utilizing the triangular inequality and (62), we obtain

$$d(\bar{u}, \bar{v}) \leq \delta_n^0 + \delta_n^1 + \dots + \delta_n^{k-1} \longrightarrow 0 \text{ as } n \longrightarrow \infty. \quad (67)$$

Therefore, $\bar{u} = \bar{v}$, which exhausts the proof. $\square$

Corollary 5. Replace the condition (f_1) in Theorem 3 by one of the following conditions:

$$\begin{aligned}(f'_1) \mathcal{R}|_{\mathcal{T}_1(\mathcal{M})} &\text{ is complete} \\ (f''_1) \mathcal{T}_1(\mathcal{M}) &\text{ is } \mathcal{R}|_{\mathcal{T}_2(\mathcal{M})}\text{-directed.}\end{aligned}$$

Then, the same conclusion of Theorem 3 follows.

Theorem 4. Suppose all the hypotheses of Theorem 3 and the following condition hold:

$$(f_3) \text{ Either } \mathcal{T}_1 \text{ or } \mathcal{T}_2 \text{ is one-to-one.}$$

Then, $\mathcal{T}_1$ and $\mathcal{T}_2$ have a unique point of coincidence.

Theorem 5. Suppose all the hypotheses of Theorem 3 and the following condition embodied in condition (e') of Theorem 3 hold:

$$(e'_3): \mathcal{T}_1 \text{ and } \mathcal{T}_2 \text{ are weakly compatible.}$$

Then, $\mathcal{T}_1$ and $\mathcal{T}_2$ have a unique common fixed point.

The proofs of Corollary 5 and of Theorems 4 and 5 can be obtained in the similar lines of Corollary 4.6 and Theorems 4.7 and 4.8, respectively, contained in [35].

Taking $\mathcal{T}_2 = I_{\mathcal{M}}$, identity mapping on $\mathcal{M}$, and Theorems 2 and 3, we get the following fixed point result which is a sharpened version corresponding to Theorem 1.

Corollary 6. Let $\mathcal{M}$ be a nonempty set equipped with a binary relation $\mathcal{R}$ and a metric d . Let $\mathcal{T}_1$ be a self-mapping

on $\mathcal{M}$ and $\mathcal{N}$ be an $\mathcal{R}$ -complete subspace of $\mathcal{M}$ such that $\mathcal{T}_1(\mathcal{M}) \subseteq \mathcal{N}$. Assume that the following conditions hold:

- (i) $\mathcal{R}$ is $\mathcal{T}_1$ -closed and locally finitely $\mathcal{T}_1$ -transitive.
- (ii) Either $\mathcal{T}_1$ is $\mathcal{R}$ -continuous or $\mathcal{R}|_{\mathcal{N}}$ is d -self-closed.
- (iii) $\mathcal{M}(\mathcal{T}_1, \mathcal{R})$ is nonempty.
- (iv) There exists $\phi \in \Phi$ such that

$$d(\mathcal{T}_1 u, \mathcal{T}_1 v) \leq \phi(d(u, v)), \quad \forall u, v \in \mathcal{M} \text{ with } (u, v) \in \mathcal{R}. \quad (68)$$

Then, $\mathcal{T}_1$ has a fixed point.

- (v) Furthermore, if $\mathcal{T}_1(\mathcal{M})$ is $\mathcal{R}^s$ -connected, then $\mathcal{T}_1$ has a unique fixed point.

5. Examples

In this section, we present some examples to support the worth of our newly proved results.

Example 1. Let $\mathcal{M} = [0, 4)$ equipped with a usual metric d and binary relation $\mathcal{R} = \{(0, 0), (1, 1), (0, 1), (1, 0), (0, 3), (2, 3)\}$. We consider a pair of self-mappings $(\mathcal{T}_1, \mathcal{T}_2)$ on $\mathcal{M}$ defined by

$$\begin{aligned} \mathcal{T}_1(u) &= \begin{cases} [u], & u \in [0, 1); \\ 1, & u \in [1, 4), \end{cases} \\ \mathcal{T}_2(u) &= \begin{cases} u, & u \in [0, 1) \setminus \{1/2\}; \\ 1, & u = \frac{1}{2}; \\ 3, & \text{otherwise.} \end{cases} \end{aligned} \quad (69)$$

Let $\mathcal{N} = \{0, 1\}$ be $\mathcal{R}$ -complete such that $\mathcal{T}_1(\mathcal{M}) = \{0, 1\} \subset \mathcal{T}_2(\mathcal{M}) = ([0, 1] \setminus \{1/2\}) \cup \{3\}$ and $\mathcal{N} \subset \mathcal{T}_2(\mathcal{M}) = ([0, 1] \setminus \{1/2\}) \cup \{3\}$. Define a mapping $\phi: [0, \infty) \rightarrow [0, \infty)$ by $\phi(t) = (1/3)t$. Clearly, $0 \in \mathcal{M}(\mathcal{T}_1, \mathcal{T}_2, \mathcal{R})$, $\phi \in \Phi$, both $\mathcal{T}_1$ and $\mathcal{T}_2$ are not continuous, and $\mathcal{R}$ is $(\mathcal{T}_1, \mathcal{T}_2)$ -closed. It is easy to see that $\mathcal{R}$ is locally finitely $\mathcal{T}_1$ -transitive relation but neither locally $\mathcal{T}_2$ -transitive nor transitive. For any $\mathcal{R}$ -preserving sequence $\{u_n\}$ in $\mathcal{N}$,

$$(u_n, u_{n+1}) \in \mathcal{R}|_{\mathcal{N}}, \quad \text{for all } n \in \mathbb{N} \text{ with } u_n \xrightarrow{d} u. \quad (70)$$

Notice that if $(u_n, u_{n+1}) \in \mathcal{R}|_{\mathcal{N}}$, for all $n \in \mathbb{N}$, then there exists $K \in \mathbb{N}$ such that $u_n = u \in \{0, 1\}$, for all $n \geq K$. So, we are able to choose a subsequence $\{u_{n_k}\}$ of the sequence $\{u_n\}$ such that $u_{n_k} = u$ (for all $k \in \mathbb{N}$), which amounts to referring that $[u_{n_k}, u] \in \mathcal{R}|_{\mathcal{N}}$ (for all $k \in \mathbb{N}$). Hence, $\mathcal{R}|_{\mathcal{N}}$ is d -self-closed. It is easy to see that $\mathcal{R}$ is $(\mathcal{T}_1, \mathcal{T}_2)$ -compatible. Now, to verify the contractive condition (d) of Theorem 2. To do this, we need to show that only for $(\mathcal{T}_2 u, \mathcal{T}_2 v) \in \{(0, 3)\}$ and for the remaining elements can be easily verified. If $(\mathcal{T}_2 u,$

$\mathcal{T}_2 v) = (0, 3)$, then $u = 0$ and $v \in [1, 4)$, which implies that $\mathcal{T}_1 0 = 0$ and $\mathcal{T}_1 v = 1$:

$$\begin{aligned} 1 &= d(\mathcal{T}_1 0, \mathcal{T}_1 v) \leq \phi(d(\mathcal{T}_2 0, \mathcal{T}_2 3)) \\ &= \phi(d(0, 3)) \\ &= \phi(3) = 1. \end{aligned} \quad (71)$$

Clearly, $\mathcal{R}$ is $(\mathcal{T}_1, \mathcal{T}_2)$ -compatible. Thus, all the hypotheses ((a), (b), (c), (d), and (e')) of Theorem 2 are satisfied; hence, $\mathcal{T}_1$ and $\mathcal{T}_2$ have a coincidence point (namely, $C(\mathcal{T}_1, \mathcal{T}_2) = \{0\}$). Notice that $\mathcal{T}_1$ and $\mathcal{T}_2$ are weakly compatible, i.e., commute at their coincidence point. Therefore, all the hypotheses of Theorem 4 are satisfied. Notice that " $u = 0$ " is the unique common fixed point of $\mathcal{T}_1$ and $\mathcal{T}_2$.

Example 2. Consider $\mathcal{M} = [0, 5)$ equipped with a usual metric d and binary relation $\mathcal{R} = \{(0, 0), (1, 2), (2, 3), (3, 0), (1, 0), (2, 0)\}$. On $\mathcal{M}$, we define two self-mappings $\mathcal{T}_1$ and $\mathcal{T}_2$ by

$$\begin{aligned} \mathcal{T}_1(u) &= \begin{cases} \left[\frac{u}{2}\right], & 0 \leq u \leq 1; \\ 1, & u = 2, \\ 3, & u = 3, \\ 2, & \text{otherwise,} \end{cases} \\ \mathcal{T}_2(u) &= \begin{cases} u, & u \in [0, 1] \setminus \{1/2, 1/3\}; \\ 2, & u = \frac{1}{2}; \\ 3, & u = \frac{1}{3}; \\ 4, & \text{otherwise,} \end{cases} \end{aligned} \quad (72)$$

where $[.]$ stands for the greatest integer function. It is easy to see that $\mathcal{R}$ is neither locally $\mathcal{T}_1$ -transitive nor transitive but locally finitely $\mathcal{T}_1$ -transitive relation (i.e., $\mathcal{R}|_{\mathcal{T}_1(\mathcal{M})}$ is 3-transitive). Let $\mathcal{N} = \{0, 1, 2, 3\}$ be $\mathcal{R}$ -complete such that $\mathcal{T}_1(\mathcal{M}) = \{0, 1, 2, 3\} \subset \mathcal{T}_2(\mathcal{M}) = ([0, 1] \setminus \{1/2, 1/3\}) \cup \{2, 3, 4\} \cap \mathcal{N}$ and $\mathcal{N} \subset \mathcal{T}_2(\mathcal{M}) = ([0, 1] \setminus \{1/2, 1/3\}) \cup \{2, 3, 4\}$. Define a mapping $\phi: [0, \infty) \rightarrow [0, \infty)$ by $\phi(t) = (3/4)t$. Then, $\phi \in \Phi$. Notice that as $0 \in \mathcal{M}(\mathcal{T}_1, \mathcal{T}_2, \mathcal{R})$, $\phi \in \Phi$, both $\mathcal{T}_1$ and $\mathcal{T}_2$ are not continuous, and $\mathcal{R}$ is $(\mathcal{T}_1, \mathcal{T}_2)$ -closed. Choose any $\mathcal{R}$ -preserving sequence $\{u_n\}$ in $\mathcal{N}$, i.e.,

$$(u_n, u_{n+1}) \in \mathcal{R}|_{\mathcal{N}}, \quad \text{for all } n \in \mathbb{N} \text{ with } u_n \xrightarrow{d} u. \quad (73)$$

As if $(u_n, u_{n+1}) \in \mathcal{R}|_{\mathcal{N}}$, for all $n \in \mathbb{N}$, then there exists $K \in \mathbb{N}$ such that $u_n = u \in \{0\}$, for all $n \geq K$. Thus, we choose a subsequence $\{u_{n_k}\}$ of the sequence $\{u_n\}$ such that $u_{n_k} = u$ (for all $k \in \mathbb{N}$), which amounts to saying that $[u_{n_k}, u] \in \mathcal{R}|_{\mathcal{N}}$ (for all $k \in \mathbb{N}$). Hence, $\mathcal{R}|_{\mathcal{N}}$ is d -self-closed. In view of $\mathcal{T}_1$ and

$\mathcal{T}_2$, it is easy to see that $\mathcal{R}$ is $(\mathcal{T}_1, \mathcal{T}_2)$ -compatible. By straightforward calculations, one can verify the underlying contractive condition (d) of Theorem 2. Clearly, all the hypotheses ((a), (b), (c), (d), and (e')) of Theorem 2 are satisfied; hence, $\mathcal{T}_1$ and $\mathcal{T}_2$ have a coincidence point (namely, $C(\mathcal{T}_1, \mathcal{T}_2) = \{0\}$). Notice that $\mathcal{T}_1$ and $\mathcal{T}_2$ commute at their coincidence point. Therefore, condition (e₃') of Theorem 5 is also satisfied. Hence, all the hypotheses of Theorem 5 are satisfied. Observe that $x = 0$ is a unique common fixed point of $\mathcal{T}_1$ and $\mathcal{T}_2$ (also ϕ lies in nonlinear class of [32]), whereas main results of Alam et al. [32] are not applicable because $\mathcal{R}$ is not locally $\mathcal{T}_1$ -transitive.

6. Conclusions

In the context of nonlinear contractions, the use of arbitrary binary relation cannot ensure the existence of coincidence results (e.g., [16]). Alam et al. [32] employed the locally $\mathcal{T}_1$ -transitive binary relation to prove their results. With a view to have further improvement, we utilized the relatively weaker notion than the locally $\mathcal{T}_1$ -transitive binary relation, namely, locally finitely $\mathcal{T}_1$ -transitive relation, but at the same time, the class of nonlinear contractions remains relatively smaller. However, in order to prove their results, Alam et al. [32] utilized Boyd–Wong nonlinear contraction defined by

$$\Omega = \left\{ \phi: [0, \infty) \longrightarrow [0, \infty): \phi(t) < t \text{ for each } t > 0 \text{ and } \limsup_{r \rightarrow t^+} \phi(r) < t \text{ for each } t > 0 \right\}, \quad (74)$$

which enlarges the class Φ (described earlier), i.e., $\Phi \subset \Omega$.

Henceforth, our newly proved results and the results due to Alam et al. [32] are independent of each other. None of them is a proper generalization of another.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All the authors read and approved the final manuscript.

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