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*Research Article*

# Discussion on Geraghty Type Hybrid Contractions

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In this manuscript, we define the notion of Geraghty type hybrid contractions in the setting of $b$-metric spaces. We prove the existence of a fixed point for such mappings whenever $b$-metric space is complete. Our observed results not only unify several existing results but also extend some known results.

## 1. Introduction and Preliminaries

The distance notion is one of the ancient and most basic concepts in the history of mathematics. In modern mathematics history, this notion was formally formulated by Frechét as "$L$-space." Later, it was redefined as "metric space" by Hausdorff. After then, this concept has been extended and generalized in several ways. From all these generalizations of metric notions, the $b$-metric is the most interesting.

In order to introduce the subject clearly, we first fix the basic concepts and notations. A function $\delta$, defined from $X \times X$ (where $X$ is a nonempty set) to nonnegative reals, is said to be a distance function, if it is symmetrical, that is $\delta(u, v) = \delta(v, u)$, for every $u, v \in X$ and $\delta(u, v) = 0$ if and only if $u = v$.

Moreover, a distance function $\delta$ is a (standard) metric in case that

$$\delta(u, v) \leq \delta(u, \nu) + \delta(\nu, v), \text{ for all } u, v, \nu \in X. \tag{1}$$

As we mentioned above, the distance notion, as well as the notion of the metric, has been extended and generalized in several directions. One of the outstanding generalizations of metric notion is named $b$-metric. Indeed, the concept of $b$-metric was considered by distinct authors, in various periods of the time, involving Bakhtin [1] and Czerwik [2]. Later, many researchers were interested in this topic, and thus, a series of interesting results were obtained, see, e.g., [3–19].

*Definition 1.* A distance function $b$ on $X$ is said to be a b-metric over constant $s \geq 1$ if the inequality (weighted triangle inequality)

$$b(u, v) \leq s[\mathrm{b}(u, \nu) + \mathrm{b}(\nu, v)], \tag{2}$$

holds for *all* $u, v, \nu \in X$.

In what follows, we consider that $(X, \mathrm{b}, s)$ denotes a $b$-metric space.

An immediate observation is that the notion of $b$-metric is more general than the concept of metric; for instance, when $s = 1$, we recover the notion of metric space. Moreover, we mention that a $b$-metric is not necessarily continuous, see, e.g., [20, 21].

*Example 2.* The function $b$ on $X = [0,\infty)$, where $b(u, v) = |u - v|^q$, $q > 1$, is a $b$-metric over $s = 2^q$, but not a metric.

*Definition 3.* On a $b$-metric space $(X, b, s)$, let $\{u_n\}$ be a sequence in $X$.

(a) The sequence $\{u_n\}$ is convergent in $(X, b, s)$ to $u$, if for every $e > 0$, there exists $n_0 \in \mathbb{N}$ such that $b(u_n, u) < e$ for all $n > n_0$. (We denote by $u_n \to u$ as $n \to \infty$ or $\lim_{n\to\infty} u_n = u$.)

(b) The sequence $\{u_n\}$ is Cauchy, if for every $e > 0$, there exists $n_0 \in \mathbb{N}$ such that $b(u_n, u_{n+l}) < e$ for all $n > n_0, l > 0$

(c) If every Cauchy sequence in $X$ converges to a point $u \in X$, then the triplet $(X, b, s)$ is said to be complete

In short, $(X^*, b, s)$ denotes a complete $b$-metric space over $s$.

Recently, Mitrovic et al. [22] introduced the following type of contractions.

*Definition 4* (see [22]). Let $(X, b, s)$ and $T : X \to X$ be a self-mapping. We say that $T$ is a $(r, a)$ -weight type contraction, if there exists $\kappa \in [0, 1)$ such that

$$b(Tu, Tv) \le \kappa \cdot M^r(T, u, v, a), \tag{3}$$

where $r \ge 0$, $a = (a_1, a_2, a_3)$,$a_i \ge 0, i = 1, 2, 3$ such that $a_1 + a_2 + a_3 = 1$ and

$$M^r(T, u, v, a) = \begin{cases} [a_1(b(u, v))^r + a_2(b(u, Tu))^r + a_3(b(v, Tv))^r]^{1/r}, & r > 0 \\ (b(u, v))^{a_1}(b(u, Tu))^{a_2}(b(v, Tv))^{a_3}, & r = 0, \end{cases} \tag{4}$$

for all $u, v \in X \setminus \mathrm{Fix}(T)$, where $\mathrm{Fix}(T) = \{\varpi \in X, T\varpi = \varpi\}$.

In 1973, Geraghty [23] introduced a class of auxiliary functions to refine the Banach contraction principle. Let $\mathcal{G}$ be the set defined as

$$\mathcal{G} = \left\{\beta_{\mathrm{b}} : [0,\infty) \to [0, 1) \mid \lim_{n\to\infty} \beta_{\mathrm{b}}(t_n) = 1 \text{ implies } \lim_{n\to\infty} t_n = 0\right\}. \tag{5}$$

**Theorem 5** (see Geraghty [23]). *On a complete metric space $(X, d)$, a mapping $T : X \to X$ admits a unique fixed point provided that there exists a function $\beta \in \mathcal{G}$ such that*

$$d(Tu, Tv) \le \beta(d(u, v))d(u, v), \text{ for any } u, v \in X. \tag{6}$$

## 2. Main Results

Let the set $\mathcal{G}_b = \{\beta_{\mathrm{b}} : [0,\infty) \to [0, (1/s)) \mid \limsup_{n\to\infty} \beta_{\mathrm{b}}(t_n) = (1/s) \text{ implies } \lim_{n\to\infty} t_n = 0.\}$.

*Definition 6.* On $(X, b, s)$, a mapping $T : X \to X$ is called Geraghty type hybrid contraction, if there exists $\beta_b \in \mathcal{G}_b$ such that

$$b(Tu, Tv) \le \beta_{\mathrm{b}}(M^r(T, u, v, a))M^r(T, u, v, a), \tag{7}$$

where $r \ge 0$, $a = (a_1, a_2, a_3) \in [0,\infty) \times [0,\infty) \times [0,\infty)$, with $a_1 < 1$, $a_1 + a_2 + a_3 = 1$ and

$$M^r(T, u, v, a) = \left\{ \begin{matrix} [a_1(b(u, v))^r + a_2(b(u, Tu))^r + a_3(b(v, Tv))^r]^{1/r}, & r > 0, & u, v \in X \\ (b(u, v))^{a_1}(b(u, Tu))^{a_2}(b(v, Tv))^{a_3}, & r = 0, & u, v \in X \setminus Fix(T). \end{matrix} \right\} \tag{8}$$

**Theorem 7.** *On $(X^*, b, s)$, a Geraghty type hybrid contraction $T : X \to X$ admits a unique fixed point $\varpi \in X$ if one of the following hypotheses is satisfied:*

*(i) $T$ is continuous at $\varpi$*

*(ii) $a_2 < 1$*

*(iii) $a_3 < 1$*

*Moreover, for any $u_0 \in X$ the sequence $\{T^n u_0\}$ converges to $\varpi$.*

*Proof.* We take an arbitrary point $u_0 \in X$. Starting from this initial point, we shall construct a recursive sequence $\{u_n\}$ with the following formula:

$$u_{n+1} = Tu_n \text{ for all } n \ge 0. \tag{9}$$

It is evident that if there exists $k_0$ such that $u_{k_0} = u_{k_0+1}$, then $u_{k_0}$ becomes a fixed point of $T$. Therefore, from now, we assume that

$$u_n \ne u_{n+1} \text{ for all } n \ge 0. \tag{10}$$

We shall discuss all possible situations.

*Case 8.* Suppose that $r > 0$. From (7), we have that

$$b(Tu_n, Tu_{n-1}) \le \beta_{\mathrm{b}}(M^r(T, u_n, u_{n-1}, a))M^r(T, u_n, u_{n-1}, a), \tag{11}$$

where

$$M^r(T, u_n, u_{n-1}, a) = [a_1(b(u_n, u_{n-1}))^r + a_2(b(u_n, u_{n+1}))^r + a_3(b(u_{n-1}, u_n))^r]^{1/r}. \tag{12}$$

It yields

$$b(u_{n+1}, u_n) \le \beta_b\left([(a_1 + a_3)(b(u_n, u_{n-1}))^r + a_2 b(u_n, u_{n+1})^r]^{1/r}\right) \cdot [(a_1 + a_3)(b(u_n, u_{n-1}))^r + a_2 b(u_n, u_{n+1})^r]^{1/r} \tag{13}$$

which is equivalent to

$$\frac{b(u_{n+1}, u_n)}{[(a_1 + a_3)(b(u_n, u_{n-1}))^r + a_2 b(u_n, u_{n+1})]^r]^{1/r}} \le \beta_b\left([(a_1 + a_3)(b(u_n, u_{n-1}))^r + a_2 b(u_n, u_{n+1})^r]^{1/r}\right) < \frac{1}{s}. \tag{14}$$

Therefore, we have

$$b(u_{n+1}, u_n) < \left[\frac{(a_1 + a_3)}{s^r - a_2}\right]^{1/r} b(u_n, u_{n-1}) \le b(u_n, u_{n-1}). \tag{15}$$

It yields $\{b(u_{n+1}, u_n)\}$ is nonincreasing sequence bounded by 0. Thus, the sequence $\{b(u_{n+1}, u_n)\}$ converges to a nonnegative real number, say $L$. We assert that $L$ is 0. On the one hand, by taking the lim sup of all sides of (13), we deduce that

$$L \le \limsup_{n\to\infty} \beta_b\left([(a_1 + a_3)(b(u_n, u_{n-1}))^r + a_2 b(u_n, u_{n+1})^r]^{1/r}\right) L < \frac{1}{s} L. \tag{16}$$

Suppose on the contrary, that $L > 0$, we obtain

$$\frac{1}{s} \le 1 \le \limsup_{n\to\infty} \beta_b\left([(a_1 + a_3)(b(u_n, u_{n-1}))^r + a_2 b(u_n, u_{n+1})^r]^{1/r}\right) < \frac{1}{s}. \tag{17}$$

Thus, the limit $\lim_{n\to\infty}[(a_1 + a_3)(b(u_n, u_{n-1}))^r + a_2 b(u_n, u_{n+1})^r]^{1/r} = 0$. Consequently, $L = 0$.

We assert that the sequence $\{u_n\}$ is $b$-Cauchy.

On contrary, supposing that the sequence $\{u_n\}$ is not $b$-Cauchy, we can find $e > 0$ and two sequences of positive integers $\{q_i\}$ and $\{p_i\}$, $p_i > q_i \ge i$ such that

$$b\left(u_{q_i}, u_{p_i}\right) \ge e \text{ and } b\left(u_{q_i}, u_{p_i-1}\right) < e, \tag{18}$$

$$\begin{aligned}
e &\le \liminf_{i\to\infty} b\left(u_{q_i}, u_{p_i}\right) \le \limsup_{i\to\infty} b\left(u_{q_i}, u_{p_i}\right) \le se \\
\frac{e}{s} &\le \liminf_{i\to\infty} b\left(u_{q_i+1}, u_{p_i}\right) \le \limsup_{i\to\infty} b\left(u_{q_i+1}, u_{p_i}\right) \le s^2 e \\
\frac{e}{s} &\le \liminf_{i\to\infty} b\left(u_{q_i}, u_{p_i+1}\right) \le \limsup_{i\to\infty} b\left(u_{q_i}, u_{p_i+1}\right) \le s^2 e \\
\frac{e}{s^2} &\le \liminf_{i\to\infty} b\left(u_{q_i+1}, u_{p_i+1}\right) \le \limsup_{i\to\infty} b\left(u_{q_i+1}, u_{p_i+1}\right) \le s^3 e.
\end{aligned} \tag{19}$$

On the other hand,

$$\begin{aligned}
\frac{e}{s} &\le b\left(u_{q_i+1}, u_{p_i}\right) = b\left(Tu_{q_i}, Tu_{p_i-1}\right) \\
&\le \beta_b\left(M^r\left(T, u_{q_i}, u_{p_i-1}, a\right)\right) M^r\left(T, u_{q_i}, u_{p_i-1}, a\right) \\
&< \frac{1}{s} M^r\left(T, u_{q_i}, u_{p_i-1}, a\right)
\end{aligned} \tag{20}$$

where

$$\begin{aligned}
M^r\left(T, u_{q_i}, u_{p_i-1}, a\right) &= \left[a_1\left(b\left(u_{q_i}, u_{p_i-1}\right)\right)^r + a_2\left(b\left(u_{q_i}, Tu_{q_i}\right)\right)^r + a_3\left(b\left(u_{p_i-1}, Tu_{p_i-1}\right)\right)^r\right]^{1/r} \\
&= \left[a_1\left(b\left(u_{q_i}, u_{p_i-1}\right)\right)^r + a_2\left(b\left(u_{q_i}, u_{q_i+1}\right)\right)^r + a_3\left(b\left(u_{p_i-1}, u_{p_i}\right)\right)^r\right]^{1/r}
\end{aligned} \tag{21}$$

Taking lim sup of (21), we find

$$\limsup_{i\to\infty} M^r\left(T, u_{q_i}, u_{p_i-1}, a\right) \le a_1^{1/r} e < e. \tag{22}$$

If we combine the observed inequalities above, in particular, (20) and (22), we have

$$\begin{aligned}
\frac{e}{s} &\le \limsup_{i\to\infty} b\left(u_{q_i+1}, u_{p_i}\right) \\
&\le \limsup_{i\to\infty} \beta_b\left(M^r\left(T, u_{q_i}, u_{p_i-1}, a\right)\right) M^r\left(T, u_{q_i}, u_{p_i-1}, a\right) \\
&< e \limsup_{i\to\infty} \beta_b\left(M^r\left(T, u_{q_i}, u_{p_i-1}, a\right)\right) < \frac{e}{s},
\end{aligned} \tag{23}$$

since $a_1 < 1$. It implies that

$$\frac{1}{s} \le \limsup_{i\to\infty} \beta_b\left(M^r\left(T, u_{q_i}, u_{p_i-1}, a\right)\right) \le \frac{1}{s}. \tag{24}$$

Since $\beta_b \in \mathcal{G}_b$, we conclude $\lim_{i\to\infty} M^r(T, u_{q_i}, u_{p_i-1}, a) = 0$. Attendantly,

$$\lim_{i\to\infty} n\left(u_{q_i}, u_{p_i-1}\right) = 0. \tag{25}$$

Under these observations, by employing the weighted triangle axiom together with (18), we get

$$e \le b\left(u_{q_i}, u_{p_i}\right) \le s\left[b\left(u_{q_i}, u_{p_i-1}\right) + b\left(u_{p_i-1}, u_{p_i}\right)\right] \to 0 as i \to \infty. \tag{26}$$

Therefore, $\{u_n\}$ is a $b$-Cauchy sequence in $(X^*, b, s)$, so we can find a point $\varpi \in X$ such that

$$\lim_{n\to\infty} u_n = \varpi. \tag{27}$$

We assert now that this point, $\varpi$ is a fixed point of $T$.

(i) Assuming that the mapping $T$ is continuous at $\varpi \in X$, since $\lim_{n\to\infty} \mathrm{b}(\varpi, u_{n+1}) = 0$, we have

$$\lim_{n\to\infty} b(Tu_n, T\varpi) = b(T\varpi, T\varpi) = 0 \tag{28}$$

and from (49), we get $b(\varpi, T\varpi) = 0$, i.e., $T\varpi = \varpi$.

For the other cases, we consider the inequality,

$$b(\varpi, T\varpi) \le s[b(\varpi, u_{n+1}) + b(Tu_n, T\varpi)], \tag{29}$$

for any $n \in \mathbb{N}$.

(ii) Suppose that $a_2 < 1$. If $T\varpi \ne \varpi$, we have

$$\begin{aligned} 0 < b(T\varpi, \varpi) &\le s[b(T\varpi, u_{n+1}) + b(u_{n+1}, \varpi)] \\ &= s[b(T\varpi, Tu_n) + b(u_{n+1}, \varpi)] \\ &\le s[\beta_b(M^r(T, \varpi, u_n, a))M^r(T, \varpi, u_n, a) + b(u_{n+1}, \varpi)] \\ &< M^r(T, \varpi, u_n, a) + sb(u_{n+1}, \varpi) \\ &= [a_1(b(\varpi, u_n))^r + a_2(b(\varpi, T\varpi))^r + a_3(b(u_n, Tu_n))^r]^{1/r} \\ &\quad + sb(u_{n+1}, \varpi) \end{aligned} \tag{30}$$

and when $n \to \infty$, we get

$$0 < b(T\varpi, \varpi) \le (a_2)^{1/r} b(T\varpi, \varpi). \tag{31}$$

Since $a_2 < 1$, we get a contradiction, that is, $T\varpi = \varpi$.

(iii) Suppose that $a_3 < 1$. Assuming that $b(T\varpi, \varpi) > 0$, we have

$$\begin{aligned} 0 < b(\varpi, T\varpi) &\le s[b(\varpi, u_{n+1}) + b(u_{n+1}, T\varpi)] \\ &= s[b(\varpi, u_{n+1}) + b(Tu_n, T\varpi)] \\ &\le s[b(\varpi, u_{n+1}) + \beta_b(M^r(T, u_n, \varpi, a))M^r(T, u_n, \varpi, a)] \\ &< sb(\varpi, u_{n+1}) + M^r(T, u_n, \varpi, a)] \le sb(\varpi, u_{n+1}) \\ &\quad + [a_1(b(u_n, \varpi))^r + a_2(b(u_n, u_{n+1}))^r + a_3(b(\varpi, T\varpi))^r]^{1/r}. \end{aligned} \tag{32}$$

Taking $n \to \infty$, we have

$$0 < b(T\varpi, \varpi) \le (a_3)^{1/r} b(T\varpi, \varpi), \tag{33}$$

which is a contradiction, since $a_3 < 1$. Therefore, $T\varpi = \varpi$.

*Case 9.* $r = 0$. Here, (7) and (8) become

$$b(Tu, Tv) \le \beta_b(I(T, u, v, a))I(T, u, v, a), \tag{34}$$

for every $u, v \in X$, where

$$I(T, u, v, a) := (b(u, v))^{a_1}(b(u, Tu))^{a_2}(b(v, Tv))^{1-a_1-a_2} \tag{35}$$

$\kappa \in [0, 1)$ and $a_1, a_2 \in (0, 1)$.

As in the proof of the case $r > 0$, we shall consider a recursive sequence $\{u_n = Tu_{n-1}\}$, starting with an arbitrary point $u \in X$ where $u_0 = u$. By using the same argument of this part of the proof, we presume that

$$u_n \ne u_{n+1} \text{ for all } n \ge 0. \tag{36}$$

Employing $u = u_{n-1}$ and $v = u_n$ in (34), we find that

$$\begin{aligned} b(u_n, u_{n+1}) = b(Tu_{n-1}, Tu_n) &\le \beta_b((b(u_{n-1}, u_n))^{a_1} \\ &\quad \cdot (b(u_{n-1}, Tu_{n-1}))^{a_2}(b(u_n, Tu_n))^{1-a_1-a_2}) \\ &\quad \cdot (b(u_{n-1}, u_n))^{a_1}(b(u_{n-1}, Tu_{n-1}))^{a_2} \\ &\quad \cdot (b(u_n, Tu_n))^{1-a_1-a_2} \\ &= \beta_b((b(u_{n-1}, u_n))^{a_1+a_2}(b(u_n, u_{n+1}))^{1-a_1-a_2}) \\ &\quad \cdot (b(u_{n-1}, u_n))^{a_1+a_2}(b(u_n, u_{n+1}))^{1-a_1-a_2} \\ &< \frac{1}{s}(b(u_{n-1}, u_n))^{a_1+a_2}(b(u_n, Tu_n))^{1-a_1-a_2} \\ &\le (b(u_{n-1}, u_n))^{a_1+a_2}(b(u_n, Tu_n))^{1-a_1-a_2} \end{aligned} \tag{37}$$

It yields that

$$\begin{aligned} (b(u_n, u_{n+1}))^{a_1+a_2} \le (b(u_{n-1}, u_n))^{a_1+a_2} iff b(u_n, u_{n+1}) \\ \le b(u_{n-1}, u_n), \end{aligned} \tag{38}$$

for each $n \in \mathbb{N}$. Attendantly, we deduce that the sequence of nonnegative numbers $\{b(u_{n-1}, u_n)\}$ is a nonincreasing sequence. Ergo, there is a real number $L \ge 0$ such that $\lim_{n\to\infty} b(u_{n-1}, u_n) = L$.

As in the previous case, we assert that $L = 0$. Supposing on the contrary, that $L > 0$, by taking lim sup in (37), we derive that

$$\begin{aligned} L \le \limsup_{n\to\infty} \beta_b((b(u_{n-1}, u_n))^{a_1}(b(u_{n-1}, Tu_{n-1}))^{a_2} \\ \cdot (b(u_n, Tu_n))^{1-a_1-a_2})L. \end{aligned} \tag{39}$$

Since $L > 0$, we obtain

$$\begin{aligned} \frac{1}{s} \le 1 \le \limsup_{n\to\infty} \beta_b((b(u_{n-1}, u_n))^{a_1}(b(u_{n-1}, u_n))^{a_2} \\ \cdot (b(u_n, u_{n+1}))^{1-a_1-a_2})\frac{1}{s}. \end{aligned} \tag{40}$$

Thus, $\lim_{n\to\infty}((b(u_{n-1}, u_n))^{a_1+a_2}(b(u_n, u_{n+1}))^{1-a_1-a_2}) = 0$ and consequently, $L = 0$.

We claim that $\{u_n\}$ is a $b$-Cauchy sequence. On contrary, if we suppose that $\{u_n\}$ is not a $b$-Cauchy sequence, that is, we can find $e > 0$ and the sequences $\{q_i\}$, $\{p_i\}$ of positive integers with $p_i > q_i \ge i$ such that

$$b\left(u_{q_i}, u_{p_i}\right) \ge e \text{ and } b\left(u_{q_i}, u_{p_i-1}\right) < e. \tag{41}$$

By the weighted triangle inequality, we have

$$e \leq b\left(u_{q_i}, u_{p_i}\right) \leq s\left[b\left(u_{q_i}, u_{q_i+1}\right) + b\left(u_{q_i+1}, u_{p_i}\right)\right]. \tag{42}$$

Since

$$\begin{aligned} b\left(u_{q_i+1}, u_{p_i}\right) &= b\left(Tu_{q_i}, Tu_{p_i-1}\right) \\ &\leq \beta_b\left(I\left(T, u_{q_i}, u_{p_i-1}, a\right)\right) I\left(T, u_{q_i}, u_{p_i-1}, a\right) \\ &< \frac{1}{s} I\left(T, u_{q_i}, u_{p_i-1}, a\right) \end{aligned} \tag{43}$$

where

$$\begin{aligned} I\left(T, u_{q_i}, u_{p_i-1}, a\right) &= \left(b\left(u_{q_i}, u_{p_i-1}\right)\right)^{a_1}\left(b\left(u_{q_i}, Tu_{q_i}\right)\right)^{a_2} \\ &\quad \cdot \left(b\left(u_{p_i-1}, Tu_{p_i-1}\right)\right)^{1-a_1-a_2} \\ &= \left(b\left(u_{q_i}, u_{p_i-1}\right)\right)^{a_1}\left(b\left(u_{q_i}, u_{q_i+1}\right)\right)^{a_2} \\ &\quad \cdot \left(b\left(u_{p_i-1}, u_{p_i}\right)\right)^{1-a_1-a_2}, \end{aligned} \tag{44}$$

taking lim sup of (43), we find

$$\limsup_{i\to\infty} I\left(T, u_{q_i}, u_{p_i-1}, a\right) = 0 \text{ and hence } \limsup_{i\to\infty} b\left(u_{q_i+1}, u_{p_i}\right) = 0. \tag{45}$$

If we combine the observed inequalities above, in particular, (42) and (45), we get that we get

$$e \leq b\left(u_{q_i}, u_{p_i}\right) \leq s\left[b\left(u_{q_i}, u_{q_i+1}\right) + b\left(u_{q_i+1}, u_{p_i}\right)\right] \longrightarrow 0, as i \longrightarrow \infty. \tag{46}$$

Therefore, the sequence $\{u_n\}$ is $b$-Cauchy in $(X^*, b, s)$, so it is convergent at a point $\varpi \in X$, that is

$$\lim_{n\to\infty} u_n = \varpi. \tag{47}$$

Now, we assert that $\varpi$ is a fixed point of $T$.

If the assumption (i) holds, since $\lim_{n\to\infty} b(\varpi, u_{n+1}) = 0$ we get

$$\lim_{n\to\infty} b(Tu_n, T\varpi) = b(T\varpi, T\varpi) = 0 \tag{48}$$

and from the inequality

$$b(\varpi, T\varpi) \leq s[b(\varpi, u_{n+1}) + b(Tu_n, T\varpi)], \tag{49}$$

for any $n \in \mathbb{N}$, we obtain $b(\varpi, T\varpi) = 0$, i.e., $T\varpi = \varpi$.

Suppose that $a_2 < 1$ or $a_3 < 1$. Assuming that $T\varpi \neq \varpi$, we have

$$\begin{aligned} 0 < b(T\varpi, \varpi) &\leq s[b(T\varpi, u_{n+1}) + b(u_{n+1}, \varpi)] \\ &= s[b(T\varpi, Tu_n) + b(u_{n+1}, \varpi)] \\ &\leq s[\beta_b(I(T, \varpi, u_n, a))I(T, \varpi, u_n, a) + b(u_{n+1}, \varpi)] \\ &< I(T, \varpi, u_n, a) + sb(u_{n+1}, \varpi) \\ &= \left[(b(\varpi, u_n))^{a_1}(b(\varpi, T\varpi))^{a_2}(b(u_n, Tu_n))^{1-a_1-a_2}\right] \\ &\quad + sb(u_{n+1}, \varpi). \end{aligned} \tag{50}$$

At the limit as $n \longrightarrow \infty$, we have $b(T\varpi, \varpi) = 0$. So, $T\varpi = \varpi$.

*Example 10.* We shall derive several distinct contractions from Definition 6. Some examples are given below. Let $T$ be a self-mapping on $X$.

(1) If $r = 2$, $a = (1/3, 1/3, 1/3)$, we obtain the following condition

$$\begin{aligned} b(Tu, Tv) &\leq \beta_b\left(\frac{1}{\sqrt{3}}\left[b^2(u, v) + b^2(u, Tu) + b^2(v, Tv)\right]^{1/2}\right) \\ &\quad \cdot \left(\frac{1}{\sqrt{3}}\left[b^2(u, v) + b^2(u, Tu) + b^2(v, Tv)\right]^{1/2}\right) \end{aligned} \tag{51}$$

(2) If $r = 1$, $a = (1/3, 1/3, 1/3)$, we obtain the following condition

$$\begin{aligned} b(Tu, Tv) &\leq \beta_b\left(\frac{1}{3}[b(u, v) + b(u, Tu) + b(v, Tv)]\right) \\ &\quad \cdot \left(\frac{1}{3}[b(u, v) + b(u, Tu) + b(v, Tv)]\right) \end{aligned} \tag{52}$$

(3) If $r = 0$, $a = (0, a_1, 1 - a_1)$ with $a_1 \in (0, 1)$, we obtain

$$\begin{aligned} b(Tu, Tv) &\leq \beta_b\left((b(u, Tu))^{a_1} b(v, Tv))^{1-a_1}\right) \\ &\quad \cdot (b(u, Tu))^{a_1}(b(v, Tv))^{1-a_1} \end{aligned} \tag{53}$$

which means that $T$ is an interpolative Kannan type Geraghty-contraction;

(4) If $r = 0$, $a = (a_1, a_2, 1 - a_1 - a_2)$ with $a_1, a_2 \in (0, 1)$, we have

$$\begin{aligned} b(Tu, Tv) &\leq \beta_b\left((b(u, v))^{a_1}(b(u, Tu))^{a_2}(b(v, Tv))^{1-a_1-a_2}\right) \\ &\quad \cdot (b(u, v))^{a_1}(b(u, Tu))^{a_2}(b(v, Tv))^{1-a_1-a_2} \end{aligned} \tag{54}$$

that is $T$ is an interpolative Reich-Rus-C$'$iric$'$ type Geraghty-contraction.

Related to these examples, we can establish some consequences, by choosing proper values for $r, a_1, a_2, a_3$ in Theorem 15.

**Corollary 11.** *Let $(X^*, b, s)$ and a self-mapping $T$ on $X$. If there exists a function $\beta_b \in \mathcal{G}_b$ such that*

$$b(Tu, Tv) \le \beta_b\left(\frac{\left[b^2(u, v) + b^2(u, Tu) + b^2(v, Tv)\right]^{1/2}}{\sqrt{3}}\right) \cdot \frac{\left[b^2(u, v) + b^2(u, Tu) + b^2(v, Tv)\right]^{1/2}}{\sqrt{3}}, \tag{55}$$

*for all $u, v \in X$ then $T$ admits a unique fixed point $\varpi \in X$.*

**Corollary 12.** *Let $(X^*, b, s)$ and a self-mapping $T$ on $X$. If there exists a function $\beta_b \in \mathcal{G}_b$ such that*

$$b(Tu, Tv) \le \beta_b\left(\frac{b(u, v) + b(u, Tu) + b(v, Tv)}{3}\right) \cdot \frac{b(u, v) + b(u, Tu) + b(v, Tv)}{3}, \tag{56}$$

*for all $u, v \in X$ then $T$ admits a unique fixed point $\varpi \in X$.*

**Corollary 13.** *Let $(X^*, b, s)$ be a complete $b$ -metric space, a self-mapping $T$ on $X$ and $a_1 \in (0, 1)$. If there exists a function $\beta_b \in \mathcal{G}_b$ such that*

$$b(Tu, Tv) \le \beta_b\left((b(u, Tu))^{a_1}(b(v, Tv))^{1-a_1}\right) \cdot \left((b(u, Tu))^{a_1}(b(v, Tv))^{1-a_1}\right), \tag{57}$$

*for all $u, v \in XFix(T)$, then $T$ admits a fixed point $\varpi \in X$.*

**Corollary 14.** *Let $(X^*, b, s)$ be a complete b-metric space, a self-mapping $T$ on $X$ and $a_1, a_2 \in (0, 1)$. If there exists a function $\beta_b \in \mathcal{G}_b$ such that*

$$b(Tu, Tv) \le \beta_b\left((b(u, v))^{a_1}(b(u, Tu)^{a_2}(b(v, Tv))^{1-a_1-a_2}\right) \cdot \left((b(u, v))^{a_1}(b(u, Tu)^{a_2}(b(v, Tv))^{1-a_1-a_2}\right), \tag{58}$$

*for all $u, v \in X \setminus Fix(T)$, then $T$ admits a fixed point $\varpi \in X$.*

## 3. Immediate Consequences

By letting $\beta_b(t) = \kappa$, we shall observe the Definition 4, [22].

**Theorem 15** (see [22]). *Let $(X^*, b, s)$. A $(r, a)$ -weight type contraction mapping $T : X \to X$ admits a fixed point $\varpi \in X$ if one of the following holds:*

*(i) $T$ is continuous at such point $\varpi$*

*(ii) $s^r a_2 < 1$*

*(iii) $s^r a_3 < 1$*

*Moreover, for any $u_0 \in X$ the sequence $\{T^n u_0\}$ converges to $\varpi$.*

We list the following corollaries.

**Corollary 16.** *On the complete $b$ -metric space $(X^*, b, s)$ let $T : X \to X$ be a mapping. If there exists $\kappa \in [0, 1)$ such that*

$$b(Tu, Tv) \le \kappa \cdot b^{a_1}(u, v) \cdot b^{a_2}(u, Tu) \cdot b^{a_3}(v, Tv), \tag{59}$$

*for all $u, v \in X \setminus Fix(T)$, $a_1, a_2, a_3 \ge 0$ and $\sum_{i=1}^{3} a_i = 1$, then $T$ has a fixed point $\varpi \in X$. $u_0 \in X$ the sequence $\{T^n u_0\}$ converges to $\varpi$.*

*Proof.* Put in Theorem 15, $r = 0$ and $a = (a_1, a_2, a_3)$.

**Corollary 17.** *On the complete $b$ -metric let $T : X \to X$ be a mapping such that*

$$b(Tu, Tv) \le \kappa \sqrt[3]{b(u, v)b(u, Tu)b(v, Tv)}, \tag{60}$$

*for all $u, v \in X \setminus Fix(T)$, where $\kappa \in [0, 1)$. Then, $T$ has a fixed point $\varpi \in X$.*

*Proof.* Put in Theorem 15, $r = 0$ and $a = (1/3, 1/3, 1/3)$.

**Corollary 18.** *Let $(X^*, b, s)$ be a complete $b$ -metric space and $T : X \to X$ be a mapping such that for every $u, v \in X \setminus Fix(T)$*

$$b(Tu, Tv) \le \frac{\kappa}{3}[b(u, v) + b(u, Tu) + b(v, Tv)], \tag{61}$$

*where $\kappa \in [0, 1)$. The mapping $T$ has a fixed point $\varpi$ provided that one of the following hold:*

*(i) $T$ is continuous at $\varpi \in X$*

*(ii) $s < 3$*

*Then, $T$ has a fixed point $\varpi$. Moreover for any $u_0 \in X$, the sequence $\{T^n u_0\}$ converges to $\varpi$.*

*Proof.* Let $r = 1$ and $a = (1/3, 1/3, 1/3)$ in Theorem 15.

**Corollary 19.** *Let $(X^*, b, s)$ be a complete $b$ -metric space and $T : X \to X$ be a mapping such that*

$$b(Tu, Tv) \le \frac{\kappa}{\sqrt{3}}\left[b^2(u, v) + b^2(u, Tu) + b^2(v, Tv)\right]^{1/2}, \tag{62}$$

*for all $u, v \in XFix(T)$, where $\kappa \in [0, 1)$. Assume that one of the following conditions hold:*

*(i) $T$ is continuous at $\varpi \in X$*

*(ii)* $s^2 < 3$

*Then, T has a fixed point* $\varpi$ *and for any* $u_0 \in X$*, the sequence* $\{T^n u_0\}$ *converges to* $\varpi$.

*Proof.* Take $r = 2$ and $a = (1/3, 1/3, 1/3)$ in Theorem 15.

**Corollary 20** (see [24]). *Let* $(X^*, b, s)$*, T be a self-mapping on X and* $a_1 \in (0, 1)$*. If there exists a function* $\beta_b \in \mathcal{G}_b$ *such that*

$$b(Tu, T\nu) \leq \kappa \cdot b^{a_1}(u, Tu)\, b^{1-a_1}(\nu, T\nu), \tag{63}$$

*for all* $u, \nu \in inX \setminus Fix(T)$*; then, T admits a unique fixed point* $\varpi \in X$.

*Proof.* Choose $\beta_b(t) = \kappa$ in Corollary 13.

**Corollary 21** (see [25]). *Let* $(X^*, b, s)$*, T be a self-mapping on X and* $a_1, a_2 \in (0, 1)$*. If there exists* $\kappa \in (0, 1)$ *such that*

$$b(Tu, T\nu) \leq \kappa \cdot b^{a_1}(u, \nu)\, b^{a_2}(u, Tu)\, b^{1-a_1-a_2}(\nu, T\nu), \tag{64}$$

*for all* $u, \nu \in inX \setminus Fix(T)$*; then, T admits a unique fixed point* $\varpi \in X$.

*Proof.* Choose $\beta_b(t) = \kappa$ in Corollary 14.

## 4. Conclusions

In this paper, we combine linear and nonlinear contractions to unify and extend the several existing results. This approach may bring new frames to the topic of metric fixed point theory. In particular, interpolative contraction may extend several results in the setting of Banach space.

We also mention that, for the case $s = 1$, we find a series of results known in the context of metric spaces, see, e.g., [25–36].

## Data Availability

No data were used to support this study.

## Conflicts of Interest

The authors declare that they have no competing interests.

## Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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