Research Article

Global Existence of Solutions to a System of Integral Equations Related to an Epidemic Model

Mohamed Jleli and Bessem Samet

Department of Mathematics, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia

Correspondence should be addressed to Bessem Samet; bsamet@ksu.edu.sa

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A system of integral equations related to an epidemic model is investigated. Namely, we derive sufficient conditions for the existence and uniqueness of global solutions to the considered system. The proof is based on Perov's fixed point theorem and some integral inequalities.

1. Introduction

Many phenomena related to infectious diseases can be modeled as an integral equation (see e.g., [1–4] and the references therein). In [3], Gripenberg investigated the large time behavior of solutions to the integral equation

$$x(t) = k \left(p(t) - \int_0^t A(t-s)x(s) ds \right) \left(f(t) + \int_0^t a(t-s)x(s) ds \right), t \geq 0, \quad (1)$$

which arises in the study of the spread of an infectious disease that does not induce permanent immunity. Namely, sufficient conditions were provided so that (1) admits non-negative, continuous, and bounded solution. Using the comparison method and some integral estimates, Pachpatte [5] established the convergence of solutions to (1) to 0 as $t \rightarrow \infty$. In [6], Brestovanská studied the integral equation

$$x(t) = \left(g_1(t) + \int_0^t A_1(t-s)F_1(s, x(s)) ds \right) \cdots \cdot \left(g_p(t) + \int_0^t A_p(t-s)F_p(s, x(s)) ds \right), \quad (2)$$

for all $t \geq 0$. Namely, sufficient criteria for the global existence and uniqueness of global solutions to (2) were derived. Moreover, under certain conditions, the convergence of solutions to (2) to 0 as $t \rightarrow \infty$ was proved. In [7], using weakly Picard technique operators in a gauge space, Olaru investigated the qualitative behavior of solutions to the integral equation

tence and uniqueness of global solutions to (2) were derived. Moreover, under certain conditions, the convergence of solutions to (2) to 0 as $t \rightarrow \infty$ was proved. In [7], using weakly Picard technique operators in a gauge space, Olaru investigated the qualitative behavior of solutions to the integral equation

$$x(t) = \left(g_1(t) + \int_0^t K_1(t, s, x(s)) ds \right) \left(g_2(t) + \int_0^t K_2(t, s, x(s)) ds \right), t \geq 0. \quad (3)$$

In this paper, we consider the system of integral equations

$$\begin{cases} x(t) = \prod_{i=1}^2 \left(f_i(t) + \int_0^t A_i(t-s)F_i(s, x(s), y(s)) ds \right), t \geq 0, \\ y(t) = \prod_{i=1}^2 \left(g_i(t) + \int_0^t B_i(t-s)G_i(s, x(s), y(s)) ds \right), t \geq 0, \end{cases} \quad (4)$$

where $f_i, A_i, g_i, B_i \in C([0, \infty))$ and $F_i, G_i \in C([0, \infty) \times \mathbb{R} \times \mathbb{R})$. Namely, we are concerned with the global existence of solutions to the considered system. Using Perov's fixed point theorem, sufficient conditions are derived for which the system (4) admits one and only one continuous global solution.

The rest of the paper is organized as follows. In Section 2, we recall some notions on fixed point theory including Perov's fixed point theorem. In Section 3, we state and prove our main result.

2. Preliminaries

Let n be a positive natural number and define the partial order $\leq_n$ in $\mathbb{R}^n$ by

$$y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \leq_n z = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix} \Leftrightarrow y_i \leq z_i, i = 1, 2, \dots, n, \quad (5)$$

for all $y, z \in \mathbb{R}^n$. We denote by $0_{\mathbb{R}^n}$ the zero vector in $\mathbb{R}^n$, i.e.,

$$0_{\mathbb{R}^n} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \quad (6)$$

Let S be a nonempty set and $d : S \times S \longrightarrow S$ be a given mapping. We say that d is a vector-valued metric on S (see, e.g., [8]), if for all $x, y, z \in S$,

- (i) $0_{\mathbb{R}^n} \leq_n d(x, y)$
- (ii) $d(x, y) = 0_{\mathbb{R}^n} \Leftrightarrow x = y$
- (iii) $d(x, y) = d(y, x)$
- (iv) $d(x, z) \leq_n d(x, y) + d(y, z)$

In this case, we say that (S, d) is a generalized metric space. In such spaces, the notions of convergent sequence, Cauchy sequence, and completeness are similar to those for usual metric spaces.

Let $M_n(\mathbb{R}_+)$ be set of square matrices of size n with nonnegative coefficients. Given $\mathcal{K} \in M_n(\mathbb{R}_+)$, we denote by $\rho(\mathcal{K})$ its spectral radius.

Lemma 1 (Perov's fixed point theorem, see [9]). *Let (S, d) be a complete generalized metric space and $\mathcal{H} : S \longrightarrow S$ be a given mapping. Suppose that there exists $\mathcal{K} \in M_n(\mathbb{R}_+)$ with $\rho(\mathcal{K}) < 1$ such that*

$$d(\mathcal{H}(x), \mathcal{H}(y)) \leq_n \mathcal{K} d(x, y), \quad (7)$$

for all $x, y \in S$. Then, the mapping $\mathcal{H}$ admits a unique fixed point in S .

3. Global Existence

The system (4) is investigated under the following assumptions:

- (i) $f_i, A_i, g_i, B_i \in C([0, \infty))$ and $F_i, G_i \in C([0, \infty) \times \mathbb{R} \times \mathbb{R})$, $i = 1, 2$

- (ii) f_i, g_i , $i = 1, 2$, are bounded functions
- (iii) $A_i, B_i \in L^1((0, \infty)) \cap L^\zeta((0, \infty))$, $i = 1, 2$, for some $\zeta > 1$
- (iv) For all $i = 1, 2$, there exist positive constants $L_1^{(i)}$ and $L_2^{(i)}$ such that

$$|F_i(t, u, v) - F_i(t, \bar{u}, \bar{v})| \leq L_1^{(i)} |u - \bar{u}| + L_2^{(i)} |v - \bar{v}|, \quad (8)$$

for all $t \geq 0$ and $(u, v), (\bar{u}, \bar{v}) \in \mathbb{R}^2$

- (v) For all $i = 1, 2$, there exist positive constants $Q_1^{(i)}$, $Q_2^{(i)}$, $\ell_1^{(i)}$, and $\ell_2^{(i)}$ such that

$$|F_i(t, u, v)| \leq Q_1^{(i)} |u|^{\ell_1^{(i)}} + Q_2^{(i)} |v|^{\ell_2^{(i)}}, \quad (9)$$

for all $t \geq 0$ and $(u, v) \in \mathbb{R}^2$

- (vi) For all $i = 1, 2$, there exist positive constants $M_1^{(i)}$ and $M_2^{(i)}$ such that

$$|G_i(t, u, v) - G_i(t, \bar{u}, \bar{v})| \leq M_1^{(i)} |u - \bar{u}| + M_2^{(i)} |v - \bar{v}|, \quad (10)$$

for all $t \geq 0$ and $(u, v), (\bar{u}, \bar{v}) \in \mathbb{R}^2$

- (vii) For all $i = 1, 2$, there exist positive constants $P_1^{(i)}$, $P_2^{(i)}$, $\sigma_1^{(i)}$, and $\sigma_2^{(i)}$ such that

$$|G_i(t, u, v)| \leq P_1^{(i)} |u|^{\sigma_1^{(i)}} + P_2^{(i)} |v|^{\sigma_2^{(i)}}, \quad (11)$$

for all $t \geq 0$ and $(u, v) \in \mathbb{R}^2$

- (viii) There exist positive constants ρ_1 and ρ_2 satisfying

$$\left[\mu + \left(Q_1^{(1)} \rho_1^{(1)} + Q_2^{(1)} \rho_2^{(1)} \right) \delta \right] \left[\mu + \left(Q_1^{(2)} \rho_1^{(2)} + Q_2^{(2)} \rho_2^{(2)} \right) \delta \right] \leq \rho_1, \quad (12)$$

$$\left[\mu + \left(P_1^{(1)} \rho_1^{(1)} + P_2^{(1)} \rho_2^{(1)} \right) \delta \right] \left[\mu + \left(P_1^{(2)} \rho_1^{(2)} + P_2^{(2)} \rho_2^{(2)} \right) \delta \right] \leq \rho_2, \quad (13)$$

where

$$\mu = \max \{ \mu_f, \mu_g \}, \quad (14)$$

$$\mu_f = \max \{ |f_1(t)|, |f_2(t)| : t \geq 0 \}, \quad (15)$$

$$\mu_g = \max \{ |g_1(t)|, |g_2(t)| : t \geq 0 \}, \quad (16)$$

$$\delta = \max_{i=1,2} \left\{ \int_0^\infty |A_i(s)| ds, \int_0^\infty |B_i(s)| ds \right\}. \quad (17)$$

Remark 2. Notice that from (i) and (ii), one has $\mu < \infty$. Moreover, by (iii), one has $\delta < \infty$.

Our main result is given by the following theorem.

Theorem 3. Under assumptions (i)–(viii), system (4) admits one and only one solution $(x^*, y^*) \in C([0, \infty)) \times C([0, \infty))$ satisfying $|x^*(t)| \leq \rho_1$ and $|y^*(t)| \leq \rho_2$, for all $t \geq 0$.

Proof. Let T be an arbitrary positive number and $I_T = [0, T]$. For $i = 1, 2$, let

$$C_{T, \rho_i} = \{x \in C(I_T) : |x(t)| \leq \rho_i, t \in I_T\}. \quad (18)$$

We introduce the mapping $\mathcal{H} : C_{T, \rho_1} \times C_{T, \rho_2} \longrightarrow C(I_T) \times C(I_T)$ defined by

$$\begin{aligned} \mathcal{H}(x, y)(t) &= (\mathcal{H}_1(x, y)(t), \mathcal{H}_2(x, y)(t)) \\ &= \left(\prod_{i=1}^2 \mathcal{H}_1^{(i)}(x, y)(t), \prod_{i=1}^2 \mathcal{H}_2^{(i)}(x, y)(t) \right), t \in I_T, \end{aligned} \quad (19)$$

where

$$\mathcal{H}_1^{(i)}(x, y)(t) = f_i(t) + \int_0^t A_i(t-s) F_i(s, x(s), y(s)) ds, \quad (20)$$

$$\mathcal{H}_2^{(i)}(x, y)(t) = g_i(t) + \int_0^t B_i(t-s) G_i(s, x(s), y(s)) ds, \quad (21)$$

for all $i = 1, 2$.

Let $(x, y) \in C_{T, \rho_1} \times C_{T, \rho_2}$. For all $i = 1, 2$ and $t \in I_T$, using (i), (ii), (iii), and (v), and taking in consideration Remark 2, one obtains

$$\begin{aligned} |\mathcal{H}_1^{(i)}(x, y)(t)| &\leq |f_i(t)| + \int_0^t |A_i(t-s)| |F_i(s, x(s), y(s))| ds \leq \mu \\ &+ \int_0^t |A_i(t-s)| \left(Q_1^{(i)} |x(s)|^{\ell_1^{(i)}} + Q_2^{(i)} |y(s)|^{\ell_2^{(i)}} \right) ds \leq \mu \\ &+ \left(Q_1^{(i)} \rho_1^{\ell_1^{(i)}} + Q_2^{(i)} \rho_2^{\ell_2^{(i)}} \right) \int_0^t |A_i(t-s)| ds = \mu \\ &+ \left(Q_1^{(i)} \rho_1^{\ell_1^{(i)}} + Q_2^{(i)} \rho_2^{\ell_2^{(i)}} \right) \int_0^\infty |A_i(\tau)| d\tau \leq \mu \\ &+ \left(Q_1^{(i)} \rho_1^{\ell_1^{(i)}} + Q_2^{(i)} \rho_2^{\ell_2^{(i)}} \right) \delta. \end{aligned} \quad (22)$$

Therefore, using (12), it holds that

$$\begin{aligned} |\mathcal{H}_1^{(i)}(x, y)(t)| &= \prod_{i=1}^2 |\mathcal{H}_1^{(i)}(x, y)(t)| \\ &\leq \left[\mu + \left(Q_1^{(1)} \rho_1^{\ell_1^{(1)}} + Q_2^{(1)} \rho_2^{\ell_2^{(1)}} \right) \delta \right] \\ &\quad \cdot \left[\mu + \left(Q_1^{(2)} \rho_1^{\ell_1^{(2)}} + Q_2^{(2)} \rho_2^{\ell_2^{(2)}} \right) \delta \right] \leq \rho_1, \end{aligned} \quad (23)$$

which yields

$$\mathcal{H}_1 \left(C_{T, \rho_1} \times C_{T, \rho_2} \right) \subset C_{T, \rho_1}. \quad (24)$$

Similarly, for all $i = 1, 2$ and $t \in I_T$, using (i), (ii), (iii), and (vii), and taking in consideration Remark 2, one obtains

$$\begin{aligned} |\mathcal{H}_2^{(i)}(x, y)(t)| &\leq |g_i(t)| + \int_0^t |B_i(t-s)| |G_i(s, x(s), y(s))| ds \\ &\leq \mu + \int_0^t |B_i(t-s)| \left(P_1^{(i)} |x(s)|^{\sigma_1^{(i)}} + P_2^{(i)} |y(s)|^{\sigma_2^{(i)}} \right) ds \\ &\leq \mu + \left(P_1^{(i)} \rho_1^{\sigma_1^{(i)}} + P_2^{(i)} \rho_2^{\sigma_2^{(i)}} \right) \int_0^t |B_i(t-s)| ds \\ &\leq \mu + \left(P_1^{(i)} \rho_1^{\sigma_1^{(i)}} + P_2^{(i)} \rho_2^{\sigma_2^{(i)}} \right) \delta. \end{aligned} \quad (25)$$

Hence, using (13), it holds that

$$\begin{aligned} |\mathcal{H}_2^{(i)}(x, y)(t)| &= \prod_{i=1}^2 |\mathcal{H}_2^{(i)}(x, y)(t)| \\ &\leq \left[\mu + \left(P_1^{(1)} \rho_1^{\sigma_1^{(1)}} + P_2^{(1)} \rho_2^{\sigma_2^{(1)}} \right) \delta \right] \\ &\quad \cdot \left[\mu + \left(P_1^{(2)} \rho_1^{\sigma_1^{(2)}} + P_2^{(2)} \rho_2^{\sigma_2^{(2)}} \right) \delta \right] \leq \rho_2, \end{aligned} \quad (26)$$

which yields

$$\mathcal{H}_2 \left(C_{T, \rho_1} \times C_{T, \rho_2} \right) \subset C_{T, \rho_2}. \quad (27)$$

Therefore, it follows from (24) and (27) that the mapping $\mathcal{H}$ maps the set $C_{T, \rho_1} \times C_{T, \rho_2}$ into itself, i.e.,

$$\mathcal{H} : C_{T, \rho_1} \times C_{T, \rho_2} \longrightarrow C_{T, \rho_1} \times C_{T, \rho_2}. \quad (28)$$

Next, let us introduce the metric

$$d_r : C(I_T) \times C(I_T) \longrightarrow \mathbb{R}, \quad (29)$$

defined by

$$d_r(x, y) = \max_{t \in I_T} e^{-rt} |x(t) - y(t)|, (x, y) \in C(I_T) \times C(I_T), \quad (30)$$

where $r > 0$ will be specified later. Moreover, we introduce the vector-valued metric

$$D_r : (C_{T,\rho_1} \times C_{T,\rho_2}) \times (C_{T,\rho_1} \times C_{T,\rho_2}) \longrightarrow \mathbb{R}^2, \quad (31)$$

defined by

$$D_r((x, y), (\bar{x}, \bar{y})) = \begin{pmatrix} d_r(x, \bar{x}) \\ d_r(y, \bar{y}) \end{pmatrix}, (x, y), (\bar{x}, \bar{y}) \in C_{T,\rho_1} \times C_{T,\rho_2}. \quad (32)$$

It can be easily seen that $(C_{T,\rho_1} \times C_{T,\rho_2}, D_r)$ is a complete generalized metric space. On the other hand, for all $(x, y), (\bar{x}, \bar{y}) \in C_{T,\rho_1} \times C_{T,\rho_2}$, and $t \in I_T$, using (22), one has

$$\begin{aligned} |\mathcal{H}_1(x, y)(t) - \mathcal{H}_1(\bar{x}, \bar{y})(t)| &= \left| \prod_{i=1}^2 \mathcal{H}_1^{(i)}(x, y)(t) - \prod_{i=1}^2 \mathcal{H}_1^{(i)}(\bar{x}, \bar{y})(t) \right| \\ &= \left| \mathcal{H}_1^{(1)}(x, y)(t) \mathcal{H}_1^{(2)}(x, y)(t) - \mathcal{H}_1^{(1)}(\bar{x}, \bar{y})(t) \mathcal{H}_1^{(2)}(\bar{x}, \bar{y})(t) \right| \\ &\leq \left| \mathcal{H}_1^{(1)}(x, y)(t) \right| \left| \mathcal{H}_1^{(2)}(x, y)(t) - \mathcal{H}_1^{(2)}(\bar{x}, \bar{y})(t) \right| \\ &\quad + \left| \mathcal{H}_1^{(2)}(\bar{x}, \bar{y})(t) \right| \left| \mathcal{H}_1^{(1)}(x, y)(t) - \mathcal{H}_1^{(1)}(\bar{x}, \bar{y})(t) \right| \\ &\leq \left[\mu + \left(Q_1^{(1)} \rho_1^{\ell_1^{(1)}} + Q_2^{(1)} \rho_2^{\ell_2^{(1)}} \right) \delta \right] \left| \mathcal{H}_1^{(2)}(x, y)(t) - \mathcal{H}_1^{(2)}(\bar{x}, \bar{y})(t) \right| \\ &\quad + \left[\mu + \left(Q_1^{(2)} \rho_1^{\ell_1^{(2)}} + Q_2^{(2)} \rho_2^{\ell_2^{(2)}} \right) \delta \right] \left| \mathcal{H}_1^{(1)}(x, y)(t) - \mathcal{H}_1^{(1)}(\bar{x}, \bar{y})(t) \right|. \end{aligned} \quad (33)$$

Moreover, using (iii), (iv), (20) and Hölder's inequality, for all $i = 1, 2$, one obtains

$$\begin{aligned} &\left| \mathcal{H}_1^{(i)}(x, y)(t) - \mathcal{H}_1^{(i)}(\bar{x}, \bar{y})(t) \right| \\ &= \left| f_i(t) + \int_0^t A_i(t-s) F_i(s, x(s), y(s)) ds - f_i(t) \right. \\ &\quad \left. - \int_0^t A_i(t-s) F_i(s, \bar{x}(s), \bar{y}(s)) ds \right| \leq \int_0^t |A_i(t-s)| |F_i(s, x(s), y(s)) \\ &\quad - F_i(s, \bar{x}(s), \bar{y}(s))| ds \leq \int_0^t |A_i(t-s)| \left(L_1^{(i)} |x(s) - \bar{x}(s)| \right. \\ &\quad \left. + L_2^{(i)} |y(s) - \bar{y}(s)| \right) ds = L_1^{(i)} \int_0^t |A_i(t-s)| e^{-rs} |x(s) \\ &\quad - \bar{x}(s)| ds + L_2^{(i)} \int_0^t |A_i(t-s)| e^{-rs} |y(s) - \bar{y}(s)| ds \\ &\leq \left(L_1^{(i)} \int_0^t |A_i(t-s)| e^{rs} ds \right) d_r(x, \bar{x}) + \left(L_2^{(i)} \int_0^t |A_i(t-s)| e^{rs} ds \right) d_r(y, \bar{y}) \\ &= \left(\int_0^t |A_i(t-s)| e^{rs} ds \right) \left(L_1^{(i)} d_r(x, \bar{x}) + L_2^{(i)} d_r(y, \bar{y}) \right) \\ &\leq \delta_\zeta \left(\int_0^t e^{r\zeta s} ds \right)^{\frac{1}{\zeta}} \left(L_1^{(i)} d_r(x, \bar{x}) + L_2^{(i)} d_r(y, \bar{y}) \right), \end{aligned} \quad (34)$$

where

$$\zeta' = \frac{\zeta}{\zeta - 1}, \quad (35)$$

$$\delta_\zeta = \max_{i=1,2} \left\{ \left(\int_0^\infty |A_i(s)|^\zeta ds \right)^{1/\zeta}, \left(\int_0^\infty |B_i(s)|^\zeta ds \right)^{1/\zeta} \right\}. \quad (36)$$

Notice that by (iii), one has $\delta_\zeta < \infty$. Hence, it holds that

$$\begin{aligned} &\left| \mathcal{H}_1^{(i)}(x, y)(t) - \mathcal{H}_1^{(i)}(\bar{x}, \bar{y})(t) \right| \leq \delta_\zeta \zeta'^{-1/\zeta'} r^{-1/\zeta'} \\ &\quad \cdot \left(e^{rt\zeta'} - 1 \right)^{1/\zeta'} \left(L_1^{(i)} d_r(x, \bar{x}) + L_2^{(i)} d_r(y, \bar{y}) \right). \end{aligned} \quad (37)$$

Therefore, by (33), one obtains

$$\begin{aligned} &|\mathcal{H}_1(x, y)(t) - \mathcal{H}_1(\bar{x}, \bar{y})(t)| \\ &\leq \left[\mu + \left(Q_1^{(1)} \rho_1^{\ell_1^{(1)}} + Q_2^{(1)} \rho_2^{\ell_2^{(1)}} \right) \delta \right] \delta_\zeta \zeta'^{-1/\zeta'} r^{-1/\zeta'} \left(e^{rt\zeta'} - 1 \right)^{1/\zeta'} \\ &\quad \times \left(L_1^{(2)} d_r(x, \bar{x}) + L_2^{(2)} d_r(y, \bar{y}) \right) \\ &\quad + \left[\mu + \left(Q_1^{(2)} \rho_1^{\ell_1^{(2)}} + Q_2^{(2)} \rho_2^{\ell_2^{(2)}} \right) \delta \right] \delta_\zeta \zeta'^{-1/\zeta'} r^{-1/\zeta'} \left(e^{rt\zeta'} - 1 \right)^{1/\zeta'} \\ &\quad \times \left(L_1^{(1)} d_r(x, \bar{x}) + L_2^{(1)} d_r(y, \bar{y}) \right) = \delta_\zeta \zeta'^{-1/\zeta'} r^{-1/\zeta'} \left(e^{rt\zeta'} - 1 \right)^{1/\zeta'} \\ &\quad \times \left(L_1^{(2)} \left[\mu + \left(Q_1^{(1)} \rho_1^{\ell_1^{(1)}} + Q_2^{(1)} \rho_2^{\ell_2^{(1)}} \right) \delta \right] \right. \\ &\quad \left. + L_1^{(1)} \left[\mu + \left(Q_1^{(2)} \rho_1^{\ell_1^{(2)}} + Q_2^{(2)} \rho_2^{\ell_2^{(2)}} \right) \delta \right] \right) d_r(x, \bar{x}) \\ &\quad + \delta_\zeta \zeta'^{-1/\zeta'} r^{-1/\zeta'} \left(e^{rt\zeta'} - 1 \right)^{1/\zeta'} \left(L_2^{(2)} \left[\mu + \left(Q_1^{(1)} \rho_1^{\ell_1^{(1)}} + Q_2^{(1)} \rho_2^{\ell_2^{(1)}} \right) \delta \right] \right. \\ &\quad \left. + L_2^{(1)} \left[\mu + \left(Q_1^{(2)} \rho_1^{\ell_1^{(2)}} + Q_2^{(2)} \rho_2^{\ell_2^{(2)}} \right) \delta \right] \right) d_r(y, \bar{y}), \end{aligned} \quad (38)$$

which yields

$$\begin{aligned} &d_r(\mathcal{H}_1(x, y), \mathcal{H}_1(\bar{x}, \bar{y})) \\ &\leq \left(1 - e^{-rT\zeta'} \right)^{1/\zeta'} \delta_\zeta \zeta'^{-1/\zeta'} r^{-1/\zeta'} (\alpha_{11} d_r(x, \bar{x}) + \alpha_{12} d_r(y, \bar{y})), \end{aligned} \quad (39)$$

where

$$\begin{aligned} \alpha_{11} &= L_1^{(2)} \left[\mu + \left(Q_1^{(1)} \rho_1^{\ell_1^{(1)}} + Q_2^{(1)} \rho_2^{\ell_2^{(1)}} \right) \delta \right] \\ &\quad + L_1^{(1)} \left[\mu + \left(Q_1^{(2)} \rho_1^{\ell_1^{(2)}} + Q_2^{(2)} \rho_2^{\ell_2^{(2)}} \right) \delta \right], \end{aligned} \quad (40)$$

$$\begin{aligned} \alpha_{12} &= L_2^{(2)} \left[\mu + \left(Q_1^{(1)} \rho_1^{\ell_1^{(1)}} + Q_2^{(1)} \rho_2^{\ell_2^{(1)}} \right) \delta \right] \\ &\quad + L_2^{(1)} \left[\mu + \left(Q_1^{(2)} \rho_1^{\ell_1^{(2)}} + Q_2^{(2)} \rho_2^{\ell_2^{(2)}} \right) \delta \right]. \end{aligned} \quad (41)$$

Similarly, using (iii), (vi), (21), and (25), one obtains

$$d_r(\mathcal{H}_2(x, y), \mathcal{H}_2(\bar{x}, \bar{y})) \leq \left(1 - e^{-rT\zeta'}\right)^{1/\zeta'} \delta_\zeta \zeta'^{-1/\zeta'} r^{-1/\zeta'} (\alpha_{21} d_r(x, \bar{x}) + \alpha_{22} d_r(y, \bar{y})), \quad (42)$$

where

$$\alpha_{21} = M_1^{(2)} \left[\mu + \left(P_1^{(1)} \rho_1^{\sigma_1^{(1)}} + P_2^{(1)} \rho_2^{\sigma_2^{(1)}} \right) \delta \right] + M_1^{(1)} \left[\mu + \left(P_1^{(2)} \rho_1^{\sigma_1^{(2)}} + P_2^{(2)} \rho_2^{\sigma_2^{(2)}} \right) \delta \right], \quad (43)$$

$$\alpha_{22} = M_2^{(2)} \left[\mu + \left(P_1^{(1)} \rho_1^{\sigma_1^{(1)}} + P_2^{(1)} \rho_2^{\sigma_2^{(1)}} \right) \delta \right] + M_2^{(1)} \left[\mu + \left(P_1^{(2)} \rho_1^{\sigma_1^{(2)}} + P_2^{(2)} \rho_2^{\sigma_2^{(2)}} \right) \delta \right]. \quad (44)$$

Therefore, it follows from (19), (39), and (42) that

$$D_r(\mathcal{H}(x, y), \mathcal{H}(\bar{x}, \bar{y})) \circ_{\mathbb{R}^2} \mathcal{H} D_r((x, y), (\bar{x}, \bar{y})), (x, y), (\bar{x}, \bar{y}) \in C_{T, \rho_1} \times C_{T, \rho_2}, \quad (45)$$

where $\mathcal{H}$ is the square matrix of size 2 defined by

$$\mathcal{H} = \left(1 - e^{-rT\zeta'}\right)^{1/\zeta'} \delta_\zeta \zeta'^{-1/\zeta'} r^{-1/\zeta'} \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}. \quad (46)$$

On the other hand, one has

$$\rho(\mathcal{H}) = \left(1 - e^{-rT\zeta'}\right)^{1/\zeta'} \delta_\zeta \zeta'^{-1/\zeta'} r^{-1/\zeta'} \rho \left[\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \right]. \quad (47)$$

Therefore, taking

$$r^{1/\zeta'} \geq \delta_\zeta \zeta'^{-1/\zeta'} \rho \left[\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \right], \quad (48)$$

one obtains

$$\rho(\mathcal{H}) \leq \left(1 - e^{-rT\zeta'}\right)^{1/\zeta'} < 1. \quad (49)$$

Then, by Lemma 1, one deduces that the mapping $\mathcal{H}$ defined by (19) admits a unique fixed point in $C_{T, \rho_1} \times C_{T, \rho_2}$, which is the unique solution to (4) in $C_{T, \rho_1} \times C_{T, \rho_2}$. On the other hand, since the real number $T > 0$ is arbitrary, it holds that (4) admits a unique continuous global solution (x^*, y^*) satisfying $|x^*(t)| \leq \rho_1$ and $|y^*(t)| \leq \rho_2$, for all $t \geq 0$. The proof is completed.

We end the paper with the following example.

Example 4. Consider the system of integral equations

$$\begin{aligned} x(t) &= \left(\frac{2}{t+1} + \int_0^t \frac{e^{-2(t-s)}}{8} \sin \left(\frac{x(s)}{4} + \frac{3y(s)}{\sqrt{s+16}} \right) ds \right) \\ &\quad \cdot \int_0^t \frac{3e^{-(t-s)^2}}{16\sqrt{\pi}} \left(\frac{x(s)}{s^2+2} + \frac{y(s)}{2} \right) ds, t \geq 0, \\ y(t) &= \left(\frac{t}{t+1} + \int_0^t \frac{e^{-(t-s)}}{16} \arctan \left(\frac{2x(s)+y(s)}{5} \right) ds \right) \\ &\quad \cdot \left(1 + \int_0^t \frac{e^{-(t-s)^2}}{16\sqrt{\pi}} \left(\frac{x(s)}{6} + \frac{2y(s)}{s^2+3} \right) ds \right), t \geq 0. \end{aligned} \quad (50)$$

System (50) is a special case of System (4), where

$$f_1(t) = \frac{2}{t+1}, f_2(t) = 0, A_1(t) = \frac{e^{-2t}}{8}, A_2(t) = \frac{3e^{-t^2}}{16\sqrt{\pi}}, \quad (51)$$

$$F_1(t, u, v) = \sin \left(\frac{u}{4} + \frac{3v}{\sqrt{t+16}} \right), F_2(t, u, v) = \frac{u}{t^2+2} + \frac{v}{2}, \quad (52)$$

$$g_1(t) = \frac{t}{t+1}, g_2(t) = 1, B_1(t) = \frac{e^{-t}}{16}, B_2(t) = \frac{e^{-t^2}}{16\sqrt{\pi}}, \quad (53)$$

$$G_1(t, u, v) = \arctan \left(\frac{2u+v}{5} \right), G_2(t, u, v) = \frac{u}{6} + \frac{2v}{t^2+3}. \quad (54)$$

Let us check the validity of assumptions (i)–(viii). It can be easily seen that

$$f_i, A_i, g_i, B_i \in C([0, \infty)), F_i, G_i \in C([0, \infty) \times \mathbb{R} \times \mathbb{R}), i = 1, 2, \quad (55)$$

$$\mu_f = \max \{|f_1(t)|, |f_2(t)| : t \geq 0\} = 2, \quad (56)$$

$$\mu_g = \max \{|g_1(t)|, |g_2(t)| : t \geq 0\} = 1, \quad (57)$$

$$\mu = \max \{\mu_f, \mu_g\} = 2. \quad (58)$$

Moreover, one has

$$\int_0^\infty |A_1(t)| dt = \frac{1}{16}, \int_0^\infty |A_1(t)|^\zeta dt = \frac{1}{2\zeta 8^\zeta} \text{ (for all } \zeta > 1), \quad (59)$$

$$\int_0^\infty |A_2(t)| dt = \frac{3}{32}, \int_0^\infty |A_2(t)|^\zeta dt = \left(\frac{3}{16\sqrt{\pi}} \right)^\zeta \frac{\sqrt{\pi}}{2\sqrt{\zeta}} \text{ (for all } \zeta > 1), \quad (60)$$

$$\int_0^\infty |B_1(t)|dt = \frac{1}{16}, \int_0^\infty |B_1(t)|^\varsigma dt = \frac{1}{\varsigma 16^\varsigma} \text{ (for all } \varsigma > 1), \quad (61)$$

$$\int_0^\infty |B_2(t)|dt = \frac{1}{32}, \int_0^\infty |B_2(t)|^\varsigma dt = \frac{\sqrt{\pi}}{2\sqrt{\varsigma}(16\sqrt{\pi})^\varsigma} \text{ (for all } \varsigma > 1), \quad (62)$$

$$\delta = \max_{i=1,2} \left\{ \int_0^\infty |A_i(s)|ds, \int_0^\infty |B_i(s)|ds \right\} = \frac{3}{32}. \quad (63)$$

Therefore, assumptions (i)–(iii) of Theorem 3 are satisfied. For all $t \geq 0$ and $(u, v), (\bar{u}, \bar{v}) \in \mathbb{R}^2$, one has

$$\begin{aligned} |F_1(t, u, v) - F_1(t, \bar{u}, \bar{v})| &= \left| \sin \left(\frac{u}{4} + \frac{3v}{\sqrt{t+16}} \right) \right. \\ &\quad \left. - \sin \left(\frac{\bar{u}}{4} + \frac{3\bar{v}}{\sqrt{t+16}} \right) \right| \leq \frac{1}{4}|u - \bar{u}| + \frac{3}{4}|v - \bar{v}| \end{aligned} \quad (64)$$

$$\begin{aligned} |F_2(t, u, v) - F_2(t, \bar{u}, \bar{v})| &= \left| \frac{u}{t^2+2} + \frac{v}{2} - \frac{\bar{u}}{t^2+2} - \frac{\bar{v}}{2} \right| \\ &\leq \frac{1}{2}|u - \bar{u}| + \frac{1}{2}|v - \bar{v}|, \end{aligned} \quad (65)$$

which shows that assumption (iv) of Theorem 3 is satisfied with

$$L_1^{(1)} = \frac{1}{4}, \quad L_2^{(1)} = \frac{3}{4}, \quad L_1^{(2)} = L_2^{(2)} = \frac{1}{2}. \quad (66)$$

Similarly, one has

$$\begin{aligned} |G_1(t, u, v) - G_1(t, \bar{u}, \bar{v})| &= \left| \arctan \left(\frac{2u+v}{5} \right) - \arctan \left(\frac{2\bar{u}+\bar{v}}{5} \right) \right| \\ &\leq \frac{2}{5}|u - \bar{u}| + \frac{1}{5}|v - \bar{v}| \end{aligned} \quad (67)$$

$$\begin{aligned} |G_2(t, u, v) - G_2(t, \bar{u}, \bar{v})| &= \left| \frac{u}{6} + \frac{2v}{t^2+3} - \frac{\bar{u}}{6} - \frac{2\bar{v}}{t^2+3} \right| \\ &\leq \frac{1}{6}|u - \bar{u}| + \frac{2}{3}|v - \bar{v}|, \end{aligned} \quad (68)$$

which shows that assumption (vi) of Theorem 3 is satisfied with

$$M_1^{(1)} = \frac{2}{5}, M_2^{(1)} = \frac{1}{5}, M_1^{(2)} = \frac{1}{6}, M_2^{(2)} = \frac{2}{3}. \quad (69)$$

For all $t \geq 0$ and $(u, v) \in \mathbb{R}^2$, one has

$$|F_1(t, u, v)| = \left| \sin \left(\frac{u}{4} + \frac{3v}{\sqrt{t+16}} \right) \right| \leq \frac{1}{4}|u| + \frac{3}{4}|v|, \quad (70)$$

$$|F_2(t, u, v)| = \left| \frac{u}{t^2+2} + \frac{v}{2} \right| \leq \frac{1}{2}|u| + \frac{1}{2}|v|, \quad (71)$$

which shows that assumption (v) of Theorem 3 is satisfied with

$$Q_1^{(1)} = \frac{1}{4}, Q_2^{(1)} = \frac{3}{4}, Q_1^{(2)} = Q_2^{(2)} = \frac{1}{2}, \ell_1^{(1)} = \ell_2^{(1)} = \ell_1^{(2)} = \ell_2^{(2)} = 1. \quad (72)$$

Similarly, one has

$$|G_1(t, u, v)| = \left| \arctan \left(\frac{2u+v}{5} \right) \right| \leq \frac{2}{5}|u| + \frac{1}{5}|v|, \quad (73)$$

$$|G_2(t, u, v)| = \left| \frac{u}{6} + \frac{2v}{t^2+3} \right| \leq \frac{1}{6}|u| + \frac{2}{3}|v|, \quad (74)$$

which shows that assumption (VII) is satisfied with

$$P_1^{(1)} = \frac{2}{5}, P_2^{(1)} = \frac{1}{5}, P_1^{(2)} = \frac{1}{6}, P_2^{(2)} = \frac{2}{3}, \sigma_1^{(1)} = \sigma_2^{(1)} = \sigma_1^{(2)} = \sigma_2^{(2)} = 1. \quad (75)$$

From the above estimates, one deduces that the system of inequalities (12) and (13) is equivalent to

$$\begin{cases} \left[2 + \frac{3}{32} \left(\frac{1}{4}\rho_1 + \frac{3}{4}\rho_2 \right) \right] \left[2 + \frac{3}{64}(\rho_1 + \rho_2) \right] \leq \rho_1, \\ \left[2 + \frac{3}{160}(2\rho_1 + \rho_2) \right] \left[2 + \frac{1}{64}(\rho_1 + 4\rho_2) \right] \leq \rho_2. \end{cases} \quad (76)$$

Taking $\rho_1 = \rho_2 = \rho > 0$, (76) reduces to

$$\begin{cases} \left(2 + \frac{3}{32}\rho \right)^2 \leq \rho, \\ \left(2 + \frac{9}{160}\rho \right) \left(2 + \frac{5}{64}\rho \right) \leq \rho. \end{cases} \quad (77)$$

On the other hand, one observes easily that

$$\left(2 + \frac{9}{160}\rho \right) \left(2 + \frac{5}{64}\rho \right) \leq \left(2 + \frac{3}{32}\rho \right)^2, \rho > 0. \quad (78)$$

Therefore, any $\rho > 0$ satisfying

$$\left(2 + \frac{3}{32}\rho \right)^2 \leq \rho, \quad (79)$$

is a solution to (77). In particular, for $\rho = 36$, one has

$$\left(2 + \frac{3}{32}\rho \right)^2 = 28.890625 < 36 = \rho. \quad (80)$$

Therefore, $\rho = 36$ is a solution to (77), which shows that assumption (viii) of Theorem 3 is satisfied with $(\rho_1, \rho_2) = (36, 36)$.

Finally, by Theorem 3, one deduces that system (50) admits one and only one solution $(x^*, y^*) \in C([0, \infty)) \times C([0, \infty))$ satisfying

$$\sup_{t \geq 0} \{|x^*(t)|, |y^*(t)|\} \leq 36. \quad (81)$$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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