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Research Article

Multiplication Operators on Orlicz Generalized Difference (sss)

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In this article, we inspect the sufficient conditions on the Orlicz generalized difference sequence space to be premodular Banach (sss). We look at some topological and geometrical structures of the multiplication operators described on Orlicz generalized difference prequasi normed (sss).

1. Introduction

The multiplication operators have a large subject of mathematics in functional analysis, namely, in eigenvalue distribution theorem, geometric structure of Banach spaces, and theory of fixed point. For more technicalities (see [1–6]), by $\mathbb{C}^{\mathbb{N}}$, c, ℓ_∞, ℓ_r, and c_0, we mean the spaces of each, convergent, bounded, r-absolutely summable and convergent to zero sequences of complex numbers. $\mathbb{N}$ displays the set of nonnegative integers. Tripathy et al. [7] popularized and measured the forward and backward generalized difference sequence spaces:

$$G\left(\Delta_n^{(m)}\right) = \left\{(w_k) \in \mathbb{C}^{\mathbb{N}} : \left(\Delta_n^{(m)} w_k\right) \in G\right\},\quad G(\Delta_n^m) = \left\{(w_k) \in \mathbb{C}^{\mathbb{N}} : (\Delta_n^m w_k) \in G\right\}, \tag{1}$$

where $m, n \in \mathbb{N}$, $G = \ell_\infty$, c, or c_0, with

$$\Delta_n^{(m)} w_k = \sum_{\nu=0}^{m} (-1)^\nu C_\nu^m w_{k+\nu n}, \text{and } \Delta_n^m w_k = \sum_{\nu=0}^{m} (-1)^\nu C_\nu^m w_{k-\nu n}, \tag{2}$$

successively. When $n = 1$, the generalized difference sequence spaces concentrated to $G(\Delta^{(m)})$ defined and examined by Et and Çolak [8]. If $m = 1$, the generalized difference sequence spaces diminished to $G(\Delta_n)$ constructed and studied by Tripathy and Esi [9]. While if $n = 1$ and $m = 1$, the generalized difference sequence spaces reduced to $G(\Delta)$ defined and investigated by Kizmaz [10].

An Orlicz function [11] is a function $\psi : [0,\infty) \to [0,\infty)$, which is convex, continuous, and nondecreasing with $\psi(0) = 0$, $\psi(u) > 0$, for $u > 0$ and $\psi(u) \to \infty$, as $u \to \infty$. In [12], an Orlicz function ψ is called to satisfy the δ_2-condition for each values of $x \geq 0$, if there is $k > 0$, such that $\psi(2x) \leq k\psi(x)$. The δ_2-condition is equivalent to $\psi(lx) \leq kl\psi(x)$, for every values of x and $l > 1$. Lindentrauss and Tzafriri [13] used the idea of an Olicz function to construct the Orlicz sequence space:

$$\ell_\psi = \left\{u \in \mathbb{C}^{\mathbb{N}} : \rho(\beta u) < \infty, \text{for some } \beta > 0\right\}, \text{where } \rho(u) = \sum_{k=0}^{\infty} \psi(|u_k|), \tag{3}$$

$(\ell_\psi, \|.\|)$ is a Banach space with the Luxemburg norm:

$$\|u\| = \inf\left\{\beta > 0 : \rho\left(\frac{u}{\beta}\right) \leq 1\right\}. \tag{4}$$

Every Orlicz sequence space includes a subspace that is isomorphic to c_0 or ℓ^q, for some $1 \le q < \infty$.

Recently, different classes of sequences have been examined the usage of Orlicz functions via Et et al. [14], Mursaleen et al. [15–17], and Alotaibi et al. [18].

Let $r = (r_j) \in \mathbb{R}^{+\mathbb{N}}$, where $\mathbb{R}^{+\mathbb{N}}$ denotes the space of sequences with positive reals, and we define the Orlicz backward generalized difference sequence space as follows:

$$\left(\ell_{\psi}(\Delta_{n+1}^{m})\right)_{\tau} = \left\{w = (w_j) \in \mathbb{C}^{\mathbb{N}} : \exists \sigma > 0 \text{ with } \tau(\sigma w) < \infty\right\}, \tag{5}$$

where $\tau(w) = \sum_{j=0}^{\infty} \psi(|\Delta_{n+1}^{m} |w_j||)$, $w_j = 0$, for $j < 0$, $\Delta_{n+1}^{m} | w_j | = \Delta_{n+1}^{m-1} | w_j | - \Delta_{n+1}^{m-1} |w_{j-1}|$, and $\Delta^0 w_j = w_j$, for all $j, n, m \in \mathbb{N}$. It is a Banach space, with

$$\|w\| = \inf\left\{\sigma > 0 : \tau\left(\frac{w}{\sigma}\right) \le 1\right\}. \tag{6}$$

When $\psi(w) = w^r$, then $\ell_{\psi}(\Delta_{n+1}^{m}) = \ell_r(\Delta_{n+1}^{m})$ investigated via many authors (see [19–21]). By $\mathfrak{B}(W, Z)$, we will denote the set of every operators which are linear and bounded between Banach spaces W and Z, and if $W = Z$, we write $\mathfrak{B}(W)$. On sequence spaces, Basarir and Kara examined the compact operators on some Euler $B(m)$-difference sequence spaces [22], some difference sequence spaces of weighted means [23], the Riesz $B(m)$-difference sequence space [24], the B-difference sequence space derived by weighted mean [25], and the m^{th} order difference sequence space of generalized weighted mean [26]. Mursaleen and Noman [27, 28] investigated the compact operators on some difference sequence spaces. The multiplication operators on $(ces(r), \|.\|)$ with the Luxemburg norm $\|.\|$ elaborated by Komal et al. [29]. İlkhan et al. [30] studied the multiplication operators on Cesáro second order function spaces. Bakery et al. [31] examined the multiplication operators on weighted Nakano (sss). The aim of this article is to explain some results of $\left(\ell_{\psi}(\Delta_{n+1}^{m})\right)_{\tau}$ equipped with the prequasi norm τ. Firstly, we accord the sufficient conditions on the Orlicz generalized difference sequence space to become premodular Banach (sss). Secondly, we provide with the necessity and sufficient conditions on the Orlicz generalized difference sequence space provided with the prequasi norm so that the multiplication operator defined on it is bounded, approximable, invertible, Fredholm, and closed range operator.

2. Preliminaries and Definitions

Definition 1 [32]. An operator $V \in \mathfrak{B}(W)$ is known as approximable if there are $D_r \in F(W)$, for every $r \in \mathbb{N}$ and $\lim_{r\to\infty} \|V - D_r\| = 0$.

By $\Upsilon(W, Z)$, we will denote the space of all approximable operators from W to Z.

Theorem 2 [32]. *Let W be a Banach space with $\dim(W) = \infty$; then,*

$$F(W) \underset{\neq}{\subset} \Upsilon(W) \underset{\neq}{\subset} \mathfrak{B}_c(W) \underset{\neq}{\subset} \mathfrak{B}(W). \tag{7}$$

Definition 3 [33]. An operator $V \in \mathfrak{B}(W)$ is named Fredholm if $\dim(R(V))^c < \infty$, $\dim(ker V) < \infty$, and $R(V)$ are closed, where $(R(V))^c$ indicates the complement of range V.

The sequence $e_j = (0, 0, \ldots, 1, 0, 0, \cdots)$ with 1 in the j^{th} coordinate, for all $j \in \mathbb{N}$, will be used in the sequel.

Definition 4 [34]. The space of linear sequence spaces $\mathbb{Y}$ is called (sss) if

(1) $e_r \in \mathbb{Y}$ with $r \in \mathbb{N}$

(2) Let $u = (u_r) \in \mathbb{C}^{\mathbb{N}}$, $v = (v_r) \in \mathbb{Y}$, and $|u_r| \le |v_r|$, for every $r \in \mathbb{N}$, then $u \in \mathbb{Y}$. This means $\mathbb{Y}$ be solid

(3) If $(u_r)_{r=0}^{\infty} \in \mathbb{Y}$, then $(u_{[r/2]})_{r=0}^{\infty} \in \mathbb{Y}$, wherever $[r/2]$ indicates the integral part of $r/2$

Definition 5 [35]. A subspace of the (sss) $\mathbb{Y}_{\tau}$ is named a premodular (sss) if there is a function $\tau : \mathbb{Y} \to [0, \infty)$ confirming the conditions:

(i) $\tau(y) \ge 0$ for each $y \in \mathbb{Y}$ and $\tau(y) = 0 \Leftrightarrow y = \theta$, where θ is the zero element of $\mathbb{Y}$

(ii) There exists $a \ge 1$ such that $\tau(\eta y) \le a|\eta|\tau(y)$, for all $y \in \mathbb{Y}$, and $\eta \in \mathbb{C}$

(iii) For some $b \ge 1$, $\tau(y + z) \le b(\tau(y) + \tau(z))$, for every $y, z \in \mathbb{Y}$

(iv) $|y_r| \le |z_r|$ with $r \in \mathbb{N}$ implies $\tau((y_r)) \le \tau((z_r))$

(v) For some $b_0 \ge 1$, $\tau((y_r)) \le \tau((y_{[r/2]})) \le b_0 \tau((y_i))$

(vi) If $y = (y_r)_{r=o}^{\infty} \in \mathbb{Y}$ and $d > 0$, then there is $r_0 \in \mathbb{N}$ with $\tau((y_r)_{r=r_0}^{\infty}) < d$

(vii) There is $t > 0$ with $\tau(v, 0, 0, 0, \cdots) \ge t|v|\tau(1, 0, 0, 0, \cdots)$, for any $v \in \mathbb{C}$

The (sss) $\mathbb{Y}_{\tau}$ is known as prequasi normed (sss) if τ administers the parts (i)-(iii) of Definition 5 and when the space $\mathbb{Y}$ is complete under τ, then $\mathbb{Y}_{\tau}$ is named a prequasi Banach (sss).

Theorem 6 [35]. *A prequasi norm (sss) $\mathbb{Y}_{\tau}$ if it is premodular (sss).*

The inequality [36], $|a_i + b_i|^{q_i} \le H(|a_i|^{q_i} + |b_i|^{q_i})$, where $q_i \ge 0$ for all $i \in \mathbb{N}$, $H = \max\{1, 2^{h-1}\}$ and $h = \sup_i q_i$, will be used in the sequel.

3. Main Results

3.1. Prequasi Norm on $\ell_\psi(\Delta^m_{n+1})$. In this section, we explain the conditions on the Orlicz backward generalized difference sequence space to form premodular Banach (sss).

Definition 7. The backward generalized difference Δ^m_{n+1} is named an absolute nondecreasing, if $|x_i| \le |y_i|$, for all $i \in \mathbb{N}$, then $|\Delta^m_{n+1}|x_i|| \le |\Delta^m_{n+1}|y_i||$.

Theorem 8. *Let ψ be an Orlicz function fulfilling the δ_2 -condition and Δ^m_{n+1} be an absolute nondecreasing, then the space $(\ell_\psi(\Delta^m_{n+1}))_\tau$ can be a premodular Banach (sss), where*

$$\tau(w) = \sum_{j=0}^{\infty} \psi\left(\left|\Delta^m_{n+1}|w_j|\right|\right), \text{for all } w \in \ell_\psi(\Delta^m_{n+1}). \tag{8}$$

Proof. (1-i) Assume $v, w \in \ell_\psi(\Delta^m_{n+1})$. Since ψ can be nondecreasing, convex, agreeable δ_2-condition, and Δ^m_{n+1} can be an absolute nondecreasing, then there is $b > 0$ such that

$$\begin{aligned} \tau(v+w) &= \sum_{i=0}^{\infty} \psi(|\Delta^m_{n+1}|v_i + w_i||) \\ &\le \sum_{i=0}^{\infty} \psi(|\Delta^m_{n+1}|v_i|| + |\Delta^m_{n+1}|w_i||) \\ &\le \frac{1}{2}\left(\sum_{i=0}^{\infty} \psi(2|\Delta^m_{n+1}|v_i||) + \sum_{i=0}^{\infty} \psi(2|\Delta^m_{n+1}|w_i||)\right. \\ &\le \frac{b}{2}(\tau(v) + \tau(w)) \le B(\tau(v) + \tau(w)) < \infty, \end{aligned} \tag{9}$$

for some $B = \max\{1, (b/2)\}$. Then, $v + w \in \ell_\psi(\Delta^m_{n+1})$.

(1) (1-ii) Suppose $\lambda \in \mathbb{C}$ and $v \in \ell_\psi(\Delta^m_{n+1})$. Since ψ is fulfilling the δ_2-condition, we obtain

$$\begin{aligned} \tau(\lambda v) &= \sum_{r=0}^{\infty} \psi(|\Delta^m_{n+1}|\lambda v_r||) \le d|\lambda| \sum_{r=0}^{\infty} \psi(|\Delta^m_{n+1}|v_r||) \\ &\le D|\lambda|\tau(v) < \infty, \end{aligned} \tag{10}$$

where $D = \max\{1, d\}$. Then, $\lambda v \in \ell_\psi(\Delta^m_{n+1})$. So, from parts (1-i) and (1-ii), the space $\ell_\psi(\Delta^m_{n+1})$ is linear. Since $e_r \in \ell_q \subseteq \ell_\psi \subseteq \ell_\psi(\Delta^m_{n+1})$, for every $r \in \mathbb{N}$ and $q \ge 1$, hence, $e_r \in \ell_\psi(\Delta^m_{n+1})$, for each $r \in \mathbb{N}$.

(2) Let $|x_i| \le |y_i|$, for every $i \in \mathbb{N}$ and $y \in \ell_\psi(\Delta^m_{n+1})$. Since ψ is nondecreasing and Δ^m_{n+1} is an absolute nondecreasing, therefore, we get

$$\tau(x) = \sum_{i=0}^{\infty} \psi(|\Delta^m_{n+1}|x_i||) \le \sum_{i=0}^{\infty} \psi(|\Delta^m_{n+1}|y_i||) = \tau(y) < \infty, \tag{11}$$

hence $x \in \ell_\psi(\Delta^m_{n+1})$

(3) Suppose $(v_r) \in \ell_\psi(\Delta^m_{n+1})$, one has

$$\tau\left(\left(v_{\left[\frac{r}{2}\right]}\right)\right) = \sum_{r=0}^{\infty} \psi\left(\left|\Delta^m_{n+1}\left|v_{\left[\frac{r}{2}\right]}\right|\right|\right) \le 2\sum_{r=0}^{\infty} \psi(|\Delta^m_{n+1}|v_r||) = 2\tau(v), \tag{12}$$

then $(v_{[r/2]}) \in \ell_\psi(\Delta^m_{n+1})$

(i) Evidently, $\tau(w) \ge 0$ and $\tau(w) = 0 \Leftrightarrow w = \theta$

(ii) There is $D \ge 1$ where $\tau(\eta w) \le D|\eta|\tau(w)$, for every $w \in \ell_\psi(\Delta^m_{n+1})$ and $\eta \in \mathbb{C}$

(iii) For some $B \ge 1$, we obtain $\tau(v + w) \le B(\tau(v) + \tau(w))$, for all $v, w \in \ell_\psi(\Delta^m_{n+1})$

(iv) Plainly from (2).

(v) From (3), we have that $b_0 = 2 \ge 1$

(vi) It is apparent that $\bar{F} = \ell_\psi(\Delta^m_{n+1})$

(vii) Since ψ is verifying the δ_2-condition, there is ζ with $0 < \zeta \le \psi(|\eta|)/|\eta|$ such that $\tau(\eta, 0,0,0, \cdots) \ge \zeta|\eta|\tau(1,0,0,0, \cdots)$, for each $\eta \ne 0$ and $\zeta > 0$, if $\eta = 0$

Therefore, the space $(\ell_\psi(\Delta^m_{n+1}))_\tau$ is premodular (sss). To show that $(\ell_\psi(\Delta^m_{n+1}))_\tau$ is a premodular Banach (sss), Suppose $x^i = (x^i_k)_{k=0}^{\infty}$ is a Cauchy sequence in $(\ell_\psi(\Delta^m_{n+1}))_\tau$, then for all $\varepsilon \in (0, 1)$, there is $i_0 \in \mathbb{N}$ such that for all $i, j \ge i_0$, we get

$$\tau\left(x^i - x^j\right) = \sum_{k=0}^{\infty} \psi\left(\left|\Delta^m_{n+1}\left|x^i_k - x^j_k\right|\right|\right) < \psi(\varepsilon). \tag{13}$$

Since ψ is nondecreasing; hence, for $i, j \ge i_0$ and $k \in \mathbb{N}$, we obtain

$$\left|\Delta^m_{n+1}\left|x^i_k\right| - \Delta^m_{n+1}\left|x^j_k\right|\right| < \varepsilon. \tag{14}$$

Hence, $(\Delta^m_{n+1}|x^j_k|)$ is a Cauchy sequence in $\mathbb{C}$ for fixed $k \in \mathbb{N}$, so $\lim_{j\to\infty} \Delta^m_{n+1} x^j_k = \Delta^m_{n+1} x^0_k$ for fixed $k \in \mathbb{N}$. Therefore, $\tau(x^i - x^0) < \psi(\varepsilon)$, for each $i \ge i_0$. Finally, to explain that $x^0 \in \ell_\psi(\Delta^m_{n+1})$, we have

$$\tau\left(x^0\right) = \tau\left(x^0 - x^n + x^n\right) \le B\left(\tau\left(x^n - x^0\right) + \tau(x^n)\right) < \infty. \tag{15}$$

So, $x^0 \in \ell_\psi(\Delta^m_{n+1})$. This implies that $(\ell_\psi(\Delta^m_{n+1}))_\tau$ is a premodular Banach (sss).

Taking into consideration (Theorem 6), we be over the following theorem.

Theorem 9. *If ψ is an Orlicz function satisfying the δ_2-condition and Δ^m_{n+1} is an absolute nondecreasing, then the space $(\ell_\psi(\Delta^m_{n+1}))_\tau$ is prequasi Banach (sss), where*

$$\tau(x) = \sum_{j=0}^{\infty} \psi\left(\left|\Delta^m_{n+1}\left|x_j\right|\right|\right), \text{for all } x \in \ell_\psi(\Delta^m_{n+1}). \tag{16}$$

Corollary 10. *If $0 < p < \infty$ and Δ^m_{n+1} is an absolute nondecreasing, then $(\ell_p(\Delta^m_{n+1}))_\tau$ is a premodular Banach (sss), where $\tau(x) = \sum_{i=0}^{\infty} | \Delta^m_{n+1} | x_i||^p$, for all $x \in \ell_p(\Delta^m_{n+1})$.*

4. Bounded Multiplication Operator on $\ell_\psi(\Delta^m_{n+1})$

Here and after, we explain some geometric and topological structures of the multiplication operator reserve on $\ell_\psi(\Delta^m_{n+1})$.

Definition 11. Let $\kappa \in \mathbb{C}^{\mathbb{N}} \cap \ell_\infty$ and W_τ be a prequasi normed (sss). An operator $V_\kappa : W_\tau \to W_\tau$ is named multiplication operator if $V_\kappa w = \kappa w = (\kappa_r w_r)_{r=0}^{\infty} \in W$, for every $w \in W$. If $V_\kappa \in \mathfrak{B}(W)$, we call it a multiplication operator generated by κ.

Theorem 12. *If $\kappa \in \mathbb{C}^{\mathbb{N}}$, ψ is an Orlicz function verifying the δ_2-condition, and Δ^m_{n+1} is an absolute nondecreasing, then $\kappa \in \ell_\infty$, if and only if, $V_\kappa \in \mathfrak{B}(\ell_\psi(\Delta^m_{n+1})_\tau)$, where $\tau(x) = \sum_{r=0}^{\infty}\psi(|\Delta^m_{n+1}|x_r||)$, for each $x \in \ell_\psi(\Delta^m_{n+1})$.*

Proof. Assume the conditions can be satisfied. Let $\kappa \in \ell_\infty$. So, there is $\varepsilon > 0$ with $|\kappa_r| \le \varepsilon$, for each $r \in \mathbb{N}$, for $x \in (\ell_\psi(\Delta^m_{n+1})_\tau$. Since Δ^m_{n+1} is an absolute nondecreasing and ψ is nondecreasing verifying the δ_2-condition, then

$$\begin{aligned}\tau(V_\kappa x) &= \tau(\kappa x) = \tau\left((\kappa_r x_r)_{r=0}^{\infty}\right) = \sum_{r=0}^{\infty}\psi(|\Delta^m_{n+1}(|\kappa_r||x_r|)|) \\ &\le \sum_{r=0}^{\infty}\psi(|\Delta^m_{n+1}(\varepsilon|x_r|)|) \le d\varepsilon\sum_{r=0}^{\infty}\psi(|\Delta^m_{n+1}|x_r||) \le D\tau(x),\end{aligned} \tag{17}$$

where $D = \max\{1, d\varepsilon\}$. This implies $V_\kappa \in \mathfrak{B}(\ell_\psi(\Delta^m_{n+1})_\tau)$. Inversely, suppose that $V_\kappa \in \mathfrak{B}(\ell_\psi(\Delta^m_{n+1})_\tau)$. Let us suppose $\kappa \notin \ell_\infty$, hence, for all $j \in \mathbb{N}$, there is $i_j \in \mathbb{N}$ so as to $\kappa_{i_j} > j$. Since Δ^m_{n+1} is an absolute nondecreasing and ψ is nondecreasing, one has

$$\begin{aligned}\tau\left(V_\kappa e_{i_j}\right) &= \tau\left(\kappa e_{i_j}\right) = \tau\left(\left(\kappa_r\left(e_{i_j}\right)_r\right)_{r=0}^{\infty}\right) \\ &= \sum_{r=0}^{\infty}\psi\left(\left|\Delta^m_{n+1}\left(\left|\kappa_r\right|\left|\left(e_{i_j}\right)_r\right|\right)\right|\right) = \psi\left(\left|\Delta^m_{n+1}\left|\kappa_{i_j}\right|\right|\right) \\ &> \psi(|\Delta^m_{n+1}|j||) = \psi(|\Delta^m_{n+1}|j||)\tau\left(e_{i_j}\right).\end{aligned} \tag{18}$$

This proves that $V_\kappa \notin \mathfrak{B}(\ell_\psi(\Delta^m_{n+1})_\tau)$. Therefore, $\kappa \in \ell_\infty$.

Theorem 13. *Let $\kappa \in \mathbb{C}^{\mathbb{N}}$ and $(\ell_\psi(\Delta^m_{n+1}))_\tau$ be a prequasi normed (sss), with $\tau(x) = \sum_{r=0}^{\infty}\psi(|\Delta^m_{n+1}|x_r||)$, for all $x \in \ell_\psi(\Delta^m_{n+1})$. Then, $|\kappa_r| = 1$, for every $r \in \mathbb{N}$, if and only if, V_κ is an isometry.*

Proof. Presume $|\kappa_r| = 1$, for each $r \in \mathbb{N}$, we have

$$\begin{aligned}\tau(V_\kappa x) &= \tau(\kappa x) = \tau\left((\kappa_r x_r)_{r=0}^{\infty}\right) \\ &= \sum_{r=0}^{\infty}\psi(|\Delta^m_{n+1}(|\kappa_r||x_r|)|) \\ &= \sum_{r=0}^{\infty}\psi(|\Delta^m_{n+1}|x_r||) = \tau(x),\end{aligned} \tag{19}$$

for each $x \in (\ell_\psi(\Delta^m_{n+1}))_\tau$. Therefore, V_κ is an isometry. Inversely, suppose that $|\kappa_i| < 1$, for some $i = i_0$, given that Δ^m_{n+1} is an absolute nondecreasing and ψ is nondecreasing, we get

$$\begin{aligned}\tau\left(V_\kappa e_{i_0}\right) &= \tau\left(\kappa e_{i_0}\right) = \tau\left(\left(\kappa_r\left(e_{i_0}\right)_r\right)_{r=0}^{\infty}\right) \\ &= \sum_{r=0}^{\infty}\psi\left(\left|\Delta^m_{n+1}\left(\left|\kappa_r\right|\left|\left(e_{i_0}\right)_r\right|\right)\right|\right) \\ &< \sum_{r=0}^{\infty}\psi\left(\left|\Delta^m_{n+1}\left|\left(e_{i_0}\right)_r\right|\right|\right) = \tau\left(e_{i_0}\right).\end{aligned} \tag{20}$$

While $|\kappa_{i_0}| > 1$, we can show that $\tau(V_\kappa e_{i_0}) > \tau(e_{i_0})$. As a result, in both cases, we obtain a contradiction. Therefore, $|\kappa_r| = 1$, for all $r \in \mathbb{N}$.

5. Approximable Multiplication Operator on $\ell_\psi(\Delta^m_{n+1})$

In this section, we investigate the sufficient conditions on the Orlicz backward generalized difference sequence space equipped with prequasi norm τ so that the multiplication operator acting on $\ell_\psi(\Delta^m_{n+1})$ is an approximable and compact.

By card(A), we denote the cardinality of the set A.

Theorem 14. *If $\kappa \in \mathbb{C}^{\mathbb{N}}$ and $(\ell_\psi(\Delta^m_{n+1}))_\tau$ is a prequasi normed (sss), where $\tau(x) = \sum_{r=0}^{\infty}\psi(|\Delta^m_{n+1}|x_r||)$, for all $x \in \ell_\psi(\Delta^m_{n+1})$, then $V_\kappa \in \Upsilon((\ell_\psi(\Delta^m_{n+1}))_\tau)$ if and only if $(\kappa_r)_{r=0}^{\infty} \in c_0$.*

Proof. Let $V_\kappa \in \Upsilon((\ell_\psi(\Delta^m_{n+1}))_\tau)$. So, $V_\kappa \in B_c((\ell_\psi(\Delta^m_{n+1}))_\tau)$, to show that $(\kappa_r)_{r=0}^{\infty} \in c_0$. Assume $(\kappa_r)_{r=0}^{\infty} \notin c_0$, therefore there is $\delta > 0$ so that $A_\delta = \{r \in \mathbb{N} : |\kappa_r| \ge \delta\}$ has card$(A_\delta) = \infty$. Suppose $a_i \in A_\delta$, for each $i \in \mathbb{N}$, then $\{e_{a_i} : a_i \in A_\delta\}$ is an infinite bounded set in $(\ell_\psi(\Delta^m_{n+1}))_\tau$. Suppose

$$\begin{aligned}\tau\left(V_\kappa e_{a_i} - V_\kappa e_{a_j}\right) &= \tau\left(\kappa e_{a_i} - \kappa e_{a_j}\right) = \tau\left(\left(\kappa_r\left(\left(e_{a_i}\right)_r - \left(e_{a_j}\right)_r\right)\right)_{r=0}^\infty\right) \\ &= \sum_{r=0}^\infty \psi\left(\left|\Delta^m_{n+1}\left|\kappa_r\left(\left(e_{a_i}\right)_r - \left(e_{a_j}\right)_r\right)\right|\right|\right) \\ &\geq \sum_{r=0}^\infty \psi\left(\left|\Delta^m_{n+1}\left|\delta\left(\left(e_{a_i}\right)_r - \left(e_{a_j}\right)_r\right)\right|\right|\right) \\ &= \tau\left(\delta e_{a_i} - \delta e_{a_j}\right),\end{aligned} \tag{21}$$

for every $a_i, a_j \in A_\delta$. This proves $\{e_{a_i} : a_i \in B_\delta\} \in \ell_\infty$ which cannot have a convergent subsequence under V_κ. This gives that $V_\kappa \notin B_c((\ell_\psi(\Delta^m_{n+1}))_\tau)$. Then, $V_\kappa \notin \Upsilon((\ell_\psi(\Delta^m_{n+1}))_\tau)$, and this gives a contradiction. So, $\lim_{i\to\infty}\kappa_i = 0$. Contrarily, assume $\lim_{i\to\infty}\kappa_i = 0$, then for all $\delta > 0$, the set $A_\delta = \{i \in \mathbb{N} : |\kappa_i| \geq \delta\}$ has card $(A_\delta) < \infty$. So, for all $\delta > 0$, the space

$$\left((\ell_\psi(\Delta^m_{n+1}))_\tau\right)_{A_\delta} = \left\{x = (x_i) \in \mathbb{C}^{A_\delta}\right\} \tag{22}$$

is finite dimensional. Then, $V_\kappa \mid ((\ell_\psi(\Delta^m_{n+1}))_\tau)_{A_\delta}$ is a finite rank operator. For all $i \in \mathbb{N}$, illustrate $\kappa_i \in \mathbb{C}^\mathbb{N}$ by

$$(\kappa_i)_j = \begin{cases} \kappa_j, & j \in A_{\frac{1}{i}} \\ 0, & \text{otherwise.} \end{cases} \tag{23}$$

Evidently, V_{κ_i} has rank $(V_{\kappa_i}) < \infty$ as dim $((\ell_\psi(\Delta^m_{n+1}))_\tau)_{A_{1/i}} < \infty$, for all $i \in \mathbb{N}$. Hence, since Δ^m_{n+1} is an absolute nondecreasing and ψ is convex and nondecreasing, we obtain

$$\begin{aligned}\tau((V_\kappa - V_{\kappa_i})x) &= \tau\left(\left(\left(\kappa_j - (\kappa_i)_j\right)x_j\right)_{j=0}^\infty\right) \\ &= \sum_{j=0}^\infty \psi\left(\left|\Delta^m_{n+1}\left(\left|\left(\kappa_j - (\kappa_i)_j\right)x_j\right|\right)\right|\right) \\ &= \sum_{j=0, j\in A_{\frac{1}{i}}}^\infty \psi\left(\left|\Delta^m_{n+1}\left(\left|\left(\kappa_j - (\kappa_i)_j\right)x_j\right|\right)\right|\right) \\ &\quad + \sum_{j=0, j\notin A_{\frac{1}{i}}}^\infty \psi\left(\left|\Delta^m_{n+1}\left(\left|\left(\kappa_j - (\kappa_i)_j\right)x_j\right|\right)\right|\right) \\ &= \sum_{j=0, j\notin A_{\frac{1}{i}}}^\infty \psi\left(\left|\Delta^m_{n+1}\left|\kappa_j x_j\right|\right|\right) \leq \frac{1}{i}\sum_{j=0, j\notin A_{\frac{1}{i}}}^\infty \psi\left(\left|\Delta^m_{n+1}\left|x_j\right|\right|\right) \\ &< \frac{1}{i}\sum_{j=0}^\infty \psi\left(\left|\Delta^m_{n+1}\left|x_j\right|\right|\right) = \frac{1}{i}\tau(x).\end{aligned} \tag{24}$$

This gives that $\|V_\kappa - V_{\kappa_i}\| \leq 1/i$, and that V_κ is a limit of finite rank operators. So, V_κ is an approximable operator.

Theorem 15. *Pick up $\kappa \in \mathbb{C}^\mathbb{N}$ and $(\ell_\psi(\Delta^m_{n+1}))_\tau$ be a prequasi normed (sss), where $\tau(x) = \sum_{r=0}^\infty \psi(|\Delta^m_{n+1}|x_r||)$, for every $x \in \ell_\psi(\Delta^m_{n+1})$. Therefore, $V_\kappa \in B_c((\ell_\psi(\Delta^m_{n+1}))_\tau)$, if and only if, $(\kappa_i)_{i=0}^\infty \in c_0$.*

Proof. Clearly, since every approximable operator is compact.

Corollary 16. *If $\kappa \in \mathbb{C}^\mathbb{N}$, ψ is an Orlicz function satisfying the δ_2-condition, and Δ^m_{n+1} is an absolute nondecreasing, then $\mathfrak{B}_c((\ell_\psi(\Delta^m_{n+1}))_\tau) \subsetneq \mathfrak{B}((\ell_\psi(\Delta^m_{n+1}))_\tau)$, where $\tau(x) = \sum_{r=0}^\infty \psi(|\Delta^m_{n+1}|x_r||)$, for all $x \in \ell_\psi(\Delta^m_{n+1})$.*

Proof. In view of I that is a multiplication operator on $(\ell_\psi(\Delta^m_{n+1}))_\tau$ generated by $\kappa = (1, 1,)$. So, $I \notin B_c((\ell_\psi(\Delta^m_{n+1}))_\tau)$ and $I \in B((\ell_\psi(\Delta^m_{n+1}))_\tau)$.

6. Fredholm Multiplication Operator on $\ell_\psi(\Delta^m_{n+1})$

In this section, we introduce the sufficient conditions on the sequence space $\ell_\psi(\Delta^m_{n+1})$ equipped with prequasi norm τ so that the multiplication operator acting on it has closed range, invertible, and Fredholm.

Theorem 17. *Let $\kappa \in \mathbb{C}^\mathbb{N}$, $(\ell_\psi(\Delta^m_{n+1}))_\tau$, be prequasi Banach (sss), where $\tau(x) = \sum_{r=0}^\infty \psi(|\Delta^m_{n+1}|x_r||)$, for all $x \in \ell_\psi(\Delta^m_{n+1})$, and $V_\kappa \in B((\ell_\psi(\Delta^m_{n+1}))_\tau)$. Then, κ be bounded away from zero on $(\ker(\kappa))^c$, if and only if, $R(V_\kappa)$ is closed.*

Proof. Suppose the sufficient condition be satisfied, so, there is $\varepsilon > 0$ with $|\kappa_i| \geq \varepsilon$, for every $i \in (\ker(\kappa))^c$, to prove that $R(V_\kappa)$ is closed. Let d be a limit point of $R(V_\kappa)$. Hence, there is $V_\kappa x_i$ in $(\ell_\psi(\Delta^m_{n+1}))_\tau$, for each $i \in \mathbb{N}$ so that $\lim_{i\to\infty} V_\kappa x_i = d$. Clearly, $(V_\kappa x_i)$ is a Cauchy sequence. As Δ^m_{n+1} is an absolute nondecreasing and ψ is nondecreasing, we have

$$\begin{aligned}\tau\left(V_\kappa x_i - V_\kappa x_j\right) &= \sum_{r=0}^\infty \psi\left(\left|\Delta^m_{n+1}\left|\kappa_r(x_i)_r - \kappa_r\left(x_j\right)_r\right|\right|\right) \\ &= \sum_{r=0, r\in(\ker(\kappa))^c}^\infty \psi\left(\left|\Delta^m_{n+1}\left|\kappa_r(x_i)_r - \kappa_r\left(x_j\right)_r\right|\right|\right) \\ &\quad + \sum_{r=0, r\notin(\ker(\kappa))^c}^\infty \psi\left(\left|\Delta^m_{n+1}\left|\kappa_r(x_i)_r - \kappa_r\left(x_j\right)_r\right|\right|\right) \\ &\geq \sum_{r=0, r\in(\ker(\kappa))^c}^\infty \psi\left(\left|\Delta^m_{n+1}\left(|\kappa_r|\left|(x_i)_r - \left(x_j\right)_r\right|\right)\right|\right) \\ &= \sum_{r=0}^\infty \psi\left(\left|\Delta^m_{n+1}\left(|\kappa_r|\left|(y_i)_r - \left(y_j\right)_r\right|\right)\right|\right) \\ &> \sum_{r=0}^\infty \psi\left(\left|\Delta^m_{n+1}\left|\varepsilon(y_i)_r - \left(y_j\right)_r\right|\right|\right) = \tau(\varepsilon(y_n - y_m)),\end{aligned} \tag{25}$$

where

$$(y_i)_r = \begin{cases} (x_i)_r, & r \in (\ker(\kappa))^c \\ 0, & r \notin (\ker(\kappa))^c \end{cases}. \quad (26)$$

This gives that (y_i) is a Cauchy sequence in $(\ell_\psi(\Delta^m_{n+1}))_\tau$. As $(\ell_\psi(\Delta^m_{n+1}))_\tau$ is complete, there is $x \in (\ell_\psi(\Delta^m_{n+1}))_\tau$ so that $\lim_{i\to\infty} y_i = x$. As V_κ is continuous, then $\lim_{i\to\infty} V_\kappa y_i = V_\kappa x$. Although $\lim_{i\to\infty} V_\kappa x_i = \lim_{i\to\infty} V_\kappa y_i = d$, therefore, $V_\kappa x = d$. So, $d \in R(V_\kappa)$. This implies that $R(V_\kappa)$ is closed. Inversely, assume $R(V_\kappa)$ be closed, hence, V_κ be bounded away from zero on $((\ell_\psi(\Delta^m_{n+1}))_\tau)_{(\ker(\kappa))^c}$. So, there is $\varepsilon > 0$ so that $\tau(V_\kappa x) \geq \varepsilon\tau(x)$, for every $x \in ((\ell_\psi(\Delta^m_{n+1}))_\tau)_{(\ker(\kappa))^c}$.

Let $B = \{r \in (\ker(\kappa))^c : |\kappa_r| < \varepsilon\}$ as Δ^m_{n+1} is an absolute nondecreasing and ψ is nondecreasing verifying the δ_2-condition, if $B \neq \varphi$; then for $i_0 \in B$, one has

$$\tau(V_\kappa e_{i_0}) = \tau\left(\left(\kappa_r(e_{i_0})_r\right)_{r=0}^\infty\right) = \sum_{r=0}^\infty \psi\left(\left|\Delta^m_{n+1}\left|\kappa_r(e_{n_0})_r\right|\right|\right) < \sum_{r=0}^\infty \psi\left(\left|\Delta^m_{n+1}\left|\varepsilon(e_{n_0})_r\right|\right|\right) \leq d\varepsilon\tau(e_{n_0}), \quad (27)$$

for some $d \geq 1$. This implies a contradiction. Therefore, $B = \varphi$ so that $|\kappa_r| \geq \varepsilon$, for each $r \in (\ker(\kappa))^c$. This completes the proof of the theorem.

Theorem 18. *Let $\kappa \in \mathbb{C}^\mathbb{N}$ and $(\ell_\psi(\Delta^m_{n+1}))_\tau$ be a prequasi Banach (sss), with $\tau(w) = \sum_{r=0}^\infty \psi(|\Delta^m_{n+1}|w_r||)$, for every $w \in \ell_\psi(\Delta^m_{n+1})$. There are $b > 0$ and $B > 0$ so that $b < \kappa_r < B$, for every $r \in \mathbb{N}$, if and only if, $V_\kappa \in B((\ell_\psi(\Delta^m_{n+1}))_\tau)$ be invertible.*

Proof. Assume the conditions be established, define $\gamma \in \mathbb{C}^\mathbb{N}$ by $\gamma_r = 1/\kappa_r$, from Theorem 12, we obtain $V_\kappa, V_\gamma \in B((\ell_\psi(\Delta^m_{n+1}))_\tau)$ and $V_\kappa . V_\gamma = V_\gamma . V_\kappa = I$. Therefore, V_γ is the inverse of V_κ. Conversely, let V_κ be invertible. Hence, $R(V_\kappa) = ((\ell_\psi(\Delta^m_{n+1}))_\tau)_\mathbb{N}$. This gives $R(V_\kappa)$ which is closed. From Theorem 17, there is $b > 0$ so that $|\kappa_r| \geq b$, for each $r \in (\ker(\kappa))^c$. Now, $\ker(\kappa) = \varphi$, else $\kappa_{r_0} = 0$, for several $r_0 \in \mathbb{N}$, we have $e_{r_0} \in \ker(V_\kappa)$. This implies a contradiction, as $\ker(V_\kappa)$ is trivial. Therefore, $|\kappa_r| \geq a$, for every $r \in \mathbb{N}$. Because V_κ is bounded, so from Theorem 12, there is $B > 0$ so that $|\kappa_r| \leq B$, for each $r \in \mathbb{N}$. Hence, we have proved that $b \leq |\kappa_r| \leq B$, for every $r \in \mathbb{N}$.

Theorem 19. *Pick up $\kappa \in \mathbb{C}^\mathbb{N}$ and $(\ell_\psi(\Delta^m_{n+1}))_\tau$ be a prequasi Banach (sss), where $\tau(w) = \sum_{r=0}^\infty \psi(|\Delta^m_{n+1}|w_r||)$, for every $w \in \ell_\psi(\Delta^m_{n+1})$. Then, $V_\kappa \in B((\ell_\psi(\Delta^m_{n+1}))_\tau)$ be the Fredholm operator, if and only if, (i) card $(\ker(\kappa)) < \infty$ and (ii) $|\kappa_r| \geq \varepsilon$, for each $r \in (\ker(\kappa))^c$.*

Proof. Assume V_κ be Fredholm, Let card $(\ker(\kappa)) = \infty$. Therefore, $e_n \in \ker(V_\kappa)$, for every $n \in \ker(\kappa)$. As e_n's is linearly independent, this implies card $(\ker(V_\kappa) = \infty$. This gives a contradiction. Hence, card $(\ker(\kappa)) < \infty$. From Theorem 17, condition (ii) is verified. Next, if the necessary conditions are satisfied, to prove that V_κ is Fredholm, from Theorem 17, condition (ii) implies that $R(V_\kappa)$ is closed. Condition (i) gives that dim $(\ker(V_\kappa)) < \infty$ and dim $((R(V_\kappa))^c) < \infty$. So, V_κ is Fredholm.

Data Availability

No data were used to support this study.

Ethical Approval

This article does not contain any studies with human participants or animals performed by any of the authors.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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