Research Article

Some Nonlinear Integral Inequalities Connected with Retarded Terms on Time Scales

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The objective of this research is to formulate a specific class of integral inequalities of Gronwall kind concerning retarded term and nonlinear integrals with time scales theory. Our results generate several new inequalities that reflect continuous and discrete form, as well as giving the unknown function an upper bound estimate. The effectiveness of such inequalities arises from the belief that it is widely relevant in unique circumstances where there is no valid utilization of various available inequalities. Applications are additionally represented to display the legitimacy of built-up hypotheses.

1. Introduction

Over the last decades, a variety of basic and critical inequalities have been inspected with improvement in the methods of differential and integral equations, which are anticipating a great deal of study in the analysis of boundedness, global existence, and stability of solutions of differential and integral equations as well as difference equations [1, 2]. Among others, the inequalities of the Gronwall-Bellman type are of unique significance for providing explicit estimates for unknown functions.

Bellman [3] has proven the integral inequality

$$\mathfrak{L}(\hbar) \leq \mathfrak{m} + \int_{\hbar_1}^{\hbar} \mathcal{G}(\mathcal{P}) \mathfrak{L}(\mathcal{P}) d\mathcal{P}, \hbar \in [\hbar_1, \hbar], \quad (1)$$

for some $\mathfrak{m} \geq 0$, which is significantly dedicated to evaluate the equilibrium and asymptotic behavior with a view to find solutions for integral equations. Such inequalities have been

remarkably strengthened over a very long period of time by verifying their importance and intrinsic potential in the diverse fields of applied sciences.

Afterward, Pachpatte [4] replaced the constant $\mathfrak{m}$ from the prior integral inequality by a nondecreasing function $\mathfrak{m}(\hbar)$ and contemplated

$$\mathfrak{L}(\hbar) \leq \mathfrak{m}(\hbar) + \mathcal{H}(\hbar) \int_0^{\hbar} \mathcal{G}(\mathcal{P}) \mathfrak{L}(\mathcal{P}) d\mathcal{P}, \hbar \in [0, \infty). \quad (2)$$

The inequality (2) encourages fresh speculations and can be classified as incredible techniques in the understanding of specific differential and integral equations.

El-Owaidey et al. [5] introduced yet another basic type of inequality

$$\mathfrak{L}(\hbar) \leq \mathfrak{m}_0 + \int_0^{\hbar} \mathcal{G}(\mathcal{P}) \left[\mathfrak{L}^{\Psi}(\mathcal{P}) + \int_0^{\mathcal{P}} \mathcal{J}(\mathfrak{x}) \mathfrak{L}(\mathfrak{x}) d\mathfrak{x} \right] d\mathcal{P}, \hbar \in [0, \infty), \quad (3)$$

in which integrand accommodates the power and allows it more challenging to determine the unknown function where $0 \leq \Psi < 1$ is included.

Several scholars researched linear and nonlinear modifications for booming such kinds of inequalities (see [6–12]). Most articles tend to do with retarded nonlinear integral inequalities. In this database, the retarded integral inequality was identified by Lipovan [13].

$$\begin{aligned} \mathfrak{Q}(\hbar) &\leq \mathfrak{m} + \int_{h_1}^{\hbar} \mathcal{G}(\mathcal{P}) \rho(\mathfrak{Q}(\mathcal{P})) d\mathcal{P} + \int_{\omega(h_1)}^{\omega(\hbar)} \mathcal{J}(\mathcal{P}) \rho(\mathfrak{Q}(\mathcal{P})) d\mathcal{P}, \hbar_1 \\ &\leq \hbar < \hbar_2, \end{aligned} \quad (4)$$

$\rho \in C(\mathbb{R}_+, \mathbb{R}_+)$, $\rho(\mathfrak{Q}) > 0$ for $\mathfrak{Q} > 0$, and the integral consists of nonlinear equations in its more standard way. Over the span of late years, multiple retarded integral inequalities have been discussed by numerous researchers [14, 15] and the references therein.

Remarkably, the dynamic inequalities predict a necessary position within the production of the basic principle of time-scale dynamic equations. Hilger [16] was named to be the primary researcher who started the growth of time scale analytics. The ultimate point is to study an equation or an inequality which can be dynamic such that a time scale $\mathbb{T}$ be a domain of an unknown function. The motivation that guides the time scale theory is to unify continuous and discrete inspection. Several authors have approached and finished proper assessment of the characterizations and utilization of different types of inequalities on a timely basis [17–21] and the references in that. Around the beginning, Bohner and Peterson [22] tested the integral inequality

$$\mathfrak{Q}(\hbar) \leq \mathfrak{m}(\hbar) + \int_{h_1}^{\hbar} \mathcal{J}(\mathcal{P}) \mathfrak{Q}(\mathcal{P}) \Delta \mathcal{P}, \hbar \in \mathbb{T}. \quad (5)$$

Later in 2010, Li [23] has assessed the consequent nonlinear integral inequality of one variable

$$\mathfrak{Q}^{\gamma}(\hbar) \leq \mathfrak{m}(\hbar) + \mathcal{H}(\hbar) \int_{h_1}^{\hbar} [\mathcal{G}(\mathcal{P}) \mathfrak{Q}(\rho(\mathcal{P})) + \mathcal{J}(\mathcal{P})] \Delta \mathcal{P}, \quad (6)$$

for $\hbar \in \hbar_1$ with initial conditions $\mathfrak{Q}(\hbar) = \Omega(\hbar)$, $\hbar \in [\beta, \hbar_1] \cap \mathbb{T}$, $Y(\rho(\hbar)) \leq (\mathfrak{m}(\hbar))^{1/\gamma}$ for $\hbar \in \hbar_1$, $\rho(\hbar) \leq \hbar_1$, where $\gamma \geq 1$ is a constant, $\rho(\hbar) \leq \hbar$, $-\infty < \beta = \inf \{\rho(\hbar), \hbar \in \mathbb{T}_0\} \leq \hbar_1$, and $\Omega(\hbar) \in C_{rd}([\beta, \hbar_1] \cap \mathbb{T}, \mathbb{R}_+)$. Besides that, Pachpatte [24] has attempted to see the augmentation of the integral inequality

$$\mathfrak{Q}(\hbar) \leq \mathfrak{m}(\hbar) \int_{h_1}^{\hbar} \mathcal{G}(\mathcal{P}) \left[\mathfrak{Q}(\mathcal{P}) + \int_{h_1}^{\mathcal{P}} \mathcal{J}(\mathcal{P}, \mathfrak{x}) \mathfrak{Q}(\mathfrak{x}) \Delta \mathfrak{x} \right] \Delta \mathcal{P}, \quad (7)$$

such that $\mathcal{J}(\mathcal{P}, \mathfrak{x}) \geq 0$, $\mathcal{J}^{\Delta}(\mathcal{P}, \mathfrak{x}) \geq 0$ for $\hbar, \mathfrak{x} \in \mathbb{T}$ and $\sigma \leq \hbar$.

Further, in 2017, Haidong [25] suggested that nonlinear integral inequality be generalized as follows

$$\begin{aligned} \mathfrak{Q}(\hbar) &\leq \mathfrak{m}(\hbar) + \mathcal{G}(\hbar) \int_{\omega(h_1)}^{\omega(\hbar)} \left[\mathcal{G}_1(\mathcal{P}) \mathfrak{Q}(\mathcal{P}) + \mathcal{G}_2(\mathcal{P}) \int_{\omega(h_1)}^{\hbar} \mathcal{J}(\mathfrak{x}) \mathfrak{Q}(\mathfrak{x}) \Delta \mathfrak{x} \right] \Delta \mathcal{P} \\ &\quad + \lambda \mathfrak{m}(T) \int_{\omega(h_1)}^{\omega(T)} \left[\mathcal{G}_1(\mathcal{P}) \mathfrak{Q}(\mathcal{P}) + \mathcal{G}_2(\mathcal{P}) \int_{\omega(h_1)}^{\hbar} \mathcal{J}(\mathfrak{x}) \mathfrak{Q}(\mathfrak{x}) \Delta \mathfrak{x} \right] \Delta \mathcal{P}, \end{aligned} \quad (8)$$

where $\lambda \geq 0$. Although a lot of studies have been conducted on integral inequalities related to time scales, there is not that much research on retarded integral inequalities on time scales has been performed. For certain cases, however, specific types of integral and differential equations via power are required to investigate time scale delay inequalities in order to thrive and meet the desired targets.

Our primary concern of this work is not only to analyze nonlinear integral inequalities with retarded term but also to explore the well-known existing results which determine the explicit bounds of the solutions of the unknown functions of the particular dynamic equations on time scales. The new speculations are used as supportive tools to exhibit the description of integral inequalities and equations. The offered dynamic integral strategy for acquiring new results is clear and effective. There are other benefits of this technique: it is fast and short. Moreover, the proposed procedure can be modified to solve various systems with nonlinear fractional partial differential equations.

The rest of the manuscript is arranged as follows. In Section 2, we have some preliminary data which is an essential element for our key studies. Section 3 is devoted to theoretical discussions on the immense solutions of the problem beneath consideration. The examples supporting the theoretical consequences are given in Section 4. Finally, a few concluding feedback and suggestions for future research are provided in Section 5 and thus completes this work.

2. Basic Material on Time Scales

Below, we interpret some basic definitions and valuable theorem regarding time scale calculus.

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$.

Definition 1 (see [22]). The forward jump operator σ on $\mathbb{T}$ be defined by $\sigma(\hbar) = \inf \{\mathcal{P} \in \mathbb{T} : \mathcal{P} > \hbar\}$ for all $\hbar \in \mathbb{T}$. If $\sigma(\hbar) = \hbar$, then $\hbar$ is said to be right-dense and right-scattered if $\sigma(\hbar) > \hbar$. The backward jump operator and left-scattered and left-dense points are defined in a similar way.

Definition 2 (see [22]). $\xi : \mathbb{T} \rightarrow \mathbb{R}_+$ is called the graininess function if $\xi(\hbar) = \sigma(\hbar) - \hbar$. Also, $\xi(\hbar) = 0$ for $\mathbb{T} = \mathbb{R}$ and $\xi(\hbar) = \mathcal{P}$, with a positive constant $\mathcal{P}$ for $\mathbb{T} = \mathcal{P}\mathbb{Z}$.

Definition 3 (see [22]). The set $\mathbb{T}^{\sharp}$ is derived from T as follows: if $\mathbb{T}$ has a left-scattered maximum $\mathfrak{m}$, then $\mathbb{T}^{\sharp} = \mathbb{T} - \mathfrak{m}$; otherwise, $\mathbb{T}^{\sharp} = \mathbb{T}$.

Definition 4 (see [22]). For some $\hbar \in \mathbb{T}^\#$ and a function $\mathfrak{Z} \in (\mathbb{T}, \mathbb{R})$, the delta derivative of $\mathfrak{Z}$ is denoted by $\mathfrak{Z}^\Delta(\hbar)$ and satisfies

$$|\mathfrak{Z}(\sigma(\hbar)) - \mathfrak{Z}(\mathcal{P}) - \mathfrak{Z}^\Delta(\hbar)(\sigma(\hbar) - \mathcal{P})| \leq \varepsilon |\sigma(\hbar) - \mathcal{P}|, \forall \varepsilon > 0, \mathcal{P} \in \mathbb{N}, \quad (9)$$

and a neighborhood of $\hbar$ is $\mathbb{N}$. Also, $\mathfrak{Z}$ is the delta differential function at $\hbar$.

Definition 5 (see [22]). An antiderivative of $\mathfrak{Z}$ is $\mathfrak{Z}_1$ if $\mathfrak{Z}_1^\Delta(\hbar) = \mathfrak{Z}(\hbar)$, $\hbar \in \mathbb{T}^\#$, whereas

$$\int_{\varsigma_1}^{\varsigma} \mathfrak{Z}(\hbar) \Delta \hbar = \mathfrak{Z}_1(\varsigma) - \mathfrak{Z}_1(\varsigma_1), \varsigma_1, \varsigma \in \mathbb{T}, \quad (10)$$

is the Cauchy integral of $\mathfrak{Z}$.

Definition 6 (see [22]). The set of all regressive and rd-continuous functions $\mathcal{G} : \mathbb{T} \rightarrow \mathbb{R}$ will be represented by $\mathfrak{R}$ provided $1 + \xi(\hbar)\mathcal{G}(\hbar) \neq 0$ for all $\hbar \in \mathbb{T}^\#$ holds.

Definition 7. If $\mathbb{T} = \mathbb{R}$ and $\mathcal{G} : \mathbb{R} \rightarrow \mathbb{R}$ are a continuous function, then the exponential function is given by

$$e_{\mathcal{G}}(\hbar, \mathcal{P}) = e^{\int_{\mathcal{P}}^{\hbar} \mathcal{G}(\tau^*) d\tau^*}, e_{\alpha}(\hbar, \mathcal{P}) = e^{\alpha(\hbar - \mathcal{P})}, e_{\alpha}(\hbar, 0) = e^{\alpha(\hbar)}, \mathcal{P}, \hbar, \alpha \in \mathbb{R}. \quad (11)$$

Theorem 8 (see [22]). If $\mathcal{G} \in \mathfrak{R}$, then

(i) $e_{\mathcal{G}}(\hbar, \hbar) \equiv 1$ and $e_0(\hbar, r) \equiv 1$

$$e_{\mathcal{G}}(\sigma(\hbar), \mathcal{P}) \equiv (1 + \xi(\hbar))\mathcal{G}(\hbar)e_{\mathcal{G}}(\hbar, \mathcal{P}), \quad \frac{1}{e_{\mathcal{G}}(\hbar, \mathcal{P})} = e_{\ominus \mathcal{G}}(\hbar, \mathcal{P}) = e_{\mathcal{G}}(\mathcal{P}, \hbar). \quad (12)$$

(ii) If $\mathcal{J} \in \mathfrak{R}^+$, then $\ominus \mathcal{J} \in \mathfrak{R}^+$

(iii) If $\mathcal{J} \in \mathfrak{R}^+$, then $e_{\mathcal{J}}(\hbar, \hbar_0) > 0$, for all $\hbar_0, \hbar \in \mathbb{T}$, where $\ominus \mathcal{J} = -(\mathcal{J}/(1 + \xi \mathcal{J}))$.

To more descriptions of the study of the time scale, we direct the reader to Bohner and Peterson [22] excellent monograph, which describes and organizes most of the time scale logic.

3. Nonlinear Powered Integral Inequalities via Retarded Term

Given the documentations all through the content for the simplicity of perusing: $\mathbb{R}$ stands for the set of real numbers, $\mathbb{C}_{rd}$ is the set of rd continuous functions, $\mathbb{R}_+ = [0, \infty)$, $\hbar_1 \in \mathbb{T}$, $\hbar_1 \geq 0$, $\mathbb{T}_0 = [\hbar_1, \infty) \cap \mathbb{T}$, and

(H1). The continuous function $\mathfrak{Z}(\hbar) \in \mathbb{C}_{rd}(\mathbb{T}_0, \mathbb{R}_+)$ is a nonnegative.

(H2). $\xi(\hbar) \in \mathbb{C}_{rd}(\mathbb{T}_0, \mathbb{R}_+)$ is a function which is increasing and differentiable on $[\hbar_1, \infty)$ so that $\xi(\hbar) \leq \hbar$, $\xi(\hbar_1) = \hbar_1$.

(H3). $\mathcal{G}(\hbar), \mathcal{J}(\hbar), \mathcal{B}(\hbar) \in \mathbb{C}_{rd}(\mathbb{T}_0, \mathbb{R}_+)$.

(H4). $\mathbf{m}(\hbar) \in \mathbb{C}_{rd}([\mathbb{T}_0, (0, \infty)))$ is nondecreasing.

(H5). $\mu, \kappa \in \mathbb{C}_{rd}(\mathbb{R}_+, \mathbb{R}_+)$ is a nondecreasing function and $\mu(\mathcal{J}) > 0$, $\kappa(\mathcal{J}) > 0$ for $\mathcal{J} > 0$.

Specifically, the fundamental lemma to be used afterwards is presented below:

Lemma 9 (see [26]). Let $\psi \geq 0$, $j \geq \vartheta \geq 0$, and $j \neq 0$, then

$$\psi^{\vartheta/j} \leq \frac{\vartheta}{j} \psi + \frac{j - \vartheta}{j}. \quad (13)$$

Proof. For $\vartheta = 0$, inequality (13) is accurate, unless $\delta = 1$ if $\vartheta > 0$, $\delta = \vartheta/j$, and

$$\psi^{\vartheta/j} \leq \frac{\vartheta}{j} Y^{\vartheta/j} \psi + \frac{j - \vartheta}{j} Y^{\vartheta/j}, Y > 0. \quad (14)$$

We obtain (13) for $Y = 1$.

Theorem 10. Suppose that (H1), (H2), (H3), and (H4) be satisfied. Moreover

$$\begin{aligned} \mathfrak{Z}(\hbar) &\leq \mathbf{m}(\hbar) + \int_{\hbar_1}^{\xi(\hbar)} \mathcal{G}(\mathcal{P}) \mathfrak{Z}(\mathcal{P}) \left(\mathfrak{Z}(\mathcal{P}) + \int_{\hbar_1}^{\mathcal{P}} \mathcal{J}(\mathfrak{x}) \mathfrak{Z}(\mathfrak{x}) \Delta \mathfrak{x} \right)^{\aleph} \Delta \mathcal{P} \\ &\quad + \int_{\hbar_1}^{\xi(\hbar)} \mathcal{G}(\mathcal{P}) \mathfrak{Z}(\mathcal{P}) \left(\int_{\hbar_1}^{\mathcal{P}} \mathcal{B}(\mathfrak{x}) \mathfrak{Z}(\mathfrak{x}) \Delta \mathfrak{x} \right) \Delta \mathcal{P}. \end{aligned} \quad (15)$$

$\forall \hbar \in \mathbb{T}_0$. Then

$$\mathfrak{Z}(\hbar) \leq \mathbf{m}(\hbar) + \frac{\mathcal{H}_1(\hbar) e_{\ominus(-Y)}(\sigma(\hbar), \hbar_1)}{1 - \mathcal{H}_1(\hbar) \int_{\hbar_1}^{\xi(\hbar)} \chi(\mathcal{P}) e_{\ominus(-Y)}(\mathcal{P}, \mathcal{P}_1) \Delta \mathcal{P}}, \quad (16)$$

where $\aleph \in (0, 1]$ is a positive constant,

$$\begin{aligned} \mathcal{H}_1(\hbar) &= \int_{\hbar_1}^{\xi(\hbar)} \mathcal{G}(\mathcal{P}) \mathcal{H}(\mathcal{P}) \left[\left(\aleph \left(\mathcal{H}(\mathcal{P}) + \int_{\hbar_1}^{\mathcal{P}} \mathcal{J}(\mathfrak{x}) \mathcal{H}(\mathfrak{x}) \Delta \mathfrak{x} \right) \right. \right. \\ &\quad \left. \left. + (1 - \aleph) + \int_{\hbar_1}^{\mathcal{P}} \mathcal{B}(\mathfrak{x}) \mathcal{H}(\mathfrak{x}) \Delta \mathfrak{x} \right) \Delta \mathcal{P}, \end{aligned} \quad (17)$$

$$Y(\hbar) = \mathcal{G}(\hbar) \left[2\aleph(\mathcal{H}(\hbar)) \left(1 + \int_{\hbar_1}^{\hbar} \mathcal{F}(\mathcal{P}) \Delta \mathcal{P} \right) + (1 - \aleph) + 2\mathcal{H}(\hbar) \int_{\hbar_1}^{\hbar} \mathcal{B}(\mathcal{P}) \Delta \mathcal{P} \right], \quad (18)$$

$$\chi(\hbar) = \mathcal{G}(\hbar) \left[\aleph \left(1 + \int_{\hbar_1}^{\hbar} \mathcal{F}(\mathcal{P}) \Delta \mathcal{P} \right) + \int_{\hbar_1}^{\hbar} \mathcal{B}(\mathcal{P}) \Delta \mathcal{P} \right]. \quad (19)$$

Proof. Let us define

$$\begin{aligned} \mathcal{Q}(\hbar) &= \int_{\hbar_1}^{\xi(\hbar)} \mathcal{G}(\mathcal{P}) \mathfrak{L}(\mathcal{P}) \left(\mathfrak{L}(\mathcal{P}) + \int_{\hbar_1}^{\mathcal{P}} \mathcal{F}(\mathfrak{x}) \mathfrak{L}(\mathfrak{x}) \Delta \mathfrak{x} \right) \Delta \mathcal{P} \\ &\quad + \int_{\hbar_1}^{\xi(\hbar)} \mathcal{G}(\mathcal{P}) \mathfrak{L}(\mathcal{P}) \left(\int_{\hbar_1}^{\mathcal{P}} \mathcal{B}(\mathfrak{x}) \mathfrak{L}(\mathfrak{x}) \Delta \mathfrak{x} \right) \Delta \mathcal{P}, \end{aligned} \quad (20)$$

therefore, (15) reaches to

$$\mathfrak{L}(\hbar) \leq \mathfrak{m}(\hbar) + \mathcal{Q}(\hbar), \quad \mathcal{Q}(\hbar_1) = 0, \quad (21)$$

by executing Lemma 9 and (21) into (20), we deduce

$$\begin{aligned} \mathcal{Q}(\hbar) &\leq \int_{\hbar_1}^{\xi(\hbar)} \mathcal{G}(\mathcal{P}) \mathfrak{L}(\mathcal{P}) \left[\aleph \left(\mathfrak{L}(\mathcal{P}) + \int_{\hbar_1}^{\mathcal{P}} \mathcal{F}(\mathfrak{x}) \mathfrak{L}(\mathfrak{x}) \Delta \mathfrak{x} \right) + (1 - \aleph) \right] \Delta \mathcal{P} \\ &\quad + \int_{\hbar_1}^{\xi(\hbar)} \mathcal{G}(\mathcal{P}) \mathfrak{L}(\mathcal{P}) \left(\int_{\hbar_1}^{\mathcal{P}} \mathcal{B}(\mathfrak{x}) \mathfrak{L}(\mathfrak{x}) \Delta \mathfrak{x} \right) \Delta \mathcal{P} \\ &\leq \int_{\hbar_1}^{\xi(\hbar)} \mathcal{G}(\mathcal{P}) [m(\mathcal{P}) + \mathcal{Q}(\mathcal{P})] \\ &\quad \cdot \left[\aleph \left((m(\mathcal{P}) + \mathcal{Q}(\mathcal{P})) + \int_{\hbar_1}^{\mathcal{P}} \mathcal{F}(\mathfrak{x}) (m(\mathfrak{x}) + \mathcal{Q}(\mathfrak{x})) \Delta \mathfrak{x} + (1 - \aleph) \right) \right] \Delta \mathcal{P} \\ &\quad + \int_{\hbar_1}^{\xi(\hbar)} \mathcal{G}(\mathcal{P}) [m(\mathcal{P}) + \mathcal{Q}(\mathcal{P})] \left[\int_{\hbar_1}^{\mathcal{P}} \mathcal{B}(\mathfrak{x}) (m(\mathfrak{x}) + \mathcal{Q}(\mathfrak{x})) \Delta \mathfrak{x} \right] \Delta \mathcal{P} \\ &\leq \int_{\hbar_1}^{\xi(\hbar)} \left[\mathcal{G}(\mathcal{P}) m(\mathcal{P}) (\aleph(m(\mathcal{P})) + \int_{\hbar_1}^{\mathcal{P}} \mathcal{F}(\mathfrak{x}) m(\mathfrak{x}) \Delta \mathfrak{x}) \right. \\ &\quad \left. + (1 - \aleph) \mathcal{G}(\mathcal{P}) m(\mathcal{P}) + \mathcal{G}(\mathcal{P}) m(\mathcal{P}) \left(\int_{\hbar_1}^{\mathcal{P}} \mathcal{B}(\mathfrak{x}) m(\mathfrak{x}) \Delta \mathfrak{x} \right) \right] \Delta \mathcal{P} \\ &\quad + \int_{\hbar_1}^{\xi(\hbar)} \left[\mathcal{G}(\mathcal{P}) \left(2(\aleph(m(\mathcal{P}))) \left(1 + \int_{\hbar_1}^{\mathcal{P}} \mathcal{F}(\mathfrak{x}) \Delta \mathfrak{x} \right) \right) \right. \\ &\quad \left. + (1 - \aleph) \mathcal{G}(\mathcal{P}) + 2\mathcal{G}(\mathcal{P}) m(\mathcal{P}) \left(\int_{\hbar_1}^{\mathcal{P}} \mathcal{B}(\mathfrak{x}) \Delta \mathfrak{x} \right) \right] \mathcal{Q}(\mathcal{P}) \Delta \mathcal{P} \\ &\quad + \int_{\hbar_1}^{\xi(\hbar)} \left[\mathcal{G}(\mathcal{P}) \left(\aleph \left(1 + \int_{\hbar_1}^{\mathcal{P}} \mathcal{F}(\mathfrak{x}) \Delta \mathfrak{x} \right) \right) \right. \\ &\quad \left. + \mathcal{G}(\mathcal{P}) \left(\int_{\hbar_1}^{\mathcal{P}} \mathcal{B}(\mathfrak{x}) \Delta \mathfrak{x} \right) \right] \mathcal{Q}^2(\mathcal{P}) \Delta \mathcal{P} \leq \mathcal{H}_1(\hbar) \\ &\quad + \int_{\hbar_1}^{\xi(\hbar)} Y(\mathcal{P}) \mathcal{Q}(\mathcal{P}) \Delta \mathcal{P} + \int_{\hbar_1}^{\xi(\hbar)} \chi(\mathcal{P}) \mathcal{Q}^2(\mathcal{P}) \Delta \mathcal{P}, \end{aligned} \quad (22)$$

where $\mathcal{H}_1(\hbar)$, $Y(\hbar)$, and $\chi(\hbar)$ are quoted in (17), (18), and

(19), respectively. Fixing $\hbar_2 \in \mathbb{T}_0$ for an arbitrary $\hbar \in [\hbar_1, \hbar_2] \cap \mathbb{T}$ and taking $\mathcal{Q}_1(\hbar)$ by

$$\mathcal{Q}_1(\hbar) \leq \mathcal{H}_1(\hbar_2) + \int_{\hbar_1}^{\xi(\hbar)} Y(\mathcal{P}) \mathcal{Q}(\mathcal{P}) \Delta \mathcal{P} + \int_{\hbar_1}^{\xi(\hbar)} \chi(\mathcal{P}) \mathcal{Q}^2(\mathcal{P}) \Delta \mathcal{P}, \quad (23)$$

(22) can be carried out as

$$\mathcal{Q}(\hbar) \leq \mathcal{Q}_1(\hbar), \quad \mathcal{Q}(\xi(\hbar)) \leq \mathcal{Q}_1(\xi(\hbar)) \leq \mathcal{Q}_1(\hbar), \quad \mathcal{Q}_1(\hbar_1) = \mathcal{H}_1(\hbar_2), \quad (24)$$

since $\mathcal{Q}_1(\hbar)$ is nondecreasing, hence we can compose (23) and (24) as

$$\mathcal{Q}_1^\Delta(\hbar) \leq Y(\xi(\hbar)) \xi^\Delta(\hbar) \mathcal{Q}_1(\hbar) + \chi(\xi(\hbar)) \xi^\Delta(\hbar) \mathcal{Q}_1^2(\hbar), \quad (25)$$

take $\mathfrak{Z}(\hbar) = \mathcal{Q}_1^{-1}(\hbar)$, then $\mathfrak{Z}^\Delta(\hbar) = -\mathcal{Q}_1^{-2}(\hbar) \mathcal{Q}_1^\Delta(\hbar)$; therefore, the last inequality with Theorem 8 yields

$$\mathfrak{Z}^\Delta(\hbar) + Y(\xi(\hbar)) \xi^\Delta(\hbar) \mathfrak{Z}(\hbar) \geq -\chi(\xi(\hbar)) \omega^\Delta(\hbar), \quad (26)$$

or

$$\begin{aligned} \left(\frac{\mathfrak{Z}(\hbar)}{e_{\ominus(Y)}(\sigma(\hbar), \hbar_1)} \right)^\Delta &= \frac{\mathfrak{Z}^\Delta(\hbar) + Y(\xi(\hbar)) \xi^\Delta(\hbar) \mathfrak{Z}(\hbar)}{e_{\ominus(Y)}(\sigma(\hbar), \hbar_1)} \\ &\geq -\chi(\xi(\hbar)) \xi^\Delta(\hbar) e_{\ominus(Y)}(\sigma(\hbar), \hbar_1). \end{aligned} \quad (27)$$

Integrating (27) and applying $\mathfrak{Z}(\hbar_1) = \mathcal{H}_1^{-1}(\hbar_2)$, we attain

$$\mathfrak{Z}(\hbar) \geq \frac{1 - \mathcal{H}_1(\hbar_2) \int_{\hbar_1}^{\hbar} \chi(\xi(\mathcal{P})) \xi^\Delta(\mathcal{P}) e_{\ominus(Y)}(\sigma(\hbar), \hbar_1) \Delta \mathcal{P}}{\mathcal{H}_1(\hbar_2) e_{\ominus(Y)}(\sigma(\hbar), \hbar_1)}. \quad (28)$$

Substituting $\mathfrak{Z}(\hbar) = \mathcal{Q}_1^{-1}(\hbar)$ in (28) and from (24), we get

$$\mathcal{Q}(\hbar) \leq \frac{\mathcal{H}_1(\hbar_2) e_{\ominus(Y)}(\sigma(\hbar), \hbar_1)}{1 - \mathcal{H}_1(\hbar_2) \int_{\hbar_1}^{\hbar} \chi(\xi(\mathcal{P})) \xi^\Delta(\mathcal{P}) e_{\ominus(Y)}(\sigma(\hbar), \hbar_1) \Delta \mathcal{P}}. \quad (29)$$

Substituting the previous value in (21), we have

$$\mathfrak{L}(\hbar_2) \leq m(\hbar_2) + \frac{\mathcal{H}_1(\hbar_2) e_{\ominus(Y)}(\sigma(\hbar), \hbar_1)}{1 - \mathcal{H}_1(\hbar_2) \int_{\hbar_1}^{\xi(\hbar)} \chi(\mathcal{P}) e_{\ominus(Y)}(\mathcal{P}, \mathcal{P}_1) \Delta \mathcal{P}}, \quad (30)$$

the arbitrariness of $\hbar_2$ in the previous inequality claims the optimal bound in (16).

Remark 11. Choose $\mathfrak{m}(\hbar) = u_0$, $\gamma = 1$, $\mathfrak{L}(\hbar) \leq \phi(u(t))$, $\mathcal{F}(\hbar) = g(t)$, and $\mathcal{G}(\hbar) = f(t)$, then Theorem 10 will become a small deviation of Theorem 2.2 studied by Abdeldaim and El-Deeb [14], if $\mathbb{T} = \mathbb{R}$ and $q(t) = 0$.

Theorem 12. The presumption (H1), (H2), (H4), and (H5), $\mathcal{G}(\hbar), \mathcal{F}(\hbar) \in \mathbb{C}_{rd}(\mathbb{T}_0, \mathbb{R}_+)$, and

$$\begin{aligned} \mathfrak{L}(\hbar) &\leq \mathfrak{m}(\hbar) + \int_{\hbar_1}^{\xi(\hbar)} \mathcal{G}(\mathcal{P})\mu(\mathfrak{L}(\mathcal{P})) \\ &\cdot \left(\mathfrak{L}^\lambda(\mathcal{P}) + \int_{\hbar_1}^{\mathcal{P}} \mathcal{F}(\mathfrak{x})\kappa(\mathfrak{L}(\mathfrak{x}))\Delta\mathfrak{x} \right)^{\aleph} \Delta\mathcal{P}, \forall \hbar \in \mathbb{T}_0, \end{aligned} \quad (31)$$

satisfy, then

$$\mathfrak{L}(\hbar) \leq \Sigma^{-1}(\bar{M}(\hbar)), \quad (32)$$

$\forall \hbar \in [\hbar_1, \hbar_2]$, where $\aleph$ and λ are constants, $\aleph > 0$, $\lambda \geq 1$, $\aleph + \lambda > 1$, and

$$\begin{aligned} \bar{M}(\hbar) &= \Sigma(m(\hbar)) + \int_{\hbar_1}^{\xi(\hbar)} \mathcal{G}(\mathcal{P})\mathcal{F}(\mathcal{P})\Delta\mathcal{P}, \mathcal{F}(\hbar) \\ &= \left(G^{-1} \left[\mathcal{P}^{-1} \left(\mathcal{P} \left(G(\mathfrak{m}^\lambda(\hbar)) + \int_{\hbar_1}^{\hbar} \mathcal{F}(\mathcal{P})\Delta\mathcal{P} \right) \right. \right. \right. \\ &\quad \left. \left. \left. + \lambda \int_{\hbar_1}^{\omega(\hbar)} \mathcal{G}(\mathcal{P})\Delta\mathcal{P} \right) \right] \right)^{\aleph}, \end{aligned} \quad (33)$$

$$G(\ell) = \int_1^\ell \frac{\Delta\mathcal{P}}{\kappa(\mathcal{P})}, \ell > 0, \quad (34)$$

$$\mathcal{P}(\ell) = \int_1^\ell \frac{\phi(G^{-1}(\mathcal{P}))\Delta\mathcal{P}}{(\mu(G(\mathcal{P}))) (G^{-1}(\mathcal{P}))^{\aleph+\lambda-1}}, \ell > 0, \quad (35)$$

$$\Sigma(\ell) = \int_1^\ell \frac{\Delta\mathcal{P}}{\mu(\mathcal{P})}, \ell > 0, \quad (36)$$

the inverses of G , $\mathcal{P}$, and Σ are G^{-1} , $\mathcal{P}^{-1}$, and Σ^{-1} , and select $\hbar_3$ in such a way that

$$\begin{aligned} &\mathcal{P} \left(G(\mathfrak{m}^\lambda(\hbar)) + \int_{\hbar_1}^{\hbar} \mathcal{F}(\mathcal{P})\Delta\mathcal{P} \right) + \lambda \int_{\hbar_1}^{\xi(\hbar)} \mathcal{G}(\mathcal{P})\Delta\mathcal{P} \\ &\leq \int_1^\ell \frac{\phi(G^{-1}(\mathcal{P}))\Delta\mathcal{P}}{(\mu(G^{-1}(\mathcal{P}))) (G^{-1}(\mathcal{P}))^{\aleph+\lambda-1}}, \xi(\mathfrak{m}(\hbar)) \\ &\quad + \int_{\hbar_1}^{\xi(\hbar)} \mathcal{G}(\mathcal{P})\mathcal{F}(\mathcal{P})\Delta\mathcal{P} \leq \frac{\Delta\mathcal{P}}{\mu(\mathcal{P})}, \mathcal{P}^{-1} \\ &\quad \times \left(\mathcal{P} \left(G(\mathfrak{m}^\lambda(\hbar)) + \int_{\hbar_1}^{\hbar} \mathcal{F}(\mathcal{P})\Delta\mathcal{P} \right) \right. \\ &\quad \left. + \lambda \int_{\hbar_1}^{\xi(\hbar)} \mathcal{G}(\mathcal{P})\Delta\mathcal{P} \right) \leq \frac{\Delta\mathcal{P}}{\phi(\mathcal{P})}. \end{aligned} \quad (37)$$

Proof. Let $\hbar_3 \in \mathbb{T}_0$ be fixed for $\hbar \in [\hbar_1, \hbar_3] \cap \mathbb{T}$ and letting

$$\begin{aligned} \mathcal{Q}_2(\hbar) &= \mathfrak{m}(\hbar_3) + \int_{\hbar_1}^{\xi(\hbar)} \mathcal{G}(\mathcal{P})\mu(\mathfrak{L}(\mathcal{P})) \\ &\cdot \left(\mathfrak{L}^\lambda(\mathcal{P}) + \int_{\hbar_1}^{\mathcal{P}} \mathcal{F}(\mathfrak{x})\kappa(\mathfrak{L}(\mathfrak{x}))\Delta\mathfrak{x} \right)^{\aleph} \Delta\mathcal{P}, \end{aligned} \quad (38)$$

(31) and (38) imply that

$$\mathfrak{L}(\hbar) \leq \mathcal{Q}_2(\hbar), \quad \mathcal{Q}_2(\hbar_1) = \mathfrak{m}(\hbar_3), \quad (39)$$

since $\mathcal{Q}_2(\hbar)$ is nondecreasing, then (38) equals to

$$\begin{aligned} \mathcal{Q}_2^\Delta(\hbar) &= \mathcal{G}(\xi(\hbar))\xi^\Delta(\hbar)\mu(\mathfrak{L}(\xi(\hbar))) \\ &\cdot \left(\mathfrak{L}^\lambda(\xi(\hbar)) + \int_{\hbar_1}^{\xi(\hbar)} \mathcal{F}(\mathcal{P})\kappa(\mathfrak{L}(\mathcal{P}))\Delta\mathcal{P} \right)^{\aleph} \\ &\leq \mathcal{G}(\xi(\hbar))\xi^\Delta(\hbar)\mu(\mathfrak{L}(\xi(\hbar))) \\ &\cdot \left(\mathcal{Q}_2^\lambda(\xi(\hbar)) + \int_{\hbar_1}^{\xi(\hbar)} \mathcal{F}(\mathcal{P})\kappa(\mathcal{Q}_2(\mathcal{P}))\Delta\mathcal{P} \right)^{\gamma} \\ &\leq \mathcal{G}(\xi(\hbar))\xi^\Delta(\hbar)\mu(\mathfrak{L}(\xi(\hbar)))\mathfrak{Z}_1^\aleph(\xi(\hbar)), \end{aligned} \quad (40)$$

where

$$\mathfrak{Z}_1(\xi(\hbar)) = \mathcal{Q}_2^\lambda(\xi(\hbar)) + \int_{\hbar_1}^{\xi(\hbar)} \mathcal{F}(\mathcal{P})\kappa(\mathcal{Q}_2(\mathcal{P}))\Delta\mathcal{P}, \quad (41)$$

$$\mathfrak{Z}_1(\xi(\hbar_1)) = \mathfrak{m}^\lambda(\hbar_3), \mathcal{Q}_2(\xi(\hbar)) \leq \mathfrak{Z}_1(\xi(\hbar)). \quad (42)$$

By the definition of $\mathfrak{Z}_1(\xi(\hbar))$, utilizing (34), (42), and $\mathfrak{Z}_1(\xi(\hbar)) > 0$, we have

$$\mathfrak{Z}_1^\Delta(\xi(\hbar))\xi^\Delta(\hbar) = \lambda\mathcal{Q}_2^{\lambda-1}(\xi(\hbar))\mathcal{Q}_2^\Delta(\xi(\hbar))\xi^\Delta(\hbar) + \mathcal{F}(\xi(\hbar))\xi^\Delta(\hbar)\kappa(\mathcal{Q}_2(\xi(\hbar))), \quad (43)$$

so that

$$\begin{aligned} \mathfrak{Z}_1^\Delta(\xi(\hbar)) &\leq \lambda\mathcal{G}(\xi(\hbar))\xi^\Delta(\hbar)\mu(\mathfrak{Z}_1(\xi(\hbar)))\mathfrak{Z}_1^{\aleph+\lambda-1}(\xi(\hbar)) \\ &\quad + \mathcal{F}(\xi(\hbar))\kappa(\mathfrak{Z}_1(\xi(\hbar))) \frac{\mathfrak{Z}_1^\Delta(\xi(\hbar))}{\kappa(\mathfrak{Z}_1(\xi(\hbar)))} \\ &\leq \frac{\lambda\mathcal{G}(\xi(\hbar))\xi^\Delta(\hbar)\mu(W(\xi(\hbar)))\mathfrak{Z}_1^{\aleph+\lambda-1}(\xi(\hbar))}{\kappa(\mathfrak{Z}_1(\xi(\hbar)))} \\ &\quad + \mathcal{F}(\xi(\hbar)). \end{aligned} \quad (44)$$

Integrating the above inequality, using (34) and (42), we get

$$\begin{aligned} G(\mathfrak{Z}_1(\xi(\hbar))) &\leq G(\mathfrak{m}^\lambda(\hbar_3)) + \int_{\hbar_1}^{\hbar} \mathcal{J}(\xi(\mathcal{P})) \Delta \mathcal{P} \\ &\quad + \int_{\hbar_1}^{\hbar} \frac{\lambda \mathcal{G}(\xi(\mathcal{P})) \xi^\Delta(\mathcal{P}) \mu(\mathfrak{Z}_1(\xi(\mathcal{P}))) \mathfrak{Z}_1^{N+\lambda-1}(\xi(\mathcal{P}))}{\kappa(\mathfrak{Z}_1(\xi(\mathcal{P})))} \Delta \mathcal{P} \\ &\leq G(\mathfrak{m}^\lambda(\hbar_3)) + \int_{\hbar_1}^{\hbar_2} \mathcal{J}(\xi(\mathcal{P})) \Delta \mathcal{P} \\ &\quad + \int_{\hbar_1}^{\hbar} \frac{\lambda \mathcal{G}(\xi(\mathcal{P})) \xi^\Delta(\mathcal{P}) \mu(\mathfrak{Z}_1(\omega(\mathcal{P}))) \mathfrak{Z}_1^{N+\lambda-1}(\xi(\mathcal{P}))}{\kappa(\mathfrak{Z}_1(\xi(\mathcal{P})))} \Delta \mathcal{P}, \end{aligned} \quad (45)$$

for $\hbar < \hbar_3$. Denoting

$$\mathcal{Q}_3(\hbar) = \int_{\hbar_1}^{\hbar} \frac{\lambda \mathcal{G}(\xi(\mathcal{P})) \xi^\Delta(\mathcal{P}) \mu(\mathfrak{Z}_1(\xi(\mathcal{P}))) \mathfrak{Z}_1^{N+\lambda-1}(\xi(\mathcal{P}))}{\kappa(\mathfrak{Z}_1(\xi(\mathcal{P})))} \Delta \mathcal{P}, \quad (46)$$

(45) and (46) give

$$\begin{aligned} \mathfrak{Z}_1(\xi(\hbar)) &\leq G^{-1}(\mathcal{Q}_3(\hbar)), \quad \mathcal{Q}_3(\hbar_1) = G(\mathfrak{m}^\lambda(\hbar_3)) \\ &\quad + \int_{\hbar_1}^{\hbar_3} \mathcal{J}(\xi(\mathcal{P})) \Delta \mathcal{P}. \end{aligned} \quad (47)$$

Differentiating (46) and applying (47), we observe that

$$\begin{aligned} \mathcal{Q}_3^\Delta(\hbar) &= \frac{\lambda \mathcal{G}(\xi(\hbar)) \xi^\Delta(\hbar) \mu(\mathfrak{Z}_1(\xi(\hbar))) \mathfrak{Z}_1^{N+\lambda-1}(\xi(\hbar))}{\kappa(\mathfrak{Z}_1(\xi(\hbar)))} \\ &\leq \frac{\lambda \mathcal{G}(\xi(\hbar)) \xi^\Delta(\hbar) \mu(G^{-1}(\mathcal{Q}_3(\hbar))) G^{-1}(\mathcal{Q}_3^{N+\lambda-1}(\hbar))}{\kappa(G^{-1}(\mathcal{Q}_3(\hbar)))}, \end{aligned} \quad (48)$$

implies

$$\frac{\kappa(G^{-1}(\mathcal{Q}_3(\hbar))) \mathcal{Q}_3^\Delta(\hbar)}{\mu(G^{-1}(\mathcal{Q}_3(\hbar))) G^{-1}(\mathcal{Q}_3^{N+\lambda-1}(\hbar))} \leq \lambda \mathcal{G}(\xi(\hbar)) \xi^\Delta(\hbar). \quad (49)$$

Inequality (49) with integration and (35) and (47) generate the approximation

$$\begin{aligned} \mathcal{P}(\mathcal{Q}_3(\hbar)) &\leq \mathcal{P}\left(G(\mathfrak{m}^\lambda(\hbar_3)) + \int_{\hbar_1}^{\hbar_3} \mathcal{J}(\xi(\mathcal{P})) \Delta \mathcal{P}\right) \\ &\quad + \lambda \int_{\hbar_1}^{\hbar} \mathcal{G}(\xi(\mathcal{P})) \xi^\Delta(\mathcal{P}) \Delta \mathcal{P}, \end{aligned} \quad (50)$$

$$\begin{aligned} \mathcal{Q}_3(\hbar) &\leq \mathcal{P}^{-1}\left(\mathcal{P}\left(G(\mathfrak{m}^\lambda(\hbar_3)) + \int_{\hbar_1}^{\hbar_2} \mathcal{J}(\xi(\mathcal{P})) \Delta \mathcal{P}\right)\right. \\ &\quad \left.+ \lambda \int_{\hbar_1}^{\xi(\hbar)} \mathcal{G}(\mathcal{P}) \Delta \mathcal{P}\right). \end{aligned} \quad (51)$$

In view of (47) and (51), we derive

$$\begin{aligned} \mathfrak{Z}_1^N(\xi(\hbar)) &\leq \left(G^{-1}\left[\mathcal{P}^{-1}\left(\mathcal{P}\left(G(\mathfrak{m}^\lambda(\hbar_3)) + \int_{\hbar_1}^{\hbar_2} \mathcal{J}(\xi(\mathcal{P})) \Delta \mathcal{P}\right)\right.\right.\right. \\ &\quad \left.\left.+ \lambda \int_{\hbar_1}^{\xi(\hbar)} \mathcal{G}(\mathcal{P}) \Delta \mathcal{P}\right)\right]\right)^N = \mathcal{F}(\hbar). \end{aligned} \quad (52)$$

Substitute (52) in (40), integrate from $\hbar_1$ to $\hbar$ in the resultant inequality, employ (36), and put $\hbar = \hbar_3$, we attain

$$\mathcal{Q}_2(\hbar) \leq \Sigma^{-1}\left(\Sigma(m(\hbar_3)) + \int_{\hbar_1}^{\xi(\hbar_2)} \mathcal{G}(\mathcal{P}) \mathcal{F}(\mathcal{P}) \Delta \mathcal{P}\right). \quad (53)$$

The acquired inequality in (32) can be produced by the arbitrary $\hbar_3$ to the last inequality and from (39).

Remark 13. As a special case on $\mathbb{T} = \mathbb{R}$, if we take $\mathcal{L}(\hbar) = u(x)$, $\mathfrak{m}(\hbar) = g(x)$, $\mathcal{G}(\hbar) = k(x, t)$, $\mathcal{J} = 0$, $N = 1$, $\mu(\mathcal{L}(\hbar)) = \phi(u(t))$, $\hbar_1 = x_0$, and $\xi(\hbar) = x$, then Theorem 12 changes to Theorem 2.1 of [27] due to Oguntuase.

Theorem 14. If $\mathcal{G}(\hbar) \in \mathbb{C}_{rd}(\mathbb{T}_0, \mathbb{R}_+)$, the conditions (H1), (H2), (H4) and

$$\begin{aligned} \mathfrak{Z}^{N+1}(\hbar) &\leq \mathfrak{m}(\hbar) + \left(\int_{\hbar_1}^{\xi(\hbar)} \mathcal{G}(\mathcal{P}) \mathfrak{Z}^N(\mathcal{P}) \Delta \mathcal{P}\right)^2 + 2 \int_{\hbar_1}^{\xi(\hbar)} \mathcal{G}(\mathcal{P}) \mathfrak{Z}^N(\mathcal{P}) \\ &\quad \cdot \left(\mathfrak{Z}^N(\mathcal{P}) + \int_{\hbar_1}^{\mathcal{P}} \mathcal{G}(\mathfrak{x}) \mathfrak{Z}^N(\mathfrak{x}) \Delta \mathfrak{x}\right) \Delta \mathcal{P}, \quad \forall \hbar \in \mathbb{T}_0, \end{aligned} \quad (54)$$

hold. Then

$$\mathfrak{Z}(\hbar) \leq \mathfrak{m}^{1/(N+1)}(\hbar) + \frac{2}{N+1} \int_{\hbar_1}^{\xi(\hbar)} \mathcal{G}(\mathcal{P}) \Pi(\mathcal{P}) e_{\ominus(-\mathcal{G})}(\sigma(\mathcal{P}), \mathcal{P}_1) \Delta \mathcal{P}, \quad (55)$$

provided with $N \in (0, 1)$ is a constant and

$$\Pi(\hbar) = \left[m^{(1-N)/(N+1)}(\hbar) + \frac{2(1-N)}{N+1} \int_{\hbar_1}^{\xi(\hbar)} \mathcal{G}(\mathcal{P}) e_{\ominus(-\mathcal{G})} \frac{1-N}{N} (\mathcal{P}, \mathcal{P}_1) \Delta \mathcal{P}\right]^{N/(1-N)}, \quad (56)$$

such that

$$\mathbf{m}^{(1-N)/(N+1)}(\bar{h}) + \frac{2(1-N)}{N+1} \int_{h_1}^{\xi(\bar{h})} \mathcal{G}(\mathcal{P}) e_{\ominus(-\mathcal{G})}^{(1-N)/N}(\mathcal{P}, \mathcal{P}_1) \Delta \mathcal{P} > 0, \forall \bar{h} \in \mathbb{T}. \quad (57)$$

Proof. Fix $\bar{h}_4 \in \mathbb{T}_0$ an arbitrary $\bar{h} \in [\bar{h}_1, \bar{h}_4] \cap \mathbb{T}$ and denote the positive and nondecreasing function $\mathcal{Q}_4(\bar{h})$ by

$$\begin{aligned} \mathcal{Q}_4^{N+1}(\bar{h}) &= \mathbf{m}(\bar{h}_4) + \left(\int_{h_1}^{\xi(\bar{h})} \mathcal{G}(\mathcal{P}) \mathfrak{L}^N(\mathcal{P}) \Delta \mathcal{P} \right)^2 + 2 \int_{h_1}^{\xi(\bar{h})} \mathcal{G}(\mathcal{P}) \mathfrak{L}^N(\mathcal{P}) \\ &\quad \cdot \left(\mathfrak{L}^N(\mathcal{P}) + \int_{h_1}^{\mathcal{P}} \mathcal{G}(\mathfrak{x}) \mathfrak{L}^N(\mathfrak{x}) \Delta \mathfrak{x} \right) \Delta \mathcal{P}. \end{aligned} \quad (58)$$

(54) restates as

$$\mathfrak{L}(\bar{h}) \leq \mathcal{Q}_4(\bar{h}), \mathfrak{L}(\xi(\bar{h})) \leq \mathcal{Q}_4(\xi(\bar{h})) \leq \mathcal{Q}_4(\bar{h}), \mathcal{Q}_4(\bar{h}_1) = \mathbf{m}^{1/(1+N)}(\bar{h}_4). \quad (59)$$

Differentiating (58) and employing (59), we get

$$\begin{aligned} (N+1) \mathcal{Q}_4^N(\bar{h}) \mathcal{Q}_4^{\Delta}(\bar{h}) &= 2 \left(\int_{h_1}^{\xi(\bar{h})} \mathcal{G}(\mathcal{P}) \mathfrak{L}^N(\mathcal{P}) \Delta \mathcal{P} \right) \mathcal{G}(\xi(\bar{h})) \xi^{\Delta}(\bar{h}) \mathfrak{L}^N(\xi(\bar{h})) \\ &\quad + 2 \mathcal{G}(\xi(\bar{h})) \xi^{\Delta}(\bar{h}) \mathfrak{L}^N(\xi(\bar{h})) \\ &\quad \cdot \left(\mathfrak{L}^N(\xi(\bar{h})) + \int_{h_1}^{\xi(\bar{h})} \mathcal{G}(\mathcal{P}) \mathfrak{L}^N(\mathcal{P}) \Delta \mathcal{P} \right), \end{aligned} \quad (60)$$

which becomes

$$\begin{aligned} \mathcal{Q}_4^{\Delta}(\bar{h}) &\leq \frac{2}{N+1} \mathcal{G}(\xi(\bar{h})) \xi^{\Delta}(\bar{h}) \left(\mathcal{Q}_4^N(\bar{h}) + 2 \int_{h_1}^{\xi(\bar{h})} \mathcal{G}(\mathcal{P}) \mathcal{Q}_4^N(\mathcal{P}) \Delta \mathcal{P} \right) \\ &\leq \frac{2}{N+1} \mathcal{G}(\xi(\bar{h})) \xi^{\Delta}(\bar{h}) \mathfrak{Z}_2(\bar{h}), \quad \mathfrak{Z}_2(\bar{h}_1) = \mathbf{m}^{N/(1+N)}(\bar{h}_4), \end{aligned} \quad (61)$$

so that

$$\mathfrak{Z}_2(\bar{h}) = \mathcal{Q}_4^N(\bar{h}) + 2 \int_{h_1}^{\xi(\bar{h})} \mathcal{G}(\mathcal{P}) \mathcal{Q}_4^N(\mathcal{P}) \Delta \mathcal{P}, \quad (62)$$

$$\mathcal{Q}_4(\bar{h}) \leq \mathfrak{Z}_2^{1/N}(\bar{h}), \quad \mathcal{Q}_4(\xi(\bar{h})) \leq \mathfrak{Z}_2^{1/N}(\xi(\bar{h})) \leq \mathfrak{Z}_2^{1/N}(\bar{h}). \quad (63)$$

Taking derivative (62) and from (63), we deduce

$$\begin{aligned} \mathfrak{Z}_2^{\Delta}(\bar{h}) &= N \mathcal{Q}_4^{N-1}(\bar{h}) \mathcal{Q}_4^{\Delta}(\bar{h}) + 2 \mathcal{G}(\xi(\bar{h})) \xi^{\Delta}(\bar{h}) \mathcal{Q}_4^N(\xi(\bar{h})) \\ &\leq \frac{2N}{N+1} \mathcal{G}(\xi(\bar{h})) \xi^{\Delta}(\bar{h}) \mathfrak{Z}_2^{(N-1)/N}(\bar{h}) \mathfrak{Z}_2(\bar{h}) + 2 \mathcal{G}(\xi(\bar{h})) \xi^{\Delta}(\bar{h}) \mathfrak{Z}_2(\bar{h}) \\ &\leq \frac{2N}{N+1} \mathcal{G}(\xi(\bar{h})) \xi^{\Delta}(\bar{h}) \mathfrak{Z}_2^{(2N-1)/N}(\bar{h}) + 2 \mathcal{G}(\xi(\bar{h})) \xi^{\Delta}(\bar{h}) \mathfrak{Z}_2(\bar{h}), \end{aligned} \quad (64)$$

or equivalently,

$$\begin{aligned} \mathfrak{Z}_2^{(1-2N)/N}(\bar{h}) \mathfrak{Z}_2^{\Delta}(\bar{h}) - 2 \mathcal{G}(\xi(\bar{h})) \xi^{\Delta}(\bar{h}) \mathfrak{Z}_2^{(1-N)/N}(\bar{h}) \\ \leq \frac{2N}{N+1} \mathcal{G}(\xi(\bar{h})) \xi^{\Delta}(\bar{h}), \end{aligned} \quad (65)$$

which by using Theorem 8, (61), and (63) leads to bound

$$\begin{aligned} \mathfrak{Z}_2(\bar{h}) &\leq e_{\ominus(-\mathcal{G})}(\sigma(\bar{h}), \bar{h}_1) \left[\mathbf{m}^{(1-N)/(N+1)}(\bar{h}_4) \right. \\ &\quad \left. + \frac{2(1-N)}{N+1} \int_{h_1}^{\bar{h}} \mathcal{G}(\xi(\mathcal{P})) \xi^{\Delta}(\mathcal{P}) e_{\ominus(-\mathcal{G})}^{(1-N)/N}(\sigma(\mathcal{P}), \mathcal{P}_1) \Delta \mathcal{P} \right]^{N/(1-N)}, \end{aligned} \quad (66)$$

where $\mathcal{G} = -2\mathcal{G}(\bar{h})$. From (61) and (66), it is noticed that

$$\begin{aligned} \mathcal{Q}_4^{\Delta}(\bar{h}) &\leq \frac{2}{N+1} \mathcal{G}(\xi(\bar{h})) \xi^{\Delta}(\bar{h}) e_{\ominus(-\mathcal{G})}(\sigma(\bar{h}), \bar{h}_1) \left[\mathbf{m}^{(1-N)/(N+1)}(\bar{h}_4) \right. \\ &\quad \left. + \frac{2(1-N)}{N+1} \int_{h_1}^{\xi(\bar{h})} \mathcal{G}(\mathcal{P}) e_{\ominus(-\mathcal{G})}^{(1-N)/N}(\mathcal{P}, \mathcal{P}_1) \Delta \mathcal{P} \right]^{N/(1-N)}, \end{aligned} \quad (67)$$

$\bar{h}_4 \in \mathbb{T}_0$ is chosen; therefore, the required estimation in (55) can be obtained by integrating the above inequality from $\bar{h}_1$ to $\bar{h}$ and then combining the obtained inequality with (59).

4. Enforcement on Theoretical Results

This segment is about to discuss the immediate utilization of Theorem 14 by assessing the boundedness and uniqueness of the retarded nonlinear integrodifferential equation on time scales. For this, let us consider

$$\begin{aligned} (\gamma+1) \mathfrak{L}^{\gamma}(\bar{h}) \mathfrak{L}^{\Delta}(\bar{h}) &= \mathcal{H}(\xi(\bar{h}), \mathfrak{L}(\xi(\bar{h}))) + \mathcal{A} \\ &\quad \cdot \left(\xi(\bar{h}), \mathfrak{L}(\xi(\bar{h})), \int_{h_1}^{\bar{h}} \mathcal{F}(\mathcal{P}, \mathfrak{L}(\mathcal{P})) \Delta \mathcal{P} \right), \mathfrak{L}(\bar{h}_1) \\ &= \bar{h}_1, \end{aligned} \quad (68)$$

$\forall \bar{h} \in \mathbb{T}_0$, where $\mathcal{H}, \mathcal{F} : \mathbb{T}_0 \times \mathbb{R} \longrightarrow \mathbb{R}, \mathcal{A} : \mathbb{T}_0 \times \mathbb{R}$

$$\times \mathbb{R} \longrightarrow \mathbb{R} \text{ and } \omega(\bar{h}) \leq \bar{h}, 0 < \omega^{\Delta}(\bar{h}) \leq \bar{h}, \omega(\bar{h}_1) = \bar{h}_1. \quad (69)$$

We suggest the boundedness on the solution of (68) in the first illustration.

For example, if $\mathfrak{L}(\bar{h})$ is the solution of (68) with conditions

$$|\mathcal{H}(\bar{h}, \mathfrak{L}(\bar{h}))| \leq \mathcal{G}^2(\bar{h}) |\mathfrak{L}^{2N}(\bar{h})|, \quad (70)$$

$$\begin{aligned} \left| \mathcal{A} \left(\bar{h}, \mathfrak{L}(\bar{h}), \int_{h_1}^{\bar{h}} \mathcal{F}(\mathcal{P}, \mathfrak{L}(\mathcal{P})) \Delta \mathcal{P} \right) \right| &\leq |\mathcal{F}(\mathcal{P}, \mathfrak{L}(\mathcal{P}))| \\ &\quad \cdot \left[|\mathfrak{L}^N(\bar{h})| + \int_{h_1}^{\bar{h}} |\mathcal{F}(\mathcal{P}, \mathfrak{L}(\mathcal{P}))| \Delta \mathcal{P} \right], \end{aligned} \quad (71)$$

then

$$|\mathfrak{L}(\hbar)| \leq \hbar_1 + \frac{2}{\aleph + 1} \int_{\hbar_1}^{\xi(\hbar)} \mathcal{G}(\mathcal{P}) \Pi_1(\mathcal{P}) e_{\odot(-\mathcal{G})}(\sigma(\mathcal{P}), \mathcal{P}_1) \Delta \mathcal{P}, \quad (72)$$

where $\mathcal{G}$, $\aleph$, and ξ are the same as in Theorem 14 and

$$\Pi_1(\hbar) = \left[\hbar_1^{1-\aleph} + \frac{2(1-\aleph)}{\aleph + 1} \int_{\hbar_1}^{\xi(\hbar)} \mathcal{G}(\mathcal{P}) e_{\mathfrak{l}(-\mathcal{G})}^{(1-\aleph)/\aleph}(\mathcal{P}, \mathcal{P}_1) \Delta \mathcal{P} \right]^{\aleph/(1-\aleph)}. \quad (73)$$

Proof. Keeping $\hbar = \mathcal{P}$ in (68) and integrating from $\hbar_1$ to $\hbar$, we have

$$\begin{aligned} \mathfrak{L}^{\aleph+1}(\hbar) &= \mathfrak{L}^{\aleph+1}(\hbar) + \int_{\hbar_1}^{\hbar} \mathcal{H}(\xi(\mathcal{P}), \mathfrak{L}(\xi(\mathcal{P}))) \Delta \mathcal{P} \\ &\quad + \int_{\hbar_1}^{\hbar} \mathcal{A} \left(\xi(\mathcal{P}), \mathfrak{L}(\xi(\mathcal{P})), \int_{\hbar_1}^{\mathcal{P}} \mathcal{F}(\mathfrak{x}, \mathfrak{L}(\mathfrak{x})) \Delta \mathfrak{x} \right) \Delta \mathcal{P}, \end{aligned} \quad (74)$$

which with the help of (70) and (71) takes the form

$$\begin{aligned} |\mathfrak{L}^{\aleph+1}(\hbar)| &= \left| \hbar_1^{\aleph+1} + \int_{\hbar_1}^{\hbar} \mathcal{H}((\xi(\mathcal{P}), \mathfrak{L}(\xi(\mathcal{P}))) \Delta \mathcal{P} \right. \\ &\quad \left. + \int_{\hbar_1}^{\hbar} \mathcal{A}(\omega(\mathcal{P}), \mathfrak{L}(\xi(\mathcal{P})), \int_{\hbar_1}^{\mathcal{P}} \mathcal{F}(\mathfrak{x}, \mathfrak{L}(\mathfrak{x})) \Delta \mathfrak{x}) \Delta \mathcal{P} \right| \\ &\leq \left| \hbar_1^{\aleph+1} \right| + \int_{\hbar_1}^{\hbar} |\mathcal{H}(\xi(\mathcal{P}), \mathfrak{L}(\xi(\mathcal{P})))| \Delta \mathcal{P} \\ &\quad + \int_{\hbar_1}^{\hbar} \left| \mathcal{A}(\xi(\mathcal{P}), \mathfrak{L}(\xi(\mathcal{P})), \int_{\hbar_1}^{\mathcal{P}} \mathcal{F}(\mathfrak{x}, \mathfrak{L}(\mathfrak{x})) \Delta \mathfrak{x}) \right| \Delta \mathcal{P} \\ &\leq \hbar_1^{\aleph+1} + \left(\int_{\hbar_1}^{\hbar} \mathcal{G}(\xi(\mathcal{P})) \mathfrak{L}^{\aleph}(\xi(\mathcal{P})) \Delta \mathcal{P} \right)^2 \\ &\quad + 2 \int_{\hbar_1}^{\hbar} G(\xi(\mathcal{P})) \mathfrak{L}^{\aleph}(\xi(\mathcal{P})) \\ &\quad \cdot \left[\mathfrak{L}^{\aleph}(\xi(\mathcal{P})) + \int_{\hbar_1}^{\mathcal{P}} \mathcal{G}(\mathfrak{x}) |\mathfrak{L}^{\aleph}(\mathfrak{x}) \Delta \mathfrak{x} \right]. \end{aligned} \quad (75)$$

Let $\bar{\mathcal{G}}(\hbar) = (\mathcal{G}(\hbar))/(\xi^{\Delta}(\xi^{-1}(\hbar)))$, then $\mathcal{G}(\hbar) \leq \bar{\mathcal{G}}(\hbar)$ for $\hbar \in \mathbb{T}_0$; therefore, (75) yields

$$\begin{aligned} |\mathfrak{L}^{\aleph+1}(\hbar)| &\leq \hbar_1^{\aleph+1} + \left(\int_{\hbar_1}^{\xi(\hbar)} \bar{\mathcal{G}}(\mathcal{P}) \mathfrak{L}^{\aleph}(\mathcal{P}) \Delta \mathcal{P} \right)^2 \\ &\quad + 2 \int_{\hbar_1}^{\xi(\hbar)} \bar{\mathcal{G}}(\mathcal{P}) \mathfrak{L}^{\aleph}(\mathcal{P}) \left[\mathfrak{L}^{\aleph}(\mathcal{P}) + \int_{\hbar_1}^{\mathcal{P}} \bar{\mathcal{G}}(\mathfrak{x}) \mathfrak{L}^{\aleph}(\mathfrak{x}) \Delta \mathfrak{x} \right]. \end{aligned} \quad (76)$$

The requisite bound (72) can be accomplished by the rea-

sonable implementation of Theorem 14 with some alterations in the previous inequality; hence, here, we remove the information.

The second example is based on the uniqueness on the solution of (66).

For example, we list the following hypotheses as below

$$|\mathcal{H}(\hbar, \mathfrak{L}_1(\hbar)) - \mathcal{H}(\hbar, \mathfrak{L}_2(\hbar))| \leq \mathcal{G}^2(\hbar) |\mathfrak{L}_1^{2\aleph}(\hbar) - \mathfrak{L}_2^{2\aleph}(\hbar)|, \quad (77)$$

$$\begin{aligned} &\left| \mathcal{A} \left(\hbar, \mathfrak{L}_1(\hbar), \int_{\hbar_1}^{\hbar} \mathcal{F}(\mathcal{P}, \mathfrak{L}_1(\mathcal{P})) \Delta \mathcal{P} \right) \right. \\ &\quad \left. - \mathcal{A} \left(\hbar, \mathfrak{L}_2(\hbar), \int_{\hbar_1}^{\hbar} \mathcal{F}(\mathcal{P}, \mathfrak{L}_2(\mathcal{P})) \Delta \mathcal{P} \right) \right| \\ &\leq |\mathcal{F}(\mathcal{P}, \mathfrak{L}_1(\mathcal{P})) - \mathcal{F}(\mathcal{P}, \mathfrak{L}_2(\mathcal{P}))| \\ &\quad \times \left[|\mathfrak{L}_1^{\aleph}(\hbar) - \mathfrak{L}_2^{\aleph}(\hbar)| + \int_{\hbar_1}^{\hbar} |\mathcal{F}(\mathcal{P}, \mathfrak{L}_1(\mathcal{P})) - \mathcal{F}(\mathcal{P}, \mathfrak{L}_2(\mathcal{P}))| \Delta \mathcal{P} \right], \end{aligned} \quad (78)$$

then (68) has at most one solution.

Proof. The solutions $\mathfrak{L}_1(\hbar)$ and $\mathfrak{L}_2(\hbar)$ of (68) transform into

$$\begin{aligned} \mathfrak{L}_1^{\aleph+1}(\hbar) - \mathfrak{L}_2^{\aleph+1}(\hbar) &= \int_{\hbar_1}^{\hbar} \mathcal{H}(\xi(\mathcal{P}), \mathfrak{L}_1(\xi(\mathcal{P}))) \Delta \mathcal{P} \\ &\quad + \int_{\hbar_1}^{\hbar} \mathcal{A} \left(\xi(\mathcal{P}), \mathfrak{L}_1(\xi(\mathcal{P})), \int_{\hbar_1}^{\mathcal{P}} \mathcal{F}(\mathfrak{x}, \mathfrak{L}_1(\mathfrak{x})) \Delta \mathfrak{x} \right) \Delta \mathcal{P} \\ &\quad - \int_{\hbar_1}^{\hbar} \mathcal{H}(\xi(\mathcal{P}), \mathfrak{L}_2(\xi(\mathcal{P}))) \Delta \mathcal{P} \\ &\quad + \int_{\hbar_1}^{\hbar} \mathcal{A} \left(\xi(\mathcal{P}), \mathfrak{L}_2(\xi(\mathcal{P})), \int_{\hbar_1}^{\mathcal{P}} \mathcal{F}(\mathfrak{x}, \mathfrak{L}_2(\mathfrak{x})) \Delta \mathfrak{x} \right) \Delta \mathcal{P}. \end{aligned} \quad (79)$$

Equivalently

$$\begin{aligned} |\mathfrak{L}_1^{\aleph+1}(\hbar) - \mathfrak{L}_2^{\aleph+1}(\hbar)| &\leq \left| \int_{\hbar_1}^{\hbar} \mathcal{H}(\xi(\mathcal{P}), \mathfrak{L}_1(\xi(\mathcal{P}))) \Delta \mathcal{P} \right. \\ &\quad \left. + \int_{\hbar_1}^{\hbar} \mathcal{A}(\xi(\mathcal{P}), \mathfrak{L}_1(\xi(\mathcal{P})), \int_{\hbar_1}^{\mathcal{P}} \mathcal{F}(\mathfrak{x}, \mathfrak{L}_1(\mathfrak{x})) \Delta \mathfrak{x}) \Delta \mathcal{P} \right. \\ &\quad \left. - \int_{\hbar_1}^{\hbar} \mathcal{H}(\xi(\mathcal{P}), \mathfrak{L}_2(\xi(\mathcal{P}))) \Delta \mathcal{P} \right. \\ &\quad \left. + \int_{\hbar_1}^{\hbar} \mathcal{A}(\xi(\mathcal{P}), \mathfrak{L}_2(\xi(\mathcal{P})), \int_{\hbar_1}^{\mathcal{P}} \mathcal{F}(\mathfrak{x}, \mathfrak{L}_2(\mathfrak{x})) \Delta \mathfrak{x}) \Delta \mathcal{P} \right| \\ &\leq \int_{\hbar_1}^{\hbar} |\mathcal{H}(\xi(\mathcal{P}), \mathfrak{L}_1(\xi(\mathcal{P}))) - \mathcal{H}(\xi(\mathcal{P}), \mathfrak{L}_2(\xi(\mathcal{P})))| \Delta \mathcal{P} \\ &\quad + \int_{\hbar_1}^{\hbar} |\mathcal{A}(\xi(\mathcal{P}), \mathfrak{L}_1(\xi(\mathcal{P})), \int_{\hbar_1}^{\mathcal{P}} \mathcal{F}(\mathfrak{x}, \mathfrak{L}_1(\mathfrak{x})) \Delta \mathfrak{x}) \\ &\quad - \mathcal{A}(\xi(\mathcal{P}), \mathfrak{L}_2(\xi(\mathcal{P})), \int_{\hbar_1}^{\mathcal{P}} \mathcal{F}(\mathfrak{x}, \mathfrak{L}_2(\mathfrak{x})) \Delta \mathfrak{x})| \Delta \mathcal{P}. \end{aligned} \quad (80)$$

Applying (77) and (78) in the above inequality, we get

$$\begin{aligned} |\mathfrak{L}_1^{N+1}(\hbar) - \mathfrak{L}_2^{N+1}(\hbar)| &\leq \left(\int_{\hbar_1}^{\xi(\hbar)} \mathcal{G}(\mathcal{P})(\mathfrak{L}_1^N(\mathcal{P}) - \mathfrak{L}_2^N(\mathcal{P})) \Delta \mathcal{P} \right)^2 \\ &+ 2 \int_{\hbar_1}^{\xi(\hbar)} \mathcal{G}(\mathcal{P})(\mathfrak{L}_1^N(\mathcal{P}) - \mathfrak{L}_2^N(\mathcal{P})) \\ &\times \left[(\mathfrak{L}_1^N(\mathcal{P}) - \mathfrak{L}_2^N(\mathcal{P})) + \int_{\hbar_1}^{\mathcal{P}} \mathcal{G}(\mathfrak{x})(\mathfrak{L}_1^N(\mathfrak{x}) - \mathfrak{L}_2^N(\mathfrak{x})) \Delta \mathfrak{x} \right]. \end{aligned} \quad (81)$$

The last inequality by making few modifications to the Theorem 14 for the function

$$|\mathfrak{L}_1^{N+1}(\hbar) - \mathfrak{L}_2^{N+1}(\hbar)|, \quad (82)$$

induces

$$[\mathfrak{L}_1^{N+1}(\hbar) - \mathfrak{L}_2^{N+1}(\hbar)] \leq 0. \quad (83)$$

Thus, $\mathfrak{L}_1(\hbar) = \mathfrak{L}_2(\hbar)$, and there exists at least one solution of (68).

5. Conclusion

Unlike some proven and defined inequalities in the literature, Theorem 10, Theorem 12, and Theorem 14 have examined some dynamic integral inequalities of the Gronwall-Bellman form in a single independent variable with a retarded and nonlinear term that can be used to overcome the qualitative properties of integral equations. Our observations may be used to solve the difficulty of measuring the explicit bounds of undefined functions and to expand and unify continuous inequalities by the use of basic technologies. We believe that the findings obtained here are of a general kind and offer many contributions to statistical data and are useful to identify the existence and uniqueness of the integrodifferential equations. As should be obvious, the provided results present a helpful resource in the study of solutions of certain delay dynamic equations on time scales. The inequalities that have been created unify some known continuous and discrete inequalities. One can say that it will be attractive for the researchers to generalize our results for further exploration.

Data Availability

The data are available on request.

Conflicts of Interest

The authors declare that there are no competing interests.

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