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Research Article

Fixed Point Results via G-Function over the Complete Partial b-Metric Space

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In this paper, we consider an auxiliary function G to combine and unify several existing fixed point theorems in the setting of the complete partial b-metric space. We consider also some examples to support the observed main results.

1. Introduction and Preliminaries

The notion of the distance has been investigated and improved from the beginning of the mathematics sciences. The first formal definition was given by Hausdorff and Frechet under the name of metric spaces. The formal definition was extended, improved, and generalized in several ways. In this paper, we shall consider the combination of notions of partial metric space and b-metric space. Partial metric space, defined by Matthews [1, 2] is the most economical way to calculate the distance in computer science. So, it is important in the setting of theoretical computer science. On the other hand, b-metric is the most interesting and real generalization of metric spaces; in this case, the triangle inequality is replaced by a modified version of triangle inequality.For more details on the advances of fixed point theory in the setting of b-metric spaces, see e.g. [13]-[27].

In this paper, we shall propose a fixed point theorem by using an auxiliary function G to combine, generalize, and unify several fixed point results in the setting of the complete partial b-metric spaces.

In [3], the authors proposed a new fixed point theorem in the setting of metric spaces.

We consider the follow sets of functions:

(1) $\mathbb{G}$ be the set of the functions $\mathscr{G} : [0,\infty)^3 \longrightarrow [0,\infty)$ that satisfy the following conditions:

(f_1) $\mathscr{G}$ is continuous,
(f_2) $\mathscr{G}(0,0,0) = 0$,
(f_3) $\max\{\tau, \upsilon\} \leq \mathscr{G}(\tau, \upsilon, \omega)$, for all $\tau, \upsilon, \omega \in [0,\infty)$.
In [3], some examples of such a function were given.

(i) $\mathscr{G}(\tau,\upsilon,\omega) = \tau + \upsilon + \omega$

(ii) $\mathscr{G}(\tau,\upsilon,\omega) = \max\{\tau,\upsilon,\omega\}$

(iii) $\mathscr{G}(\tau,\upsilon,\omega) = (\tau+\upsilon)(1+\omega)$

(2) Φ be the set of functions $\psi : [0,\infty) \longrightarrow [0,\infty)$ that satisfy the following conditions:

(b_1) ψ is nondecreasing,
(b_2) $\sum_{i\geq 1}\psi^i(u) < \infty$ for each $u > 0$. (Here, by ψ^i, we denote the ith iterate ohψ.)

We mention that the functions $\psi \in \Phi$ are called (c) -comparison functions. Moreover, it is not difficult to check that $\phi(u) < u$ for every $u > 0$.

(3) $\Gamma = \{\gamma : X \longrightarrow [0,\infty) \mid \gamma \text{ is lower semicontinuous}\}$

Theorem 1 (see [3]). *Let (X, d) be a complete metric space, a lower semicontinuous function $\gamma : X \longrightarrow [0,\infty)$, and a* *self-mapping $T : X \longrightarrow X$. If there exist $\psi \in \Phi$ and $\mathcal{G} \in \mathbb{G}$ such that*

$$\mathcal{G}(d(Tx,Ty),\gamma(Tx),\gamma(Ty)) \leq \psi\left(\max\left\{\begin{array}{c}\mathcal{G}(d(x,y),\gamma(x),\gamma(y)),\\ \dfrac{\mathcal{G}(d(x,Tx),\gamma(Tx),\gamma(x))+\mathcal{G}(d(y,Ty),\gamma(Ty),\gamma(y))}{2}\end{array}\right\}\right), \quad (1)$$

for every $x, y \in X$, then T has a unique fixed point.

Let X be a nonempty set.

(i) A function $b : X \times X \longrightarrow [0,\infty)$ is a b-metric on X if for a given real number $s \geq 1$ and for all $x, y, z \in X$ the following conditions hold:

$$\begin{aligned}&(b_1)\ b(x,y)=0 \Leftrightarrow x=y,\\ &(b_2) b(x,y)=b(y,x),\\ &(b_3) b(x,y)\leq s[b(x,z)+b(z,y)].\end{aligned} \quad (2)$$

The triplet $(X, b, s \geq 1)$ is called a b-metric space.

(ii) A function $\rho : X \times X \longrightarrow [0,\infty)$ is a partial metric on X if for all $x, y, z \in X$ the following conditions hold:

$$\begin{aligned}&(\rho_1)\ x=y \Leftrightarrow \rho(x,x)=\rho(y,y)=\rho(x,y) \Leftrightarrow x=y,\\ &(\rho_2)\ \rho(x,x)\leq \rho(x,y),\\ &(\rho_3)\ \rho(x,y)=\rho(y,x)\\ &(\rho_4)\ \rho(x,y)\leq \rho(x,z)+\rho(z,y)-\rho(z,z).\end{aligned} \quad (3)$$

The pair (X, ρ) is said to be a partial metric space.

Combining these two concepts, Shukla [4] introduced the notion of partial b-metric space as follows.

(iii) A function $\rho_b : X \times X \longrightarrow [0,\infty)$ is a partial b-metric on X if for all $x, y, z \in X$ the following conditions hold:

$$\begin{aligned}&\left(\rho_{b_1}\right)\ x=y \Leftrightarrow \rho_b(x,x)=\rho_b(y,y)=\rho_b(x,y),\\ &\left(\rho_{b_2}\right)\ \rho_b(x,x)\leq \rho_b(x,y),\\ &\left(\rho_{b_3}\right)\ \rho_b(x,y)=\rho_b(y,x),\\ &\left(\rho_{b_4}\right)\ \rho_b(x,y)\leq s[\rho_b(x,z)+\rho_b(z,y)]-\rho_b(z,z).\end{aligned} \quad (4)$$

The triplet $(X, \rho_b, s \geq 1)$ is said to be a partial b-metric space.

On a partial b-metric space $(X, \rho_b, s \geq 1)$ a sequence $\{x_n\}$ is said to be

(i) convergent to $x \in X$ if $\lim_{n\to\infty}\rho_b(x_n, x) = b(x, x)$ (the limit of a convergent sequence is not necessarily unique)

(ii) Cauchy if $\lim_{n,m\to\infty}\rho_b(x_n, x_p)$ exists and its finite

Moreover, the partial b-metric space is complete if for every Cauchy sequence $\{ax_n\}$ there exists $x \in X$ such that

$$\lim_{n,p\to\infty}\rho_b\left(x_n,x_p\right)=\lim_{n\to\infty}\rho_b(x_n,x)=\rho_b(x,x). \quad (5)$$

Let $(X, \rho_b, s \geq 1)$ be a partial b-metric space. We say that a self-mapping T on X is continuous if for every sequence $\{x_n\}$ in X which converges to a point $x \in X$ we have

$$\lim_{n\to\infty}\rho_b(Tx_n,Tx)=\lim_{n\to\infty}\rho_b\left(Tx_n,Tx_{n+j}\right)=\rho_b(Tx,Tx). \quad (6)$$

In [5], the authors introduced the following new notions.

(i) On a partial b-metric space, a sequence $\{x_n\}$ is a 0 -Cauchy sequence if $\lim_{n\to\infty}\rho_b(x_n, x_p) = 0$

(ii) The space $(X, \rho_b, s \geq 1)$ is said to be 0-complete if for each 0-Cauchy sequence $\{x_n\}$ in X, there exists a point $x \in X$ such that

$$\lim_{n,p\to\infty}\rho_b\left(x_n,x_p\right)=\lim_{n\to\infty}\rho_b(x_n,x)=\rho_b(x,x)=0. \quad (7)$$

Moreover, they proved that if the partial b-metric space $(X, \rho_b, s \geq 1)$ is complete, then it is 0-complete.

For a better understanding of the connections between these spaces (partial metric space, b-metric space, and partial b-metric space), we mention some papers that can be consulted [6–12].

Let Φ_b be the set of functions $\phi : [0,\infty) \longrightarrow [0,\infty)$ that satisfy the following conditions:

$(\phi_1)\phi$ is nondecreasing,

$(\phi_2)\sum_{i\geq 1} s^i\phi^i(u) < \infty$ for each $u > 0$. (Here, by ϕ^i, we denote the ith iterate oh ϕ.)

2. Main Results

The following is the main result of the paper.

Theorem 2. *Let $(X, \rho_b, s \geq 1)$ be a 0-complete partial b -metric space, a function $\gamma \in \Gamma$, $\mathcal{G} \in \mathbb{G}$, and a self-mapping $T : X \longrightarrow X$. If there exists $\phi \in \Phi_b$ such that*

$$\mathcal{G}(\rho_b(Tx,Ty),\gamma(Tx),\gamma(Ty)) \leq \phi\left(\max\left\{\begin{array}{c}\mathcal{G}(\rho_b(x,y),\gamma(x),\gamma(y)),\\ \dfrac{\mathcal{G}(\rho_b(x,Tx),\gamma(Tx),\gamma(x))+\mathcal{G}(d(y,Ty),\gamma(Ty),\gamma(y))}{2s}\end{array}\right\}\right), \quad (8)$$

for every $x, y \in X$ *. If* T *is continuous or* ρ_b *is continuous, then* T *has a unique fixed point.*

Proof. Starting with a point $x_0 \in X$, we consider the sequence $\{x\}$ defined by $x_n = Tx_{n-1}$, $n \in \mathbb{N}$. Without losing the generality, we can assume that for any $n \in \mathbb{N}$, we have $b(x_n, x_{n+1}) > 0$. Indeed, on the contrary, if there exists a positive integer j_0 such that $x_{j_0} = x_{j_0+1}$, we get that x_{j_0} is a fixed point of T, because due to the way the sequence was $\{x\}$ defined, it follows that $x_{j_0} = Tx_{j_0}$. Moreover, using this remark, we can easily see that

$$\mathcal{G}(\rho_b(x_n, x_{n+1}), \gamma(x_n), \gamma(x_{n+1})) > 0, \quad \text{for every } n \in \mathbb{N}. \tag{9}$$

Again supposing that $\mathcal{G}(\rho_b(x_{j_0}, x_{j_0+1}), \gamma(x_{j_0}), \gamma(x_{j_0+1})) = 0$ for some j_0 from (f_3), we have

$$\begin{aligned} 0 < \rho_b\left(x_{j_0}, x_{j_0+1}\right) &\le \max\left\{\rho_b\left(x_{j_0}, x_{j_0+1}\right), \gamma\left(x_{j_0}\right)\right\} \\ &\le \mathcal{G}\left(\rho_b\left(x_{j_0}, x_{j_0+1}\right), \gamma\left(x_{j_0}\right), \gamma\left(x_{j_0+1}\right)\right), \end{aligned} \tag{10}$$

which is a contradiction. Taking $x = x_n$ and $y = x_{n+1}$ in (8) we get

$$\begin{aligned} &\mathcal{G}(\rho_b(x_{n+1}, x_{n+2}), \gamma(x_{n+1}), \gamma(x_{n+2})) \\ &\quad\le \mathcal{G}(\rho_b(Tx_n, Tx_{n+1}), \gamma(Tx_n), \gamma(Tx_{n+1}) \\ &\quad\le \phi\left(\max\left\{\begin{array}{c} \mathcal{G}(\rho_b(x_n, x_{n+1}), \gamma(x_n), \gamma(x_{n+1})), \\ \dfrac{\mathcal{G}(\rho_b(x_n, Tx_n), \gamma(x_n), \gamma(Tx_n)) + \mathcal{G}(\rho_b(x_{n+1}, Tx_{n+1}), \gamma(x_{n+1}), \gamma(Tx_{n+1}))}{2s} \end{array}\right\}\right) \\ &\quad\le \phi\left(\max\left\{\begin{array}{c} \mathcal{G}(\rho_b(x_n, x_{n+1}), \gamma(x_n), \gamma(x_{n+1})), \\ \dfrac{\mathcal{G}(\rho_b(x_n, x_{n+1}), \gamma(x_n), \gamma(x_{n+1})) + \mathcal{G}(\rho_b(x_{n+1}, x_{n+2}), \gamma(x_{n+1}), \gamma(x_{n+2}))}{2s} \end{array}\right\}\right) \\ &\quad\le \phi\left(\max\left\{\begin{array}{c} \mathcal{G}(\rho_b(x_n, x_{n+1}), \gamma(x_n), \gamma(x_{n+1})), \\ \mathcal{G}(\rho_b(x_{n+1}, x_{n+2}), \gamma(x_{n+1}), \gamma(x_{n+2})) \end{array}\right\}\right). \end{aligned} \tag{11}$$

There are two possibilities, namely,

$$\begin{aligned} &\max\{\mathcal{G}(\rho_b(x_n, x_{n+1}), \gamma(x_n), \gamma(x_{n+1})), \\ &\mathcal{G}(\rho_b(x_{n+1}, x_{n+2}), \gamma(x_{n+1}), \gamma(x_{n+2}))\} \\ &\quad= \mathcal{G}(d(x_{n+1}, x_{n+2}), \gamma(x_{n+1}), \gamma(x_{n+2})), \end{aligned} \tag{12}$$

which leads us (since $\phi(u) < u$ for any $u > 0$) to

$$\begin{aligned} &\mathcal{G}(d(x_{n+1}, x_{n+2}), \gamma(x_{n+1}), \gamma(x_{n+2})) \\ &\quad\le \phi(\mathcal{G}(\rho_b(x_{n+1}, x_{n+2}), \gamma(x_{n+1}), \gamma(x_{n+2}))) \\ &\quad< \mathcal{G}(\rho_b(x_{n+1}, x_{n+2}), \gamma(x_{n+1}), \gamma(x_{n+2})). \end{aligned} \tag{13}$$

But, this is a contradiction, and then

$$\begin{aligned} &\max\{\mathcal{G}(d(x_n, x_{n+1}), \gamma(x_n), \gamma(x_{n+1})), \\ &\mathcal{G}(d(x_{n+1}, x_{n+2}), \gamma(x_{n+1}), \gamma(x_{n+2}))\} \\ &\quad= \mathcal{G}(d(x_n, x_{n+1}), \gamma(x_n), \gamma(x_{n+1})). \end{aligned} \tag{14}$$

Therefore, by (11) and taking into account (f_3), we have

$$\begin{aligned} \rho_b(x_{n+1}, x_{n+2}) &\le \max\{\rho_b(x_{n+1}, x_{n+2}), \gamma(x_{n+1})\} \\ &\le \mathcal{G}(\rho_b(x_{n+1}, x_{n+2}), \gamma(x_{n+1}), \gamma(x_{n+2})) \\ &\le \phi(\mathcal{G}(\rho_b(x_n, x_{n+1}), \gamma(x_n), \gamma(x_{n+1}))), \\ &\qquad \text{for every } n \in \mathbb{N} \cup 0. \end{aligned} \tag{15}$$

Consequently, for every $n \in \mathbb{N}$, we obtain

$$\begin{aligned} \rho_b(x_n, x_{n+1}) &\le \phi(\mathcal{G}(\rho_b(x_{n-1}, x_n), \gamma(x_{n-1}), \gamma(x_n))) \\ &\le \phi^n(\mathcal{G}(\rho_b(x_0, x_1), \gamma(x_0), \gamma(x_1))). \end{aligned} \tag{16}$$

Let $p, m \in \mathbb{N}$ such that $p < m$. By applying the (triangle-type inequality) (ρ_{b4}), we have

$$\begin{aligned} \rho_b(x_p, x_m) &\le s\left[\rho_b(x_p, x_{p+1}) + \rho_b(x_{p+1}, x_m)\right] - \rho_b(x_{p+1}, x_{p+1}) \\ &\le s\left[\rho_b(x_p, x_{p+1}) + \rho_b(x_{p+1}, x_m)\right] \\ &\le s\rho_b(x_p, x_{p+1}) + s^2\left[\rho_b(x_{p+1}, x_{p+2})\right. \\ &\quad \left.+ s\left[\rho_b(x_{p+2}, x_m)\right] - \rho_b(x_{p+2}, x_{p+2})\right) \\ &\le s\rho_b(x_p, x_{p+1}) + s^2\left[\rho_b(x_{p+1}, x_{p+2})\right. \\ &\quad \left.+ s\left[\rho_b(x_{p+2}, x_m)\right] \cdots \le s\rho_b(x_p, x_{p+1})\right. \\ &\quad + s^2\rho_b(x_{p+1}, x_{p+2}) + \cdots + s^{m-p-1}\rho_b(x_{m-1}, x_m), \end{aligned} \tag{17}$$

and (17) leads us to

$$\begin{aligned}\rho_b(x_p, x_m) &\leq \frac{1}{s^{p-1}}\sum_{i=p}^{m-1} s^i\phi^i(\mathcal{G}(\rho_b(x_0, x_1), \gamma(x_0), \gamma(x_1))) \\ &< \frac{1}{s^{p-1}}\sum_{i=p}^{m-1} s^i\phi^i(\mathcal{G}(\rho_b(x_0, x_1), \gamma(x_0), \gamma(x_1))) \\ &= \frac{1}{s^{p-1}}\left[S_{m-1} - S_{p-1}\right],\end{aligned} \tag{18}$$

where $S_n = \sum_{i=0}^{n} s^i\phi^i(\mathcal{G}(\rho_b(x_0, x_1), \gamma(x_0), \gamma(x_1)))$. Keeping in mind (ϕ_2), we deduce that there exists $S_n \longrightarrow S$ as $n \longrightarrow \infty$, and from (18), we get

$$\lim_{p,m\to\infty}\rho_b(x_p, x_m) = 0. \tag{19}$$

Consequently, $\{x_n\}$ is a 0-Cauchy sequence in a 0-complete partial b-metric space, and then there exists $\varsigma \in X$ such that

$$\lim_{p,m\to\infty}\rho_b(x_p, x_m) = \lim_{p\to\infty}\rho_b(x_p, \varsigma) = \rho_b(\varsigma, \varsigma) = 0. \tag{20}$$

Moreover, by (f_3) together with (16), we have

$$\begin{aligned}\gamma(x_n) &\leq \max\{\rho_b(x_n, x_{n+1}), \gamma(x_n)\} \\ &\leq \mathcal{G}(\rho_b(x_n, x_{n+1}), \gamma(x_n), \gamma(x_{n+1})) \\ &\leq \phi^n(\mathcal{G}(\rho_b(x_0, x_1), \gamma(x_0), \gamma(x_1))\end{aligned} \tag{21}$$

and using (ϕ_2)

$$\lim_{n\to\infty}\gamma(x_n) = 0. \tag{22}$$

Plus, by (ϕ_1) and (20),

$$\gamma(\varsigma) = 0. \tag{23}$$

We claim that this point ς is in fact a fixed point of the mapping T. If the mapping T is continuous, then by (6), we have

$$\rho_b(\varsigma, \varsigma) = \lim_{p\to\infty}\rho_b(Tx_p, T\varsigma) = \lim_{p\to\infty}\rho_b(Tx_p, Tx_{p+i}) = 0. \tag{24}$$

Thus, applying the triangle inequality (ρ_4),

$$\rho_b(\varsigma, T\varsigma) \leq s[\rho_b(\varsigma, x_{n+1}) + \rho_b x_{n+1}, T\varsigma] - \rho_b(x_{n+1}, x_{n+1}), \tag{25}$$

and together with (20) and (24), letting $n \longrightarrow \infty$, we get $\rho_b(\varsigma, T\varsigma) = 0$, that is, ς is a fixed point of T.

Let assume now that ρ_b is continuous, that is, $\lim_{n\to\infty}\rho_b(x_n, T\varsigma) = \rho_b(\varsigma, T\varsigma)$ Replacing x by x_n and y by ς in (8), we have (for every $n \in \mathbb{N}$)

$$\begin{aligned}&\mathcal{G}(\rho_b(x_{n+1}, T\varsigma), \gamma(x_{n+1}), \gamma(T\varsigma)) = \mathcal{G}(\rho_b(Tx_n, T\varsigma), \gamma(Tx_n), \gamma(T\varsigma)) \\ &\leq \phi\left(\max\left\{\begin{array}{c}\mathcal{G}(\rho_b(x_n, \varsigma), \gamma(x_n), \gamma(\varsigma), \\ \dfrac{\mathcal{G}(\rho_b(x_n, Tx_n), \gamma(x_n), \gamma(Tx_n)) + \mathcal{G}(\rho_b(\varsigma, T\varsigma), \gamma(\varsigma), \gamma(T\varsigma))}{2s}\end{array}\right\}\right) \\ &\leq \phi\left(\max\left\{\begin{array}{c}\mathcal{G}(\rho_b(x_n, \varsigma), \gamma(x_n), \gamma(\varsigma)), \\ \dfrac{\mathcal{G}(\rho_b(x_n, x_{n+1}), \gamma(x_n), \gamma(x_{n+1})) + \mathcal{G}(\rho_b(\varsigma, T\varsigma), \gamma(\varsigma), \gamma(T\varsigma))}{2s}\end{array}\right\}\right) \\ &< \max\left\{\begin{array}{c}\mathcal{G}(\rho_b(x_n, \varsigma), \gamma(x_n), \gamma(\varsigma)), \\ \dfrac{\mathcal{G}(\rho_b(x_n, x_{n+1}), \gamma(x_n), \gamma(x_{n+1})) + \mathcal{G}(\rho_b(\varsigma, T\varsigma), \gamma(\varsigma), \gamma(T\varsigma))}{2s}\end{array}\right\}.\end{aligned} \tag{26}$$

Letting $n \longrightarrow \infty$ and taking into account (f_1), we have

$$\begin{aligned}&\mathcal{G}(\rho_b(\varsigma, T\varsigma), 0, \gamma(T\varsigma)) \\ &= \lim_{n\to\infty}\mathcal{G}(\rho_b(x_{n+1}, T\varsigma), \gamma(x_{n+1}), \gamma(T\varsigma)) \\ &< \lim_{n\to\infty}\max\left\{\begin{array}{c}\mathcal{G}(\rho_b(x_n, \varsigma), \gamma(x_n), \gamma(\varsigma)), \\ \dfrac{\mathcal{G}(\rho_b(x_n, x_{n+1}), \gamma(x_n), \gamma(x_{n+1})) + \mathcal{G}(\rho_b(\varsigma, T\varsigma), \gamma(\varsigma), \gamma(T\varsigma))}{2s}\end{array}\right\} \\ &= \max\left\{\mathcal{G}(0,0,0), \frac{\mathcal{G}(0,0,0) + \mathcal{G}(\rho_b(\varsigma, T\varsigma), 0, \gamma(T\varsigma))}{2}\right\} \\ &= \frac{\mathcal{G}(\rho_b(\varsigma, T\varsigma), 0, \gamma(T\varsigma))}{2s}.\end{aligned} \tag{27}$$

Consequently, $\mathcal{G}(\rho_b(\varsigma, T\varsigma), 0, \gamma(T\varsigma)) = 0$. But, taking (f_3) into account, we get

$$0 \leq \max\{\rho_b(\varsigma, T\varsigma), 0\} \leq \mathcal{G}(\rho_b(\varsigma, T\varsigma), 0, \gamma(T\varsigma)) = 0, \tag{28}$$

which means $\rho_b(\varsigma, T\varsigma) = 0$. Thus, $T\varsigma = \varsigma$.

As a last step, we claim that ς is the unique fixed point of T. Supposing on the contrary, that there exists another point $\upsilon \in X$ such that $T\varsigma = \varsigma \neq \upsilon = T\upsilon$. First of all, applying (8) with $x = \upsilon = y$, we have

$$\mathcal{G}(0, \gamma(\upsilon), \gamma(\upsilon)) \leq \phi(\mathcal{G}(0, \gamma(\upsilon), \gamma(\upsilon))) < \mathcal{G}(0, \gamma(\upsilon), \gamma(\upsilon)), \tag{29}$$

which implies that $\gamma(\upsilon) = \gamma(T\upsilon) = 0$. Let now $x = \varsigma$ and $y = \upsilon$ in (8). We have

$$\begin{aligned}&\mathcal{G}(\rho_b(\varsigma, \upsilon), 0, 0) = \mathcal{G}(\rho_b(T\varsigma, T\upsilon), \gamma(T\varsigma), \gamma(T\upsilon)) \\ &\leq \phi\left(\max\left\{\begin{array}{c}\mathcal{G}(\rho_b(\varsigma, \upsilon), \gamma(\varsigma), \gamma(\upsilon)), \\ \dfrac{\mathcal{G}(\rho_b(\varsigma, T\varsigma), \gamma(\varsigma), \gamma(T\varsigma)) + \mathcal{G}(\rho_b(\upsilon, T\upsilon), \gamma(\upsilon), \gamma(T\upsilon))}{2s}\end{array}\right\}\right) \\ &= \phi\left(\max\left\{\begin{array}{c}\mathcal{G}(\rho_b(\varsigma, \upsilon), \gamma(\varsigma), \gamma(\upsilon)), \\ \dfrac{\mathcal{G}(0, \gamma(\varsigma), \gamma(T\varsigma)) + \mathcal{G}(0, \gamma(\upsilon), \gamma(T\upsilon))}{2s}\end{array}\right\}\right) \\ &\leq \phi(\mathcal{G}(\rho_b(\varsigma, \upsilon), 0, 0)) < \mathcal{G}(\rho_b(\varsigma, \upsilon), 0, 0).\end{aligned} \tag{30}$$

This is a contradiction. Therefore, $\rho_b(\varsigma, \upsilon) = 0$, that is, T admits a unique fixed point.

In particular, letting $G(\tau, \upsilon, \omega) = \tau + \upsilon + \omega$, for $\tau, \upsilon, \omega \in [0, \infty)$, we can omit the continuity conditions of the mapping T or the partial b-metric ρ_b.

Theorem 3. *Let $(X, \rho_b, s \geq 1)$ be a 0-complete partial b -metric space, a function $\gamma \in \Gamma$, $G \in \mathscr{F}$, and a self-mapping $T : X \longrightarrow X$. If there exists $\phi \in \Phi_b$ such that*

$$\rho_b(Tx, Ty) + \gamma(Tx) + \gamma(Ty)) \leq \phi\left(\max\left\{\begin{array}{c}\rho_b(x, y) + \gamma(x) + \gamma(y)),\\ \dfrac{\rho_b(x, Tx) + \gamma(Tx) + \gamma(x) + \rho_b(y, Ty) + \gamma(Ty) + \gamma(y))}{2s}\end{array}\right\}\right), \tag{31}$$

for every $x, y \in X$, then T has a unique fixed point.

Proof. Of course, since the function $G(\tau, \upsilon, \omega) = \tau + \upsilon + \omega \in \mathscr{F}$, by Theorem 2, we have that the sequence $\{x_n\}$ defined as $x_n = Tx_{n-1}$ is convergent to a point $\varsigma \in X$, and moreover, (22) and (23) hold. We claim that this point ς is a fixed point of T. For this purpose, by (31), for $x = \varsigma$ and $y = \varsigma$, we get

$$\begin{aligned}
\rho_b(\varsigma, T\varsigma) + \gamma(x_{n+1}) + \gamma(T\varsigma)) &\leq s[\rho_b(\varsigma, Tx_n) + \rho_b(Tx_n, T\varsigma)] \\
&\quad - \rho_b(Tx_n, Tx_n) + \gamma(x_{n+1}) + \gamma(T\varsigma)) \\
&\leq s\rho_b(\varsigma, x_{n+1}) + s[\rho_b(Tx_n, T\varsigma) + \gamma(Tx_n) + \gamma(T\varsigma)] \\
&\leq s\rho_b(\varsigma, x_{n+1}) + s\phi\left(\max\left\{\rho_b(x_n, \varsigma) + \gamma(x_n) + \gamma(\varsigma), \frac{\rho_b(x_n, x_{n+1}) + \gamma(x_n) + \gamma(x_{n+1})) + \rho_b(\varsigma, T\varsigma) + \gamma(\varsigma) + \gamma(T\varsigma)}{2s}\right\}\right) \\
&< s\rho_b(\varsigma, x_{n+1}) + s\max\left\{\rho_b(x_n, \varsigma) + \gamma(x_n) + \gamma(\varsigma), \frac{\rho_b(x_n, x_{n+1}) + \gamma(x_n) + \gamma(x_{n+1})) + \rho_b(\varsigma, T\varsigma) + \gamma(\varsigma) + \gamma(T\varsigma))}{2s}\right\}.
\end{aligned} \tag{32}$$

Letting $n \longrightarrow \infty$, in the above inequality and keeping in mind (19), (22), and (23), we get

$$\rho_b(\varsigma, T\varsigma) + \gamma(T\varsigma) \leq s\phi\left(\frac{\rho_b(\varsigma, T\varsigma) + \gamma(T\varsigma)}{2s}\right) < \frac{\rho_b(\varsigma, T\varsigma) + \gamma(T\varsigma)}{2}, \tag{33}$$

which is a contradiction. Therefore, $\rho_b(\varsigma, T\varsigma) = 0$, that is, $T\varsigma = \varsigma$.

As in the previous theorem, supposing that there exists υ, another fixed point of T, by (31), we have

$$2\gamma(\upsilon) = \rho_b(T\upsilon, T\upsilon) + \gamma(\upsilon) + \gamma(T\upsilon) \leq \phi(2\gamma(\upsilon)) < 2\gamma(\upsilon), \tag{34}$$

which is a contradiction. Thus, $\gamma(\upsilon) = 0$ and taking $x = \varsigma$ and $y = \upsilon$ in (31), we have

$$\begin{aligned}
\rho_b(\varsigma, \upsilon) = \rho_b(\varsigma, \upsilon) + \gamma(\varsigma) + \gamma(\upsilon) &= \rho_b(\varsigma, \upsilon) + \gamma(\varsigma) + \gamma(\upsilon) \\
&\leq \phi\left(\max\left\{\rho_b(\varsigma, \upsilon) + \gamma(\varsigma) + \gamma(\upsilon), \frac{\rho_b(\varsigma, T\varsigma) + \gamma(\varsigma) + \gamma(T\varsigma) + \rho_b(\upsilon, \upsilon) + \gamma(\upsilon) + \gamma(T\upsilon)}{2s}\right\}\right) \\
&= \phi(\rho_b(\varsigma, \upsilon)) < \rho_b(\varsigma, \upsilon).
\end{aligned} \tag{35}$$

But, this is a contraction, so $\rho_b(\varsigma, \upsilon) = 0$ which proves the uniqueness of the fixed point.

Example 4. Let the set $X = [0, 1]$ and the function $\rho_b : X \times X \longrightarrow [0, \infty)$ be defined by $\rho_b(x, 1/2) = \rho_b(1/2, x) = 1$ for any $x \in X$ and $\rho_b(x, y) = (\max\{x, y\})^2$, otherwise. It easy to see that ρ_b is a partial b -metric space, with $s = 2$. Moreover, since $\lim_{n,m\to\infty}\rho_b(x_n, x_m) = \lim_{n,m\to\infty}(\max\{x_n, x_m\})^2 = 0$ implies $\lim_{n,m\to\infty}x_n = 0$, we have

$$\lim_{n\to\infty}\rho_b(x_n, 0) = \rho_b(0, 0) = 0, \tag{36}$$

which shows that (X, ρ_b, s) is 0-complete. On the other hand, taking, for example, the sequence $\{x_n\}$ in X, where $x_n = n/(2n+1)$, we have $\lim_{n,m\to\infty}\rho_b(x_n, x_m) = 1/4$, but $\lim_{n\to\infty}\rho_b(x_n, 1/2) = 1 = \rho_b(1/2, 1/2) = 1$. Thus, the space (X, ρ_b, s) is not complete.

Let the mapping $T : X \longrightarrow X$ be defined as

$$Tx = \begin{cases} \dfrac{x}{4}, & \text{if } x \in [0, 1), \\ \dfrac{1}{2}, & \text{if } x = 1. \end{cases} \tag{37}$$

Choosing $\phi(u) = u/2$ and $\gamma(u) = u$, we have

(i) If $x = y = 1$, then

$$\begin{aligned} &\rho_b(T1, T1) + \gamma(T1) + \gamma(T1)) \\ &\quad = \rho_b\left(\frac{1}{2}, \frac{1}{2}\right) + 2\gamma\left(\frac{1}{2}\right) = \frac{5}{4} < \frac{3}{2} \\ &\quad = \phi\left(\max\left\{\begin{array}{c} \rho_b(1, 1) + \gamma(1) + \gamma(1)), \\ \dfrac{\rho_b(1, T1) + \gamma(T1) + \gamma(1))}{2} \end{array}\right\}\right) \end{aligned} \tag{38}$$

(ii) If $x = 1$, $y \in [0, 1)$, then

$$\begin{aligned} &\rho_b(Tx, T1) + \gamma(Tx) + \gamma(T1)) \\ &\quad = \rho_b\left(\frac{x}{4}, \frac{1}{2}\right) + \gamma\left(\frac{x}{4}\right) + \gamma\left(\frac{1}{2}\right) = \frac{3+x}{4} < \frac{2+x}{2} \\ &\quad \le \phi\left(\max\left\{\begin{array}{c} \rho_b(x, 1) + \gamma(x) + \gamma(1)), \\ \dfrac{\rho_b(x, Tx) + \gamma(Tx) + \gamma(x) + \rho_b(1, T1) + \gamma(T1) + \gamma(1))}{4} \end{array}\right\}\right) \end{aligned} \tag{39}$$

(iii) If $x, y \in [0, 1)$, then

$$\begin{aligned} &\rho_b(Tx, Ty) + \gamma(Tx) + \gamma(Ty)) \\ &\quad = \rho_b\left(\frac{x}{4}, \frac{y}{4}\right) + \gamma\left(\frac{x}{4}\right) + \gamma\left(\frac{y}{4}\right) \le \frac{x^2 + 4x + 4y}{16} \le \frac{x^2 + x + y}{2} \\ &\quad \le \phi\left(\max\left\{\begin{array}{c} \rho_b(x, y) + \gamma(x) + \gamma(y)), \\ \dfrac{\rho_b(x, Tx) + \gamma(Tx) + \gamma(x) + \rho_b(Ty, y) + \gamma(Ty) + \gamma(y))}{4} \end{array}\right\}\right) \end{aligned} \tag{40}$$

(We considered here $\max\{x, y\} = x$. The case $\max\{x, y\} = y$ is similar.)

Consequently, by Theorem 3, the mapping T admits a unique fixed point.

3. Conclusion

In this paper, we investigate the uniqueness and the existence of a fixed point for certain contraction via the G-function in one of the most general frames and complete the partial b-metric space. Regarding that the metric fixed point theory has a key role in the solution of not only differential equations and fractional differential equations but also integral equations, our results can be applied in these problems.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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