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Research Article

On $\mathfrak{R}$-Partial b-Metric Spaces and Related Fixed Point Results with Applications

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In this paper, we introduce the notion of $\mathfrak{R}$-partial b-metric spaces and prove some related fixed point results in the context of this notion. We also discuss an example to validate our result. Finally, as applications, we evince the importance of our work by discussing some fixed point results on graphical-partial b-metric spaces and on partially-ordered-partial b-metric spaces.

1. Introduction and Preliminaries

Due to the fact that fixed point theory plays a very crucial role for different mathematical models to obtain their solution existence and has a wide range of applications in different fields related to mathematics, this theory has intrigued many researchers.

By the inception of the Banach fixed point theorem [1], researchers are continuously trying to get the generalizations of this classical result through different methodologies. For instance, Czerwik [2] introduced the notion of b-metric spaces, with a triangle inequality weaker than that of metric spaces, in a view to generalize the Banach contraction principle. Moving on the same sequel, Matthews [3] introduced the notion of a partial metric space, which was a part of the study for denotational semantics of dataflow networks and gave a generalized version of the Banach contraction principle. The concept of partial metric spaces was further extended to partial b-metric spaces by Shukla in [4]. A number of researchers took keen interest in the generalized version of the metric spaces some work is available in [5–27].

Recently, Gordgi et al. [28] introduced the notion of orthogonal sets and gave a new extension for the classical Banach contraction principle. More details can be found in [29, 30].

After looking into the structure of orthogonal metric spaces, introduced by [29, 30], and the binary relation used with a metric, [31, 32], we introduce the notion of $\mathfrak{R}$-partial b-metric spaces. We are also improving and generalizing the concept of orthogonal contractions in the sense of $\mathfrak{R}$-partial b-metric spaces and establish some fixed point theorems for the proposed contractions.

Throughout this paper, we denote by $\mathbb{N}$,$\mathbb{R}$,$\mathbb{Z}$, and $\mathbb{R}^+$ the set of natural numbers, real numbers, integer numbers, and nonnegative real numbers, respectively.

Definition 1 (see [2]. *Let H be a nonempty set and $s \geq 1$. Suppose a mapping $d : H \times H \longrightarrow \mathbb{R}^+$ satisfies the following conditions for all $h,l,z \in H$:*

$(bM1)d(h, l) = 0$ if and only if $h = l$;
$(bM2)d(h, l) = d(l, h)$;
$(bM3)d(h, l) \leq s[d(h, z) + d(z, l)]$.
Then d is called a b-metric on H, and (H, d) is called a b-metric space with coefficient s.

Definition 2 (see [3]. *Let H be a nonempty set. Let $p : H \times H \longrightarrow \mathbb{R}^+$ satisfy the following for all $h,l,z \in H$:*
$(pM1)h = l$ if and only if $p(h, h) = p(h, l) = p(l, l)$;
$(pM2)p(h, h) \leq p(h, l)$;
$(pM3)p(h, l) = p(l, h)$;
$(pM4)p(h, l) \leq p(h, z) + p(z, l) - p(z, z)$.
Then (H, p) is called a partial metric space.

Definition 3 [4]. *A partial b -metric on $H \neq \varnothing$ is a function $\sigma : H \times H \longrightarrow \mathbb{R}^+$ such that for all $h,l,z \in H$, and for some $s \geq 1$, we have*
$(\sigma 1)h = l$ if and only if $\sigma(h, h) = \sigma(h, l) = \sigma(l, l)$;
$(\sigma 2)\sigma(h, h) \leq \sigma(h, l)$;
$(\sigma 3)\sigma(h, l) = \sigma(l, h)$;
$(\sigma 4)\sigma(h, l) \leq s[\sigma(h, z) + \sigma(z, l)] - \sigma(z, z)$.
A partial b-metric space is denoted with (H, σ, s). The number s is called the coefficient of (H, σ, s).

Remark 4 (see [4]. *It is clear that every partial metric space is a partial b -metric space with coefficient $s = 1$ and every b -metric space is a partial b -metric space with the same coefficient and a zero self-distance. However, the converse of this fact need not hold.*

Example 1 [4]. *Let $H = \mathbb{R}^+$, $p > 1$ be a constant and $\sigma : H \times H \longrightarrow \mathbb{R}^+$ be defined by*

$$\sigma(h, l) = |h - l|^p + (\max\{h, l\})^p \quad \text{for all} \quad h, l \in H. \quad (1)$$

Then, (H, σ, s) is a partial b-metric space with coefficient $s = 2^p > 1$, but it is neither a b-metric nor a partial metric space.

Definition 5 [33]. *Let H be a nonempty set. A subset $\mathfrak{R}$ of H^2 is called a binary relation on H . Then, for any $h, l \in H$, we say that "h is $\mathfrak{R}$ -related to l" , that is, $h\mathfrak{R}l$, or "h relates to l under $\mathfrak{R}$" if and only if $(h, l) \in \mathfrak{R}$. $(h, l) \notin \mathfrak{R}$ means that "h is not $\mathfrak{R}$ -related to l" or "h is not related to l under $\mathfrak{R}$".*

Definition 6 [33]. *A binary relation $\mathfrak{R}$ defined on a nonempty set H is called (a) reflexive if $(h, h) \in \mathfrak{R} \forall h \in H$;*
(b) irreflexive if $(h, h) \notin \mathfrak{R}$ for some $h \in H$;
(c) symmetric if $(h, l) \in \mathfrak{R}$ implies $(l, h) \in \mathfrak{R} \forall h, l \in H$;
(d) antisymmetric if $(h, l) \in \mathfrak{R}$ and $(l, h) \in \mathfrak{R}$ imply $h = l$ $\forall h,l \in H$;
(e) transitive if $(h, l) \in \mathfrak{R}$ and $(l, z) \in \mathfrak{R}$ imply $(h, z) \in \mathfrak{R}$ $\forall h,l, z \in H$;
(f) preorder if $\mathfrak{R}$ is reflexive and transitive;
(g) partial order if $\mathfrak{R}$ is reflexive, antisymmetric, and transitive.

Definition 7 [32]. *Let H be a nonempty set and let $\mathfrak{R}$ be a binary relation on H.*

(a) A sequence $\{h_n\}$ is called an $\mathfrak{R}$-sequence if

$$(\forall n \in \mathbb{N}, h_n \mathfrak{R} h_{n+1}). \quad (2)$$

(b) A map $T : H \longrightarrow H$ is $\mathfrak{R}$-preserving if

$$\forall h, l \in H, h\mathfrak{R}l \quad \text{implies} \quad Th\mathfrak{R}Tl. \quad (3)$$

Definition 8 [32]. *Let (H, d) be a metric space and $\mathfrak{R}$ be a binary relation on H . Then, $(H, d, \mathfrak{R})$ is called an $\mathfrak{R}$ -metric space.*

Definition 9 [31]. *A mapping $T : H \longrightarrow H$ is $\mathfrak{R}$ -continuous at $h_0 \in H$ if for each $\mathfrak{R}$ -sequence $\{h_n\}_{n\in\mathbb{N}}$ in H with $h_n \longrightarrow h_0$, we get $T(h_n) \longrightarrow T(h_0)$. Thus, T is $\mathfrak{R}$ -continuous on H if T is $\mathfrak{R}$ -continuous at each $h_0 \in H$.*

Definition 10 [31]. *A map $T : H \longrightarrow H$ is an $\mathfrak{R}$ -contraction, if*

$$d(Th, Tl) \leq kd(h, l), \quad (4)$$

for all $h, l \in H$ with $h\mathfrak{R}l$, where $0 < k < 1$.

Khalehoghli et al. [31] extended the result of Banach in the following way.

Theorem 11 [31]. *If T is an $\mathfrak{R}$ -preserving and $\mathfrak{R}$ -continuous $\mathfrak{R}$ -contraction on an $\mathfrak{R}$ -complete $\mathfrak{R}$ -metric space with $h_0 \in H$ such that $h_0\mathfrak{R}l$ for each $l \in H$. Then, T has a unique fixed point.*

2. Main Results

Let us begin this section with the definition of $\mathfrak{R}$-partial b-metric spaces.

Definition 12. Let $H \neq \varnothing$ and $\mathfrak{R}$ be a reflexive binary relation on H , denoted as $(H, \mathfrak{R})$. A map $\sigma_{\mathfrak{R}} : H \times H \longrightarrow \mathbb{R}^+$ is called an $\mathfrak{R}$ -partial b -metric on the set H , if the following conditions are satisfied for all $h, l, z \in H$ with either ($h\mathfrak{R}l$ or $l\mathfrak{R}h$), either ($h\mathfrak{R}z$ or $z\mathfrak{R}h$) and either ($z\mathfrak{R}l$ or $l\mathfrak{R}z$):
$(\sigma_{\mathfrak{R}}1)h = l$ if and only if $\sigma_{\mathfrak{R}}(h, h) = \sigma_{\mathfrak{R}}(h, l) = \sigma_{\mathfrak{R}}(l, l)$;
$(\sigma_{\mathfrak{R}}2)\sigma_{\mathfrak{R}}(h, h) \leq \sigma_{\mathfrak{R}}(h, l)$;
$(\sigma_{\mathfrak{R}}3)\sigma_{\mathfrak{R}}(h, l) = \sigma_{\mathfrak{R}}(l, h)$;
$(\sigma_{\mathfrak{R}}4)\sigma_{\mathfrak{R}}(h, l) \leq s[\sigma_{\mathfrak{R}}(h, z) + \sigma_{\mathfrak{R}}(z, l)] - \sigma_{\mathfrak{R}}(z, z)$, where $s \geq 1$.
Then, $(H, \mathfrak{R}, \sigma_{\mathfrak{R}}, s)$ is called $\mathfrak{R}$-partial b-metric space with the coefficient $s \geq 1$.

Remark 13. In the above definition, a set H is endowed with a reflexive binary relation $\mathfrak{R}$ and $\sigma_{\mathfrak{R}} : H \times H \longrightarrow \mathbb{R}^+$ satisfies $(\sigma_{\mathfrak{R}}1)$ - $(\sigma_{\mathfrak{R}}4)$ only for those elements which are comparable under the reflexive binary relation $\mathfrak{R}$. Hence, the $\mathfrak{R}$ -partial b -metric may not be a partial b -metric, but the converse is true.

The following simplest example shows that the $\mathfrak{R}$-partial b-metric with $s \geq 1$ need not to be a partial b-metric with $s \geq 1$.

Example 2. Let $H = \{-1, -2, 1, 2\}$ and let the binary relation be defined by $h\mathfrak{R}l$ if and only if $h = l$ or $h, l > 0$. It is easy to prove that $\sigma_{\mathfrak{R}}(h, l) = \max\{|h|, |l|\}$ is an $\mathfrak{R}$ -partial b -metric on H with $s \geq 1$, but $\sigma_{\mathfrak{R}}$ is not a partial b -metric on H with $s \geq 1$. Indeed, for $h = -2$ and $l = 2$, we have $\sigma_{\mathfrak{R}}(h, h) = \sigma_{\mathfrak{R}}(h, l) = \sigma_{\mathfrak{R}}(l, l) = 2$.

In the coming definitions, let $(H, \mathfrak{R}, \sigma_{\mathfrak{R}}, s)$ be an $\mathfrak{R}$ -partial b-metric space with the coefficient $s \geq 1$.

Definition 14. Let $\{h_n\}$ be an $\mathfrak{R}$ -sequence in $(H, \mathfrak{R}, \sigma_{\mathfrak{R}}, s)$, that is, $h_n \mathfrak{R} h_{n+1}$ or $h_{n+1} \mathfrak{R} h_n$ for each $n \in \mathbb{N}$. Then

(i) $\{h_n\}$ is a convergent sequence to some $h \in H$ if $\lim_{n\to\infty} \sigma_{\mathfrak{R}}(h_n, h) = \sigma_{\mathfrak{R}}(h, h)$ and $h_n \mathfrak{R} h$ for each $n \geq k$

(ii) $\{h_n\}$ is Cauchy if $\lim_{n,m\to\infty} \sigma_{\mathfrak{R}}(h_n, h_m)$ exists and is finite

Definition 15. $(H, \mathfrak{R}, \sigma_{\mathfrak{R}}, s)$ is said to be $\mathfrak{R}$ -complete if for every Cauchy $\mathfrak{R}$ -sequence in H , there is $h \in H$ with $\lim_{n,m\to\infty} \sigma_{\mathfrak{R}}(h_n, h_m) = \lim_{n\to\infty} \sigma_{\mathfrak{R}}(h_n, h) = \sigma_{\mathfrak{R}}(h, h)$ and $h_n \mathfrak{R} h$ for each $n \geq k$.

Definition 16. We say that $T : H \longrightarrow H$ is an R -property map, if for any iterative $\mathfrak{R}$ -sequence $\{h_n : h_n = T^n h, h \in H\}$ in $(H, \mathfrak{R}, \sigma_{\mathfrak{R}}, s)$ with $\lim_{n\to\infty} \sigma_{\mathfrak{R}}(h_n, h) = \sigma_{\mathfrak{R}}(h, h)$, $h_n \mathfrak{R} h$ for some $n \geq k$ and $\lim_{n\to\infty} \sigma_{\mathfrak{R}}(h_n, Th) \leq \sigma_{\mathfrak{R}}(h, h)$, we have that $h\mathfrak{R}Th$ or $Th\mathfrak{R}h$.

Definition 17. We say that $T : H \longrightarrow H$ is $\mathfrak{R}$ -0-continuous at $h \in H$ if for each $\mathfrak{R}$ -sequence $\{h_n\}$ in $(H, \mathfrak{R}, \sigma_{\mathfrak{R}}, s)$ with $\lim_{n\to\infty} \sigma_{\mathfrak{R}}(h_n, h) = 0$, we have $\lim_{n\to\infty} \sigma_{\mathfrak{R}}(Th_n, Th) = 0$. Also, T is $\mathfrak{R}$ -0-continuous on H if T is $\mathfrak{R}$ -0-continuous for each $h \in H$.

The following results help us to ensure the existence of fixed points for self maps. Throughout, we assume that $\mathfrak{R}$ is a preorder relation.

Theorem 18. *Let $(H, \mathfrak{R}, \sigma_{\mathfrak{R}}, s)$ be an $\mathfrak{R}$ -complete $\mathfrak{R}$ -partial b -metric space with the coefficient $s \geq 1$ and let $h_0 \in H$ be such that $h_0 \mathfrak{R} l$ for each $l \in H$. Let $T : H \longrightarrow H$ be an $\mathfrak{R}$ -preserving and an R -property map satisfying the following*

$$\sigma_{\mathfrak{R}}(Th, Tl) \leq k\sigma_{\mathfrak{R}}(h, l) \quad \text{for all} \quad h, l \in H \quad \text{with} \quad h\mathfrak{R}l, \tag{5}$$

where $k \in [0, 1/s)$. Then, T has a fixed point $h^* \in H$ and $\sigma_{\mathfrak{R}}(h^*, h^*) = 0$.

Proof. As $h_0 \in H$ is such that $h_0 \mathfrak{R} l$ for each $l \in H$, then by using the $\mathfrak{R}$-preserving nature of T, we construct an $\mathfrak{R}$ -sequence $\{h_n\}$ such that $h_n = Th_{n-1} = T^n h_0$ and $h_{n-1} \mathfrak{R} h_n$ for each $n \in \mathbb{N}$. We consider $h_n \neq h_{n+1}$ for each $n \in \mathbb{N} \cup \{0\}$. Thus, by (5), we get

$$\sigma_{\mathfrak{R}}(h_n, h_{n+1}) = \sigma_{\mathfrak{R}}(Th_{n-1}, Th_n) \leq k\sigma_{\mathfrak{R}}(h_{n-1}, h_n), \tag{6}$$

for all $n \in \mathbb{N}$. This inequality yields

$$\sigma_{\mathfrak{R}}(h_n, h_{n+1}) \leq k^n \sigma_{\mathfrak{R}}(h_0, h_1), \tag{7}$$

for all $n \in \mathbb{N}$. To discuss the Cauchy criteria, we will consider an arbitrary integer $n \geq 1, m \geq 1$ with $m > n$ and use $\sigma_{\mathfrak{R}_4}$ along (7) in the following way.

$$\begin{aligned}
\sigma_{\mathfrak{R}}(h_n, h_m) &\leq s[\sigma_{\mathfrak{R}}(h_n, h_{n+1}) + \sigma_{\mathfrak{R}}(h_{n+1}, h_m)] - \sigma_{\mathfrak{R}}(h_{n+1}, h_{n+1}) \\
&\leq s\sigma_{\mathfrak{R}}(h_n, h_{n+1}) + s^2[\sigma_{\mathfrak{R}}(h_{n+1}, h_{n+2}) + \sigma_{\mathfrak{R}}(h_{n+2}, h_m)] \\
&\quad - \sigma_{\mathfrak{R}}(h_{n+2}, h_{n+2}) \leq s\sigma_{\mathfrak{R}}(h_n, h_{n+1}) + s^2\sigma_{\mathfrak{R}}(h_{n+1}, h_{n+2}) \\
&\quad + s^3\sigma_{\mathfrak{R}}(h_{n+2}, h_{n+3}) + \cdots + s^{m-n}\sigma_{\mathfrak{R}}(h_{m-1}, h_m) \\
&\leq sk^n\sigma_{\mathfrak{R}}(h_0, h_1) + s^2k^{n+1}\sigma_{\mathfrak{R}}(h_0, h_1) \\
&\quad + s^3k^{n+2}\sigma_{\mathfrak{R}}(h_0, h_1) + \cdots + s^{m-n}k^{m-1}\sigma_{\mathfrak{R}}(h_0, h_1) \\
&\leq sk^n\left[1 + sk + (sk)^2 + \cdots\right]\sigma_{\mathfrak{R}}(h_0, h_1) \\
&= \frac{sk^n}{1 - sk}\sigma_{\mathfrak{R}}(h_0, h_1).
\end{aligned} \tag{8}$$

As $k \in [0, 1/s)$ and $s \geq 1$, it follows from the above inequality that

$$\lim_{n,m\to\infty} \sigma_{\mathfrak{R}}(h_n, h_m) = 0. \tag{9}$$

Therefore, $\{h_n\}$ is a Cauchy $\mathfrak{R}$-sequence. Since H is $\mathfrak{R}$ -complete, there exists $h^* \in H$ such that $\lim_{n,m\to\infty} \sigma_{\mathfrak{R}}(h_n, h_m) = \lim_{n\to\infty} \sigma_{\mathfrak{R}}(h_n, h^*) = \sigma_{\mathfrak{R}}(h^*, h^*)$ and $h_n \mathfrak{R} h^*$ for each $n \geq k$ (for some value of k). Thus, from above, we obtain $0 = \lim_{n,m\to\infty} \sigma_{\mathfrak{R}}(h_n, h_m) = \lim_{n\to\infty} \sigma_{\mathfrak{R}}(h_n, h^*) = \sigma_{\mathfrak{R}}(h^*, h^*)$ and $h_n \mathfrak{R} h^*$ for each $n \geq k$. As $h_n \mathfrak{R} h^*$ for each $n \geq k$, from (5), we get

$$\sigma_{\mathfrak{R}}(Th_n, Th^*) \leq k\sigma_{\mathfrak{R}}(h_n, h^*). \tag{10}$$

This inequality and the above findings imply

$$\lim_{n\to\infty} \sigma_{\mathfrak{R}}(h_{n+1}, Th^*) \leq \sigma_{\mathfrak{R}}(h^*, h^*) = 0. \tag{11}$$

As T is an $\mathfrak{R}$-property map, so we get $h^* \mathfrak{R} Th^*$ or $Th^* \mathfrak{R} h^*$. Without any loss of generality, we take $h^* \mathfrak{R} Th^*$. Thus, by using $\sigma_{\mathfrak{R}_4}$ with (5), we get the following for each $n \geq k$

$$\begin{aligned}
\sigma_{\mathfrak{R}}(h^*, Th^*) &\leq s\sigma_{\mathfrak{R}}(h^*, h_{n+1}) + s\sigma_{\mathfrak{R}}(h_{n+1}, Th^*) \\
&\quad - \sigma_{\mathfrak{R}}(h_{n+1}, h_{n+1}) \leq s\sigma_{\mathfrak{R}}(h^*, h_{n+1}) \\
&\quad + s\sigma_{\mathfrak{R}}(Th_n, Th^*) \leq s\sigma_{\mathfrak{R}}(h^*, h_{n+1}) \\
&\quad + sk\sigma_{\mathfrak{R}}(h_n, h^*).
\end{aligned} \tag{12}$$

When n tends to infinity, the above inequality yields $\sigma_{\mathfrak{R}}(h^*, Th^*) = 0$. Hence, we get $\sigma_{\mathfrak{R}}(h^*, Th^*) = 0$, $\sigma_{\mathfrak{R}}(h^*, h^*) =$

0 and $\sigma_{\mathfrak{R}}(Th^*, Th^*) = 0$. Therefore, $h^* = Th^*$, that is, h^* is a fixed point of T.

Remark 19. Note that the fixed point of T is unique if in the above theorem we add (I): for each fixed points h^* and l^* of T, we have $h^*\mathfrak{R}l^*$ or $l^*\mathfrak{R}h^*$.

Since h^* and l^* are fixed points of T such that $h^*\mathfrak{R}l^*$. Then, we have $T^n h^* = h^*, T^n l^* = l^*$ for all $n \in \mathbb{N}$. By the nature of h_0, we obtain

$$h_0 \mathfrak{R} h^* \text{and} h_0 \mathfrak{R} l^*. \tag{13}$$

Since T is $\mathfrak{R}$-preserving, we have

$$T^n h_0 \mathfrak{R} T^n h^* \text{and} T^n h_0 \mathfrak{R} T^n l^*, \tag{14}$$

for all $n \in \mathbb{N}$. Therefore, by the triangle inequality and (5), we get

$$\begin{aligned} \sigma_{\mathfrak{R}}(h^*, l^*) = \sigma_{\mathfrak{R}}(T^n h^*, T^n l^*) &= s[\sigma_{\mathfrak{R}}(T^n h^*, T^n h_0) \\ &+ \sigma_{\mathfrak{R}}(T^n h_0, T^n l^*)] - \sigma_{\mathfrak{R}}(T^n h_0, T^n h_0) \\ &\le s k^n \sigma_{\mathfrak{R}}(h^*, h_0) + s k^n \sigma_{\mathfrak{R}}(h_0, l^*). \end{aligned} \tag{15}$$

Taking limit as $n \longrightarrow \infty$ in the above inequality, we obtain

$$\sigma_{\mathfrak{R}}(h^*, l^*) = 0, \tag{16}$$

and so

$$h^* = l^*. \tag{17}$$

Remark 20. Note that the condition "let $h_0 \in H$ be such that $h_0 \mathfrak{R} l$ for each $l \in H$" of Theorem18may be replaced with "let $h_0 \in H$ be such that $h_0 \mathfrak{R} T h_0$."

Example 3. Let $H = \mathbb{R}$ and define $\sigma_{\mathfrak{R}} : H \times H \longrightarrow \mathbb{R}^+$ by

$$\sigma_{\mathfrak{R}}(h, l) = \begin{cases} |h - l|^2 & \text{if} \quad h, l \ge 0, \\ 0 & \text{otherwise.} \end{cases} \tag{18}$$

The relation on H is defined by $h\mathfrak{R}l$ if and only if $h = l$ or $h, l \ge 0$. Clearly, $(H, \mathfrak{R}, \sigma_{\mathfrak{R}}, 4)$ is an $\mathfrak{R}$-complete partial b-metric space. Define a map $T : H \longrightarrow H$ by

$$Th = \begin{cases} \dfrac{h}{4} & \text{if} \quad h \ge 0, \\ 0 & \text{otherwise.} \end{cases} \tag{19}$$

Then, it is very simple to verify the following:

(1) If $h = l$, then $Th = Tl$. While if $h, l \ge 0$, then $Th, Tl \ge 0$. Thus, T is an $\mathfrak{R}$-preserving map

(2) Suppose that for any iterative $\mathfrak{R}$-sequence $\{h_n\}$ in H with $\lim_{n\to\infty} \sigma_{\mathfrak{R}}(h_n, h) = \sigma_{\mathfrak{R}}(h, h)$, $h_n \mathfrak{R} h$ for some $n \ge k$, and $\lim_{n\to\infty} \sigma_{\mathfrak{R}}(h_n, Th) \le \sigma_{\mathfrak{R}}(h, h)$, then we get $h\mathfrak{R}Th$

(3) Consider $h_0 \ge 0$ any real number, then $Th_0 \ge 0$. Thus, we have $h_0, Th_0 \ge 0$, that is, $h_0 \mathfrak{R} T h_0$

(4) For each $h, l \in H$ with $h\mathfrak{R}l$, we have

case (a) $h = l$:

$$\sigma_{\mathfrak{R}}(Th, Tl) = 0 = \frac{1}{16} \times 0 = \frac{1}{16} \sigma_{\mathfrak{R}}(h, l). \tag{20}$$

case (b) $h, l \ge 0$:

$$\sigma_{\mathfrak{R}}(Th, Tl) = \left|\frac{h}{4} - \frac{l}{4}\right|^2 = \frac{1}{16}|h - l|^2 = \frac{1}{16} \sigma_{\mathfrak{R}}(h, l). \tag{21}$$

Hence, by Theorem 18, T must has a fixed point.

Example 4. Let $H = \mathbb{R}$ and define $\sigma_{\mathfrak{R}} : H \times H \longrightarrow \mathbb{R}^+$ by

$$\sigma_{\mathfrak{R}}(h, l) = \begin{cases} |h - l|^2 + (\max\{h, l\})^2 & \text{if} \quad h, l \ge 0, \\ 0 & \text{otherwise.} \end{cases} \tag{22}$$

The relation on H is defined by $h\mathfrak{R}l$ if and only if $h = l$ or $h, l \ge 0$.

Clearly, $(H, \mathfrak{R}, \sigma_{\mathfrak{R}}, 4)$ is an $\mathfrak{R}$-complete partial b-metric space. Define a map $T : H \longrightarrow H$ by

$$Th = \begin{cases} \dfrac{h}{6} & \text{if} \quad h \ge 0, \\ -1 & \text{otherwise.} \end{cases} \tag{23}$$

Then, one can verify the following:

(1) If $h = l$, then $Th = Tl$. While if $h, l \ge 0$, then $Th, Tl \ge 0$. Thus, T is an $\mathfrak{R}$-preserving map

(2) Suppose that for any iterative $\mathfrak{R}$-sequence $\{h_n\}$ in H with $\lim_{n\to\infty} \sigma_{\mathfrak{R}}(h_n, h) = \sigma_{\mathfrak{R}}(h, h)$, $h_n \mathfrak{R} h$ for some $n \ge k$, and $\lim_{n\to\infty} \sigma_{\mathfrak{R}}(h_n, Th) \le \sigma_{\mathfrak{R}}(h, h)$, then we get $h\mathfrak{R}Th$

(3) If $h_0 \ge 0$ be some real number, then $Th_0 \ge 0$. Thus, we get $h_0, Th_0 \ge 0$, that is, $h_0 \mathfrak{R} T h_0$

(4) For each $h, l \in H$ with $h\mathfrak{R}l$, we have

Case (a) If $h = l \ge 0$, then $Th = Tl \ge 0$. Thus,

$$\begin{aligned} \sigma_{\mathfrak{R}}(Th, Tl) = 0 + \left(\max\left\{\frac{h}{6}, \frac{l}{6}\right\}\right)^2 &= \frac{1}{36}(\max\{h, l\})^2 \\ &= \frac{1}{36} \sigma_{\mathfrak{R}}(h, l). \end{aligned} \tag{24}$$

Case (b) If $h = l < 0$, then $Th = Tl = -1$. Thus,

$$\sigma_{\mathfrak{R}}(Th, Tl) = 0 = \sigma_{\mathfrak{R}}(h, l). \tag{25}$$

Case (c) If $h, l \geq 0$, then $Th, Tl \geq 0$. Thus,

$$\sigma_{\mathfrak{R}}(Th, Tl) = \left|\frac{h}{6} - \frac{l}{6}\right|^2 + \left(\max\left\{\frac{h}{6}, \frac{l}{6}\right\}\right)^2 = \frac{1}{36}\sigma_{\mathfrak{R}}(h, l). \tag{26}$$

Hence, by Theorem 18, T must has a fixed point.

Remark 21. Note that the function $\sigma_{\mathfrak{R}}$ defined in the above example is neither a metric nor a b -metric nor a partial b -metric on $\mathbb{R}$. Indeed, $\sigma_{\mathfrak{R}}(4, 1) = 25$, $\sigma_{\mathfrak{R}}(4,-1) = 0$, $\sigma_{\mathfrak{R}}(-1, 1) = 0$, $\sigma_{\mathfrak{R}}(-1,-1) = 0$, that is, $(\sigma 4)$ and $(bM3)$ do not exist.

Theorem 22. *Let $(H, \mathfrak{R}, \sigma_{\mathfrak{R}}, s)$ be an $\mathfrak{R}$ -complete $\mathfrak{R}$ -partial b -metric space with the coefficient $s \geq 1$ and let $h_0 \in H$ be such that $h_0\mathfrak{R}l$ for each $l \in H$. Let $T : H \longrightarrow H$ be an $\mathfrak{R}$ -preserving and $\mathfrak{R}$ -0-continuous map satisfying the following*

$$\sigma_{\mathfrak{R}}(Th, Tl) \leq k \max\{\sigma_{\mathfrak{R}}(h, l), \sigma_{\mathfrak{R}}(h, Th), \sigma_{\mathfrak{R}}(l, Tl)\}, \tag{27}$$

for all $h, l \in H$ with $h\mathfrak{R}l$, $h\mathfrak{R}Th$, and $l\mathfrak{R}Tl$, where $k \in [0, 1/s)$. Also, let for each $\mathfrak{R}$-sequence $\{h_n\}$ in H with $h_n\mathfrak{R}a$ and $h_n\mathfrak{R}b$, we have either $a\mathfrak{R}b$ or $b\mathfrak{R}a$. Then, T has a fixed point $h^* \in H$ and $\sigma_{\mathfrak{R}}(h^*, h^*) = 0$.

Proof. As $h_0 \in H$ is such that $h_0\mathfrak{R}l$ for each $l \in H$, then by using the $\mathfrak{R}$-preserving nature of T, we obtain an $\mathfrak{R}$ -sequence $\{h_n\}$ such that $h_n = Th_{n-1} = T^n h_0$ and $h_{n-1}\mathfrak{R}h_n$ for each $n \in \mathbb{N}$. We take $h_n \neq h_{n+1}$ for each $n \in \mathbb{N} \cup \{0\}$. Then by (27), for each $n \in \mathbb{N}$, we get

$$\begin{aligned}\sigma_{\mathfrak{R}}(h_n, h_{n+1}) &= \sigma_{\mathfrak{R}}(Th_{n-1}, Th_n)\\ &\leq k \max\{\sigma_{\mathfrak{R}}(h_{n-1}, h_n), \sigma_{\mathfrak{R}}(h_{n-1}, Th_{n-1}), \sigma_{\mathfrak{R}}(h_n, Th_n)\}\\ &= k \max\{\sigma_{\mathfrak{R}}(h_{n-1}, h_n), \sigma_{\mathfrak{R}}(h_{n-1}, h_n), \sigma_{\mathfrak{R}}(h_n, h_{n+1})\}\\ &= k \max\{\sigma_{\mathfrak{R}}(h_{n-1}, h_n), \sigma_{\mathfrak{R}}(h_n, h_{n+1})\}.\end{aligned} \tag{28}$$

If $\max\{\sigma_{\mathfrak{R}}(h_{n-1}, h_n), \sigma_{\mathfrak{R}}(h_n, h_{n+1})\} = \sigma_{\mathfrak{R}}(h_n, h_{n+1})$, then from the above inequality, we obtain that $\sigma_{\mathfrak{R}}(h_n, h_{n+1}) \leq k \sigma_{\mathfrak{R}}(h_n, h_{n+1}) < \sigma_{\mathfrak{R}}(h_n, h_{n+1})$, which is a contradiction. Therefore, we must have $\max\{\sigma_{\mathfrak{R}}(h_{n-1}, h_n), \sigma_{\mathfrak{R}}(h_n, h_{n+1})\} = \sigma_{\mathfrak{R}}(h_{n-1}, h_n)$. Again, from the above inequality, we have

$$\sigma_{\mathfrak{R}}(h_n, h_{n+1}) \leq k\sigma_{\mathfrak{R}}(h_{n-1}, h_n) \forall n \in \mathbb{N}. \tag{29}$$

On repeating this process, we obtain

$$\sigma_{\mathfrak{R}}(h_n, h_{n+1}) \leq k^n \sigma_{\mathfrak{R}}(h_0, h_1) \forall n \in \mathbb{N}. \tag{30}$$

For $m,n \in \mathbb{N}$ with $m > n$, by $\sigma_{\mathfrak{R}}4$, we obtain

$$\begin{aligned}\sigma_{\mathfrak{R}}(h_n, h_m) &\leq s[\sigma_{\mathfrak{R}}(h_n, h_{n+1}) + \sigma_{\mathfrak{R}}(h_{n+1}, h_m)] - \sigma_{\mathfrak{R}}(h_{n+1}, h_{n+1})\\ &\leq s\sigma_{\mathfrak{R}}(h_n, h_{n+1}) + s^2[\sigma_{\mathfrak{R}}(h_{n+1}, h_{n+2})\\ &\quad + \sigma_{\mathfrak{R}}(h_{n+2}, h_m)] - \sigma_{\mathfrak{R}}(h_{n+2}, h_{n+2})\\ &\leq s\sigma_{\mathfrak{R}}(h_n, h_{n+1}) + s^2\sigma_{\mathfrak{R}}(h_{n+1}, h_{n+2})\\ &\quad + s^3\sigma_{\mathfrak{R}}(h_{n+2}, h_{n+3}) + \cdots + s^{m-n}\sigma_{\mathfrak{R}}(h_{m-1}, h_m).\end{aligned} \tag{31}$$

Using (30) in the above inequality, we obtain

$$\begin{aligned}\sigma_{\mathfrak{R}}(h_n, h_m) &\leq sk^n\sigma_{\mathfrak{R}}(h_0, h_1) + s^2k^{n+1}\sigma_{\mathfrak{R}}(h_0, h_1)\\ &\quad + s^3k^{n+2}\sigma_{\mathfrak{R}}(h_0, h_1) + \cdots + s^{m-n}k^{m-1}\sigma_{\mathfrak{R}}(h_0, h_1)\\ &\leq sk^n\left[1 + sk + (sk)^2 + \cdots\right]\sigma_{\mathfrak{R}}(h_0, h_1)\\ &= \frac{sk^n}{1 - sk}\sigma_{\mathfrak{R}}(h_0, h_1).\end{aligned} \tag{32}$$

As $k \in [0, 1/s)$ and $s \geq 1$, it follows from the above inequality that

$$\lim_{n,m\to\infty}\sigma_{\mathfrak{R}}(h_n, h_m) = 0. \tag{33}$$

Therefore, $\{h_n\}$ is a Cauchy $\mathfrak{R}$-sequence. Since H is $\mathfrak{R}$-complete, there exists $h^* \in H$ such that $\lim_{n,m\to\infty}\sigma_{\mathfrak{R}}(h_n, h_m) = \lim_{n\to\infty}\sigma_{\mathfrak{R}}(h_n, h^*) = \sigma_{\mathfrak{R}}(h^*, h^*)$ and $h_n\mathfrak{R}h^*$ for each $n \geq k$. Thus, from above, we obtain $0 = \lim_{n,m\to\infty}\sigma_{\mathfrak{R}}(h_n, h_m) = \lim_{n\to\infty}\sigma_{\mathfrak{R}}(h_n, h^*) = \sigma_{\mathfrak{R}}(h^*, h^*)$ and $h_n\mathfrak{R}h^*$ for each $n \geq k$. Since T is $\mathfrak{R}$-0-continuous, one gets that $\lim_{n\to\infty}\sigma_{\mathfrak{R}}(h_n, h^*) = 0$, which leads to $\lim_{n\to\infty}\sigma_{\mathfrak{R}}(Th_n, Th^*) = 0$. Obviously, we have $Th_n\mathfrak{R}Th^*$ for each $n \geq k$. Thus, $h_n\mathfrak{R}Th^*$ for each $n > k$. Since $h_n\mathfrak{R}h^*$ and $h_n\mathfrak{R}Th^*$ for each $n > k$, we have either $h^*\mathfrak{R}Th^*$ or $Th^*\mathfrak{R}h^*$. By using $\sigma_{\mathfrak{R}}4$, we get the following for each $n > k$:

$$\begin{aligned}\sigma_{\mathfrak{R}}(h^*, Th^*) &\leq s\sigma_{\mathfrak{R}}(h^*, h_{n+1}) + s\sigma_{\mathfrak{R}}(h_{n+1}, Th^*)\\ &\quad - \sigma_{\mathfrak{R}}(h_{n+1}, h_{n+1}).\end{aligned} \tag{34}$$

When n tends to infinity, the above inequality yields $\sigma_{\mathfrak{R}}(h^*, Th^*) = 0$. Hence, we get $\sigma_{\mathfrak{R}}(h^*, Th^*) = 0$, $\sigma_{\mathfrak{R}}(h^*, h^*) = 0$, and $\sigma_{\mathfrak{R}}(Th^*, Th^*) = 0$. Therefore, we say that $h^* = Th^*$, i.e., h^* is a fixed point of T.

Remark 23. Note that the fixed point of T is unique if in the above result, we add the condition: for each fixed points h^* and l^* of T , we have $h^*\mathfrak{R}l^*$ or $l^*\mathfrak{R}h^*$.

Since $h^* = Th^*$, we have $l^* = Tl^*$ and $h^*\mathfrak{R}l^*$. From (27), we get

$$\begin{aligned}\sigma_{\mathfrak{R}}(h^*, l^*) &= \sigma_{\mathfrak{R}}(Th^*, Tl^*)\\ &\leq k \max\{\sigma_{\mathfrak{R}}(h^*, l^*), \sigma_{\mathfrak{R}}(h^*, Th^*), \sigma_{\mathfrak{R}}(l^*, Tl^*)\}\\ &= k \max\{\sigma_{\mathfrak{R}}(h^*, l^*), \sigma_{\mathfrak{R}}(h^*, h^*), \sigma_{\mathfrak{R}}(l^*, l^*)\}\\ &= k\sigma_{\mathfrak{R}}(h^*, l^*) < \sigma_{\mathfrak{R}}(h^*, l^*).\end{aligned} \tag{35}$$

It is a contradiction in the case $\sigma_{\mathfrak{R}}(h^*, l^*) \neq 0$. Therefore, we must have $\sigma_{\mathfrak{R}}(h^*, l^*) = 0$, that is, $h^* = l^*$.

3. Applications to Graphical Partial b-Metric Spaces and Partially-Ordered-Partial b-Metric Spaces

In this section, we define a directed graph G on H, denoted by $G = (V(H), E(H))$, with the vertex set $V(H) = H$ and the edge set $E(H)$ such that $E(H) \subset H \times H$ and $\{(h, h): h \in H\} \subset E(H)$. Also, $E(H)$ has no parallel edge. Note that hPl denotes the path between h and l, that is, there exists a finite sequence $\{k_i\}_{i=0}^{j}$, for some finite j, such that $k_0 = h$, $k_j = l$, and $(k_i, k_{i+1}) \in E(H)$ for $i \in \{0, 1, \cdots, j-1\}$.

Definition 24. Let $H \neq \emptyset$ be associated the above-defined G, denoted as (H, G). A map $\sigma_G : H \times H \to \mathbb{R}^+$ is called a G-partial b-metric on the set H, if the following conditions are satisfied for all $h, l, z \in H$ with hPl and $z \in hPl$:

$(\sigma_G 1)$ $h = l$ if and only if $\sigma_G(h, h) = \sigma_G(h, l) = \sigma_G(l, l)$;
$(\sigma_G 2)$ $\sigma_G(h, h) \leq \sigma_G(h, l)$;
$(\sigma_G 3)$ $\sigma_G(h, l) = \sigma_G(l, h)$;
$(\sigma_G 4)$ $\sigma_G(h, l) \leq s[\sigma_G(h, z) + \sigma_G(z, l)] - \sigma_G(z, z)$, where $s \geq 1$.

Then, (H, G, σ_G, s) is called a G-partial b-metric space with the coefficient $s \geq 1$.

Remark 25. If hPl and $z \in hPl$, then we get hPz and zPl. Also note if hPz and zPl, then we have hPl.

Thus, P is a preorder relation on H. Therefore, (H, G, σ_G, s) is also an $\mathfrak{R}$-partial b-metric space.

Definition 26. Let $\{h_n\}$ be a G-sequence in (H, G, σ_G, s), that is, $h_n P h_{n+1}$ or $h_{n+1} P h_n$ for each n. Then, we say that

(i) $\{h_n\}$ is a convergent sequence to $h \in H$ if $\lim_{n\to\infty}\sigma_G(h_n, h) = \sigma_G(h, h)$ and $h_n Ph$ for each $n \geq k$

(ii) $\{h_n\}$ is Cauchy if $\lim_{n,m\to\infty}\sigma_G(h_n, h_m)$ exists and is finite

Definition 27. (H, G, σ_G, s) is said to be G-complete if for each Cauchy G-sequence in H there is $h \in H$ with $\lim_{n,m\to\infty}\sigma_G(h_n, h_m) = \lim_{n\to\infty}\sigma_G(h_n, h) = \sigma_G(h, h)$ and $h_n Ph$ for each $n \geq k$.

Note that for a map $T : H \to H$, the G-0-continuity and G-property are defined in the same way as explained in the last section.

Theorem 28. *Let (H, G, σ_G, s) be a G-complete G-partial b-metric space with the coefficient $s \geq 1$ and let $h_0 \in H$ be such that $h_0 Pl$ for each $l \in H$. Let $T : H \longrightarrow H$ be an edge preserving (if $(h, l) \in E(H)$, then $(Th, Tl) \in E(H)$) and a G-property map satisfying the following*

$$\sigma_G(Th, Tl) \leq k\sigma_G(h, l) \quad \text{for all} \quad h, l \in H \quad \text{with} \quad hPl, \tag{36}$$

where $k \in [0, 1/s)$. Then, T has a fixed point $h^* \in H$ and $\sigma_G(h^*, h^*) = 0$.

By Remark 25, we know that P is a preorder relation on H and (H, G, σ_G, s) is an $\mathfrak{R}$-partial b-metric space. Also, an edge preserving map is path preserving. Thus, all the conditions of Theorem 18 hold. Hence, T has a fixed point.

In the following, we obtain partially-ordered-partial b-metric spaces from $\mathfrak{R}$-partial b-metric spaces, by considering $^\circ$ as a partial order on H.

Definition 29. Let $H \neq \emptyset$ be associated with a partial order $^\circ$, denoted as (H°). Given a map $\sigma_\circ : H \times H \to \mathbb{R}^+$. If the following conditions are satisfied for all $h, l, z \in H$ with $h^\circ l$ and $h^\circ z^\circ l$:

$(\sigma_\circ 1)$ $h = l$ if and only if $\sigma_\circ(h, h) = \sigma_\circ(h, l) = \sigma_\circ(l, l)$;
$(\sigma_\circ 2)$ $\sigma_\circ(h, h) \leq \sigma_\circ(h, l)$;
$(\sigma_\circ 3)$ $\sigma_\circ(h, l) = \sigma_\circ(l, h)$;
$(\sigma_\circ 4)$ $\sigma_\circ(h, l) \leq s[\sigma_\circ(h, z) + \sigma_\circ(z, l)] - \sigma_\circ(z, z)$, where $s \geq 1$,

then $(H, G, \sigma_\circ, s)$ is called a partially-ordered-partial b-metric space with the coefficient $s \geq 1$.

As we discussed in the above, we state the following result.

Theorem 30. *Let $(H, G, \sigma_\circ, s)$ be an $^\circ$-complete partially-ordered-partial b-metric space with the coefficient $s \geq 1$ and let $h_0 \in H$ be such that $h_0^\circ l$ for each $l \in H$. Let $T : H \longrightarrow H$ be order preserving (if $h^\circ l$ then $Th^\circ Tl$), and an $^\circ$-property map satisfying the following:*

$$\sigma_\circ(Th, Tl) \leq k\sigma_\circ(h, l) \quad \text{for all} \quad h, l \in H \quad \text{with} \quad h^\circ l, \tag{37}$$

where $k \in [0, 1/s)$. Then, T has a fixed point $h^* \in H$ and $\sigma_\circ(h^*, h^*) = 0$.

Remark 31. $\preccurlyeq-$ completeness is defined in the same way as G-completeness.

4. Conclusion

By combining the concepts of orthogonality and the binary relation, we introduced the notion of $\mathfrak{R}$-partial b-metric spaces. We presented some related fixed point results. Some illustrated examples and an application to graphical partial

b-metric spaces and partially-ordered-partial b-metric spaces have been provided. As perspectives, it would be interesting to consider in this setting more generalized contraction mappings involving simulation functions or more control functions.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no competing interests regarding the publication of this paper.

Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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