

Research Article

Unified Framework of Approximating and Interpolatory Subdivision Schemes for Construction of Class of Binary Subdivision Schemes

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Received 3 November 2020; Revised 25 November 2020; Accepted 1 December 2020; Published 17 December 2020

Academic Editor: Ali Aral

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In this paper, a generalized algorithm to develop a class of approximating binary subdivision schemes is presented. The proposed algorithm is based on three-point approximating binary and four-point interpolating binary subdivision schemes. It contains a parameter which classifies members of the new class of subdivision schemes. A set of efficient properties, for instance, polynomial generation and reproduction, support, continuity, and Hölder continuity, is discussed. Moreover, applications of the proposed subdivision schemes are given in order to demonstrate their variety, flexibility, and visual performance.

1. Introduction

Subdivision is a competent way of producing smooth curves or surfaces in geometric modeling and computer graphics. It repeatedly refines the initial polygonal shape. After each split average step, we get closer to the limit curve, which is the limit of an infinite series. A nice property of subdivision schemes is that they are simple and local, which means that local change in initial data will only have a local effect in the resulting object. Subdivision schemes have become celebrated because of their simplicity and efficiency. There are generally two main categories of subdivision schemes: interpolatory and approximating. For interpolating subdivision schemes, limit curve always passes through initial control points while for approximating subdivision schemes it may

or may not. Subdivision schemes play an integral role in computer graphics due to their wide range of applications in many fields such as engineering, medical science, space science, graphic visualization, and image processing. Differential equations are used for mathematical modeling of many phenomena. Different techniques are being used to solve boundary value problems [1] and nonlinear problems [2]. Nowadays, subdivision schemes are also becoming a popular tool to numerically solve boundary value problems [3]. Subdivision algorithms are also a major field in many multiscale techniques applied in data compression. In some applications, the given data need not be reproduced at each step of the subdivision process, which needs the applications of subdivision schemes. Several researchers in the area of continuous geometry have been established classical subdivision

schemes for various kinds of initial control data. In geometric modeling and engineering, practical applications of subdivision curves are restricted due to their shortcomings, and to overcome these shortcomings, a lot of work has been carried out [4–17].

Deslauriers and Dubuc [18] presented a family of interpolating binary subdivision schemes. They used Lagrange interpolating polynomial for construction of schemes. Horman and Sabin (HS) [19] proposed a family of binary subdivision schemes having cubic precision. Some members of the HS family are interpolating and some are approximating. Mustafa et al. [20] offered a family of binary subdivision schemes which has alternating primal and dual symbols. Ashraf et al. [21] discussed a family of binary subdivision schemes based on Lane-Riesenfeld algorithm. Members of the proposed family have quintic precision. Mustafa and Bari [22] developed a family of univariate subdivision schemes for curve generation and data fitting. Asghar and Mustafa [23] presented a unified framework of stationary and nonstationary subdivision schemes. Keeping in view this practice, we present a generalized algorithm to develop a new class of approximating binary subdivision schemes. Ghaffar et al. [24–29] constructed geometric continuity conditions for the construction of free-form generalized subdivision curves with single shape parameter. These free-form complex shape adjustable generalized curves can be obtained by using shape-adjustable generalized subdivision schemes. These newly proposed approaches not only take over the benefits of classical subdivision curve and surface schemes but also resolve the issue of shape adjustability of subdivision curves and surfaces with the help of tension control shape parameters. They modeled some complex curves and surfaces using higher continuity conditions. The proposed masks of the schemes provide an alternative approach to generate the complex curves using higher continuity conditions with simple and straightforward calculation for the proposed algorithm because they are blended with linear polynomials rather than trigonometric functions. In 2020, Ashraf et al. [17, 30, 31] proposed a new approach using the generalized hybrid subdivision curve with shape parameters to solve the problem in construction of some symmetric curves and surfaces. These curves are easily modified by the changing the values of shape parameters.

In this paper, we offer a Lane-Riesenfeld-like algorithm to derive a class of binary approximating subdivision schemes. Our algorithm is based on the well-known four-point interpolating binary subdivision scheme [18], which is C^1 continuous, and three-point approximating binary subdivision scheme [32], which has C^3 continuity. Considering ϕ smoothing stages as in the Lane-Riesenfeld algorithm, our proposed algorithm allows us to derive a class of univariate subdivision schemes. In fact, each member of the proposed class is enumerated by ϕ , and higher values of ϕ give schemes with wider masks and support, higher continuity, higher Holder regularity, and higher degree of polynomial generation. The first member of the proposed class (corresponding to $\phi = 0$) coincides with the three-point approximating binary subdivision scheme [32]. The proposed class of schemes generates schemes of higher continuities and

visually more smooth limit curves as compared to existing families of schemes. The content of the paper is structured as follows. In Section 2, fundamental definitions and concepts are given. Section 3 presents a generalized algorithm for construction of new class of subdivision schemes. Section 4 is devoted for properties of proposed schemes, such as continuity, Hölder continuity, and support of basic limit function. Geometrical analysis and some beautiful examples of limit curve are given in Section 5. Section 6 presents a summary of the paper.

2. Preliminaries

Let the initial data be given by a set of control points $\mathbf{G}^0 = \{g_i^0 \in \mathbb{R}, i \in \mathbb{Z}\}$, and the set of control points at refinement level $h (h \geq 0, h \in \mathbb{N})$ is given by $\mathbf{G}^h = \{g_i^h \in \mathbb{R}, i \in \mathbb{Z}\}$. Define $\mathbf{G}^{h+1} = \{g_i^{h+1} \in \mathbb{R}, i \in \mathbb{Z}\}$ recursively by the following binary refinement rules:

$$g_i^{h+1} = \sum_{k \in \mathbb{Z}} b_{i-2k} g_k^h, \quad i \in \mathbb{Z}, \quad (1)$$

where the finite set $\mathbf{B} = \{b_i, i \in \mathbb{Z}\}$ is called mask. The recursive algorithm associated with the repeated application of (1) is called subdivision scheme and denoted by S . The Laurent polynomial or symbol of the scheme S is defined as

$$B(z) = \sum_{l \in \mathbb{Z}} b_l z^l. \quad (2)$$

Theorem 1 (see [33]). *If a binary scheme S is convergent, then the mask $\mathbf{B} = \{b_i, i \in \mathbb{Z}\}$ satisfies*

$$\sum_{l \in \mathbb{Z}} b_{2l} = \sum_{l \in \mathbb{Z}} b_{2l+1} = 1. \quad (3)$$

The symbol of a convergent scheme can be also be written as

$$B(z) = B_{\text{even}}(z^2) + z B_{\text{odd}}(z^2), \quad (4)$$

with $B_{\text{even}}(z) = \sum_{l \in \mathbb{Z}} b_{2l} z^l$ and $B_{\text{odd}}(z) = \sum_{l \in \mathbb{Z}} b_{2l+1} z^l$.

Theorem 2 (see [33]). *A binary scheme S associated with the symbol*

$$B(z) = \frac{(z+1)^{m+1}}{2^m} L(z) \quad (5)$$

is said to be C^m continuous if the subdivision scheme associated with the symbol $L(z)$ is contractive.

Proposition 3 (see [34]). *A binary scheme S generates polynomials of degree m if and only if*

$$B(1) = 2, B(-1) = 0 \text{ and } B^{(j)}(-1) = 0, j = 1, 2, \dots, m. \quad (6)$$

TABLE 1: Mask of the R_φ schemes corresponding to different values of parameter φ .

φ	Scheme	Mask
0	3-point	$\frac{1}{16} [1, 5, 10, 10, 5, 1]$
1	5-point	$\frac{1}{256} [-1, 4, 44, 124, 170, 124, 44, 4, -1]$
2	6-point	$\frac{1}{4096} [1, -13, -17, 309, 1338, 2478, 1338, 309, -17, -13, 1]$
3	8-point	$\frac{1}{65536} [-1, 22, -91, -580, 1303, 12362, 31557, 41928, 31557, 12362, 1303, -580, -91, 22, -1]$

Proposition 4 (see [34]). *A binary scheme S reproduces polynomials of degree n with respect to parametrization $\{e_j^h = (j + \tau)/2^h\}_{j \in \mathbb{Z}}$ with $\tau = B^{(j)}(1)/2$, if and only if it generates polynomials of degree n and*

$$B^{(j)}(1) = 2 \prod_{i=0}^{j-1} (\tau - i), \quad j = 1, 2, \dots, n. \quad (7)$$

3. Algorithm for Construction of Schemes

The well-known four-point interpolating binary subdivision scheme [18] is given by

$$\begin{cases} g_{2i}^{h+1} = g_i^h, \\ g_{2i+1}^{h+1} = -\frac{1}{16} g_{i-1}^h + \frac{9}{16} g_i^h + \frac{9}{16} g_{i+1}^h - \frac{1}{16} g_{i+2}^h. \end{cases} \quad (8)$$

By considering (4), the symbol of the even part of scheme (8) is as follows:

$$P_{\text{even}}(z) = \left(\frac{z+1}{2}\right) \left(\frac{-z^2 + 10z - 1}{8}\right). \quad (9)$$

Now, consider the three-point approximating binary subdivision scheme [32]

$$\begin{cases} g_{2i}^{h+1} = \frac{1}{16} g_{i-1}^h + \frac{10}{16} g_i^h + \frac{5}{16} g_{i+1}^h, \\ g_{2i+1}^{h+1} = \frac{5}{16} g_{i-1}^h + \frac{10}{16} g_i^h + \frac{1}{16} g_{i+1}^h. \end{cases} \quad (10)$$

The symbol of scheme (10) is given by

$$T(z) = 2 \left(\frac{z+1}{2}\right)^5. \quad (11)$$

Let us now present the class of subdivision schemes, namely, $\mathbf{R} = \{R_\varphi : \varphi \geq 0, \varphi \in \mathbb{N}\}$. The symbol of the scheme R_φ is obtained by applying symbol of the even part of scheme (8) φ -times on symbol of scheme (10) and given by

$$R_\varphi(z) = (P_{\text{even}}(z))^\varphi T(z). \quad (12)$$

So by (9), (11), and (12), we have

$$R_\varphi(z) = 2 \left(\frac{z+1}{2}\right)^{\varphi+5} \left(\frac{-z^2 + 10z - 1}{8}\right)^\varphi, \quad (13)$$

where $\{\varphi \geq 0, \varphi \in \mathbb{N}\}$. The members of the class R of subdivision schemes can be categorized by varying $\varphi = 0, 1, 2, \dots$, in (13). By taking $\varphi = 0$ in (13), we get three-point approximating binary scheme [32]. Table 1 presents mask of some members of the proposed class.

4. Properties of the Proposed Schemes

In this section, we present some desirable properties of class \mathbf{R} of subdivision schemes, comprising of polynomial generation and reproduction, support, continuity, and Hölder continuity.

4.1. Polynomial Generation and Reproduction. If a subdivision scheme generates polynomials of degree up to d_G , then the polynomial generation degree of the scheme is d_G . Also, if the initial data $\mathbf{G}^0 = \{g_i^0, i \in \mathbb{Z}\}$ is sampled from a polynomial $\hat{\mathbf{P}}$ of degree d_R and the scheme yields precisely the same polynomial in the limit, then the reproduction degree d_R is the maximal degree of polynomials that can be reproduced by the scheme. Clearly, the reproduction degree is always less than or equal to the generation degree. Now, we establish few results about polynomial generation and polynomial reproduction of the proposed subdivision schemes.

Proposition 5. *R_φ -scheme generates space of polynomials up to degree $\varphi + 4$.*

Proof. Since symbol of R_φ -scheme satisfies the conditions

$$R_\varphi(1) = 2, R_\varphi(-1) = 0 \text{ and } R_\varphi^j(-1) = 0, j = 1, 2, \dots, \varphi + 4, \quad (14)$$

so by Proposition 3, R_φ -scheme has $\varphi + 4$ polynomial generation degree.

In the view of Conti and Hormann [35], the standard parametrization $e_j^h = j/2^h$ at level $h \in \mathbb{N}$ is not appropriate to analyze a subdivision scheme to reproduce space of polynomials, and the relative shift $\tau_h = (e_0^h - e_0^{h+1})/2^{h+1}$

TABLE 2: Support, degree of polynomial generation (d_G), degree of polynomial reproduction (d_R), continuity (C), and HC of R_φ -scheme for $\varphi = 0, 1, 2$, and 3.

φ	Support	d_G	d_R	C	Hölder continuity	
					LB	UB
0	5	4	1	3	4	4
1	8	5	1	4	4.678	4.678
2	11	6	1	5	5.299	5.332
3	14	7	1	5	5.871	5.968

between the parameterizations at iteration level h and $h + 1$ is important for polynomial reproduction of degree $d_R \geq 1$. By applying a more suitable parametrization $e_j^h = (j + \tau_\varphi)/2^h$ with shift parameter $\tau_\varphi = R_\varphi^{(1)}(1)/2 = (3\varphi + 5)/2$, we have the following result.

Proposition 6. R_φ -scheme reproduces linear polynomial with respect to parametrization $\{e_j^h = (j + \tau_\varphi)/2^h\}_{j \in \mathbb{Z}}$ with shift $\tau_\varphi = (3\varphi + 5)/2$.

Proof. It can be easily verified that

$$\begin{aligned} R_\varphi^{(1)}(1) &= 2\tau_\varphi = 3\varphi + 5, \\ R_\varphi^{(j)}(1) &= 2 \prod_{i=0}^{j-1} (\tau_\varphi - j), \quad j = 0, 1. \end{aligned} \quad (15)$$

Thus, by Propositions 4 and 5, R_φ -scheme reproduces polynomial of degree one.

Table 2 presents the degree of polynomial generation and reproduction of some of the proposed R_φ -schemes. It is observed that the degree of polynomial generation is increasing linearly with the value of parameter φ .

4.2. Support. The support of a subdivision scheme quantifies how much one vertex brought change in its neighboring vertices, and its measure represents local support of the limit curve. Basic limit function (BLF) of a convergent subdivision scheme is a limit function of the initial data $\mathbf{G}^0 = \{g_i^0, i \in \mathbb{Z}\}$ which is of the form

$$g_i^0 = \begin{cases} 1, & i = 0, \\ 0, & i \neq 0. \end{cases} \quad (16)$$

By following [36], we determine that support of BLF of R_φ -scheme is $3\varphi + 5$. BLF generated by the proposed R_0 and R_1 schemes are demonstrated in Figure 1.

4.3. Continuity Analysis. Continuity of a subdivision scheme is an essential parameter on which efficiency of a scheme

depends. To investigate continuity of our proposed class, we follow the approach as given in [33] and use the symbol of R_φ -scheme.

Theorem 7. The R_φ -scheme has $C^{\varphi+3-\nu_\varphi}$ continuity, where $\varphi = 1, 2, 3, \dots$, and $\nu_\varphi = \lfloor (\varphi - 1)/2 \rfloor$ (floor function).

Proof. The symbol of R_φ -scheme (13) can be simplified as

$$R_\varphi(z) = \frac{(z+1)^{\varphi+4-\nu_\varphi}}{2^{\varphi+3-\nu_\varphi}} r_\varphi(z), \quad (17)$$

with

$$r_\varphi(z) = \left(\frac{z+1}{2}\right)^{1+\nu_\varphi} \left(\frac{-z^2 + 10z - 1}{8}\right)^\varphi, \quad \varphi = 1, 2, 3, \dots, \quad (18)$$

where $\nu_\varphi = \lfloor (\varphi - 1)/2 \rfloor$.

Let S_{r_φ} be the subdivision scheme associated with the symbol $r_\varphi(z)$. The scheme S_{r_φ} is contractive provided that $\|S_{r_\varphi}\|_\infty = \max \{ \sum_{l \in \mathbb{Z}} |r_{2l}|, \sum_{l \in \mathbb{Z}} |r_{2l+1}| \} < 1$. So, by Theorem 2, R_φ -scheme has $C^{\varphi+3-\nu_\varphi}$ continuity.

In Theorem 7, we discuss continuity of R_φ -scheme for $\varphi = 1, 2, 3, \dots$. It is to be noted that R_0 -scheme has C^3 continuity which is analyzed in [32].

Corollary 8. The R_1 -scheme has C^4 continuity.

Proof. By letting $\varphi = 1$, the symbol of R_1 -scheme from (17) and (18) is given by

$$R_1(z) = \frac{(z+1)^5}{2^4} r_1(z), \quad (19)$$

with

$$r_1(z) = \frac{1}{16} (-z^3 + 9z^2 + 9z - 1). \quad (20)$$

Let S_{r_1} be the scheme corresponding to the symbol $r_1(z)$. The scheme S_{r_1} is contractive, as $\|S_{r_1}\|_\infty = \max \{10/16, 10/16\} = (10/16) < 1$. So, by Theorem 7, R_1 -scheme has C^4 continuity.

Similarly, for different values of parameter φ , continuity of R_φ -scheme can be easily computed by using Theorem 7.

4.4. Hölder Continuity Analysis. Continuity of a subdivision scheme is related to the existence of derivative of subdivision curve. For example, subdivision curve is said to be C^m continuous if the m^{th} derivative of the curve exists and is continuous everywhere in the given interval. On the other hand,

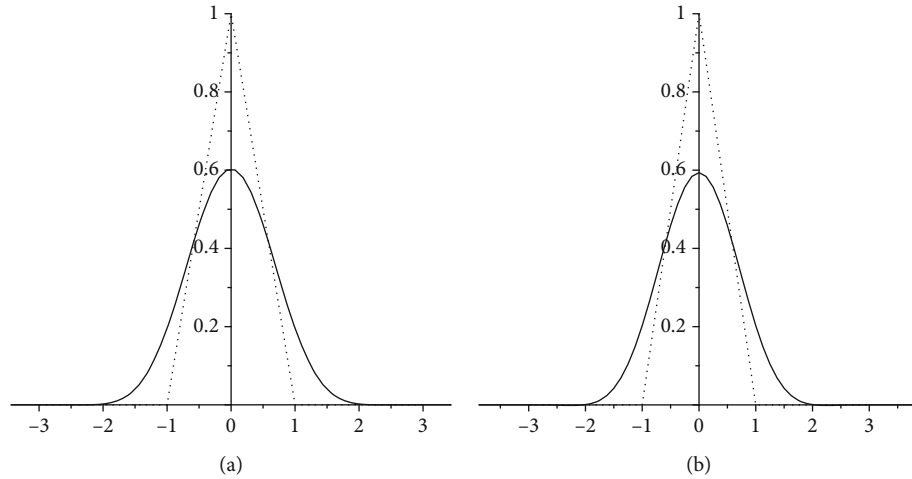


FIGURE 1: (a, b) Basic limit functions generated by the proposed schemes R_0 and R_1 , respectively.

Hölder continuity (HC) of a subdivision scheme tells how continuous the highest continuous derivative is. Therefore, it is also important to find HC of subdivision schemes along with continuity. Lower bound (LB) on HC of the proposed class is calculated by using an interesting property of symbol of R_φ -scheme, i.e., odd coefficients in $R_\varphi(z)$ are nonnegative and even coefficients are nonpositive.

Theorem 9. *LB on the HC of R_φ -scheme is $\varphi + 5 - \log_2((3/2)^\varphi + 1)$, where $\varphi = 0, 1, 2, \dots$.*

Proof. By (13), symbol of R_φ -scheme can be expressed as

$$R_\varphi(z) = \left(\frac{z+1}{2}\right)^{\varphi+5} U_\varphi(z), \quad (21)$$

where $U_\varphi(z) = (a(z))^\varphi b(z)$, $a(z) = (-z^2 + 10z - 1)/8$, and $b(z) = 2$. So LB on HC of R_φ -scheme is given by $\varphi + 5 - \log_2 \|U_\varphi\|$. As we know $\|U_\varphi\| = \max(u^\circ, u_\circ)$, where u° is the sum of odd and u_\circ is the sum of even coefficients of $U_\varphi(z)$. We can write coefficients of $U_\varphi(z)$ in the following manner:

$$\begin{pmatrix} u^\circ \\ u_\circ \end{pmatrix} = \begin{pmatrix} a^\circ & a_\circ \\ a_\circ & a^\circ \end{pmatrix}^\varphi \begin{pmatrix} b^\circ \\ b_\circ \end{pmatrix}. \quad (22)$$

Thus, we have

$$\begin{pmatrix} u^\circ \\ u_\circ \end{pmatrix} = \begin{pmatrix} \frac{5}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{5}{4} \end{pmatrix}^\varphi \begin{pmatrix} 0 \\ 2 \end{pmatrix}. \quad (23)$$

By eigenvalue decomposition, we have

$$\begin{pmatrix} u^\circ \\ u_\circ \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{3}{2} & 0 \\ 0 & 1 \end{pmatrix}^\varphi \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \quad (24)$$

which implies that

$$\begin{pmatrix} u^\circ \\ u_\circ \end{pmatrix} = \begin{pmatrix} -\left(\frac{3}{2}\right)^\varphi & +1 \\ \left(\frac{3}{2}\right)^\varphi & +1 \end{pmatrix}^\varphi. \quad (25)$$

Thus, we have

$$\|U_\varphi\| = \left(\frac{3}{2}\right)^\varphi + 1. \quad (26)$$

Consequently, LB on HC of R_φ -scheme is $\varphi + 5 - \log_2((3/2)^\varphi + 1)$, where $\varphi = 0, 1, \dots$.

Upper bound (UB) on HC of R_φ -scheme is as follows.

Theorem 10. *UB on HC of R_φ -scheme is $\varphi + 5 - \log_2(\zeta_\varphi)$, where $\varphi = 0, 1, \dots$, and ζ_φ be the joint spectral radius of the matrices Q_0 and Q_1 which are obtained by using symbol of R_φ -scheme.*

Proof. By (13), symbol of R_φ -scheme can be expressed as

$$R_\varphi(z) = \left(\frac{z+1}{2}\right)^{\varphi+5} Q_\varphi(z), \quad (27)$$

where $Q_\varphi(z) = 2((-z^2 + 10z - 1)/8)^\varphi$. Let q_0, q_1, \dots, q_d be the nonzero real coefficients of $Q_\varphi(z)$. Also, Q_0 and Q_1 are the matrices of order $d \times d$ defined by

$$(Q_0)_{mn} = q_{d+m-2n}, \text{ and } (Q_1)_{mn} = q_{d+m-2n+1}, \quad (28)$$

where $m, n = 1, 2, \dots, d$.

Let us denote joint spectral radius of both matrices Q_0 and Q_1 by ζ_φ . Then, by Rioul [37] and Dyn [33], UB on HC of R_φ -scheme is given by $\varphi + 5 - \log_2(\zeta_\varphi)$.

For different values of parameter φ , upper and lower bounds on the HC of R_φ -schemes can be straightforwardly computed by using Theorems 9 and 10. Table 2 summarizes the continuity and HC of the proposed class of subdivision schemes. It clearly indicates that as we go up for higher values of parameter φ , continuity and HC of R_φ -schemes also increase. Moreover, newly generated R_φ -schemes have higher order of continuity and HC as compared to their parent subdivision schemes.

5. Geometrical Analysis of Proposed Schemes

The shape of an object is generally controlled by a control polygon. The purpose of applying a subdivision scheme on the control polygon is to generate visually smooth curves. Figure 2 presents the behavior of some of the proposed schemes. R_0 , R_1 , and R_2 schemes are applied on the same initial polygon, and limit curves are obtained after three iterations. It is evident that the proposed class offers more choices to meet different designing needs.

5.1. Subdivision Rules for Endpoints. For closed curves, the subdivision rules of R_0 , R_1 , R_2 , and R_3 schemes can be defined by their corresponding Laurent polynomial from (13). The limit curves generated by these schemes are C^3 , C^4 , C^5 , and C^5 continuous, respectively. In case of dealing with open polygons, these rules can be used to improve the interior of the curve, while it is quite troublesome to improve the first and last edges with the help of subdivision rules of the original proposed schemes. So to handle the endpoints of an open polygon, we need to supply additional points which are not usually required in case of a closed polygon. Let $g_0^h g_1^h$ be the first edge of the open polygon $\{G^h = g_k^h : k = 0, \dots, 2^h m\}$. Now, we define an additional control point g_{-1}^h , as an extrapolatory rule in the nonrefined polygon G^h , and then we can compute the point g_1^{h+1} through the proposed schemes by applying subdivision to the subpolygon $\{g_{-1}^h, g_0^h, g_1^h, g_2^h\}$. We select the point as $g_{-1}^h = 2g_0^h - g_1^h$. The first edge of an open control polygon $\{G^h = g_k^h : k = 0, 1, \dots, 2^h m\}$ can be refined by using the following rules.

- (i) Refinement rules of the proposed three-point scheme R_0 are given by

$$\begin{aligned} g_0^{h+1} &= \frac{5}{4}g_0^h - \frac{1}{4}g_1^h, \\ g_1^{h+1} &= \frac{3}{4}g_0^h + \frac{1}{4}g_1^h. \end{aligned} \quad (29)$$

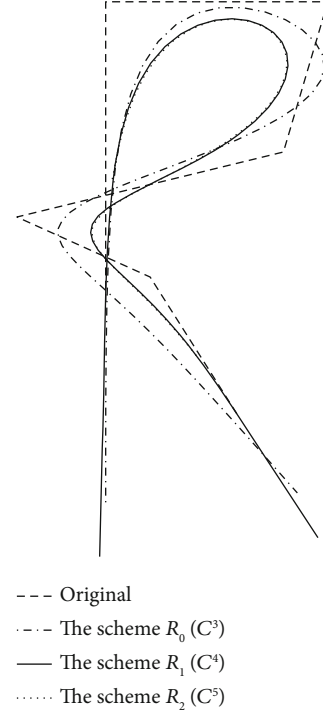


FIGURE 2: Behavior of the proposed R_0 , R_1 , and R_2 schemes after three iterations.

- (ii) Refinement rules of the proposed five-point scheme R_1 are given by

$$\begin{aligned} g_0^{h+1} &= \frac{380}{256}g_0^h - \frac{120}{256}g_1^h - \frac{4}{256}g_2^h, \\ g_1^{h+1} &= \frac{255}{256}g_0^h + \frac{2}{256}g_1^h - \frac{1}{256}g_2^h, \\ g_2^{h+1} &= \frac{132}{256}g_0^h + \frac{120}{256}g_1^h + \frac{4}{256}g_2^h, \\ g_3^{h+1} &= \frac{42}{256}g_0^h + \frac{171}{256}g_1^h + \frac{44}{256}g_2^h - \frac{1}{256}g_3^h. \end{aligned} \quad (30)$$

- (iii) Refinement rules of the proposed six-point scheme R_2 are given by

$$\begin{aligned} g_0^{h+1} &= \frac{3070}{4096}g_0^h + \frac{1029}{4096}g_1^h - \frac{4}{4096}g_2^h + \frac{1}{4096}g_3^h, \\ g_1^{h+1} &= \frac{1306}{4096}g_0^h + \frac{2495}{4096}g_1^h + \frac{308}{4096}g_2^h - \frac{13}{4096}g_3^h, \\ g_2^{h+1} &= \frac{283}{4096}g_0^h + \frac{2491}{4096}g_1^h + \frac{1338}{4096}g_2^h - \frac{17}{4096}g_3^h + \frac{1}{4096}g_4^h, \\ g_3^{h+1} &= -\frac{15}{4096}g_0^h + \frac{1337}{4096}g_1^h + \frac{2478}{4096}g_2^h + \frac{309}{4096}g_3^h - \frac{13}{4096}g_4^h. \end{aligned} \quad (31)$$

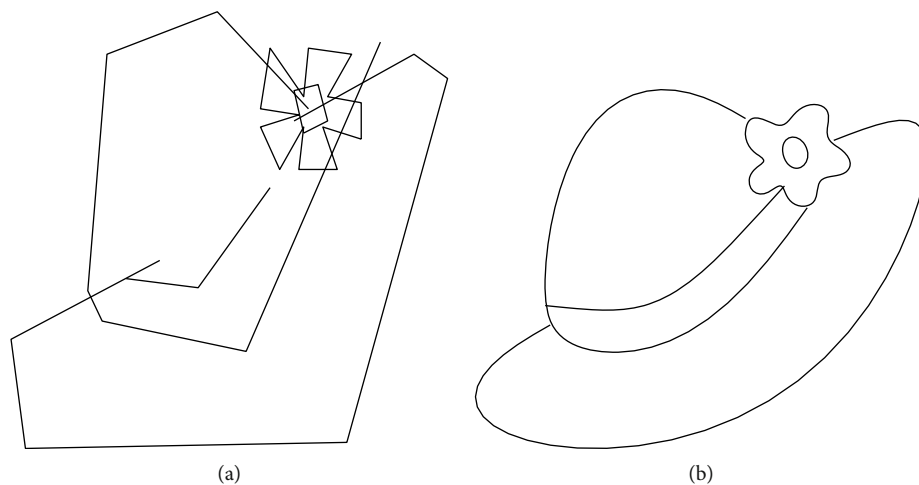


FIGURE 3: Application of R_0 -scheme: (a) initial polygon and (b) the limit curve generated by R_0 -scheme at the third subdivision level.

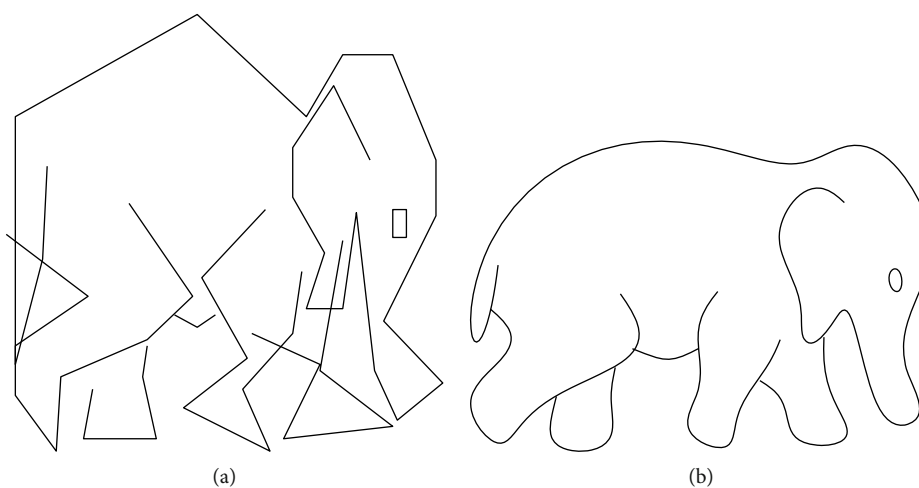


FIGURE 4: Application of R_0 -scheme: (a) initial polygon and (b) the limit curve generated by R_0 -scheme at the third subdivision level.

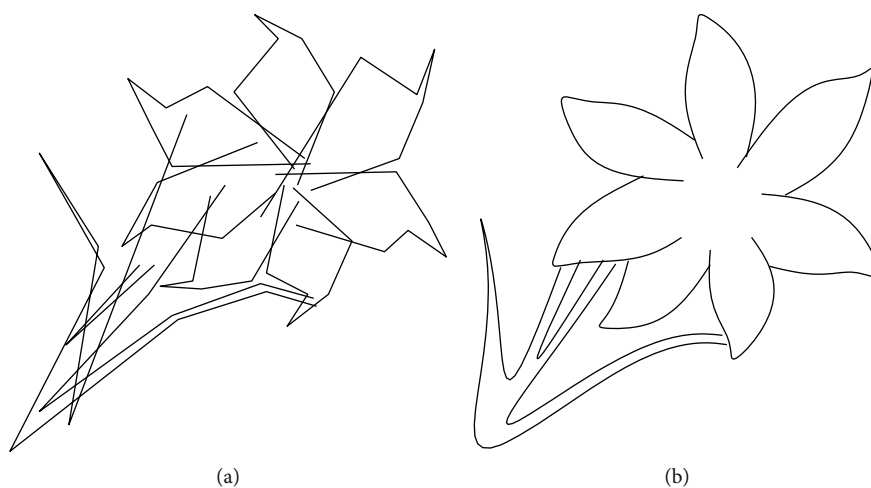


FIGURE 5: Application of R_1 -scheme: (a) initial polygon and (b) the limit curve generated by R_1 -scheme at the third subdivision level.

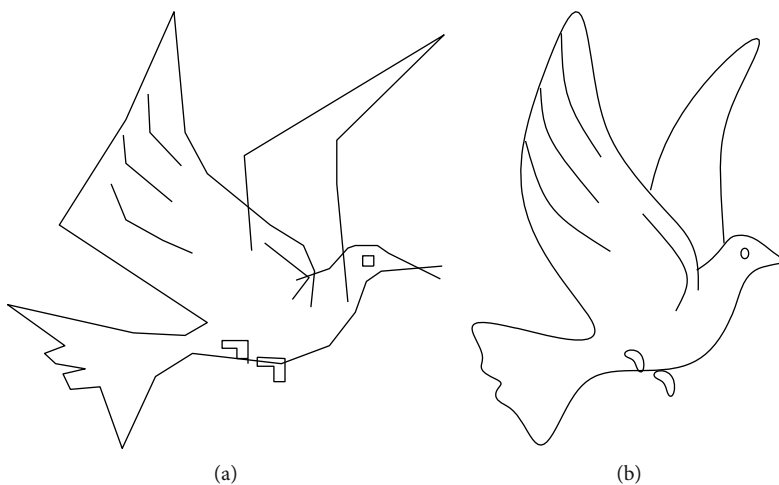


FIGURE 6: Application of R_1 -scheme: (a) initial polygon and (b) the limit curve generated by R_1 -scheme at the third subdivision level.

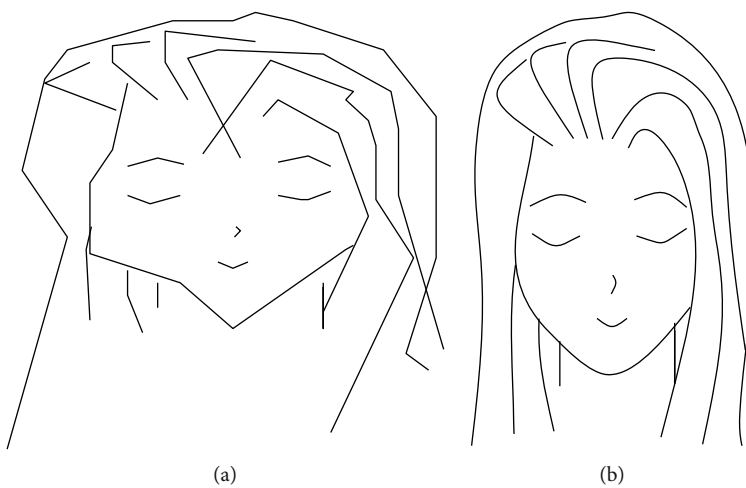


FIGURE 7: Application of R_2 -scheme: (a) initial polygon and (b) the limit curve generated by R_2 -scheme at the third subdivision level.

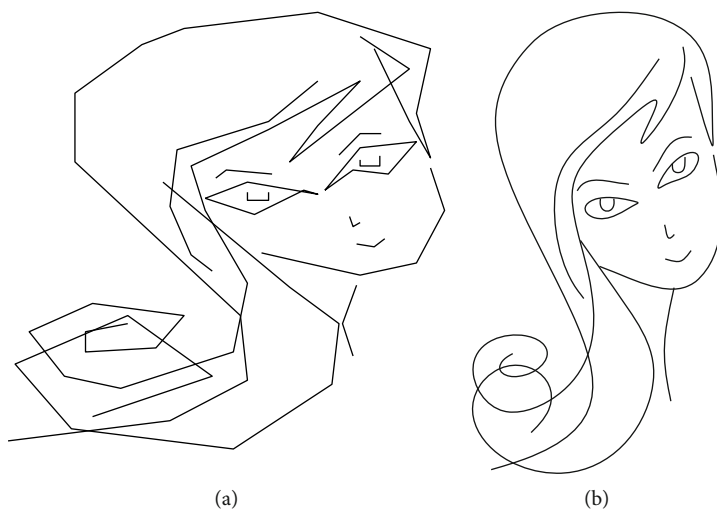


FIGURE 8: Application of R_3 -scheme: (a) initial polygon and (b) the limit curve generated by R_3 -scheme at the third subdivision level.

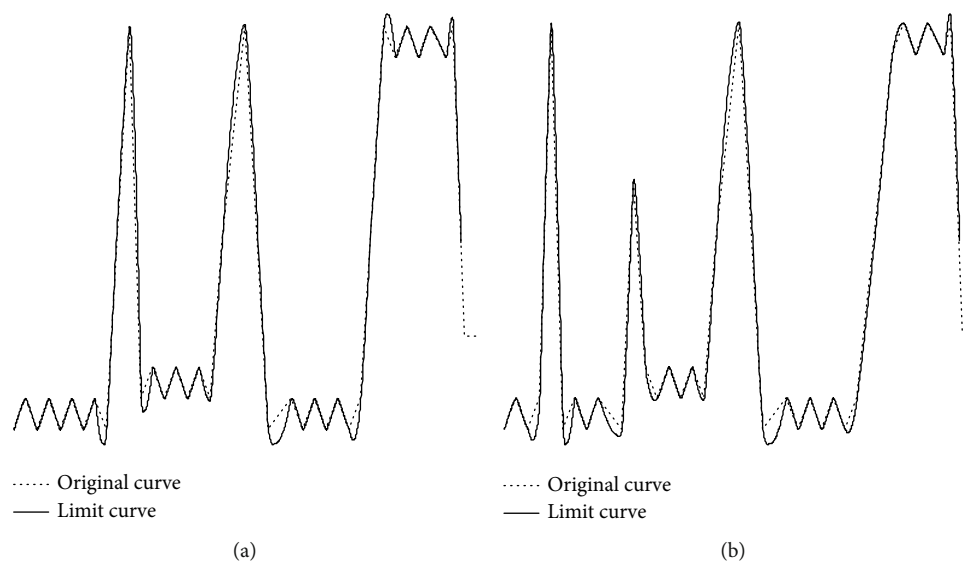


FIGURE 9: Application of R_3 -scheme: (a, b) the initial polygon along with sharp features of limit curve generated by R_3 -scheme at the third subdivision level.

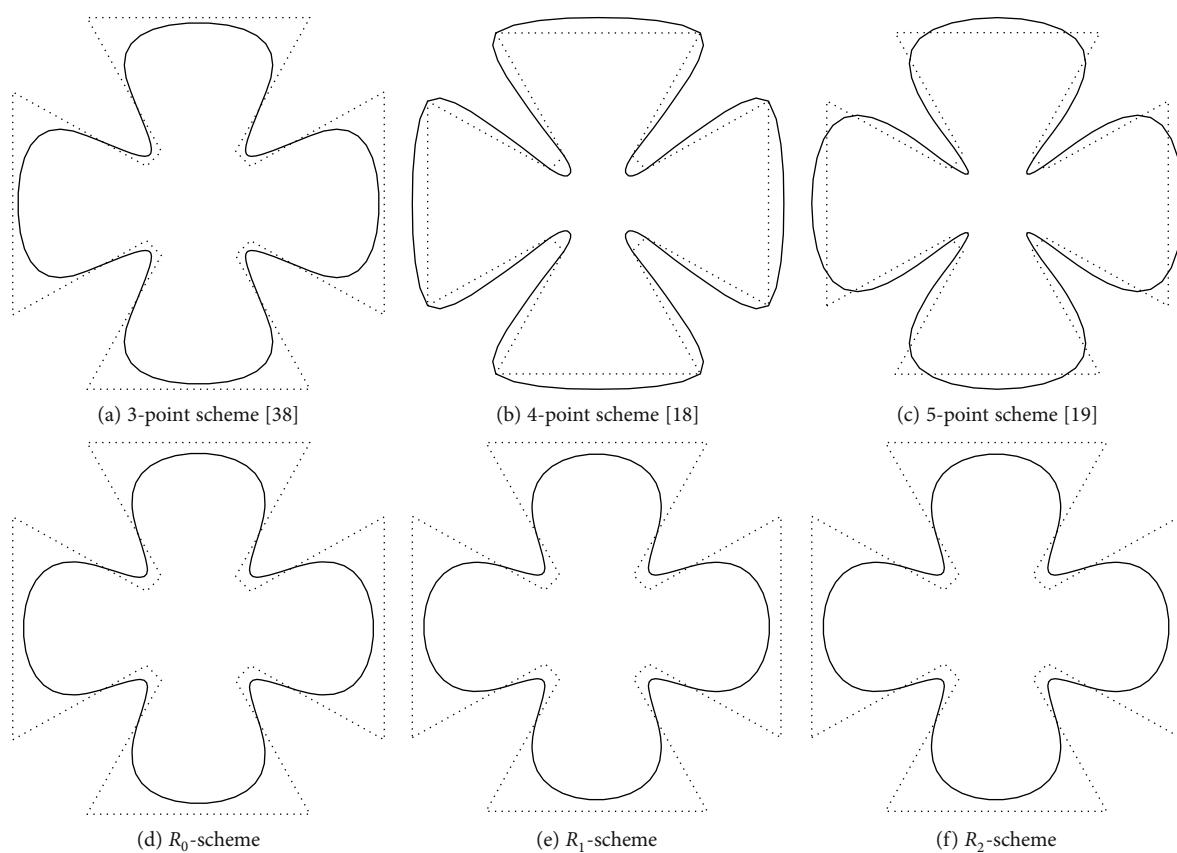


FIGURE 10: Comparison of the existing and proposed subdivision schemes at the third subdivision level.

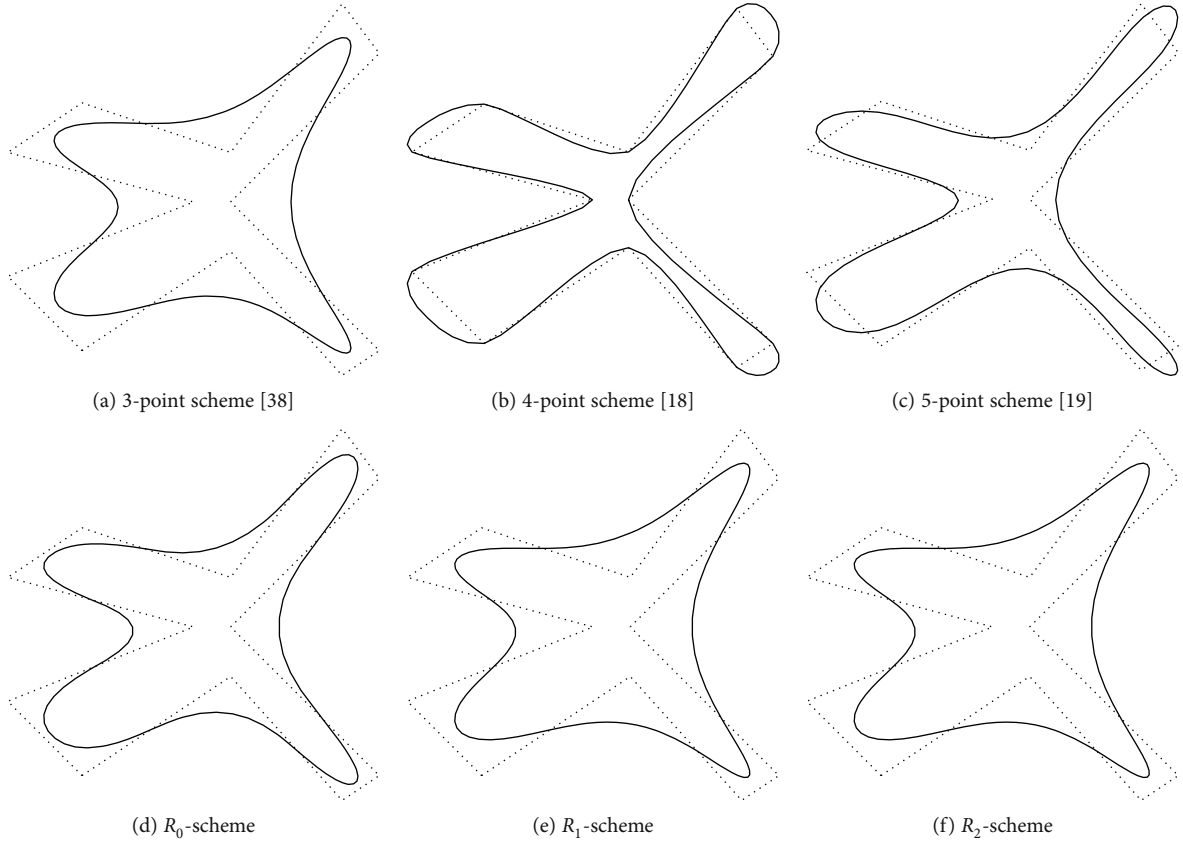


FIGURE 11: Comparison of the existing and proposed subdivision schemes at the third subdivision level.

(iv) Refinement rules of the proposed eight-point scheme R_3 are given by

$$\begin{aligned}
 g_0^{h+1} &= \frac{65066}{65536} g_0^h + \frac{1028}{65536} g_1^h - \frac{580}{65536} g_2^h + \frac{22}{65536} g_3^h, \\
 g_1^{h+1} &= \frac{33976}{65536} g_0^h + \frac{30254}{65536} g_1^h + \frac{1394}{65536} g_2^h - \frac{90}{65536} g_3^h \\
 &\quad - \frac{1}{65536} g_4^h, \\
 g_2^{h+1} &= \frac{11246}{65536} g_0^h - \frac{42508}{65536} g_1^h + \frac{12340}{65536} g_2^h - \frac{580}{65536} g_3^h \\
 &\quad + \frac{22}{65536} g_4^h, \\
 g_3^{h+1} &= \frac{1119}{65536} g_0^h + \frac{31648}{65536} g_1^h + \frac{31558}{65536} g_2^h + \frac{1303}{65536} g_3^h \\
 &\quad - \frac{91}{65536} g_4^h - \frac{1}{65536} g_5^h, \\
 g_4^{h+1} &= \frac{536}{65536} g_0^h - \frac{12340}{65536} g_1^h + \frac{41928}{65536} g_2^h + \frac{12362}{65536} g_3^h \\
 &\quad - \frac{580}{65536} g_4^h + \frac{22}{65536} g_5^h, \\
 g_5^{h+1} &= -\frac{93}{65536} g_0^h + \frac{1304}{65536} g_1^h + \frac{31557}{65536} g_2^h + \frac{31557}{65536} g_3^h \\
 &\quad + \frac{1303}{65536} g_4^h - \frac{91}{65536} g_5^h - \frac{1}{65536} g_6^h.
 \end{aligned} \tag{32}$$

Similarly, we can refine the final edges of the open polygon.

5.2. Applications and Comparison. Geometrical performance of R_0 , R_1 , R_2 , and R_3 schemes is depicted through several examples. The proposed schemes have good continuity and present smooth limit curves. Figures 3(a) and 4(a) present initial control polygons of cap and elephant, respectively, while Figures 3(b) and 4(b) are the limit curves obtained by applying three iterations of R_0 -scheme on these initial polygons. Figures 5(a) and 6(a) present initial control polygons of flower and bird, respectively, while Figures 5(b) and 6(b) are the limit curves obtained by applying three iterations of R_1 -scheme on these initial polygons. Figures 7(a) and 8(a) present initial control polygons of face of girls, while Figures 7(b) and 8(b) are the limit curves obtained by applying three iterations of R_2 -scheme on these initial polygons, respectively.

Figure 9 represents the initial polygon along with sharp features of limit curve generated by R_3 -scheme at the third subdivision level. Figures 10 and 11 present comparison of some existing subdivision schemes (3-point scheme [38], 4-point scheme [18], and 5-point scheme [19]) with the proposed subdivision schemes (R_0 , R_1 , and R_2 schemes). We have chosen two different initial polygons, and limit curves are generated after three subdivision levels. It is clear from the figures that the proposed schemes generate smooth limit curves.

6. Conclusion

Subdivision is an efficient way of constructing smooth curves or surfaces in geometric modeling and computer graphics. In

this paper, we have presented an elegant way of constructing a class of approximating binary subdivision schemes by using two well-known binary subdivision schemes. Several examples are provided to illustrate that the proposed schemes give wide choice to geometric designers for generation of smooth geometric models as per their own needs. Comparison with some existing schemes is also given. Moreover, several important properties like polynomial reproduction and generation, support of BLF, continuity, and HC of the proposed scheme are discussed. Geometrical analysis of the limit curve is also carried out.

Data Availability

No data were used in this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed equally, and they read and approved the final manuscript for publication.

Acknowledgments

Taif University Researchers Supporting Project number (TURSP-2020/20), Taif University, Taif, Saudi Arabia.

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