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Research Article

Infinite Existence Solutions of Fractional Systems with Lipschitz Nonlinearity

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The paper deals with the existence of infinitely many solutions of a class of perturbed nonlinear fractional p-Laplacian differential systems using one control parameter combined with the variational method.

1. Introduction

Fractional differential equations (FDEs) involve fractional derivatives of the form $(d^{\alpha}/dx^{\alpha})(\alpha > 0)$, where is not necessarily an integer. They are generalizations of the ordinary differential equations to a random (noninteger) order. FDEs have attracted considerable interest due to their ability to model complex phenomena in several fields of science, engineering, physics, biology, and economics (see [1–7]). In summary, many improvements have been made in the theory of partial calculus and partial differential equations and partial and ordinary differential equations (see [8–18], [2, 5]). Numerous studies have explored the existence and solutions of different nonlinear elementary and boundary value problems through the use of various nonlinear analysis tools and techniques (see, for example, [7, 19–38]). Some of these ways are the fixed point theorems, critical point theory, monotone iterative methods, coincidence degree theory, and variational methods (see [30]).

Motivated by the above, the interest of this paper is the infinite existence solutions of the following fractional system

$$\begin{cases} {}_tD_T^{\alpha}\left(\Phi_p\left({}_0D_t^{\alpha}u(t)\right)\right) = \lambda F_u(t, u(t), v(t)) + h_1(u_1), & \text{a.e.} t \in [0, T], \\ {}_tD_T^{\beta}\left(\Phi_p\left({}_0D_t^{\beta}v(t)\right)\right) = \lambda F_v(t, u(t), v(t)) + h_2(u_2), & \text{a.e.} t \in [0, T], \\ u(0) = u(T) = 0, \quad v(0) = v(T) = 0, \end{cases} \tag{1}$$

where λ is a positive real parameter, $\alpha, \beta \epsilon (0\,; 1]$, and ${}_0D_t^{\alpha}$, ${}_tD_T^{\alpha}$ and ${}_0D_t^{\beta}$, ${}_tD_T^{\beta}$ are the left and right Riemann-Liouville fractional derivatives of order α, β, respectively, $\Phi_p(s) = |s|^{p-2}s, p > 1$, $(H_0) F : [0, T] \times \mathbb{R}^2 \to \mathbb{R}$, where $F(\cdot, u, v)$ is continuous in $[0, T]$ for any $(u, v) \in \mathbb{R}^2$, $F(t, \cdot, \cdot)$ is a C^1 function in $\mathbb{R}^2$, and F_s is the partial derivative of F with respect to s, and $h_i : \mathbb{R} \longrightarrow \mathbb{R}$ are two Lipschitz continuous functions of order $(p-1)$ with Lipschitzian constants $L_i > 0$ for $1 \le i \le 2$, i.e.,

$$|h_i(x_1) - h_i(x_2)| \le L_i |x_1 - x_2|^{p-1}. \quad (2)$$

2. Preliminaries

We give some basic lemmas and notations and construct a variational framework in order to apply critical point theory to prove the existence of an infinite number of solutions to the system (1).

Let X be a real Banach space, and in addition, let Y_X denote the class of all functionals

$$\phi = X \longrightarrow \mathbb{R}, , \quad (3)$$

that possess the following property:

If $\{w_n\}$ is a sequence in X converge weakly to $w \in X$ with $\lim_{n\to\infty} \inf \phi(w_n) \le \phi(w)$; thus, $\{w_n\}$ has a subsequence converge strongly to w.

For offer, if X is uniformly convex and $S : [0,+\infty) \longrightarrow \mathbb{R}$ is a continuous strictly increasing function, then the functional $w \longrightarrow S(\|w\|)$ belongs toY_X.

Definition 1 (see Kilbas et al. [4] chapter 2, p. 87). Let u be a function defined on [a, b]. The right and left Riemann-Liouville fractional derivatives of order > 0 for a function u are defined by

$$_aD_t^\alpha u(t) := \frac{d^n}{dt^n} {}_aD_t^{\alpha-n} u(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-s)^{n-\alpha-1} u(s) ds, \quad (4)$$

$$_aD_b^\alpha u(t) := (-1)^n \frac{d^n}{dt^n} {}_aD_b^{\alpha-n} u(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^b (t-s)^{n-\alpha-1} u(s) ds, \quad (5)$$

for all $t \in [a, b]$, provided the right-hand sides are pointwise defined on $[a, b]$, where $n - 1 \le \alpha < n$ and $n \in \mathbb{N}$.

Here, $\Gamma(\alpha)$ is the standard gamma function given by

$$\Gamma(\alpha) := \int_0^{+\infty} z^{\alpha-1} e^{-z} dz. \quad (6)$$

Set $AC^n([a, b], \mathbb{R})$ the functions space $u : [a, b] \longrightarrow \mathbb{R}$ such that

$$u \in C^{n-1}([a, b], \mathbb{R}), \quad (7)$$

with

$$u^{(n-1)} \in AC^n([a, b], \mathbb{R}). \quad (8)$$

As usual, $C^{n-1}([a, b], \mathbb{R})$ denotes the mapping set having $(n-1)$ times continuously differentiable on $[a, b]$. In particularly, we have

$$AC([a, b], \mathbb{R}) := AC^1([a, b], \mathbb{R}). \quad (9)$$

Definition 2 (see [31]). Let $0 < \alpha \le 1$, for $1 < p < \infty$ the derivative fractional space

$$E_\alpha^p = \{u(t) \in L^p([0, T], \mathbb{R})_0 D_t^\alpha u(t) \in L^p([0, T], \mathbb{R}), u(0) = u(T) = 0\}. \quad (10)$$

Thus, for all $u \in E_\alpha^p$, we de ne the norm for E_α^p as follows:

$$\|u\|_\alpha = \left(\int_0^T |u(t)|^p dt + \int_0^T |{}_0D_t^\alpha u(t)|^p dt \right)^{1/p}. \quad (11)$$

Lemma 3 (see [3]). *Let $0 < \alpha \le 1$ and $1 < p < \infty$. For any $u \in E_\alpha^p$, we have*

$$\|u\|_{L^p} \le \frac{T^\alpha}{\Gamma(\alpha+1)} \|{}_0D_t^\alpha u\|_{L^p}. \quad (12)$$

Also, if $\alpha > p$ and $1/p + 1/q = 1$, then

$$\|u\|_\infty \le \frac{T^{\alpha-1/p}}{\Gamma(\alpha)((\alpha-1)q+1)^{1/q}} \|{}_0D_t^\alpha u\|_{L^p}. \quad (13)$$

Under the result of Lemma 3, we note that

$$\|u\|_{L^p} \le \frac{T^{\alpha-1/p}}{\Gamma(\alpha+1)} \|{}_0D_t^\alpha u\|_{L^p}, \quad (14)$$

for $0 < \alpha \le 1$, and

$$\|u\|_\infty \le \frac{T^{\alpha-1/p}}{\Gamma(\alpha)((\alpha-1)q+1)^{1/q}} \|{}_0D_t^\alpha u\|_{L^p}, \quad (15)$$

for $\alpha >\ p$ and $1/p + 1/q = 1$.

Under (14), we can see that (11) is equivalent to the following norm:

$$\|u\|_\alpha = \left(\int_0^T |{}_0D_t^\alpha u(t)|^p dt \right)^{1/p}, \quad \forall u \in E_\alpha^p. \quad (16)$$

For $0 < \beta \le 1$, $1 < p < \infty$. Analogous to the space E_α^p, we define the fractional derivative space E_α^p as

$$\left\{ v(t) \in L^p([0, T], \mathbb{R})_0 D_t^\beta v(t) \in L^p([0, T], \mathbb{R}), v(0) = v(T) = 0 \right\}. \quad (17)$$

Then, for any $v \in E_\beta^p$, the norm of E_β^p is defined by

$$\|v\|_\beta = \left(\int_0^T |v(t)|^p dt + \int_0^T \left|{}_0D_t^\beta v(t)\right|^p dt \right)^{1/p}, \quad \forall v \in E_\beta^p. \quad (18)$$

Similar with (14) and (15), we get

$$\|v\|_{L^p} \le \frac{T^\beta}{\Gamma(\beta+1)}\|{}_0D_t^\alpha v\|_{L^p}, \tag{19}$$

for $0<\beta\le 1$, and

$$\|v\|_\infty \le \frac{T^{\beta-1/p}}{\Gamma(\beta)((\beta-1)q+1)^{1/q}}\|{}_0D_t^\alpha v\|_{L^p}.. \tag{20}$$

Moreover, if $0<\beta\le 1$ and $1/p+1/q=1$, then, based upon (19), the weighted norm

$$\|v\|_\beta = \left(\int_0^T \left|{}_0D_t^\beta v(t)\right|^p dt\right)^{1/p}, , \tag{21}$$

is equivalent to (18), for every $v\in E_\beta^p$.

In the following discussion, for any $u\in E_\alpha^p$, $v\in E_\beta^p$ denote the space of $X=E_\alpha^p\times E_\beta^p$ with the norm

$$\|(u,v)\|_X = \left(\|u\|_\alpha^p + \|v\|_\beta^p\right)^{1/p}, \quad \forall (u,v)\in X, \tag{22}$$

where $\|u\|_\alpha$ and $\|u\|_\beta$ are defined in (16) and (21), respectively.

Clearly, X is embedded compactly on

$$C^0([0,T],\mathbb{R})\times C^0([0,T],\mathbb{R}). \tag{23}$$

Lemma 4 (see [33]). *For $0<\alpha$, $\beta\le 1$ and $1<p<\infty$. The derivative fractional space X is a reflexive separable Banach space.*

Lemma 5. *Assume that $1/p<\alpha\le 1$ and the sequence $\{u_n\}$ converge weakly to u in E_α^p, i.e., $u_n \rightharpoonup u$. Then, $\{u_n\}$ converges strongly to u in $C([0,T],\mathbb{R})$, i.e., $\|u_n-u\|_\infty \longrightarrow 0$, as $n\longrightarrow +\infty$.*

Definition 6 (see [3]). We point out to a weak solution to the system (1), for all $(u,v)\in X$ such that

$$\begin{aligned}&\int_0^T \Phi_p({}_0D_t^\alpha u(t))\,{}_0D_t^\alpha x(t)dt + \int_0^T \Phi_p({}_0D_t^\alpha v(t))\,{}_0D_t^\beta y(t)dt\\ &\quad - \int_0^T h_1(u(t))x(t)dt - \int_0^T h_2(v(t))y(t)dt\\ &\quad - \lambda\int_0^T (F_u(t,u(t),v(t))x(t) + F_v(t,u(t),v(t))y(t))dt = 0,\end{aligned} \tag{24}$$

for all $(x,y)\in X$.

We define for all $x\in\mathbb{R}$:

$$H_i(x) = \int_0^x h_i(z)dz, \Theta_i(x) = \int_0^T H_i(x(s))ds \quad \text{for all } i=1,2, \tag{25}$$

for every $t\in[0,T]$.

Lemma 7. *Let $h_1,h_2:\mathbb{R}\to\mathbb{R}$ satisfy (2) and $H_i(x),\Theta_i(x)$, $i=1,2$, defined by (25). Thus, $\Theta(u,v)\colon X\longrightarrow\mathbb{R}$ defined by*

$$\Theta(u,v) = \Theta_1(u)+\Theta_2(v) = \int_0^T H_1(u(t))dt + \int_0^T H_2(v(t))dt, \tag{26}$$

is a Gâteaux function weakly sequentially differentiable over X with

$$\Theta'(u,v)(x,y) = \int_0^T h_1(u(t))x(t)dt + \int_0^T h_2(v(t))y(t)dt, \quad \textit{for all } (x,y)\in X. \tag{27}$$

Proof. Assume that

$$\{(u_n,v_n)\}\subset X, (u_n,v_n)\rightharpoonup (u,v) \text{ in } X, \tag{28}$$

as $n\to+\infty$. According to Lemma 5 that (u_n,v_n) converges uniformly to (u,v) on $[0,T]$. Then, there exists $c_1,c_2>0$ such that $\|u_n\|_\infty\le c_1$ and $\|v_n\|_\infty\le c_2$ for any $n\in\mathbb{N}$.

Then,

$$\begin{aligned}|H_1(u_n(t)) - H_1(u(t))| &\le L_1\left|\int_{u(t)}^{u_n(t)} |s|^{p-1}ds\right|\\ &\le \frac{L_1}{p}\left(|u_n(t)|^p + |u(t)|^p\right) \le \frac{L_1}{p}\left(c_1^p + \|u(t)\|_\infty^p\right),\end{aligned} \tag{29}$$

$$\begin{aligned}|H_2(v_n(t)) - H_2(v(t))| &\le L_2\left|\int_{v(t)}^{v_n(t)} |s|^{p-1}ds\right|\\ &\le \frac{L_2}{p}\left(|v_n(t)|^p + |v(t)|^p\right) \le \frac{L_2}{p}\left(c_2^p + \|v(t)\|_\infty^p\right),\end{aligned} \tag{30}$$

for any $n\in\mathbb{N}$ and $t\in[0,T]$. Furthermore, $H_1(u_n(t))\longrightarrow H_1(u(t))$ and $H_2(v_n(t))\longrightarrow H_2(v(t))$ at every $t\in[0,T]$, and by the Lebesgue Convergence Theorem

$$\begin{aligned}\Theta(u_n,v_n) = \int_0^T H_1(u_n(t))dt + \int_0^T H_2(v_n(t))dt \longrightarrow \int_0^T H_1(u(t))dt\\ + \int_0^T H_2(v(t))dt = \Theta(u,v).\end{aligned} \tag{31}$$

Now we prove the Gâteaux differentiability of Θ. Assume that $u, x \in E_\alpha^p$ and $s \neq 06$; thus,

$$\left|\frac{\Theta_1(u+sx)-\Theta_1(u)}{s}-\int_0^T h_1(u(t))x(t)dt\right| \leq \int_0^T\left|\frac{H_1(u+sx)-H_1(u)}{s}-h_1(u(t))x(t)\right|dt = \int_0^T |h_1(u(t))+s\zeta(t)x(t)-h_1(u(t))||x(t)|dt \leq L_1\|x\|_\infty^p|s|, \tag{32}$$

where

$$0<\zeta(t)<1, \qquad t\in[0,T]. \tag{33}$$

Thus

$$\Theta_1 : E_\alpha^p \longrightarrow \mathbb{R}, \tag{34}$$

is a Gâteaux differentiable for all $u \in E_\alpha^p$.

Likewise, we have

$$\Theta_2 : E_\beta^p \longrightarrow \mathbb{R}, \tag{35}$$

which is a Gâteaux differentiable for all $v \in E_\alpha^p$.

Therefore,

$$\Theta : X \longrightarrow \mathbb{R}, \tag{36}$$

is a Gâteaux differentiable for all $(u,v)\in X$ with its derivative

$$\Theta'(u,v)(x,y) == \int_0^T h_1(u(t))x(t)dt + \int_0^T h_2(v(t))y(t)dt, (x,y)\in X. \tag{37}$$

For any three elements (u_1,v_1), (u_2,v_2), and (x,y) of X, it is easy to see that

$$\left(\Theta'(u_1,v_1)-\Theta'(u_2,v_2)\right)(x,y) = \int_0^T (h_1(u_1)-h_1(u_2)x(t))dt + \int_0^T (h_2(v_1)-h_2(v_2)y(t))dt \leq L_1\int_0^T |u_1-u_2|^{p-1}|x(t)|dt + L_2\int_0^T |v_1-v_2|^{p-1}|y(t)|dt \leq \frac{L_1T^{\alpha-1/p}}{\Gamma(\alpha)((\alpha-1)q+1)^{1/p}} \cdot \|u_1-u_2\|_\infty^{p-1}\|x\|_\alpha + \frac{L_2T^{\beta-1/p}}{\Gamma(\beta)((\beta-1)q+1)^{1/p}} \cdot \|v_1-v_2\|_\infty^{p-1}\|y\|_\alpha, \tag{38}$$

which implies

$$\left\|\Theta'(u_1,v_1)-\Theta'(u_2,v_2)\right\|_X \leq T^*\left(\|u_1-u_2\|_\infty^{p-1}+\|v_1-v_2\|_\infty^{p-1}\right), \tag{39}$$

where

$$T^* := \max\left\{\frac{L_1T^{\alpha-1/p}}{\Gamma(\alpha)((\alpha-1)q+1)^{1/q}}, \frac{L_2T^{\beta-1/p}}{\Gamma(\beta)((\beta-1)q+1)^{1/q}}\right\}. \tag{40}$$

Hence, $\Theta' : X \longrightarrow X^*$ is a compact operator.

Similarly to the proof of Theorem 5.1 of [4], we have

Lemma 8 (see [36]). *Let $1/p<\alpha$, $\beta\leq 1$, and $(u,v)\in X$. If (u,v) is a nontrivial weak solution of problem (1), then (u,v) is also a nontrivial solution of problem (1).*

Our analysis is mainly based on the following critical points theorem of Bonanno and Molica Bisci [36], which is a more precise result of Ricceri ([37], Theorem 2.5).

Lemma 9 (see [[36], Theorem 2.1]). *Let X be a reflexive real Banach space. Let $\phi,\Psi : X\longrightarrow\mathbb{R}$ be two Gâteaux differentiable functionals such that ϕ is sequentially weakly lower semicontinuous, strongly continuous, and coercive and Ψ is sequentially weakly upper semicontinuous. For every $r>\inf_X\phi$, put*

$$\varphi(r) = \inf_{u\in\phi^{-1}(]-\infty,r])}\frac{\sup_{v\in\phi^{-1}(]-\infty,r])}\Psi(v)-\Psi(u)}{r-\phi(u)}, \tag{41}$$

$$\gamma =: \lim_{r\to+\infty}\inf\varphi(r), \delta =: \lim_{r\to(\inf_X\phi)^+}\inf\varphi(r). \tag{42}$$

Then,

(1) If $\gamma<+\infty$ and $\lambda\in]0,1/\gamma[$, the following alternative holds: either the functional $\phi-\lambda\Psi$ has a global minimum or there exists a sequence $\{u_n\}$ of local minima $\phi-\lambda\Psi$ such that $\lim_{n\to+\infty}\phi(u_n)=+\infty$

(2) If $\gamma<+\infty$ and $\lambda\in]0,1/\delta[$, the following alternative holds: either there exists a global minimum of ϕ or the following alternative holds: either there exists a global minimum of $\phi-\lambda\Psi$ or there exists a sequence $\{u_n\}$ of pairwise distinct local minima of $\phi-\lambda\Psi$, with $\lim_{n\to+1}\phi(u_n)=\inf_X\phi$, which weakly converges to a global minimum of ϕ

3. Main Results

Here, we prove our main results.

Setting

$$k := \min\left\{1-\frac{L_1T^{p\alpha}}{(\Gamma(\alpha+1))^p}, 1-\frac{L_2T^{p\beta}}{(\Gamma(\beta+1))^p}\right\}, \tag{43}$$

$$\rho := \max\left\{1+\frac{L_1T^{p\alpha}}{(\Gamma(\alpha+1))^p}, 1+\frac{L_2T^{p\beta}}{(\Gamma(\beta+1))^p}\right\}, \tag{44}$$

$$M=\max\left\{\frac{T^{p\alpha-1}}{(\Gamma(\alpha))^p((\alpha-1)q+1)^{p/q}},\frac{T^{p\beta-1}}{(\Gamma(\beta))^p((\beta-1)q+1)^{p/q}}\right\}. \tag{45}$$

For a given constant $\theta\in(1/p,0)$, set

$$P(\alpha,\theta)=\frac{1}{p(\theta T)^p}\left\{\int_0^{\theta T}t^{p(1-\alpha)}dt+\int_{\theta T}^{(1-\theta)T}\left(t^{1-\alpha}-(t-\theta T)^{1-\alpha}\right)^p dt+\int_{(1-\theta)T}^{T}\left[\left(t^{1-\alpha}-(t-\theta T)^{1-\alpha}\right)-(t-((1-\theta)T))^{1-\alpha}\right]^p\right\}, \tag{46}$$

$$Q(\beta,\theta)=\frac{1}{p(\theta T)^p}\left\{\int_0^{\theta T}t^{p(1-\beta)}dt+\int_{\theta T}^{(1-\theta)T}\left(t^{1-\beta}-(t-\theta T)^{1-\beta}\right)^p dt+\int_{(1-\theta)T}^{T}\left[\left(t^{1-\beta}-(t-\theta T)^{1-\beta}\right)-(t-((1-\theta)T))^{1-\beta}\right]^p\right\}. \tag{47}$$

For any $d>0$, we denote by $\Omega(d)$ the set

$$\left\{(x,y)\in\mathbb{R}^2:\frac{1}{p}|x|^p+\frac{1}{p}|y|^p\le d\right\}. \tag{48}$$

Theorem 10. *Suppose that $k>0$ and (H0) hold. In addition,*
(H1) $1/p\le\alpha,\beta<1$
(H2) $F(t,x,y)\ge 0$ for any $(t,x,y)\,[0,T]\in[0,+\infty)[0,+\infty)$
(H3) there exists $\theta\in(0,1/p)$ where if we set

$$A_\infty=\lim_{\xi\to+\infty}\inf\frac{\int_0^T\sup_{|x|+|y|\le\xi}F(t,x,y)dt}{\xi^p}, \tag{49}$$

$$B_\infty=\lim_{\xi\to+\infty}\sup\frac{\int_{\theta T}^{(1-\theta)T}F(t,\Gamma(2-\alpha)\xi,\Gamma(2-\beta)\xi)dt}{\xi^p}, \tag{50}$$

one has

$$A_\infty<\frac{k}{2pM\rho\Delta}B_\infty, \tag{51}$$

where

$$\Delta=\max\{P(\alpha,\theta),Q(\beta,\theta)\}, \tag{52}$$

and M is given in (45).
Then, for every

$$\lambda\in\left]\frac{\rho\Delta}{B_\infty},\frac{k}{2pMA_\infty}\right[. \tag{53}$$

(1) has an unbounded sequence in X (weak solutions).

Proof. Our goal is to apply a portion (1) of Lemma 9 to problem (1). First, by taking

$$X=E_\alpha^p\times E_\beta^p, \tag{54}$$

endowed with $\|(u,v)\|_X$ similar to what is considered in (22). We define the following functional

$$I_\lambda(u,v)=\phi(u,v)-\lambda\Psi(u,v), \tag{55}$$

for all $(u,v)\in X$, where

$$\phi(u,v)=\frac{1}{p}\|u\|_\alpha^p+\frac{1}{p}\|v\|_\beta^p-\Theta(u,v), \tag{56}$$

$$\Psi(u,v)=\int_0^T F(t,u(t),v(t))dt. \tag{57}$$

Since X is embedded compact in

$$C^0([0,T],\mathbb{R})\times C^0([0,T],\mathbb{R}), \tag{58}$$

it is well known that is a well-defined Gâteaux differentiable functional whose Gâteaux derivative at the point $(u,v)\in X$ is the functional $\Psi'(u,v)\in X*$, given by

$$\Psi'(u,v)(x,y)=\int_0^T(F_u(t,u(t),v(t)))x(t)dt+\int_0^T(F_v(t,u(t),v(t)))y(t)dt, \tag{59}$$

for every $(x,y)\in X$.

We claim that the functional Ψ is a sequentially weakly upper semicontinuous functional on X. Indeed, for fixed $(u,v)\in X$, suppose that $\{(u_n,v_n)\}\subset X$, $(u_n,v_n)\rightharpoonup(u,v)$ in X as $n\longrightarrow+\infty$. Then, (u_n,v_n) converges uniformly to (u,v) on $[0,T]$. Hence,

$$\lim_{n\to+\infty}\sup\Psi(u_n,v_n)\le\int_0^T\lim_{n\to+\infty}\sup F(t,u_n(t),v_n(t))dt=\int_0^T F(t,u,(t),v(t))dt=\Psi(u,v), \tag{60}$$

which implies that it is sequentially weakly upper semicontinuous. Hence, the claim is true.

Concerning the functional ϕ, we can show that what is defined by (56) is a sequentially weakly lower semicontinuous, strongly continuous, and coercive functional on X. In fact since (2) holds for every $x_1,x_2\in\mathrm{R}$ and $h_1(0)=h_2(0)=0$, one has $|h_i(x)|\le L_i|x|^{p-1}$, $i=1,2$, for all $x\in\mathbb{R}$. It follows from

(14), (20), and Lemma 5 that

$$\begin{aligned}\phi(u,v) &\geq \frac{1}{p}\|u\|_\alpha^p + \frac{1}{p}\|v\|_\beta^p - \left|\int_0^T H_1(u(t))dt\right| - \left|\int_0^T H_2(v(t))dt\right| \\ &\geq \frac{1}{p}\|u\|_\alpha^p + \frac{1}{p}\|v\|_\beta^p - \frac{L_1}{p}\int_0^T |u(t)|^p dt - \frac{L_2}{p}\int_0^T |v(t)|^p dt \\ &\geq \left(\frac{1}{p} - \frac{L_1 T^{p\alpha}}{p(\Gamma(\alpha+1))^p}\right)\|u\|_\alpha^p + \left(\frac{1}{p} - \frac{L_2 T^{p\beta}}{p(\Gamma(\beta+1))^p}\right)\|v\|_\beta^p,\end{aligned} \tag{61}$$

for all $(u,v) \in X$ and similarly

$$\begin{aligned}\phi(u,v) &\leq \frac{1}{p}\|u\|_\alpha^p + \frac{1}{p}\|v\|_\beta^p + \left|\int_0^T H_1(u(t))dt\right| + \left|\int_0^T H_2(v(t))dt\right| \\ &\leq \left(\frac{1}{p} - \frac{L_1 T^{p\alpha}}{p(\Gamma(\alpha+1))^p}\right)\|u\|_\alpha^p + \left(\frac{1}{p} - \frac{L_2 T^{p\beta}}{p(\Gamma(\beta+1))^p}\right)\|v\|_\beta^p \\ &\leq \frac{\rho}{p}\left(\|u\|_\alpha^p + \|v\|_\beta^p\right),\end{aligned} \tag{62}$$

for all $(u,v) \in X$. So ϕ is coercive.

Moreover, $\phi + \Theta$ is a continuous functional on X, and Θ, from Lemma 5, is Gâteaux differentiable sequentially weakly continuous and therefore continuous on X, then ϕ is a continuous functional on X. It is not difficult to verify that the functional is a Gâteaux differentiable functional with the differential

$$\begin{aligned}\phi'(u,v)(x,y) &= \int_0^T \Phi_p({}_0D_t^\alpha u(t))\,{}_0D_t^\alpha x(t)dt \\ &+ \int_0^T \Phi_p({}_0D_t^\alpha v(t))\,{}_0D_t^\beta y(t)dt \\ &- \int_0^T h_1(u(t))x(t)dt - \int_0^T h_2(v(t))y(t)dt.\end{aligned} \tag{63}$$

Furthermore, ϕ is also sequentially weakly lower semicontinuous on X since Θ is sequentially weakly lower semicontinuous, and if $(u_n, v_n) \rightharpoonup (u,v)$ in X then

$$\begin{aligned}\lim_{n\to+\infty}\inf \phi(u_n,v_n) &= \lim_{n\to+\infty}\inf\left(\frac{1}{p}\|u\|_\alpha^p + \frac{1}{p}\|v\|_\beta^p\right) \\ &- \lim_{n\to+\infty}\Theta(u_n,v_n) \geq \frac{1}{p}\|u\|_\alpha^p + \frac{1}{p}\|v\|_\beta^p - \Theta(u,v) = \phi(u,v).\end{aligned} \tag{64}$$

It is easy to show that the critical points of the functional I_λ and the weak solutions of the problem (1) are the same, and by Lemma 9, we prove our result. According to

$$\|u\|_\infty = \max_{[0,T]}|u(t)| \text{ and } \|u\|_\infty = \max_{[0,T]}|u(t)|, \tag{65}$$

taking (13) and (20) into account, one has

$$\max_{rt\in[0,T]}|u(t)|^p \leq M\|u\|_\alpha^p \text{and} \max_{t\in[0,T]}|v(t)|^p \leq M\|v\|_\beta^p \tag{66}$$

for every $(u,v) \in X$.

Hence,

$$\max_{t\in[0,T]}\left(|u(t)|^p + |v(t)|^p\right) \leq M\left(\|u\|_\alpha^p + \|v\|_\beta^p\right). \tag{67}$$

So, for every $r > 0$, from the definition of and by using (61), one has

$$\begin{aligned}\phi^{-1}(]-\infty,r]) &:= \{(u,v) \in X : \phi(u,v) \leq r\} \\ &\subseteq \left\{(u,v) \in X : \frac{1}{p}\|u\|_\alpha^p + \frac{1}{p}\|v\|_\beta^p \leq \frac{r}{k}\right\} \\ &\subseteq \left\{(u,v) \in X : \frac{(\Gamma(\alpha))^p((\alpha-1)q+1)^{p/q}}{T^{p\alpha-1}}\|u\|_\infty^p\right. \\ &\quad \left. + \frac{(\Gamma(\beta))^p((\beta-1)q+1)^{p/q}}{T^{p\beta-1}}\|u\|_\infty^p \leq \frac{r}{k}\right\} \\ &\subseteq \left\{(u,v) \in X : \frac{1}{p}|u|^p + \frac{1}{p}|v|^p \leq \frac{Mr}{k}, \text{ for all } t \in [0,T]\right\}.\end{aligned} \tag{68}$$

Set

$$\varphi(r) = \inf_{(u,v)\in\phi^{-1}(]-\infty,r])} \frac{\sup_{(x,y)\in\phi^{-1}(]-\infty,r])}\Psi(x,y) - \Psi(u,v)}{r - \phi(u,v)}. \tag{69}$$

Note that $\phi(0,0) = 0$, and from the condition $(H1)$, $\Psi(0,0) \geq 0$. Hence, for every $r > 0$,

$$\begin{aligned}\varphi(r) &= \inf_{(u,v)\in\phi^{-1}(]-\infty,r])} \frac{\left(\sup_{(x,y)\in\phi^{-1}(]-\infty,r])}\Psi(x,y)\right) - \Psi(u,v)}{r - \phi(u,v)} \\ &\leq \frac{\sup_{(x,y)\in\phi^{-1}(]-\infty,r])}\Psi(x,y)}{r},\end{aligned} \tag{70}$$

and it follows from (68) that

$$\varphi(r) \leq \frac{1}{r}\sup_{\Omega(Mr/k)}\int_0^T F(t,u,v)dt, \tag{71}$$

where

$$\Omega\left(\frac{Mr}{k}\right) = \left\{(u,v) \in X : \frac{1}{p}|u(t)|^p + \frac{1}{p}|v(t)|^p \leq \frac{Mr}{k}, \forall t \in [0,T]\right\}. \tag{72}$$

Let $\{\xi_n\}$ be a sequence of positive numbers such that $\xi_n \longrightarrow +\infty$ and

$$\lim_{\xi\to+\infty} \inf \frac{\int_0^T \sup_{|x|+|y|\leq\xi} F(t,x,y)\,dt}{\xi^p} = A_\infty < +\infty. \quad (73)$$

Put $r_n = (k/p2^pM)\xi_n^p$ for all $n \in \mathbb{N}$. Let $(u,v) \in \phi^1(]1-\infty, r_n])$, by (68) one has

$$\frac{1}{p}|u(t)|^p + \frac{1}{p}|v(t)|^p \leq \frac{M}{k} r_n, \quad \forall t \in [0,T], \quad (74)$$

which implies

$$|u(t)| \leq \sqrt[p]{\frac{pM}{k} r_n} \text{ and } |v(t)| \leq \sqrt[p]{\frac{pM}{k} r_n}. \quad (75)$$

Hence, for n large enough $(r_n > 1)$

$$|u(t)| + |v(t)| \leq 2\sqrt[p]{\frac{pM}{k} r_n} = \xi_n. \quad (76)$$

Thus, for all $n \in \mathbb{N}$,

$$\begin{aligned}\varphi(r_n) &= \frac{p2^pM}{k\xi_n^p} \sup_{\{(u,v)\in X: |u(t)|+|v(t)|<\xi_n, \forall t\in[0,T]\}} \int_0^T F(t,u(t),v(t))\,dt \\ &\leq \frac{p2^pM}{k} \cdot \frac{\int_0^T \sup_{|x|+|y|<\xi_n} F(t,x,y)\,dt}{\xi_n^p}.\end{aligned} \quad (77)$$

Let

$$\gamma := \lim_{r\to+\infty} \inf \varphi(r). \quad (78)$$

Then,

$$\begin{aligned}\gamma &\leq \lim_{n\to+\infty} \inf \varphi(r_n) \leq \frac{p2^pM}{k} \cdot \lim_{n\to+\infty} \frac{\int_0^T \sup_{|x|+|y|<\xi_n} F(t,x,y)\,dt}{\xi_n^p} \\ &= \frac{p2^pM}{k} A_\infty < +\infty.\end{aligned} \quad (79)$$

Hence, $\Lambda \sqsubseteq]0, 1/\gamma[$.

For $\lambda \in \Lambda$, we shall show that the functional I_λ is unbounded from below.

Indeed, since $B_\infty/\rho^\Delta > 1/\lambda$, we can choose a sequence $\{\eta_n\}$ of positive numbers and $\varepsilon > 0$ such that $\eta_n \longrightarrow +\infty$ and

$$\frac{1}{\lambda} < \varepsilon < \frac{1}{\rho^\Delta} \cdot \frac{\int_{\theta T}^{(1-\theta)T} F(t, \Gamma(2-\alpha)\eta_n, \Gamma(2-\beta)\eta_n)\,dt}{\eta_n^p}, \quad (80)$$

for n large enough.

For all $n \in \mathbb{N}$, and $(0, 1/p)$ define $\omega_n(t) = (\omega_{1,n}(t), \omega_{2,n}(t))$ by setting

$$\omega_{1,n}(t) = \begin{cases} \dfrac{\Gamma(2-\alpha)}{\theta T} t, & t \in [0, \theta T[, \\ \Gamma(2-\alpha)\eta_n, & t \in [\theta T, (1-\theta)T], \\ \dfrac{\Gamma(2-\alpha)\eta_n}{\theta T}(T-t), & t \in](1-\theta)T, T], \end{cases} \quad (81)$$

$$\omega_{2,n}(t) = \begin{cases} \dfrac{\Gamma(2-\beta)\eta_n}{\theta T} t, & t \in [0, \theta T[, \\ \Gamma(2-\alpha)\beta\eta_n, & t \in [\theta T, (1-\theta)T], \\ \dfrac{\Gamma(2-\beta)\eta_n}{\theta T}(T-t), & t \in](1-\theta)T, T]. \end{cases} \quad (82)$$

Clearly $\omega_{i,n}(0) = \omega_{i,n}(T) = 0$ and $\omega_{i,n} \in L^p([0,T])$ for $i = 1, 2$. A direct calculation shows that

$${}_0D_t^\alpha \omega_{1,n}(t) = \begin{cases} \dfrac{\eta_n}{\theta T} t^{1-\alpha}, & t \in [0, \theta T[, \\ \dfrac{\eta_n}{\theta T}\left(t^{1-\alpha} - (t-\theta T)^{1-\alpha}\right), & t \in [\theta T, (1-\theta)T], \\ \dfrac{\eta_n}{\theta T}\left(t^{1-\alpha} - (t-\theta T)^{1-\alpha} - (t-(1-\theta)T)^{1-\alpha}\right), & t \in](1-\theta)T, T], \end{cases} \quad (83)$$

$${}_0D_t^\beta \omega_{2,n}(t) = \begin{cases} \dfrac{\eta_n}{\theta T} t^{1-\beta}, & t \in [0, \theta T[, \\ \dfrac{\eta_n}{\theta T}\left(t^{1-\beta} - (t-\theta T)^{1-\beta}\right), & t \in [\theta T, (1-\theta)T], \\ \dfrac{\eta_n}{\theta T}\left(t^{1-\beta} - (t-\theta T)^{1-\beta} - (t-(1-\theta)T)^{1-\beta}\right), & t \in](1-\theta)T, T]. \end{cases} \quad (84)$$

Furthermore,

$$\begin{aligned}\int_0^T \left|{}_0D_t^\alpha \omega_{1,n}(t)\right|^p dt &= \int_0^{\theta T} + \int_{\theta T}^{(1-\theta)T} + \int_{(1-\theta)T}^T \left|{}_0D_t^\alpha \omega_{1,n}(t)\right|^p dt \\ &= \frac{\eta_n^P}{(\theta T)^p}\left\{\int_0^{\theta T} t^{p(1-\alpha)} dt + \int_{\theta T}^{(1-\theta)T} \left(t^{1-\alpha} - (t-\theta T)^{1-\alpha}\right)^p dt \right. \\ &\quad \left. + \int_{(1-\theta)T}^T \left[\left(t^{1-\alpha} - (t-\theta T)^{1-\alpha}\right) - \left(t - ((1-\theta)T)^{1-\alpha}\right)\right]^p\right\} \\ &= pP(\alpha,\theta)\eta_n^p,\end{aligned} \quad (85)$$

$$\begin{aligned}\int_0^T \left|{}_0D_t^\beta \omega_{2,n}(t)\right|^p dt &= \int_0^{\theta T} + \int_{\theta T}^{(1-\theta)T} + \int_{(1-\theta)T}^T \left|{}_0D_t^\beta \omega_{1,n}(t)\right|^p dt \\ &= \frac{\eta_n^P}{(\theta T)^p}\left\{\int_0^{\theta T} t^{p(1-\beta)} dt + \int_{\theta T}^{(1-\theta)T} \left(t^{1-\beta} - (t-\theta T)^{1-\beta}\right)^p dt \right. \\ &\quad \left. + \int_{(1-\theta)T}^T \left[\left(t^{1-\beta} - (t-\theta T)^{1-\beta}\right) - \left(t - ((1-\theta)T)^{1-\beta}\right)\right]^p\right\} \\ &= pQ(\beta,\theta)\eta_n^p.\end{aligned} \quad (86)$$

Thus, $\omega_n \in X$, and

$$\|\omega_{1,n}(t)\|^p = \int_0^T |{}_0D_t^\alpha \omega_{1,n}(t)|^p dt = pP(\alpha,\theta)\eta_n^p,$$
$$\|\omega_{2,n}(t)\|^p = \int_0^T |{}_0D_t^\alpha \omega_{1,n}(t)|^p dt = pQ(\beta,\theta)\eta_n^p. \quad (87)$$

This and (61) imply that

$$\begin{aligned}\Phi(\omega_{1,n},\omega_{2,n}) &= \frac{1}{p}\|\omega_{1,n}(t)\|^p + \frac{1}{p}\|\omega_{2,n}(t)\|^p - \Theta(\omega_{1,n},\omega_{2,n}) \\ &\le \frac{\rho}{p}\left(\|\omega_{1,n}(t)\|^p + \|\omega_{2,n}(t)\|^p\right) \\ &= \rho(P(\alpha,\theta)+Q(\beta,\theta))\eta_n^p \le \rho\Delta\eta_n^p.\end{aligned} \quad (88)$$

From $(H2)$, we have

$$\begin{aligned}\Psi(\omega_{1,n},\omega_{2,n}) &= \int_0^{\theta T} + \int_{\theta T}^{(1-\theta)^T} + \int_{(1-\theta)^T}^T F(t,\omega_{1,n},\omega_{2,n})dt \\ &\ge \int_{\theta T}^{(1-\theta)^T} F(t,\omega_{1,n},\omega_{2,n})dt \\ &= \int_{\theta T}^{(1-\theta)^T} F(t,\Gamma(2-\alpha)\eta_n,\Gamma(2-\beta)\eta_n)dt.\end{aligned} \quad (89)$$

According to (80), (88), and (89), we have

$$\begin{aligned}I_\lambda(\omega_{1,n},\omega_{2,n}) &= \phi(\omega_{1,n},\omega_{2,n}) - \lambda\Psi(\omega_{1,n},\omega_{2,n}) \\ &\le \rho(P(\alpha,\theta)+Q(\beta,\theta))\eta_n^p \\ &\quad - \lambda\int_{\theta T}^{(1-\theta)T} F(t,\Gamma(2-\alpha)\eta_n,\Gamma(2-\beta)\eta_n)dt \\ &\le \rho\Delta(1-\lambda\varepsilon)\eta_n^p,\end{aligned} \quad (90)$$

for n large enough. Taking into account the choice of ε, the above inequality shows that

$$\lim_{n\to+\infty} I_\lambda(\omega_{1,n},\omega_{1,n}) = -\infty, \quad (91)$$

which implies that the functional I_λ is unbounded from below and the claim follows.

By using the case (1) of Lemma 9, the functional I_λ has a sequence $\{(u_n,v_n)\}$ of critical points such that

$$\Phi(u_n,v_n) \longrightarrow +\infty. \quad (92)$$

From (22) and (61), we get

$$\|(u_n,v_n)\|_X \ge \sqrt[p]{\frac{p\Phi(u_n,v_n)}{\rho}}, \quad (93)$$

which implies $\|(u_n,v_n)\|_X \longrightarrow +\infty$ and the proof of Theorem 10 is complete.

Theorem 11. *Assume that $k > 0$ and (H0) and (H2) hold. Furthermore, (H4) $F(t,0,0)=0$ for all $t \in [0,T]$.*

(H5) There exists $\theta \in (0,1/p)$ such that, if we put

$$A_0 = \lim_{\xi\to 0^+}\inf \frac{\int_0^T \sup_{|x|+|y|\le\xi} F(t,x,y)dt}{\xi^p}, \quad (94)$$

$$B_0 = \lim_{\xi\to 0^+}\sup \frac{\int_{\theta T}^{(1-\theta)T} F(t,\Gamma(2-\alpha)\xi,\Gamma(2-\beta)\xi)dt}{\xi^p}, \quad (95)$$

one has

$$A_0 < \frac{k}{2pM\rho\Delta}B_0, \quad (96)$$

where $\Delta = \max\{P(\alpha,\theta), Q(\beta,\theta)\}$ and M is given in (45).

Then, for every

$$\lambda \in \Lambda' := \left]\frac{\rho\Delta}{B_0}, \frac{k}{2pMA_0}\right[, \quad (97)$$

(1) has a sequence $\{(u_n,v_n)\}$ of weak solutions such that $(u_n,v_n) \rightharpoonup (0,0)$.

Proof. Our goal is to apply part (2) of Lemma 9 to ϕ and Ψ defined in (48) and (51), respectively.

As it has been pointed out before, the functionals ϕ and Ψ satisfy the assumption regularity required in Lemma 9.

Since $F(t,0,0)=0$ for all $t \in [0,T]$, then

$$\min_{(u,v)\in X}\phi(u,v) = \phi(0,0) = 0. \quad (98)$$

Let $\{\xi_n\}$ be a sequence of positive numbers such that $\xi_n \longrightarrow 0$ and

$$\lim_{n\to+\infty}\frac{\int_0^T \sup_{|x|+|y|\le\xi_n} F(t,x,y)dt}{\xi_n^p} = A_0 < +\infty. \quad (99)$$

Setting $r_n = (k/p2^pM)\xi_n^p$ for all $n \in \mathbb{N}$, and working as in the proof of Theorem 10, we can show that

$$\begin{aligned}\delta &= \lim_{r\to(\inf_X\Phi)}\inf\varphi(r) \le \frac{p2^pM}{k}\cdot\lim_{n\to+\infty}\frac{\int_0^T \sup_{|x|+|y|\le\xi_n} F(t,x,y)dt}{\xi_n^p} \\ &= \frac{p2^pM}{k}A_0,\end{aligned} \quad (100)$$

and so $\Lambda' \subset (0,1/\delta)$.

Now fix λ as in the conclusion, then

$$\frac{1}{\lambda} < \frac{1}{\rho\Delta} \lim_{\xi \to 0^+} \sup \frac{\int_{\theta T}^{(1-\theta)T} F(t, \Gamma(2-\alpha)\xi, \Gamma(2-\beta)\xi) dt}{\xi^p}, \quad (101)$$

and there exist a sequence $\{\tau_n\}$ of positive numbers and a constant ε_1 such that $\tau_n \leq 1/n$ and

$$\begin{aligned} &\lim_{n\to+\infty} \frac{\int_{\theta T}^{(1-\theta)T} F(t, \Gamma(2-\alpha)\tau_n, \Gamma(2-\beta)\tau_n) dt}{\tau_n^p} \\ &= \lim_{\xi \to 0^+} \sup \frac{\int_{\theta T}^{(1-\theta)T} F(t, \Gamma(2-\alpha)\xi, \Gamma(2-\beta)\xi) dt}{\xi^p}, \end{aligned} \quad (102)$$

and in addition

$$\frac{1}{\lambda} < \varepsilon_1 < \frac{1}{\rho\Delta} \lim_{n\to+\infty} \frac{\int_{\theta T}^{(1-\theta)T} F(t, \Gamma(2-\alpha)\tau_n, \Gamma(2-\beta)\tau_n) dt}{\tau_n^p}. \quad (103)$$

For all $n \in \mathbb{N}$, and $\theta \in (0, 1/p)$ define $\omega_n(t) = (\omega_{1,n}(t), \omega_{2,n}(t))$ by setting

$$\omega_{1,n}(t) = \begin{cases} \dfrac{\Gamma(2-\alpha)\tau_n}{\theta T} t, & t \in [0, \theta T[, \\ \Gamma(2-\alpha)\tau_n, & t \in [\theta T, (1-\theta)T], \\ \dfrac{\Gamma(2-\alpha)\tau_n}{\theta T}(T-t), & t \in](1-\theta)T, \mathrm{T}], \end{cases} \quad (104)$$

$$\omega_{2,n}(t) = \begin{cases} \dfrac{\Gamma(2-\beta)\tau_n}{\theta T} t, & t \in [0, \theta T[, \\ \Gamma(2-\beta)\tau_n, & t \in [\theta T, (1-\theta)T], \\ \dfrac{\Gamma(2-\beta)\tau_n}{\theta T}(T-t), & t \in](1-\theta)T, \mathrm{T}]. \end{cases} \quad (105)$$

Clearly $\omega_{i,n}(0) = \omega_{i,n}(T) = 0$ for $i = 1, 2,$ and $\{\omega_n\}$ converges strongly to $(0, 0)$ in X.

By the same argument as in Theorem 10, we have

$$\begin{aligned} I_\lambda(\omega_{1,n}, \omega_{2,n}) - \lambda\Psi(\omega_{1,n}, \omega_{2,n}) &\leq \rho(P(\alpha,\theta) + Q(\beta,\theta))\tau_n^p \\ &- \lambda \int_{\theta T}^{(1-\theta)T} F\left(t, \Gamma(2-\alpha)\tau_n, \Gamma(2-\beta)_{\tau_n}\right) dt \\ &\leq \rho\Delta(1 - \lambda\varepsilon_1)\tau_n^p < 0 = I_\lambda(0, 0), \end{aligned} \quad (106)$$

for n large enough. This together with the fact that $\|\omega_n\|_X = \|\omega_{1,n}, \omega_{2,n}\|_X \longrightarrow 0$ shows that I_λ has no local minimum at zero, and the claim follows.

The alternative of Lemma 9 case (2) ensures the existence of sequence $\{(u_n, v_n)\}$ of pairwise distinct local minima of I_λ which weakly converges to $(0, 0)$. This completes the proof of Theorem 11.

Finally, we present an example to illustrate our main results.

Example 12. Consider the following fractional differential system:

$$\begin{cases} {}_tD_1^{0,6}\left(\Phi_3\left({}_0D_t^{0,6}u(t)\right)\right) = \lambda F_u(t, u(t), v(t)) + \left(\sin\left(\frac{u_1}{2}\right)\right)^2, & \text{a.e.t} \in [0, T], \\ {}_tD_1^{0,75}\left(\Phi_3\left({}_0D_t^{0,75}u(t)\right)\right) = \lambda F_v(t, u(t), v(t)) + \left(\arctan\left(\frac{u_2}{3}\right)\right)^2, & \text{a.e.t} \in [0, T], \\ u(0) = u(1) = 0, & v(0) = v(1) = 0, \end{cases} \quad (107)$$

where $T = 1$, $\alpha = 0, 6$, $\beta = 0, 75$, and $h_1(u_1) = (\sin(u_1/2))^2$, $h_2(u_2) = (\arctan(u_2/3))^2$. Moreover, for all $(t, u, v) \in [0, 1] \times \mathbb{R}^2$ put

$$F(t, u(t), v(t)) = \left(1 + t^2\right) H(u, v), \quad (108)$$

where

$$H(u, v) = \begin{cases} \xi_{n+1}^3 \exp\left(\dfrac{-1}{1 - (u - 0.8873\xi_{n+1})^2 + (v - 0.9064\xi_{n+1})^2}\right), & (u, v) \in \Omega, \\ 0, & (u, v) \in \mathbb{R}^2 \setminus \Omega, \end{cases} \quad (109)$$

where

$$\Omega = \cup_{n\geq 1}\left\{(u, v) \colon (u - 0.8873\xi_{n+1})^2 + (v - 0.9064\xi_{n+1}) < 1\right\}, \quad (110)$$

and $\xi_1 = 1$, $\xi_{n+1} = n(\xi_n)^{4/3} + 1$ for all $n \in \mathbb{N}$.

Clearly, $h_1, h_2 : \mathbb{R} \longrightarrow \mathbb{R}$ are two Lipschitz continuous functions of order 2 with Lipschitzian constants $L_1 = 1/2$, $L_2 = 1/3$ and $h_1(0) = h_2(0) = 0$, $F(t, 0, 0) = 0$ for all $t \in [0, 1]$. With the aid of direct computation we have that

$$M \approx 1.8925, k \approx 0.2991, \rho \approx 1.7009. \quad (111)$$

Let $\theta = 1/3$, then we have

$$\begin{aligned} P\left(\alpha, \frac{1}{3}\right) &= P\left(0, 6, \frac{1}{3}\right) \approx 0.3366, Q\left(\beta, \frac{1}{3}\right) = Q\left(0, 75, \frac{1}{3}\right) \\ &\approx 0.3745. \end{aligned} \quad (112)$$

Then, $\Delta \approx 0.3745$. Thus, all conditions of Theorem 10 are satisfied.

In fact, the conditions $(H0)$, $(H1)$, and $(H2)$ hold. For all $n \in \mathbb{N}$.

Restriction of $H(u, v)$ on Ω attains its maximum in $(0.8873_{\xi n+1}, 0.9064_{\xi n+1})$ and

$$H\left(0.8873_{\xi n+1}, 0.9064_{\xi n+1}\right) = \xi_{n+1}^3 \exp(-1). \quad (113)$$

In addition,

$$\sup_{|u|+|v|\leq 0,8873\xi_{n+1}^{-1}} H(u,v)=\xi_n^3\exp(-1), \tag{114}$$

and so

$$\begin{aligned} B_\infty &= \lim\sup_{n\to+\infty}\frac{\int_{1/3}^{2/3}H(0.8873\xi_{n+1},0.9064\xi_{n+1})dt}{\xi_n^2+1}\\ &=\lim_{n\to+\infty}\frac{\xi_{n+1}^3\exp(-1)}{\xi_n^2+1}=+\infty, \end{aligned} \tag{115}$$

$$\begin{aligned} A_\infty &= \lim_{n\to+\infty}\inf\frac{\int_0^1\sup_{|u|+|v|\leq(0.8873\xi_{n+1})-1}H(u,v)dt}{(0.8873\xi_{n+1})^2}\\ &=\lim_{n\to+\infty}\frac{\xi_n^3\exp(-1)}{(0.8873\xi_{n+1})^2}=0<\frac{k}{2pM\rho\Delta}B_0, \end{aligned} \tag{116}$$

which implies that the condition ($H3$) holds. Hence, owing to Theorem 10, for each $\lambda\in(0;+\infty)$, the coupled system (107) has an unbounded sequence of weak solutions.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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