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Research Article

New Sequential Fractional Differential Equations with Mixed-Type Boundary Conditions

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In this paper, we introduce new sequential fractional differential equations with mixed-type boundary conditions $\begin{cases} ({}^CD^q + k^CD^{q-1})u(t) = f(t, u(t), {}^CD^{q-1}u(t)), t \in (0,1), \\ \alpha_1 u(0) + \beta_1 u(1) + \gamma_1 I^r u(\eta) = \varepsilon_1, \eta \in (0,1), \\ \alpha_2 u'(0) + \beta_2 u'(1) + \gamma_2 I^r u'(\eta) = \varepsilon_2, \end{cases}$ where $q \in (1,2]$ is a real number, $k, r > 0, \alpha_i, \beta_i, \gamma_i, \varepsilon_i \in \mathbb{R}, i = 1, 2, {}^CD^q$ is the Caputo fractional derivative, and the boundary conditions include antiperiodic and Riemann-Liouville fractional integral boundary value cases. Our approach to treat the above problem is based upon standard tools of fixed point theory and some new inequalities of norm form. Some existence results are obtained and well illustrated through the aid of examples.

1. Introduction

In this paper, we focus on sequential fractional differential equations with mixed-type boundary conditions.

$$\begin{cases} \left({}^CD^q + k^CD^{q-1}\right)u(t) = f\left(t, u(t), {}^CD^{q-1}u(t)\right), & t \in (0,1), \\ \alpha_1 u(0) + \beta_1 u(1) + \gamma_1 I^r u(\eta) = \varepsilon_1, & \eta \in (0,1), \\ \alpha_2 u'(0) + \beta_2 u'(1) + \gamma_2 I^r u'(\eta) = \varepsilon_2, \end{cases} \tag{1}$$

where $q \in (1,2]$ is a real number and $k, r > 0, \alpha_i, \beta_i, \gamma_i, \varepsilon_i \in \mathbb{R}$, $i = 1, 2, {}^CD^q$ is the Caputo fractional derivative of order q. The nonlinearity term f contains the unknown function and its lower order fractional derivatives. The new boundary conditions include antiperiodic and Riemann-Liouville fractional integral boundary value cases which can be regarded as the linear combination of the values of the unknown function and its first derivatives at the end points of interval, and the Riemann-Liouville fractional integral value of the unknown function and its first derivatives at an interior point of interval.

Fractional differential equations have attracted significant attention for their wide application in many fields of engineering and applied sciences (see [1–10]). Sequential fractional differential equations as an importance branch have also received wide attention; for instance, see [11–16]. Motivated by the HIV infection model and its application background in [12], the existence and uniqueness of solutions for the following sequential fractional differential system are obtained by means of Leray-Schauder's alternative and Banach's contraction principle

$$\begin{cases} \left({}^CD^p + \lambda_1^C D^{p-1}\right)u(t) = f_1(t, u(t), v(t)), & t \in (0,1), \\ \left({}^CD^p + \lambda_2^C D^{p-1}\right)v(t) = f_2(t, u(t), v(t)), & t \in (0,1), \\ u(0) = u'(0) = 0, u(1) = av(\xi), \\ v(0) = v'(0) = 0, v(1) = bu(\eta), \end{cases} \tag{2}$$

where $\lambda_i > 0$ $(i = 1, 2)$ is a parameter; $2 < p$, $q \leq 3$, ${}^CD^p$, and ${}^CD^q$ are the Caputo fractional derivatives; and the nonlinearity terms $f_1, f_2 : [0, 1] \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ are the given continuous function.

Antiperiodic boundary conditions arise in the mathematical problems of certain physical phenomena and processes. Recently, many scholars paid attention to solvability for fractional differential equations involving antiperiodic boundary conditions (see [17–21]). For example, in [21], the authors considered the nonlinear antiperiodic boundary value problems

$$\begin{cases} \left({}^CD^q + k{}^CD^{q-1}\right)u(t) = f(t, u(t)), q \in (2, 3], \quad t \in (0, T), \\ \alpha_1 u(0) + \gamma_1 u(T) = a, \alpha_2 u'(0) + \gamma_2 u'(T) = b, \alpha_3 u''(0) + \gamma_3 u''(T) = c, \end{cases} \tag{3}$$

where ${}^CD^q$ is the Caputo fractional derivatives of order q, α_i, γ_i $(i = 1,2,3)$, $a, b, c \in \mathbb{R}$, $k > 0$, f is a continuous function.

Integral boundary conditions are believed to be more reasonable than the local boundary conditions, which can describe modeling of blood flow, cellular systems, population dynamics, heat transmission, etc. There are a number of results about fractional differential equations and partial differential equations with integral boundary condition; we refer the reader to see [17, 20, 22–43] and the references cited therein. In [20], the authors discussed the following fractional differential equation with integral boundary conditions given by

$$\begin{cases} \left({}^CD^{\alpha+\beta} + k{}^CD^{\beta}\right)x(t) = f(t, x(t)), \quad t \in (0, 1), \\ x(0) = 0, x(1) = \mu \displaystyle\int_0^1 x(s)ds, {}^CD^{\alpha}x(0) + {}^CD^{\alpha}x(1) = 0, \end{cases} \tag{4}$$

where ${}^CD^{\alpha}$ and ${}^CD^{\beta}$ are the Caputo fractional derivatives; $0 < \alpha < 1$, $1 < \beta \leq 2$, $k > 0$, and $\mu > 0$ are real numbers; and f is a given continuous function.

Observing the results of the above literature, an interesting and important question is whether antiperiodic and integral boundary conditions can be unified in a system. If we have unified the conditions, how can we obtain the existence of the solutions? Through a literature search, the sequential fractional differential equation (1) has not been given up to now.

Now in this paper, we shall discuss the problem (1) by using the standard tools of fixed point theory and some new inequalities of norm from.

2. Preliminary and Lemmas

In this paper, we provide some necessary definitions and lemmas of the Caputo fractional calculus; for more information, see the books [1–3].

Definition 1. The Caputo derivative of fractional order $\alpha > 0$ for a n-times continuously differentiable function $f : [0, \infty) \longrightarrow \mathbb{R}$ is defined as

$${}^CD^{\alpha}f(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-t)^{n-\alpha-1} f^{(n)}(x) \mathrm{d}t, \quad n-1 < \alpha < n, \tag{5}$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of number α.

Definition 2. The Riemann-Liouville fractional integral of order α for a function f is defined as

$$I^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t) \mathrm{d}t, \quad \alpha > 0, \tag{6}$$

where $\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1} e^{-t} dt$.

Lemma 3. *The Caputo fractional differential equation ${}^CD^{\alpha}u(t) = 0$ has the general solution for $\alpha > 0$*

$$u(t) = c_1 + c_2 t + c_3 t^2 + \cdots + c_n t^{n-1}, \tag{7}$$

where $c_i \in \mathbb{R}$, $i = 1, 2, \cdots, n$, and n is given as in Definition 1.

Lemma 4. *Let $h(t) \in C[0, 1]$ and $u \in C^2[0, 1]$. Then, the following sequential fractional differential equations*

$$\begin{cases} \left({}^CD^q + k{}^CD^{q-1}\right)u(t) = h(t), \quad 0 < t < 1, 1 < q \leq 2, \\ \alpha_1 u(0) + \beta_1 u(1) + \gamma_1 I^r u(\eta) = \epsilon_1, \quad r > 0, 0 < \eta < 1, \\ \alpha_2 u'(0) + \beta_2 u'(1) + \gamma_2 I^r u'(\eta) = \epsilon_2, \end{cases} \tag{8}$$

have a unique solution

$$u(t)=\frac{\Delta_1-\Delta_2e^{-kt}}{\Delta_2\Delta_3}\Phi(h(1),h(\eta))+\frac{1}{\Delta_2}\psi(h(1),h(\eta))+\int_0^t e^{-k(t-s)}I^{q-1}h(s)ds, \quad (9)$$

where

$$\Delta_1=\alpha_1+\beta_1e^{-k}+\gamma_1\int_0^\eta\frac{(\eta-s)^{r-1}}{\Gamma(r)}e^{-ks}ds,$$
$$\Delta_2=\alpha_1+\beta_1+\frac{\gamma_1\eta^r}{\Gamma(r+1)},$$
$$\Delta_3=-k\alpha_2-ke^{-k}\beta_2-k\gamma_2\int_0^\eta\frac{(\eta-s)^{r-1}}{\Gamma(r)}e^{-ks}ds, \Phi(h(1),h(\eta))$$
$$=-\beta_2I^{q-1}h(1)-k\beta_2\int_0^1e^{-k(1-s)}I^{q-1}h(s)ds$$
$$+\gamma_2\left[\int_0^\eta\frac{(\eta-s)^{r-1}}{\Gamma(r)}I^{q-1}h(s)ds\text{-}k\int_0^\eta\frac{(\eta-s)^{r-1}}{\Gamma(r)}\cdot\left(\int_0^s e^{-k(s-m)}I^{q-1}h(m)dm\right)ds\right]-\epsilon_2, \Psi(h(1),h(\eta))$$
$$=-\gamma_1\int_0^\eta\frac{(\eta-s)^{r-1}}{\Gamma(r)}\left(\int_0^s e^{-k(s-m)}I^{q-1}h(m)dm\right)ds-\beta_1\int_0^1e^{-k(1-s)}I^{q-1}h(s)ds+\epsilon 1. \quad (10)$$

Proof. Using Lemma 3, the general solution of the fractional differential equations $({}^CD^q+k{}^CD^{q-1})u(t)=h(t)$ can be written as

$$u(t)=A_0e^{-kt}+A_1+\int_0^t e^{-k(t-s)}\left(I^{q-1}h(s)\right)ds, \quad (11)$$

where $A_0, A_1\in\mathbb{R}$. Differentiating (11) with respect to t, we get

$$u'(t)=-kA_0e^{-kt}+I^{q-1}h(t)-k\int_0^t e^{-k(t-s)}I^{q-1}h(s)ds. \quad (12)$$

Applying the boundary condition (8) in (11) and (12), we obtain

$$\Delta_1A_0+\Delta_2A_1=\Psi(h(1),h(\eta)), \quad \Delta_3A_0+\Phi(h(1),h(\eta))=0. \quad (13)$$

A simultaneous solution of equation (13) leads to

$$A_0=-\frac{1}{\Delta_3}\Phi(h(1),h(\eta)), \quad A_1=\frac{\Delta_1}{\Delta_2\Delta_3}\Phi(h(1),h(\eta))+\frac{1}{\Delta_2}\Psi(h(1),h(\eta)). \quad (14)$$

Substituting A_0 and A_1 to (11), we obtain the desired solution in (9). The converse of the lemma follows by direct computation. The proof is completed.

Remark 5. Caputo fractional differentiating (11) with respect to t, we obtain

$${}^CD^{q-1}u(t)=\frac{kA_0}{\Gamma(2-q)}\int_0^t(t-s)^{1-q}e^{-ks}ds+\frac{1}{\Gamma(2-q)}\cdot\int_0^t(t-s)^{1-q}I^{q-1}h(s)ds-\frac{k}{\Gamma(2-q)}\cdot\int_0^t(t-s)^{1-q}\left(\int_0^s e^{-k(s-m)}I^{q-1}h(m)dm\right)ds, \quad (15)$$

where A_0 is defined as (14).

Set $C[0,1]$ is all the continuous functions on $[0,1]$, $C_{q-1}[0,1]=\{u\in C[0,1]\colon {}^CD^{q-1}u\in C[0,1]\}$. Let $E=(C_{q-1}[0,1],\|\cdot\|_{q-1})$ denotes the Banach space endowed with the norm defined by $\|u\|_{q-1}=\|u\|+\|{}^CD^{q-1}u\|=\sup_{0\le t\le 1}|u(t)|+\sup_{0\le t\le 1}|{}^CD^{q-1}u(t)|$. For the convenience of the proofs in the next main results, we first give the bounds for integrals arising from the sequel, which are very important for us to establish the existence of solutions for problem (1).

Lemma 6. *Suppose that $h\in C([0,1],\mathbb{R})$. Then, we have*

(i) $|\Phi(h(1),h(\eta))|\le[(|\beta_2|(2-e^{-k})/\Gamma(q))+(|\gamma_2|\eta^{q+r}(2-e^{-k\eta})/\Gamma(q)\Gamma(r+1))]\|h\|+|\epsilon_2|$

(ii) $|\Psi(h(1),h(\eta))|\le[(|\beta_1|(1-e^{-k})/k\Gamma(q))+(|\gamma_1|\eta^{q+r}(1-e^{-k\eta})/k\Gamma(q)\Gamma(r+1))]\|h\|+|\epsilon_1|$

Proof.

(i) Obviously, we have

$$\left|I^{q-1}h(s)\right|\le\frac{1}{\Gamma(q\text{-}1)}\int_0^s(s-\omega)^{q-2}\mathrm{d}\omega\|h\|=\frac{s^q}{\Gamma(q)}\|h\|, \quad \left|I^{q-1}h(1)\right|\le\frac{1}{\Gamma(q)}\|h\|,$$
$$\left|\int_0^s e^{-k(s-m)}I^{q-1}h(m)dm\right|\le\frac{s^q}{\Gamma(q)}\int_0^s e^{-k(s-m)}dm\|h\|=\frac{s^q}{\Gamma(q)}\frac{1-e^{-ks}}{k}\|h\|,$$
$$\left|\int_0^1 e^{-k(1-s)}I^{q-1}h(s)ds\right|\le\frac{1}{\Gamma(q)}\int_0^1 e^{-k(1-s)}ds\|h\|=\frac{1-e^{-k}}{k\Gamma(q)}\|h\|, \quad (16)$$
$$\left|\int_0^\eta\frac{(\eta-s)^{r\text{-}1}}{\Gamma(r)}I^{q-1}h(s)ds\right|\le\frac{\eta^q}{\Gamma(q)}\int_0^\eta\frac{(\eta-s)}{\Gamma(r)}ds\|h\|\le\frac{\eta^{q+r}}{\Gamma(q)\Gamma(r+1)}\|h\|.$$

Furthermore,

$$\left|\int_0^\eta \frac{(\eta-s)^{r-1}}{\Gamma(r)}\left(\int_0^s e^{-k(s-m)}\right)I^{q-1}h(m)dm\right| \le \frac{\eta^{q+r}\left(1-e^{-k\eta}\right)}{k\Gamma(q)\Gamma(r+1)}\|h\|. \tag{17}$$

Hence,

$$\begin{aligned}
&|\Phi(h(1),h(\eta))| \\
&\quad\le \left|\beta_2 I^{q-1}h(1)\right| + k\left|\beta_2\int_0^1 e^{-k(1-s)}I^{q-1}h(s)ds\right| \\
&\qquad + |\gamma_2|\left[\left|\int_0^\eta \frac{(\eta-s)^{r-1}}{\Gamma(r)}I^{q-1}h(s)ds\right|\right. \\
&\qquad + k\left.\left|\int_0^\eta \frac{(\eta-s)^{r-1}}{\Gamma(r)}\left(\int_0^s e^{-k(s-m)}I^{q-1}h(m)dm\right)ds\right|\right] \\
&\qquad + |\epsilon_2| \le \left\{\frac{|\beta_2|}{\Gamma(q)} + \frac{|\beta_2|\left(1-e^{-k}\right)}{\Gamma(q)}\right. \\
&\qquad + \left.\frac{|\gamma_2|\eta^{q+r}}{\Gamma(q)\Gamma(r+1)} + \frac{|\gamma_2|\eta^{q+r}\left(1-e^{-k\eta}\right)}{\Gamma(q)\Gamma(r+1)}\right\}\|h\| + |\epsilon_2|.
\end{aligned} \tag{18}$$

When the proof of (ii) is similar to (i), we omit it.

In view of Lemma 4 and Remark 5, replacing $h(t)$ by $f(t,u(t),{}^C D^{q-1}u(t))$ in (9) and (15), we transform the solution of problem (1) into the fixed point of operator equation $u=\mathbb{F}u$, where operator $\mathbb{F}: E \longrightarrow E$ is defined as

$$\begin{aligned}
(\mathbb{F}u)(t) = {} & \frac{\Delta_1-\Delta_2 e^{-kt}}{\Delta_2\Delta_3}\Phi\left(f\left(1,u(1),{}^C D^{q-1}u(1)\right),\right. \\
& \left.f\left(\eta,u(\eta),{}^C D^{q-1}u(\eta)\right)\right) \\
& + \frac{1}{\Delta_2}\Psi\left(f\left(1,u(1),{}^C D^{q-1}u(1)\right),\right. \\
& \left.f\left(\eta,u(\eta),{}^C D^{q-1}u(\eta)\right)\right) \\
& + \int_0^t e^{-k(t-s)}I^{q-1}f\left(s,u(s){}^C D^{q-1}u(s)\right)ds,
\end{aligned} \tag{19}$$

$$\begin{aligned}
\left({}^C D^{q-1}\mathbb{F}u\right)(t) = {} & -\frac{k}{\Gamma(2-q)}\Phi\left(f\left(1,u(1),{}^C D^{q-1}u(1)\right),\right. \\
& \left.f\left(\eta,u(\eta),{}^C D^{q-1}u(\eta)\right)\right)\int_0^t (t-s)^{1-q}e^{-ks}ds \\
& + \frac{1}{\Gamma(2-q)}\int_0^t (t-s)^{1-q}I^{q-1}f\left(s,u(s),\right. \\
& \left.{}^C D^{q-1}u(s)\right)ds - \frac{k}{\Gamma(2-q)}\int_0^t (t-s)^{1-q} \\
& \cdot\left(\int_0^s e^{-k(s-m)}I^{q-1}f\left(m,u(m),\right.\right. \\
& \left.\left.{}^C D^{q-1}u(m)\right)dm\right)ds.
\end{aligned} \tag{20}$$

Lemma 7 (see [44]). *Let $T: E \to E$ be completely continuous (i.e., a map restricted to any bounded set in E is compact). Let*

$$\mathbb{F} = \{x\in E : x=\lambda T(x), \textit{for some } 0<\lambda<1\}. \tag{21}$$

Then, either set $\mathbb{F}$ is unbounded or $\mathbb{F}$ has at least one fixed point.

Lemma 8 (see [45]). *Let E be a Banach space, $D\subset E$ be closed and $F: D\longrightarrow D$ a strict contraction, i.e., $|Fx-Fy|\le k|x-y|$ for some $k\in(0,1)$ and all $x,y\in D$. Then, F has a unique fixed point.*

3. Main Results

Before starting and introducing the main results, we list our assumption for f:

(i) H0: $f:[0,1]\times\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{R}$ is continuous

(ii) H1: there exist positive constants $a_{11},a_{12},a_{13}\in\mathbb{R}^+$ such that

$$|f(t,u,v)| \le a_{11}|u|^{\sigma_1} + a_{12}|v|^{\sigma_2} + a_{13}, \quad \forall t\in[0,1], u,v\in\mathbb{R}, 0<\sigma_1,\sigma_2<1 \tag{22}$$

(iii) H2: there exist positive constants $a_{21},a_{22},a_{23}\in\mathbb{R}^+$ such that

$$|f(t,u,v)| \le a_{21}|u|^{\tau_1} + a_{22}|v|^{\tau_2} + a_{23}, \quad \forall t\in[0,1], u,v\in\mathbb{R}, 0<\tau_1,\tau_2>1 \tag{23}$$

(iv) H3: there exist positive constants $a_{31},a_{32},a_{33}\in\mathbb{R}^+$ such that

$$|f(t,u,v)| \le a_{31}|u| + a_{32}|v| + a_{33}, \quad \forall t\in[0,1], u,v\in\mathbb{R} \tag{24}$$

(v) H4: there exist positive constants $a_{41},a_{42},a_{43}\in\mathbb{R}^+$ such that

$$|f(t,u,v)| \le a_{41}|u|^{\rho_1} + a_{42}(t)|v|^{\rho_2} + a_{43}, \quad \forall t\in[0,1], u,v\in\mathbb{R}, 0<\rho_1<1<\rho_2 \tag{25}$$

(vi) H5: there exist positive constants $a_{51},a_{52}\in\mathbb{R}^+$ such that

$$|f(t,u,v) - f(t,\bar{u},\bar{v})| \le a_{51}|u-\bar{u}| + a_{52}|v-\bar{v}|, \quad \forall t\in[0,1], u,v,\bar{u},\bar{v}\in\mathbb{R} \tag{26}$$

For convenience, we introduce the following symbols:

$$L_1 = \frac{|\beta_2|\left(2 - e^{-k}\right)}{\Gamma(q)} + \frac{|\gamma_2|\eta^{q+r}\left(2 - e^{-k\eta}\right)}{\Gamma(q)\Gamma(r+1)},$$
$$L_2 = \frac{|\beta_1|\left(1 - e^{-k}\right)}{k\Gamma(q)} + \frac{|\gamma_1|\eta^{q+r}\left(1 - e^{-k\eta}\right)}{k\Gamma(q)\Gamma(r+1)},$$
$$L_3 = \frac{(|\Delta_1| + |\Delta_2|)L_1 + |\Delta_3|L_2}{|\Delta_2\Delta_3|} + \frac{1}{k\Gamma(q)} + \frac{k\Gamma(q)L_1 + 2}{\Gamma(3-q)\Gamma(q)},$$
$$L_4 = \frac{(|\Delta_1| + |\Delta_2|)|\varepsilon_2| + |\Delta_3\varepsilon_1|}{|\Delta_2\Delta_3|} + \frac{k|\varepsilon_2|}{\Gamma(3-q)}. \tag{27}$$

Theorem 9. *Suppose that (H0) and (H1) hold. Then, problem (1) has at least one solution.*

Proof. We first define a ball in E as $B_R = \{u | u \in E, \|u\|_{q-1} \le R\}$, where

$$R \ge \max\left\{(3L_3a_{11})^{1/(1-\sigma_1)}, (3L_3a2)^{1/(1-\sigma_1)}, 3(3L_3a_{13} + L_4)\right\}. \tag{28}$$

Then, we show that $T : B_R \longrightarrow B_R$. For $\forall u \in B_R$, using Lemma 6 and the condition (H1), we have

$$\begin{aligned}
|(\mathbb{F})(t)|s \le{}& \frac{|\Delta_1| + |\Delta_2|e^{-kt}}{|\Delta_2||\Delta_3|}\left|\Phi\left(f\left(1, u(1), {}^CD^{q-1}u(1)\right),\right.\right.\\
&\left.\left. f\left(\eta, u(\eta), {}^CD^{q-1}u(\eta)\right)\right)\right| + \frac{1}{|\Delta_2|}\left|\Psi\left(f\left(1, u(1),\right.\right.\right.\\
&\left.\left.\left.{}^CD^{q-1}u(1)\right), f\left(\eta, u(\eta), {}^CD^{q-1}u(\eta)\right)\right)\right|\\
&+ \int_0^t e^{-k(t-s)} I^{q-1}\left|f\left(s, u(s), {}^CD^{q-1}u(s)\right)\right| ds\\
\le{}& \frac{|\Delta_1| + |\Delta_2|}{|\Delta_2\Delta_3|}\left[L_1\left(a_{11}R^{\sigma_1} + a_{12}R^{\sigma_2} + a_{13}\right) + |\varepsilon_2|\right]\\
&+ \frac{1}{|\Delta_2|}\left[L_2\left(a_{11}R^{\sigma_1} + a_{12}R^{\sigma_2} + a_{13}\right) + |\varepsilon_1|\right]\\
&+ \frac{1}{k\Gamma(q)}\left[a_{11}R^{\sigma_1} + a_{12}R^{\sigma_2} + a_{13}\right]\\
\le{}& \left[\frac{(|\Delta_1| + |\Delta_2|)L_1 + |\Delta_3|L_2}{|\Delta_2\Delta_3|} + \frac{1}{k\Gamma(q)}\right]\\
&\cdot\left(a_{11}R^{\sigma_1} + a_{12}R^{\sigma_2} + a_{13}\right)\\
&+ \frac{(|\Delta_1| + |\Delta_2|)|\epsilon_2| + |\Delta_3\epsilon_1|}{|\Delta_2\Delta_3|},
\end{aligned} \tag{29}$$

$$\begin{aligned}
\left|\left({}^CD^{q-1}\mathbb{F}u\right)(t)\right| \le{}& \frac{k}{\Gamma(3-q)}\left|\Phi\left(f\left(1, u(1), {}^CD^{q-1}u(1)\right),\right.\right.\\
&\left.\left. f\left(\eta, u(\eta), {}^CD^{q-1}u(\eta)\right)\right)\right|\\
&+ \frac{1}{\Gamma(3-q)\Gamma(q)}\left|f\left(s, u(s), {}^CD^{q-1}u(s)\right)\right|\\
&+ \frac{1}{\Gamma(3-q)\Gamma(q)}\left|f\left(s, u(s), {}^CD^{q-1}u(s)\right)\right|\\
\le{}& \frac{kL_1\Gamma(q) + 2}{\Gamma(3-q)\Gamma(q)}\left(a_{11}R^{\sigma_1} + a_{12}R^{\sigma_2} + a_{13}\right)\\
&+ \frac{k|\varepsilon_2|}{\Gamma(3-q)}.
\end{aligned} \tag{30}$$

From (29) and (30), we obtain

$$\|\mathbb{F}u\|_{q-1} \le L_3a_{11}R^{\sigma 1} + L_3a_{12}R^{\sigma 2} + L_4 \le \frac{1}{3}R + \frac{1}{3}R = R. \tag{31}$$

This means $\mathbb{F} : B_R \longrightarrow B_R$.

From the formula (19), it is easy to know that operators $\mathbb{F}(u)(t), {}^CD^{q-1}\mathbb{F}(u)(t)$ are continuous on $[0, 1]$. Now, we show that operator $\mathbb{F}$ is equicontinuous. Set

$$M = \max_{t\in[0,1]}\left\{\left|f\left(t, u(t), {}^CD^{q-1}u(t)\right)\right|\right\}, \quad \forall u \in B_R. \tag{32}$$

Let $t_1, t_2 \in [0, 1] (t_1 < t_2)$, we have

$$\begin{aligned}
&\left|(\mathbb{F}u)(t_2) - (\mathbb{F}u)(t_1)\right|\\
&\le \frac{\left|e^{-kt_2} - e^{-kt_1}\right|}{|\Delta_3|}\left|\Phi\left(f\left(1, u(1), {}^CD^{q-1}u(1)\right),\right.\right.\\
&\quad\left.\left. f\left(\eta, u(\eta), {}^CD^{q-1}u(\eta)\right)\right)\right| + \int_0^{t_1}\left|e^{-k(t_2-s)} - e^{-k(t_1-s)}\right|\\
&\quad\cdot\left|I^{q-1}f\left(s, u(s), {}^CD^{q-1}u(s)\right)\right| ds\\
&\quad+ \int_{t_1}^{t_2}\left|e^{-k(t_2-s)}\right|\left|I^{q-1}f\left(s, u(s), {}^CD^{q-1}u(s)\right)\right| ds\\
&\le \frac{L_1M + |\varepsilon_2|}{|\Delta_3|}\left(e^{-kt_1} - e^{-kt_2}\right)\\
&\quad+ \frac{M}{\Gamma(q)}\int_0^{t_1}\left|e^{-k(t_2-s)} - e^{-k(t_1-s)}\right| ds + \frac{M}{\Gamma(q)}\int_{t_1}^{t_2} e^{-k(t_2-s)} ds\\
&\le \frac{L_1M + |\varepsilon_2|}{|\Delta_3|}\left(e^{-kt_1} - e^{-kt_2}\right)\\
&\quad+ \frac{M}{\Gamma(q)}\left[2\left(1 - e^{-k(t_2-t_1)}\right) + \left(e^{-kt_1} - e^{-kt_2}\right)\right],
\end{aligned} \tag{33}$$

$$\begin{aligned}
&\left|\left({}^{C}D^{q-1}\mathbb{F}\right)(u)(t_2)-\left({}^{C}D^{q-1}\mathbb{F}\right)(u)(t_1)\right| \\
&\quad \le \frac{k}{\Gamma(2-q)}\left|\Phi\left(f\left(1,u(1),{}^{C}D^{q-1}u(1)\right),\right.\right. \\
&\qquad \left.\left. f\left(\eta,u(\eta),{}^{C}D^{q-1}u(\eta)\right)\right)\right| \times \left|\int_0^{t_2}(t_2-s)^{1-q}e^{-ks}ds\right. \\
&\qquad \left. -\int_0^{t_1}(t_1-s)^{1-q}e^{-ks}ds\right| + \frac{1}{\Gamma(2-q)} \\
&\qquad \cdot\left|\int_0^{t_2}(t_2-s)^{1-q}I^{q-1}f\left(s,u(s),{}^{C}D^{q-1}u(s)\right)ds\right. \\
&\qquad \left. -\int_0^{t_1}(t_1-s)^{1-q}I^{q-1}f\left(s,u(s),{}^{C}D^{q-1}u(s)\right)ds\right| \\
&\qquad + \frac{k}{\Gamma(2-q)}\left|\int_0^{t_2}(t_2-s)^{1-q}\left(\int_0^{s}e^{-k(s-m)}I^{q-1}f\left(m,u(m),\right.\right.\right. \\
&\qquad \left.\left. {}^{C}D^{q-1}u(m)\right)dm\right)ds-\int_0^{t_1}(t_1-s)^{1-q}\left(\int_0^{s}e^{-k(s-m)}I^{q-1}f\right. \\
&\qquad \left.\left. \cdot\left(m,u(m),{}^{C}D^{\alpha-1}u(m)\right)dm\right)ds\right| \\
&\quad \le \frac{kL_1M+k|\varepsilon_2|}{\Gamma(2-q)}\left\{\left|\int_0^{t_1}\left[(t_2-s)^{1-q}-(t_1-s)^{1-q}\right]e^{-ks}ds\right|\right. \\
&\qquad \left. +\left|\int_{t_1}^{t_2}(t_2-s)^{1-q}e^{-ks}ds\right|\right\}+\frac{2M}{\Gamma(q)\Gamma(2-q)} \\
&\qquad \cdot\left\{\left|\int_0^{t_1}\left[(t_2-s)^{1-q}-(t_1-s)^{1-q}\right]ds\right|+\left|\int_{t_1}^{t_2}(t_2-s)^{1-q}ds\right|\right\}.
\end{aligned} \tag{34}$$

In (33) and (34), letting $t_1 \longrightarrow t_2$, then,

$$\left|(\mathbb{F}u)(t_2)-(\mathbb{F}u)(t_1)\right| \longrightarrow 0, \left|\left({}^{C}D^{q-1}\mathbb{F}u\right)(t_2)-\left({}^{C}D^{q-1}\mathbb{F}u\right)(t_1)\right| \longrightarrow 0, \tag{35}$$

That is, as $t_1 \longrightarrow t_2$,

$$\left\|(\mathbb{F}u)(t_2)-(\mathbb{F}u)(t_2)\right\|_{q-1} \longrightarrow 0. \tag{36}$$

Therefore, $\mathbb{F}(B_R)$ is an equicontinuous set. Furthermore, it is uniformly bounded because of $\mathbb{F}(B_R)\subset B_R$. Applying the Arzelà-Ascoli theorem, we can infer that $\mathbb{F}$ is a completely continuous operator.

Consider $V=\{u\in B_R \mid u=\mu\mathbb{F}u, 0<\mu<1\}$ and show that V is bounded. For $u\in V$, we know $\|u\|_{q-1}<\|(\mathbb{F}u)\|_{q-1}\le R$. By Lemma 7, problem (1) has at least one solution in BR.

Theorem 10. *Suppose that (H0) and (H2) hold. If $3(L_3a_{23}+L_4)\le \min\{(3L_3a_{21})^{1/(1-\tau_1)},(3L_3a_{22})^{1/(1-\tau_2)}\}$, then problem (1) has at least one solution.*

Proof. The proof is similar to Theorem 9. We just need to make sure that R satisfies $3(L_3a_{23}+L_4)\le R\le \min\{(3L_3a_{21})^{1/(1-\tau_1)},(3L_3a_{22})^{1/(1-\tau_2)}\}$ in B_R.

Theorem 11. *Suppose that (H0) and (H3) hold. If $L_3(a_{31}+a_{32})<1$, then problem (1) has at least one solution.*

Proof. The proof is similar to Theorem 9, we omit it.

Theorem 12. *Suppose that (H0) and (H4) hold. If $\max\{(3L_3a_{41})^{1/(1-\rho_1)},3(L_3a_{43}+L_4)\}\le(3L_3a_{42})^{1/(1-\rho_2)}$, then problem (1) has at least one solution.*

Proof. The proof is similar to Theorem 10, we omit it.

Theorem 13. *Suppose that (H0) and (H5) hold. If $L_3(a_{51}+a_{52})<1$, then problem (1) has a unique solution.*

Proof. Define $\sup_{t\in[0,1]}|f(t,0,0)|=M<\infty$, such that $r\ge(L_3M+L_4)/(1-L_3(a_{51}+a_{52}))$.

First, we show that $T(B_r)\subset B_r$, where $B_r=\{u\mid u\in E:\|u\|_{\alpha-1}\le r\}$. For $\forall u\in B_r$, by direct calculation, we have

$$\begin{aligned}
|(\mathbb{F}u)(t)| &\le \frac{|\Delta_1|+|\Delta_2|}{|\Delta_2\Delta_3|}\left|\Phi\left(f\left(1,u(1),{}^{C}D^{q-1}u(1)\right)-f(1,0,0)\right.\right. \\
&\quad +f(1,0,0),f\left(\eta,u(\eta),{}^{C}D^{q-1}u(\eta)\right)-f(\eta,0,0) \\
&\quad \left.+f(\eta,0,0)0\right|+\frac{1}{|\Delta_2|}\left|\Psi\left(f\left(1,u(1),{}^{C}D^{q-1}u(1)\right)\right.\right. \\
&\quad -f(1,0,0)+f(1,0,0),f\left(\eta,u(\eta),{}^{C}D^{q-1}u(\eta)\right) \\
&\quad \left.\left.-f(\eta,0,0)+f(\eta,0,0)\right)\right|+\int_0^{t}e^{-k(t-s)}I^{q-1} \\
&\quad \cdot\left[\left|f\left(s,u(s),{}^{C}D^{q-1}u(s)\right)-f(s,0,0)\right|+|f(s,0,0)|\right]ds \\
&\le \frac{|\Delta_1|+|\Delta_2|}{|\Delta_2\Delta_3|}\left[L_1\left(a_{51}\|u\|+a_{52}\left\|{}^{C}D^{q-1}u\right\|+M\right)+|\varepsilon_2|\right] \\
&\quad +\frac{1}{|\Delta_2|}\left[L_2\left(a_{51}\|u\|+a_{52}\left\|{}^{C}D^{q-1}u\right\|+M\right)+|\varepsilon_1|\right] \\
&\quad +\frac{1}{k\Gamma(q)}\left|\left[a_{51}\|u\|+a_{52}\left\|{}^{C}D^{q-1}u\right\|+M\right]\right. \\
&\le \left[\frac{(|\Delta_1|+|\Delta_2|)L_1+|\Delta_3|L_2}{|\Delta_2\Delta_3|}+\frac{1}{k\Gamma(q)}\right]\left[(a_{51}+a_{52})\|u\|_{q-1}\right. \\
&\quad \left.+M\right]+\frac{(|\Delta_1|+|\Delta_2|)|\varepsilon_2|+|\Delta_3\varepsilon_1|}{|\Delta_2\Delta_3|},
\end{aligned} \tag{37}$$

$$\begin{aligned}
\left|\left({}^{C}D^{q-1}\mathbb{F}u\right)(t)\right| &\le \frac{k}{\Gamma(3-q)}\left|\Phi\left(f\left(1,u(1),{}^{C}D^{q-1}u(1)\right)\right.\right. \\
&\quad -f(1,0,0)+f(1,0,0),f\left(\eta,u(\eta),\right. \\
&\quad \left.\left.\left.{}^{C}D^{q-1}u(\eta)\right)-f(\eta,0,0)+f(\eta,0,0)\right)\right| \\
&\quad +\frac{2}{\Gamma(3-q)\Gamma(q)}\left[\left|f\left(s,u(s),{}^{C}D^{q-1}u(s)\right)\right.\right. \\
&\quad \left.\left.-f(s,0,0)\right|+\left|f(s,0,0)\right|\right]
\end{aligned}$$

$$\leq \frac{k}{\Gamma(3-q)}\left[L_1\left(a_{51}\|u\|+a_{52}\left\|{}^{C}D^{q-1}u\right\|+M\right)+|\varepsilon_2|\right]+\frac{2}{\Gamma(3-q)\Gamma(q)}\left[a_{51}\|u\|+a_{52}\left\|{}^{C}D^{q-1}u\right\|+M\right]\leq\frac{k\Gamma(q)L_1+2}{\Gamma(3-q)\Gamma(q)}\cdot\left[\left(a_{51}+a_{52}\right)\|u\|_{q-1}+M\right]+\frac{k|\varepsilon_2|}{\Gamma(3-q)}. \quad (38)$$

Combining (37) and (38), we obtain

$$\|(\mathbb{F}u)\|_{\alpha-1}\leq L_3(a_{51}+a_{52})\|u\|_{q-1}+L_3M+L_4\leq r. \quad (39)$$

Now, for any $u, v\in B_r$, we have

$$\begin{aligned}&|(\mathbb{F}u)(t)-(\mathbb{F}v)(t)|\\&\leq\frac{|\Delta_1|+|\Delta_2|}{|\Delta_2\Delta_3|}\left|\Phi\left(f\left(1,u(1),{}^{C}D^{q-1}u(1)\right),f\left(\eta,u(\eta),{}^{C}D^{\alpha-1}u(\eta)\right)\right)-\Phi\left(f\left(1,v(1),{}^{C}D^{\alpha-1}v(1)\right),f\left(\eta,v(\eta),{}^{C}D^{\alpha-1}v(\eta)\right)\right)\right|+\frac{1}{|\Delta_2|}\left|\Psi\left(f\left(1,u(1),{}^{C}D^{\alpha-1}u(1)\right),f\left(\eta,u(\eta),{}^{C}D^{\alpha-1}u(\eta)\right)\right)\right.\\&\left.-\Psi\left(f\left(1,v(1),{}^{C}D^{\alpha-1}v(1)\right),f\left(\eta,v(\eta),{}^{C}D^{\alpha-1}v(\eta)\right)\right)\right|+\int_0^t e^{-k(t-s)}\left(I^{q-1}\left|f\left(s,u(s),{}^{C}D^{\alpha-1}u(s)\right)-f\left(s,v(s),{}^{C}D^{\alpha-1}v(s)\right)\right|ds\right.\\&\leq\left[\frac{(|\Delta_1|+|\Delta_2|)L_1+|\Delta_3|L_2}{|\Delta_2\Delta_3|}+\frac{1}{k\Gamma(q)}\right]\cdot\left[a_{51}\|u-v\|+a_{52}\left\|{}^{C}D^{\alpha-1}u-{}^{C}D^{\alpha-1}v\right\|\right]\\&\leq\left[\frac{(|\Delta_1|+|\Delta_2|)L_1+|\Delta_3|L_2}{|\Delta_2\Delta_3|}+\frac{1}{k\Gamma(q)}\right]\cdot(a_{51}+a_{52})\|u-v\|_{\alpha-1}.\end{aligned} \quad (40)$$

Similar to (40), one has

$$\left|\left({}^{C}D^{\alpha-1}\mathbb{F}u\right)(t)-\left({}^{C}D^{\alpha-1}\mathbb{F}v\right)(t)\right|\leq\frac{k\Gamma(q)L_1+2}{\Gamma(3-q)\Gamma(q)}(a_{51}+a_{52})\|u-v\|_{\alpha-1}. \quad (41)$$

Therefore,

$$\|(\mathbb{F}u)-(\mathbb{F}v)\|_{\alpha-1}\leq L_3(a_{51}+a_{52})\|u-v\|_{\alpha-1}. \quad (42)$$

Since $L_3(a_{51}+a_{52})<1$, $\mathbb{F}$ is a contraction operator. Using Lemma 8, operator $\mathbb{F}$ has a unique fixed point which is the unique solution of problem (1).

In order to illustrate our main results, we consider the following sequential fractional differential equations:

$$\begin{cases}\left({}^{C}D^{1.5}+{}^{C}D^{0.5}u(t)=f\left(t,u(t),{}^{C}D^{0.5}u(t)\right),\right. & t\in(0,1),\\ 0.1u(0)+0.2u(1)-I^2u(0.5)=0.01,\\ 0.2u'(0)+0.1u'(1)+2I^2u'(0.5)=0.02,\end{cases} \quad (43)$$

where $q=1.5, k=1, r=2, \eta=0.5, \alpha_1=0.1, \beta_1=0.2, \gamma_1=-1, \varepsilon_1=0.01, \alpha_2=0.2, \beta_2=0.1, \gamma_1=2$, and $\varepsilon_2=0.02$.

Example 1. Let $f(t,u(t),{}^{C}D^{0.5}u(t))=0.01u(t)^{0.2}+0.02\,({}^{C}D^{0.5}u(t))^{0.1}+0.15(1-t)^2$. So we have

$$\left|f\left(t,u(t),{}^{C}D^{0.5}u(t)\right)\right|\leq 0.01|u(t)|^{0.2}+0.02\left|{}^{C}D^{0.5}u(t)\right|^{0.1}+0.15, \quad (44)$$

where $a_{11}=0.01, a_{12}=0.02, a_{13}=0.15, \sigma_1=0.2$, and $\sigma_2=0.1$. Theorem 9 implies that problem (43) has at least one solution.

Example 2. Let $f(t,u(t),{}^{C}D^{0.5}u(t))=0.01u(t)^{1.2}+0.02\,({}^{C}D^{0.5}u(t))^{1.1}+0.01(1-t)^2$. By direct calculation, we obtain that

$$\begin{aligned}\Delta_1&=0.093438, \Delta_2=0.246808, \Delta_3\\&=-0.407603, L_1=0.323142, L_2\\&=0.161275, L_3=5.785852, L_4=0.130729.\end{aligned} \quad (45)$$

In the meantime, we have

$$\left|f\left(t,u(t),{}^{C}D^{0.5}u(t)\right)\right|\leq 0.01|u(t)|^{1.2}+0.02\left|{}^{C}D^{0.5}u(t)\right|^{1.1}+0.01, \quad (46)$$

where $a_{21}=0.01, a_{22}=0.22, a_{23}=0.01, \tau_1=1.2$, and $\tau_2=1.1$. So we obtain $3(L_3a_{23}+L_4)=0.565763<\min\{(3L_3a_{21})^{1/(1-\tau_1)}, (3L_3a_{22})^{1/(1-\tau_2)}\}=1.111599$. Theorem 10 implies that problem (43) has at least one solution.

Example 3. Let $f(t,u(t),{}^{C}D^{0.5}u(t))=0.01u(t)+0.02\,{}^{C}D^{0.5}u(t)+0.01(1-t)^2$. So we have

$$\begin{aligned}&\left|f\left(t,u(t),{}^{C}D^{0.5}u(t)\right)-f\left(t,v(t),{}^{C}D^{0.5}v(t)\right)\right|\\&\leq 0.01|u(t)-v(t)|+0.02\left|{}^{C}D^{0.5}u(t)-{}^{C}D^{0.5}v(t)\right|,\end{aligned} \quad (47)$$

where $a_{51}=0.01$ and $a_{52}=0.02$. Combining with the calculation result of Example 2, we can obtain that $L_3(a_{51}+a_{52})=0.173576<1$. Hence, Theorem 13 implies that problem (43) has a unique solution.

4. Conclusions

We have obtained some existence results for new sequential fractional differential equations by using some nonlinear growth conditions

$$|f(t,u(v))| \le b_{11}|u|^{\theta_1} + b_{12}|v|^{\theta_1} + b_{13}, \quad \theta_1, \theta_2 \in (0, +\infty), \tag{48}$$

which is different from the existing linear condition. Obviously, these results are easy to verify and apply (see Example 1 and Example 3).

On the other hand, we note that our results contain some special types of results by fixing the parameters in the given problem (1). For instance, let $\alpha_1 = 1$, $\beta_1 = 1$, $\gamma_1 = 0$, $\alpha_2 = 1$, $\beta_2 = 1$, and $\gamma_2 = 0$, then the results of this paper are the following sequential fractional differential equations with the boundary value conditions of the form $u(0) + u(1) = \epsilon_1$ and $u'(0) + u'(1) = \epsilon_2$. Further, letting $\alpha_1 = 0$, $\beta_1 = 1$, $\gamma_1 = -1$, $\epsilon_1 = 0$, $\alpha_2 = 0$, $\beta_2 = 1$, $\gamma_2 = -1$, and $\epsilon_2 = 0$, we obtain the results for the boundary conditions $u(1) = I^r u(\eta)$ and $u'(1) = I^r u'(\eta)$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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