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Research Article

Investigations for a Type of Variable Coefficient Fractional Subdiffusion Equation with Multidelay

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A difference scheme is constructed for a type of variable coefficient time fractional subdiffusion equation with multidelay. Stability and convergence results of the scheme are obtained, and theoretical results are proved by two numerical tests.

1. Introduction

This paper will consider the following variable coefficient time fractional subdiffusion equation with arbitrary multi-delay

$$\begin{aligned} &\frac{\partial^\alpha u}{\partial t^\alpha} - (D(x)u_x)_x = f\left(u(x,t), u(x,t-s_1), u\left(x,t-s_q\right), x, t\right), \\ &\qquad (x,t) \in (0,1)\times(0,T], \\ &u(x,t) = \phi(x,t), x\in[0,1], t\in[-s,0], \\ &u(0,t) = \gamma(t), u(1,t) = \beta(t), t\in(0,T], \end{aligned} \tag{1}$$

where $0 < c_0 \le D(x) \le c_1$ is the variable diffusion coefficient, $s_j > 0, j = 1, 2, \cdots, q$ are delay terms, and $s = \max_{1\le i\le q}\{s_i\}$; time fractional partial derivative $(\partial^\alpha u/\partial t^\alpha)(0<\alpha<1)$ is defined as

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\xi)^{-\alpha}\frac{\partial u(x,\xi)}{\partial \xi}d\xi, \tag{2}$$

where $\Gamma(\cdot)$ is the gamma function.

For simplicity, we consider the following equation with arbitrary two different delays:

$$\begin{aligned} &\frac{\partial^\alpha u}{\partial t^\alpha} - (D(x)u_x)_x = f(u(x,t), u(x,t-s_1), u(x,t-s_2), x, t), \\ &\qquad (x,t)\in(0,1)\times(0,T], \end{aligned} \tag{3}$$

$$u(x,t) = \phi(x,t), x\in[0,1], \quad t\in[-s,0], \tag{4}$$

$$u(0,t) = \gamma(t), u(1,t) = \beta(t), t\in(0,T], \tag{5}$$

where $s_2 > s_1 > 0$ are two delay terms and satisfy $s_1 = (m_1 + \delta_1)\tau$, $s_2 = (m_2+\delta_2)\tau$, $m_2 > m_1 > 0$ are integers, and $\delta_1, \delta_2 \in [0,1]$, $s = (m_2+1)\tau$.

Applications of delay differential equations (DDEs) can be found in many fields, such as electrotechnics, population dynamics, biology, and economics [1–5]. However, one can obtain the analytical solutions of DDEs in few cases. The researchers turn to numerical methods for solving DDEs in general cases [6–9]. Most studies about DDEs consider one delay; in fact, more than one delays should be considered in some systems. Tian and Kuang considered H-method and GP-stability for DDEs with several delay terms [10]; the applications of such DDEs can be found in [11–14].

Fractional delay differential equations (FDDEs) are widely used in automatic control, population dynamics, finance, etc. [15–17]. Numerical solutions for FDDEs can be found in [18–25], where [18–20] considered numerical solutions for FDDEs, and [21–25] considered numerical solutions for fractional delay partial differential equations (FDPDEs). However, the researchers seldom considered the numerical methods for FDDEs with multidelays; some studies considered such FDDEs without theoretical analysis.

In this paper, there is an effective difference scheme for solving systems (3)–(5) with multidelay. Some stability and convergence results of the scheme are obtained by mathematical proof, and the theoretical results are proved by two numerical tests.

We organize the paper as the follows. Section 2 constructs a numerical scheme to solve (3)–(5). Section 3 provides the stability and convergence results by proof. Section 4 presents two numerical tests. Section 5 gives a brief discussion.

2. The Construction of the Second Finite Difference Method

The following assumptions are assumed to be true in this paper.

(H1) Assume $D(x)$ to be sufficiently smooth function which fulfills $0 < c_0 \le D(x) \le c_1$

(H2) Suppose that $f(u(x,t), u(x,t-s_1), u(x,t-s_2), x, t)$ is sufficiently smooth and satisfies [26, 27]

$$|f(\mu+\varepsilon_1, \nu+\varepsilon_2, \iota+\varepsilon_3, x, t) - f(\mu, \nu, \iota, x, t)| \le c_2|\varepsilon_1| + c_3|\varepsilon_2| + c_4|\varepsilon_3|, \quad (6)$$

where $\varepsilon_1, \varepsilon_2, \varepsilon_3$ are arbitrary real numbers and $c_2 > 0, c_3 > 0$, and $c_4 > 0$ are three constants.

Taking $M > 0$ and $N > 0$ to be two integers, we have space and time stepsize $h = 1/M$ and $\tau = T/N$, respectively, and discrete points $x_i = ih$ and $t_k = k\tau$. Assume $\Omega_{h\tau} = \Omega_h \times \Omega_\tau$, $\Omega_h = \{x_i \mid 0 \le i \le M\}$, $\Omega_\tau = \{t_k | -(m_2+1) \le k \le N\}$, and $U_i^k = u(x_i, t_k)$. Define the following grid function space on $\Omega_{h\tau}$:

$$\mathscr{W} = \left\{v_i^k \mid 0 \le i \le M, -(m_2+1) \le k \le N\right\}[rgb]1.00,0.00,0.00, \quad (7)$$

We have

$$\delta_x v_{i+1/2}^k = \frac{v_{i+1}^k - v_i^k}{h}, \delta_x^2 v_i^k = \frac{v_{i+1}^k - 2v_i^k + v_{i-1}^k}{h^2}, \delta_x(D\delta_x v)_i^k = \left(D_{i+1/2}\delta_x v_{i+1/2}^k - D_{i-1/2}\delta_x v_{i-1/2}^k\right)/h,$$
$$\delta_t^\alpha v^k = \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\left[d_0 v^k - \sum_{j=1}^{k-1}\left(d_{k-j-1} - d_{k-j}\right)v^j - d_{k-1}v^0\right], \quad (8)$$

where $d_j = (j+1)^{1-\alpha} - j^{1-\alpha}, j \ge 0$.

We introduce the following two Lemmas,

Lemma 1 [28]. *Suppose $0 < \alpha < 1$, $y \in C^2[0, t_k]$; it holds that*

$$\left|\frac{1}{\Gamma(1-\alpha)}\int_0^{t_k}\frac{y'(s)ds}{(t_k-s)^\alpha} - \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \cdot \left[d_0 y(t_k) - \sum_{j=1}^{k-1}\left(d_{k-j-1} - d_{k-j}\right)y(t_j) - d_{k-1}y(t_0)\right]\right| \le \frac{1}{\Gamma(2-\alpha)}\left[\frac{1-\alpha}{12} + \frac{2^{2-\alpha}}{2-\alpha} - \left(1+2^{-\alpha}\right)\right]\max_{0\le t\le t_k}|y''(t)|\tau^{2-\alpha}. \quad (9)$$

Lemma 2 [29]. *Assume $0 < \alpha < 1$; then, it holds that*

(i) d_j decrease monotonically with j increases, and $0 < d_j \le 1$

(ii) $d_0 = 1$, $\sum_{j=1}^{k-1}(d_{k-j-1} - d_{k-j}) = d_0 - d_{k-1}$

Considering (3) at the point (x_i, t_k), we have

$$\frac{\partial^\alpha u}{\partial t^\alpha}(x_i, t_k) - \frac{\partial}{\partial x}\left(D(x)\frac{\partial u(x,t)}{\partial x}\right)\Big|_{(x_i,t_k)} = f(u(x_i,t_k), u(x_i, t_k - s_1), u(x_i, t_k - s_2), x_i, t_k), 0 \le i \le M, 0 \le k \le N. \quad (10)$$

From Lemma 1, we obtain

$$\frac{\partial^\alpha u}{\partial t^\alpha}(x_i, t_k) = \delta_t^\alpha U_i^k + r_i^k, \quad (11)$$

where

$$r_i^k = \frac{1}{\Gamma(2-\alpha)}\left[\frac{1-\alpha}{12} + \frac{2^{2-\alpha}}{2-\alpha} - \left(1+2^{-\alpha}\right)\right]\max_{0\le t\le t_k}\left|\frac{\partial^2 u(x_i,t_k)}{\partial t^2}\right|\tau^{2-\alpha}. \quad (12)$$

From Taylor's expansion, we have

$$D_{i+1/2}\delta_x U_{i+1/2}^k = D_{i+1/2}\frac{\partial u}{\partial x}(x_{i+1/2}, t_k) + \frac{h^2}{24}D_{i+1/2}\frac{\partial^3 u}{\partial x^3}(x_{i+1/2}, t_k) + O(h^3), \quad (13)$$

$$D_{i-1/2}\delta_x U_{i-1/2}^k = D_{i-1/2}\frac{\partial u}{\partial x}(x_{i-1/2}, t_k) + \frac{h^2}{24}D_{i-1/2}\frac{\partial^3 u}{\partial x^3}(x_{i-1/2}, t_k) + O(h^3). \quad (14)$$

Subtracting (13) and (14) and dividing the result by h, we obtain

$$\begin{aligned}\delta_x(D\delta_x U)_i^k &= \frac{1}{h}\left(D_{i+1/2}\delta_x U_{i+1/2}^k - D_{i-1/2}\delta_x U_{i-1/2}^k\right)\\ &= \frac{1}{h}\left(D_{i+1/2}\frac{\partial u}{\partial x}(x_{i+1/2},t_k) - D_{i-1/2}\frac{\partial u}{\partial x}(x_{i-1/2},t_k)\right)\\ &\quad + O(h^2) = \frac{\partial}{\partial x}\left(D(x)\frac{\partial u(x,t)}{\partial x}\right)\bigg|_{(x_i,t_k)} + O(h^2),\end{aligned} \tag{15}$$

$$\begin{aligned}&f(u(x_i,t_k),u(x_i,t_k-s_1),u(x_i,t_k-s_2),x_i,t_k)\\ &\quad = f\left(2U_i^{k-1}-U_i^{k-2},U_i^{k-m_1-\delta_1},U_i^{k-m_2-\delta_2},x_i,t_k\right)\\ &\quad + \tau^2\frac{\partial^2 u}{\partial t^2}\left(x_i,\eta_i^k\right)f_\mu\left(\rho_k,U_i^{k-m_1-\delta_1},U_i^{k-m_2-\delta_2},x_i,t_k\right).\end{aligned} \tag{16}$$

where $U_i^{k-m_1-\delta_1} = (1-\delta_1)U_i^{k-m_1} + \delta_1 U_i^{k-m_1+1}$, $U_i^{k-m_2-\delta_2} = (1-\delta_2)U_i^{k-m_2} + \delta_2 U_i^{k-m_2+1}$, $\eta_i^k \in (t_k - s_2, t_k)$, and ρ_k is a real number between $u(x_i,t_k)$ and $2U_i^{k-1} - U_i^{k-2}$.

Substituting (11), (15), and (16) into (10), we obtain

$$\begin{aligned}&\delta_t^\alpha U_i^k - \delta_x(D\delta_x U)_i^k\\ &\quad = f\left(2U_i^{k-1}-U_i^{k-2},U_i^{k-m_1-\delta_1},U_i^{k-m_2-\delta_2},x_i,t_k\right) + R_i^k,\end{aligned} \tag{17}$$

where

$$\begin{aligned}R_i^k &= r_i^k + \tau^2\frac{\partial^2 u}{\partial t^2}\left(x_i,\eta_i^k\right)f_\mu\left(\rho_k,U_i^{k-m_1-\delta_1},U_i^{k-m_2-\delta_2},x_i,t_k\right)\\ &\quad + O(h^2).\end{aligned} \tag{18}$$

From Assumptions (H1) and (H2), we can easily obtain

$$\left|R_i^k\right| \le C_R\left(\tau^{2-\alpha} + h^2\right), \quad 0 \le i \le M, 1 \le k \le N. \tag{19}$$

We have the discretization of (4) and (5):

$$U_i^k = \phi(x_i,t_k), \quad 0 \le i \le M, -(m_2+1) \le k \le 0, \tag{20}$$

$$U_0^k = \gamma(t_k), \quad U_M^k = \beta(t_k), \quad 1 \le k \le N. \tag{21}$$

Replacing U_i^k by u_i^k in (17), (20), and (21), and omitting R_i^k, we have the following numerical scheme:

$$\begin{aligned}&\delta_t^\alpha u_i^k - \delta_x(D\delta_x u)_i^k\\ &\quad = f\left(2u_i^{k-1}-u_i^{k-2},u_i^{k-m_1-\delta_1},u_i^{k-m_2-\delta_2},x_i,t_k\right),\end{aligned} \tag{22}$$

$$u_i^k = \phi(x_i,t_k), \quad 0 \le i \le M, -(m_2+1) \le k \le 0, \tag{23}$$

$$u_0^k = \gamma(t_k), \quad u_M^k = \beta(t_k), \quad 1 \le k \le N. \tag{24}$$

3. The Solvability, Convergence, and Stability of the Difference Scheme

Assume the following to be defined on Ω_h

$$\mathscr{V}_{h,0} = \{u \mid u = (u_0,u_1,\cdots,u_M), u_0 = u_M = 0\}. \tag{25}$$

If $u, v \in \mathscr{V}_{h,0}$, the following inner products and corresponding norms are introduced

$$\begin{aligned}(u,v) &= h\sum_{i=1}^{M-1} u_i v_i,\\ \|u\| &= \sqrt{(u,u)},\\ \|u\|_\infty &= \max_{1\le i\le M-1}|u_i|,\\ \langle\delta_x u,\delta_x v\rangle &= h\sum_{i=0}^{M-1}(\delta_x u_{i+1/2})(\delta_x v_{i+1/2}),\\ |\delta_x u|_1 &= \sqrt{\langle\delta_x u,\delta_x u\rangle},\\ \langle\delta_x u,\delta_x v\rangle_D &= h\sum_{i=0}^{M-1}D(x_{i+1/2})(\delta_x u_{i+1/2})(\delta_x v_{i+1/2}),\\ |\delta_x u|_{1D} &= \sqrt{\langle\delta_x u,\delta_x u\rangle_D}.\end{aligned} \tag{26}$$

We can obtain Lemma 3 from Assumption (H1):

Lemma 3. *For $\forall u \in \mathscr{V}_{h,0}$, we have $\sqrt{c_0}|\delta_x u|_1 \le |\delta_x u|_{1D} \le \sqrt{c_1}|\delta_x u|_1$.*

Lemma 4 [30]. *For $\forall u \in \mathscr{V}_{h,0}$, one can obtain*

$$\|u\|_\infty \le |\delta_x u|_1/2, \ \|u\| \le |\delta_x u|_1/\sqrt{6}. \tag{27}$$

We also introduce Lemma 5 to be utilized in the following proof.

Lemma 5 [30]. *Assuming the sequence $\{F^k \ge 0 \mid k \ge 0\}$ satisfies*

$$F^{k+1} \le A + B\tau\sum_{i=1}^{k} F^l, \quad k = 0,1,\cdots, \tag{28}$$

then

$$F^{k+1} \le A\exp(Bk\tau), \quad k = 0,1,2,\cdots, \tag{29}$$

where $A \ge 0$ and $B \ge 0$ are two constants.

Theorem 6. *Under the assumptions of (H1) and (H2), the solution of the difference schemes (22)–(24) is unique.*

Proof. Difference schemes (22)–(24) can be reformed as follows:

When $k = 1$, we have

$$-\frac{D_{i-1/2}}{h^2}u_{i-1}^1 + \left(\frac{1}{\lambda} + \frac{D_{i-1/2} + D_{i+1/2}}{h^2}\right)u_i^1 - \frac{D_{i+1/2}}{h^2}u_{i+1}^1 = \frac{1}{\lambda}u_i^0 + f_i^1, \tag{30}$$

When $k \geq 2$, we have

$$\begin{aligned} &-\frac{D_{i-1/2}}{h^2}u_{i-1}^k + \left(\frac{1}{\lambda} + \frac{D_{i-1/2} + D_{i+1/2}}{h^2}\right)u_i^k - \frac{D_{i+1/2}}{h^2}u_{i+1}^k \\ &\quad = \frac{1}{\lambda}\left[\sum_{j=1}^{k-1}(d_{k-j-1} - d_{k-j})u_i^j + d_{k-1}u_i^0\right] + f_i^k, \end{aligned} \tag{31}$$

where $\lambda = \tau^\alpha \Gamma(2-\alpha)$, $f_i^k = f(2u_i^{k-1} - u_i^{k-2}, u_i^{k-m_1-\delta_1}, u_i^{k-m_2-\delta_2}, x_i, t_k)$, $k = 1, 2, \cdots$. From (30) and (31), we can see that the schemes (22)–(24) are a linear tridiagonal system with strictly diagonally dominant coefficient matrix. Thus, schemes (22)–(31) have a unique solution.

Denote $e_i^k = U_i^k - u_i^k$, $0 \leq i \leq M$, $-(m_2+1) \leq k \leq N$, subtracting (22)–(24) from (17), (20), and (21), respectively, the following error equations can be obtained:

$$\delta_t^\alpha e_i^k - \delta_x(D\delta_x e)_i^k = p_i^k + R_i^k, \tag{32}$$

$$e_i^k = 0, \quad 0 \leq i \leq M, -(m_2+1) \leq k \leq 0, \tag{33}$$

$$e_0^k = 0, \quad e_M^k = 0, \quad 1 \leq k \leq N. \tag{34}$$

where $p_i^k = f(2U_i^{k-1} - U_i^{k-2}, U_i^{k-m_1-\delta_1}, U_i^{k-m_2-\delta_2}, x_i, t_k) - f(2u_i^{k-1} - u_i^{k-2}, u_i^{k-m_1-\delta_1}, u_i^{k-m_2-\delta_2}, x_i, t_k)$.

Theorem 7. *Denote $u(x,t)$ to be the solution of systems (3)–(5), and (22)–(24) has numerical solution $\{u_i^k \mid 0 \leq i \leq M, -(m_2+1) \leq k \leq N\}$. Then, we have*

$$\|e^k\|_\infty \leq C\left(\tau^{2-\alpha} + h^2\right), 1 \leq k \leq N, \tag{35}$$

where C is a positive constant independent of h and τ.

Proof. Multiplying (32) with $h\delta_t^\alpha e_i^k$, one can have

$$\begin{aligned} &h\sum_{i=1}^{M-1}\left(\delta_t^\alpha e_i^k\right)\left(\delta_t^\alpha e_i^k\right) - h\sum_{i=1}^{M-1}\left(\delta_x(D\delta_x e)_i^k\right)\left(\delta_t^\alpha e_i^k\right) \\ &\quad = h\sum_{i=1}^{M-1}p_i^k\left(\delta_t^\alpha e_i^k\right) + h\sum_{i=1}^{M-1}\left(R_i^k\right)\left(\delta_t^\alpha e_i^k\right), \\ &\qquad 1 \leq i \leq M-1, 0 \leq k \leq N-1. \end{aligned} \tag{36}$$

Then each term of (36) will be estimated.

$$h\sum_{i=1}^{M-1}\left(\delta_t^\alpha e_i^k\right)\left(\delta_t^\alpha e_i^k\right) = \left\|\delta_t^\alpha e_i^k\right\|^2. \tag{37}$$

By the discrete Green formula, we have

$$\begin{aligned} &-h\sum_{i=1}^{M-1}\left(\delta_x(D\delta_x e)_i^k\right)\left(\delta_t^\alpha e_i^k\right) \\ &= h\sum_{i=0}^{M-1}D_{i+1/2}\left(\delta_x e_{i+1/2}^k\right)\left(\delta_t^\alpha \delta_x e_{i+1/2}^k\right) \\ &= \frac{1}{\lambda}h\sum_{i=0}^{M-1}D_{i+1/2}\left(\delta_x e_{i+1/2}^k\right)\Bigg[\delta_x e_{i+1/2}^k \\ &\quad - \sum_{j=1}^{k-1}(d_{k-j-1} - d_{k-j})\delta_x e_{i+1/2}^j - d_{k-1}\delta_x e_{i+1/2}^0\Bigg] \\ &= \frac{1}{\lambda}\Bigg\{\left|\delta_x e^k\right|_{1D}^2 - \sum_{j=1}^{k-1}(d_{k-j-1} - d_{k-j})\left\langle\delta_x e^k, \delta_x e^j\right\rangle_D \\ &\quad - d_{k-1}\left\langle\delta_x e^k, \delta_x e^0\right\rangle_D\Bigg\} \\ &\geq \frac{1}{\lambda}\Bigg\{\left|\delta_x e^k\right|_{1D}^2 - \sum_{j=1}^{k-1}(d_{k-j-1} - d_{k-j})\frac{\left|\delta_x e^k\right|_{1D}^2 + \left|\delta_x e^j\right|_{1D}^2}{2} \\ &\quad - d_{k-1}\frac{\left|\delta_x e^k\right|_{1D}^2 + \left|\delta_x e^0\right|_{1D}^2}{2}\Bigg\} \\ &= \frac{1}{\lambda}\left\{\frac{\left|\delta_x e^k\right|_{1D}^2}{2} - \sum_{j=1}^{k-1}(d_{k-j-1} - d_{k-j})\frac{\left|\delta_x e^j\right|_{1D}^2}{2}\right\}. \end{aligned} \tag{38}$$

From the Cauchy-Schwarz inequality, we have

$$\begin{aligned} h\sum_{i=1}^{M-1}\left(R_i^k\right)\left(\delta_t^\alpha e_i^k\right) &\leq h\sum_{i=1}^{M-1}\left(\frac{\left(R_i^k\right)^2}{2\varepsilon} + \frac{\varepsilon}{2}\left(\delta_t^\alpha e_i^k\right)^2\right) \\ &= \frac{1}{2\varepsilon}\|R_i^k\|^2 + \frac{\varepsilon}{2}\|\delta_t^\alpha e_i^k\|^2. \end{aligned} \tag{39}$$

From the Cauchy-Schwarz inequality and Assumption (H2), we have

$$\begin{aligned} h\sum_{i=1}^{M-1}p_i^k\left(\delta_t^\alpha e_i^k\right) &\leq \frac{\varepsilon}{2}\|\delta_t^\alpha e_i^k\|^2 + \frac{1}{2\varepsilon}h\sum_{i=1}^{M-1}\left(p_i^k\right)^2 \\ &\leq \frac{\varepsilon}{2}\|\delta_t^\alpha e_i^k\|^2 + \frac{1}{2\varepsilon}h\sum_{i=1}^{M-1}\Big[c_3\left|2e_i^{k-1} - e_i^{k-2}\right| \\ &\quad + c_4\left|e_i^{k-m_1-\delta_1}\right| + c_5\left|e_i^{k-m_2-\delta_2}\right|\Big]^2 \\ &\leq \frac{\varepsilon}{2}\|\delta_t^\alpha e_i^k\|^2 + \frac{3c_3^2}{2\varepsilon}h\sum_{i=1}^{M-1}\left(2e_i^{k-1} - e_i^{k-2}\right)^2 \\ &\quad + \frac{3c_4^2}{2\varepsilon}h\sum_{i=1}^{M-1}\left(e_i^{k-m_1-\delta_1}\right)^2 \\ &\quad + \frac{3c_5^2}{2\varepsilon}h\sum_{i=1}^{M-1}\left(e_i^{k-m_2-\delta_2}\right)^2 \end{aligned}$$

$$\leq \frac{\varepsilon}{2}\|\delta_t^\alpha e_i^k\|^2 + \frac{24c_3^2}{2\varepsilon}\left(\|e^{k-1}\|^2+\|e^{k-2}\|^2\right) + \frac{3c_4^2}{2\varepsilon}\left(\max\left\{\|e^{k-m_1}\|^2,\|e^{k-m_1+1}\|^2\right\}\right) + \frac{3c_5^2}{2\varepsilon}\left(\max\left\{\|e^{k-m_2}\|^2,\|e^{k-m_2+1}\|^2\right\}\right). \quad (40)$$

Inserting (37)–(40) into (36), we obtain

$$\|\delta_t^\alpha e_i^k\|^2 + \frac{1}{\lambda}\left\{\frac{\left|\delta_x e^k\right|_{1D}^2}{2} - \sum_{j=1}^{k-1}\left(d_{k-j-1}-d_{k-j}\right)\frac{\left|\delta_x e^j\right|_{1D}^2}{2}\right\} \leq \frac{1}{2\varepsilon}\|R_i^k\|^2 + \frac{\varepsilon}{2}\|\delta_t^\alpha e_i^k\|^2 + \frac{\varepsilon}{2}\|\delta_t^\alpha e_i^k\|^2 + \frac{24c_3^2}{2\varepsilon}\left(\|e^{k-1}\|^2 + \|e^{k-2}\|^2\right) + \frac{3c_4^2}{2\varepsilon}\left(\max\left\{\|e^{k-m_1}\|^2, \|e^{k-m_1+1}\|^2\right\}\right) + \frac{3c_5^2}{2\varepsilon}\left(\max\left\{\|e^{k-m_2}\|^2, \|e^{k-m_2+1}\|^2\right\}\right). \quad (41)$$

Multiplying (41) by 2λ, and letting $\varepsilon = 1$, we have

$$\left|\delta_x e^k\right|_{1D}^2 \leq \sum_{j=1}^{k-1}\left(d_{k-j-1}-d_{k-j}\right)\left|\delta_x e^j\right|_{1D}^2 + \lambda\|R_i^k\|^2 + 24\lambda c_3^2\left(\|e^{k-1}\|^2+\|e^{k-2}\|^2\right) + 3\lambda c_4^2\left(\max\left\{\|e^{k-m_1}\|^2,\|e^{k-m_1+1}\|^2\right\}\right) + 3\lambda c_5^2\left(\max\left\{\|e^{k-m_2}\|^2,\|e^{k-m_2+1}\|^2\right\}\right). \quad (42)$$

From Lemmas 3 and 4, and noticing (19), we obtain

$$\left|\delta_x e^k\right|_1^2 \leq \frac{c_1}{c_0}\sum_{j=1}^{k-1}\left(d_{k-j-1}-d_{k-j}\right)\left|\delta_x e^j\right|_1^2 + \frac{\lambda}{c_0}C_R^2\left(\tau^{2-\alpha}+h^2\right)^2 + \frac{4\lambda c_3^2}{c_0}\left(\left|\delta_x e^{k-1}\right|_1^2 + \left|\delta_x e^{k-2}\right|_1^2\right) + \frac{\lambda c_4^2}{2c_0}\left(\max\left\{\left|\delta_x e^{k-m_1}\right|_1^2, \left|\delta_x e^{k-m_1+1}\right|_1^2\right\}\right) + \frac{\lambda c_5^2}{2c_0}\left(\max\left\{\left|\delta_x e^{k-m_2}\right|_1^2, \left|\delta_x e^{k-m_2+1}\right|_1^2\right\}\right), \quad (43)$$

denoting

$$C_k = \frac{1}{c_0}\Gamma(2-\alpha)\max\left\{C_R^2, 4c_3^2, c_4^2/2, c_5^2/2\right\} > 0. \quad (44)$$

Noticing $\lambda = \tau^\alpha\Gamma(2-\alpha)$, one can obtain $\tau^\alpha < 1$ from $0 < \tau < 1$. By (43), one can have

$$\left|\delta_x e^k\right|_1^2 \leq \frac{c_1}{c_0}\sum_{j=1}^{k-1}\left(d_{k-j-1}-d_{k-j}\right)\left|\delta_x e^j\right|_1^2 + C_k\left[\left(\tau^{2-\alpha}+h^2\right)^2 + \left|\delta_x e^{k-1}\right|_1^2 + \left|\delta_x e^{k-2}\right|_1^2 + \max\left\{\left|\delta_x e^{k-m_1}\right|_1^2, \left|\delta_x e^{k-m_1+1}\right|_1^2\right\} + \max\left\{\left|\delta_x e^{k-m_2}\right|_1^2, \left|\delta_x e^{k-m_2+1}\right|_1^2\right\}\right]. \quad (45)$$

From Lemmas 2 and 5, we have

$$\left|\delta_x e^k\right|_1^2 \leq C_k \exp\left(4C_k + \frac{c_1}{c_0}\sum_{j=1}^{k-1}\left(d_{k-j-1}-d_{k-j}\right)\right)\left(\tau^{2-\alpha}+h^2\right)^2 = C_k \exp\left(4C_k + \frac{c_1}{c_0}\left(1-d_{k-1}\right)\right)\left(\tau^{2-\alpha}+h^2\right)^2 \leq C_1\left(\tau^{2-\alpha}+h^2\right)^2, \quad (46)$$

where $C_1 = C_k\exp\left(4C_k + (c_1/c_0)\right)$. From Lemma 4, we have

$$\|e^k\|_\infty \leq \sqrt{C_1\left(\tau^{2-\alpha}+h^2\right)^2} = C\left(\tau^{2-\alpha}+h^2\right). \quad (47)$$

We complete the proof.

To discuss the stability of the difference schemes (22)–(24), we consider the following problem:

$$\frac{\partial^\alpha v}{\partial t^\alpha} - \left(D(x)v_x\right)_x = f\left(v(x,t), v(x,t-s_1), v(x,t-s_2), x, t\right), \quad (x,t)\in(0,1)\times(0,T], \quad (48)$$

$$v(x,t) = \phi(x,t) + \psi(x,t), \quad x\in[0,1], \quad t\in[-s,0], \quad (49)$$

$$v(0,t) = \gamma(t), \quad u(1,t) = \beta(t), \quad t\in(0,T], \quad (50)$$

where $\psi(x,t)$ is the perturbation caused by $\phi(x,t)$. The following difference scheme solving for (48)–(50) can be obtained:

$$\delta_t^\alpha u_i^k - \delta_x\left(D\delta_x u\right)_i^k = f\left(2u_i^{k-1} - u_i^{k-2}, u_i^{k-m_1-\delta_1}, u_i^{k-m_2-\delta_2}, x_i, t_k\right), \quad (51)$$

$$u_i^k = \phi(x_i, t_k) + \psi_i^k, \quad 0\leq i\leq M, -(m_2+1)\leq k\leq 0, \quad (52)$$

$$u_0^k = \gamma(t_k), \quad u_M^k = \beta(t_k), \quad 1\leq k\leq N. \quad (53)$$

Table 1: Simulation results of Example 10, where $h = 1/2000$ and $\alpha = 0.4,0.6,0.8$.

τ	$\alpha = 0.4$ $e(h,\tau)$	Rate_τ	$\alpha = 0.6$ $e(h,\tau)$	Rate_τ	$\alpha = 0.8$ $e(h,\tau)$	Rat_{e_τ}
1/50	4.244e-005	*	1.752e-004	*	6.464e-004	*
1/100	1.609e-005	1.399	6.903e-005	1.344	2.851e-004	1.181
1/150	8.950e-006	1.446	3.963e-005	1.369	1.758e-004	1.192
1/200	6.401e-006	1.165	2.724e-005	1.303	1.253e-004	1.178

Denote

$$\omega_i^k = v_i^k - u_i^k, \quad 0 \le i \le M, -(m_2+1) \le k \le N. \tag{54}$$

where $\bar{C}$ is a bounded constant independent of h and τ.

Definition 8. Assume u_i^k satisfy (22)–(24) and v_i^k satisfy (51)–(53), a numerical scheme for (3)–(5) is stable if we have

$$\left\|\omega^k\right\|_\infty \le \bar{C} \max_{-(m_2+1)\le j\le 0} \left|\psi^j\right|_1, \tag{55}$$

Similar to the proof of Theorem 7, the following stability result can be obtained.

Theorem 9. *Under the same condition as Theorem 7, numerical schemes (22)–(24) satisfy* $\|\omega^k\|_\infty \le \widehat{C} \max_{-(m_2+1)\le j\le 0} |\psi^j|_1$.

4. Numerical Test

Now, we present the following numerical tests to testify the schemes (22)–(24). Introducing the following notations,

$$\begin{aligned} e(h,\tau) &= \max_{1\le k\le N} \left\|U^k - u^k\right\|_\infty, \\ \mathrm{Rate}_\tau &= \frac{\log\left(e(h,\tau_1)/e(h,\tau_2)\right)}{\log\left(\tau_1/\tau_2\right)}, \\ \mathrm{Rate}_h &= \frac{\log\left(e(h_1,\tau)/e(h_2,\tau)\right)}{\log\left(h_1/h_2\right)}. \end{aligned} \tag{56}$$

Table 2: Simulation results of Example 10, where $\tau = 1/2000$ and $\alpha = 0.4,0.6,0.8$.

h	$\alpha = 0.4$ $e(h,\tau)$	Rate_h	$\alpha = 0.6$ $e(h,\tau)$	Rate_h	$\alpha = 0.8$ $e(h,\tau)$	Rate_h
1/10	6.213e-002	*	6.072e-002	*	5.889e-002	*
1/20	1.534e-002	2.018	1.500e-002	2.017	1.455e-002	2.017
1/40	3.842e-003	1.997	3.758e-003	1.997	3.654e-003	1.994
1/80	9.599e-004	2.001	9.396e-004	2.000	9.185e-004	1.992

when considering for Rate_τ, h should be fixed and small enough. While considering for Rate_h, τ should be fixed and small enough.

$$\begin{cases} \dfrac{\partial^\alpha u}{\partial t^\alpha} - \dfrac{\partial u}{\partial x}\left(\left(x^2+1\right)\dfrac{\partial u}{\partial x}\right) = \dfrac{u(x,t-0.1)}{1+u^2(x,t-0.05)} + G(x,t), \\ (x,t) \in (0,1)\times(0,T], \\ u(x,t) = t^{2+\alpha}\cos(2\pi x), \quad x\in[0,1], \quad t\in[-0.1,0], \\ u(0,t) = t^{2+\alpha}, \quad u(1,t) = t^{2+\alpha}, \quad t\in(0,1]. \end{cases} \tag{57}$$

Example 10. Let $D(x) = x^2 + 1$.

Equation (57) has exact solution $u(x,t) = t^{2+\alpha}\cos(2\pi x)$, where

$$\begin{aligned} G(x,t) = &\left[\frac{\Gamma(3+\alpha)}{\Gamma(3)}t^2 + 4\pi^2\left(x^2+1\right)t^{2+\alpha}\right]\cos(2\pi x) \\ &+ 4\pi x t^{2+\alpha}\sin(2\pi x) \\ &- \frac{(t-0.1)^{2+\alpha}\cos(2\pi x)}{1+(t-0.05)^{4+2\alpha}\cos^2(2\pi x)}. \end{aligned} \tag{58}$$

Table 1 presents the maximum errors in the temporal directions, where $\alpha = 0.4,0.6,0.8$, and $h = 1/2000$. From the results, we can draw a conclusion that the convergence order in the temporal directions is $2-\alpha$. Table 2 presents the maximum errors in the spatial directions, where $\alpha = 0.4,0.6,0.8$, and $\tau = 1/2000$. From the results, we can draw a conclusion that the convergence order in the spatial directions is 2.

Figure 1 gives the error planes of Example 10, where $\tau = h = 1/100$, and $\alpha = 0.1,0.3,0.7,0.9$. Figure 1 tells us that bigger α can bring bigger error.

Example 11. Let $D(x) = x^2 + 1$

$$\begin{cases} \dfrac{\partial^\alpha u}{\partial t^\alpha} - \dfrac{\partial u}{\partial x}\left(x^2\dfrac{\partial u}{\partial x}\right) = -u(x,t)^2 + u(x,t-0.1) + u(x,t-0.2) + G(x,t), (x,t)\in(0,1)\times(0,T], \\ u(x,t) = e^x x^2(1-x)^2 t^{2+\alpha}, \quad x\in[0,1], \quad t\in[-0.2,0], \\ u(0,t) = 0, \quad u(1,t) = 0, \quad t\in(0,1]. \end{cases} \tag{59}$$

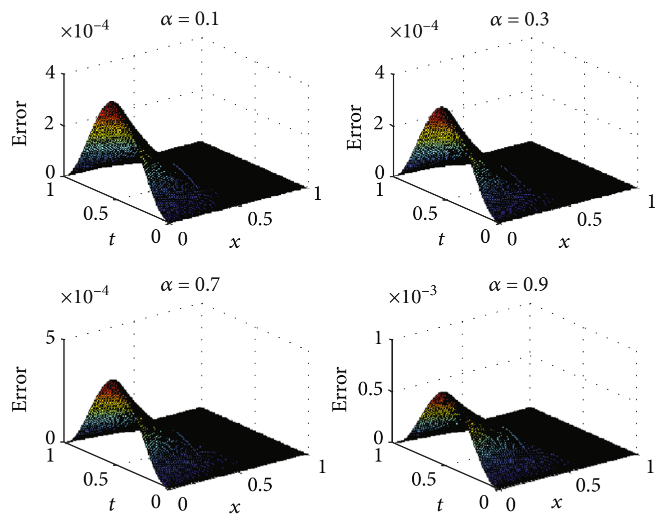

Figure 1: Error planes of Example 10, where $\tau = h = 1/100$.

Table 3: Simulation results of Example 11, where $h = 1/2000$ and $\alpha = 0.3, 0.5, 0.8$.

τ	$\alpha = 0.3$ $e(h,\tau)$	Rate_τ	$\alpha = 0.5$ $e(h,\tau)$	Rate_τ	$\alpha = 0.8$ $e(h,\tau)$	Rate_τ
1/10	9.029e-005	*	2.277e-004	*	9.049e-004	*
1/20	2.712e-005	1.735	7.924e-005	1.523	3.941e-004	1.199
1/40	8.183e-006	1.729	2.763e-005	1.520	1.713e-004	1.202
1/80	2.530e-006	1.693	9.707e-006	1.509	7.451e-005	1.201

Table 4: Simulation results of Example 11, where $\tau = 1/2000$ and $\alpha = 0.3, 0.5, 0.8$.

h	$\alpha = 0.3$ $e(h,\tau)$	Rate_h	$\alpha = 0.5$ $e(h,\tau)$	Rate_h	$\alpha = 0.8$ $e(h,\tau)$	Rate_h
1/8	8.064e-003	*	7.917e-003	*	7.624e-003	*
1/16	2.100e-003	1.941	2.057e-003	1.945	1.971e-003	1.952
1/32	5.272e-004	1.994	5.163e-004	1.994	4.957e-004	1.991
1/64	1.319e-004	1.998	1.293e-004	1.998	1.250e-004	1.987

Equation (59) has exact solution $u(x,t) = e^x x^2 (1-x)^2 t^{2+\alpha}$, where

$$
\begin{aligned}
G(x,t) = {} & e^{2x} x^4 (1-x)^4 t^{4+2\alpha} - e^x x^2 (1-x)^2 (t-0.1)^{2+\alpha} \\
& - e^x x^2 (1-x)^2 (t-0.2)^{2+\alpha} \\
& + \frac{\Gamma(3+\alpha)}{2} t^2 e^x x^2 (1-x)^2 \\
& - e^x t^{2+\alpha} \left(6x^2 - 18x^3 + 5x^4 + 8x^5 + x^6\right).
\end{aligned}
\tag{60}
$$

Tables 3 and 4 show the computational results for a different α of Example 11. When $h = 1/2000$, the maximum errors and convergence orders are provided in Table 3. It can be shown that the $2-\alpha$ order accuracy in temporal direction is verified, while when $\tau = 1/2000$, from Table 4, we can see that the 2 order accuracy in spatial direction is verified.

5. Conclusion

This paper provides a finite difference scheme for solving a type of variable coefficient time fractional subdiffusion equation with multidelay. The unconditional stability and the global convergence of the scheme in the maximum norm are proved. Numerical experiments were provided to support the theoretical results and testify the efficiency of the difference scheme.

Data Availability

The author declares that the readers can access the data supporting the conclusions of the study online directly.

Conflicts of Interest

The author declares that there is no conflicts of interests regarding the publication of this article.

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References

[1] R. D. Driver, *Ordinary and Delay Differential Equations*, Springer-Verlag, New York, 1977.

[2] K. Yang and L. Chrlotte, *Delay Differential Equations: With Applications in Population Dynamics*, Academic Press, New York, 1993.

[3] J. Wu, *Theory and Applications of Partial Functional Differential Equations*, Springer, New York, 1996.

[4] J. E. Forde, *Delay Differential Equation Models in Mathematical Biology*, University of Michigan, Ann Arbor, MI, 2005.

[5] Q. Huang, D. Qiumei, and J. Z. Li, "Numerical investigations of a class of biological models on unbounded domain,"

Numerical Mathematics: Theory, Methods and Applications, vol. 12, no. 1, pp. 169–186, 2019.

[6] D. Li and C. Zhang, "Nonlinear stability of discontinuous Galerkin methods for delay differential equations," *Applied Mathematics Letters*, vol. 23, no. 4, pp. 457–461, 2010.

[7] D. Li and C. Zhang, "Superconvergence of a discontinuous Galerkin method for first-order linear delay differential equations," *Journal of Computational Mathematics*, vol. 29, no. 5, pp. 574–588, 2011.

[8] D. Li and C. Zhang, "error estimates of discontinuous Galerkin methods for delay differential equations," *Applied Numerical Mathematics*, vol. 82, pp. 1–10, 2014.

[9] H. Qin, Q. Zhang, and S. Wan, "The continuous Galerkin finite element methods for linear neutral delay differential equations," *Applied Mathematics and Computation*, vol. 346, pp. 76–85, 2019.

[10] H. J. Tian and J. X. Kuang, "The stability of the H-methods in the numerical solution of delay differential equations with several delay terms," *Journal of Computational and Applied Mathematics*, vol. 58, pp. 171–181, 1995.

[11] C. T. Baker, G. A. Bocharov, and A. Rihan, "A report on the use of delay differential equations in numerical modeling in the biosciences," *MCCM Technical Report, Manchester*, vol. 343, pp. 2–6, 1999.

[12] M. Z. Liu and M. N. Spiker, "The stability of the H-methods in the numerical solution of delay differential equations," *IMA Journal of Numerical Analysis*, vol. 10, no. 1, pp. 31–48, 1990.

[13] J. X. Kuang, "The PL-stability of block H-methods," *Mathematica Numerica Sinica-Chinese Edition*, vol. 2, pp. 135–140, 1997.

[14] Q. Zhang, C. Zhang, and L. Wang, "The compact and Crank-Nicolson ADI schemes for two-dimensional semilinear multi-delay parabolic equations," *Journal of Computational and Applied Mathematics*, vol. 306, pp. 217–230, 2016.

[15] C. Hwang and Y. C. Cheng, "A numerical algorithm for stability testing of fractional delay systems," *Automatica*, vol. 42, no. 5, pp. 825–831, 2006.

[16] L. C. Davis, "Modification of the optimal velocity traffic model to include delay due to driver reaction time," *Physica A*, vol. 319, pp. 557–567, 2002.

[17] Y. Kuang, *Delay Differential Equations with Applications in Population Biology*, Academic Press, Boston (MA), 1993.

[18] S. Bhalekar and V. Daftardar-Gejji, "A predictor-corrector scheme for solving nonlinear delay differential equations of fractional order," *Journal of Fractional Calculus and Applications*, vol. 1, pp. 1–8, 2011.

[19] V. Daftardar-Gejji and Y. Sukale, "Solving fractional delay differential equations: a new approach," *Fractional Calculus and Applied Analysis*, vol. 18, pp. 400–418, 2015.

[20] Z. Wang, X. Huang, and J. Zhou, "A numerical method for delayed fractional-order differential equations: based on G-L definition," *Applied Mathematics & Information Sciences*, vol. 7, no. 2L, pp. 525–529, 2013.

[21] F. A. Rihan, "Computational methods for delay parabolic and time-fractional partial differential equations," *Numerical Methods for Partial Differential Equations*, vol. 26, no. 6, pp. 1556–1571, 2010.

[22] M. Sakara, F. Uludag, and F. Erdogan, "Numerical solution of time-fractional nonlinear PDEs with proportional delays by homotopy perturbation method," *Applied Mathematical Modelling*, vol. 40, no. 13-14, pp. 6639–6649, 2016.

[23] Q. F. Zhang, M. H. Ran, and D. H. Xu, "Analysis of the compact difference scheme for the semilinear fractional partial differential equation with time delay," *Applicable Analysis*, vol. 96, pp. 1–18, 2016.

[24] Q. Zhang and T. Li, "Asymptotic stability of compact and linear $$\theta $$θ-Methods for space fractional delay generalized diffusion equation," *Journal of Scientific Computing*, vol. 81, no. 3, pp. 2413–2446, 2019.

[25] Q. Zhang, L. Liu, and C. Zhang, "Compact scheme for fractional diffusion-wave equation with spatial variable coefficient and delays," *Applicable Analysis*, vol. 2020, pp. 1–22, 2020.

[26] J. Xie, D. Deng, and H. Zheng, "A compact difference scheme for one-dimensional nonlinear delay reaction-diffusion equations with variable coefficient," *International Journal of Applied Mathematics*, vol. 47, no. 1, pp. 14–19, 2017.

[27] F. Wu, X. Cheng, D. Li, and J. Duan, "A two-level linearized compact ADI scheme for two-dimensional nonlinear reaction-diffusion equations," *Computers & Mathematics with Applications*, vol. 75, no. 8, pp. 2835–2850, 2018.

[28] Z. Sun and X. Wu, "A fully discrete difference scheme for a diffusion-wave system," *Applied Numerical Mathematics*, vol. 56, no. 2, pp. 193–209, 2006.

[29] S. Chen, F. Liu, P. Zhuang, and V. Anh, "Finite difference approximations for the fractional Fokker-Planck equation," *Applied Mathematical Modelling*, vol. 33, no. 1, pp. 256–273, 2009.

[30] Z. Z. Sun, *The Numerical Methods for Partial Equations*, Science Press, Beijing, 2005, (in Chinese).