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*Research Article*

# Existence of Solutions for Fractional Boundary Value Problems with a Quadratic Growth of Fractional Derivative

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In this paper, we deal with two fractional boundary value problems which have linear growth and quadratic growth about the fractional derivative in the nonlinearity term. By using variational methods coupled with the iterative methods, we obtain the existence results of solutions. To the best of the authors' knowledge, there are no results on the solutions to the fractional boundary problem which have quadratic growth about the fractional derivative in the nonlinearity term.

## 1. Introduction

It has been seen that fractional differential equations have better effects in many realistic applications than the classical ones. Qualitative theory and its applications in physics, engineering, economics, biology, and ecology are extensively discussed and demonstrated in [1–4] and the references therein. Some recent contributions to the theory of fractional differential equation can be seen in [5–10].

Some classical tools such as fixed point theorems [5], the method of upper and lower solutions, and monotone iterative technique [11, 12] have been widely used to study the fractional differential equation. Recently, the study of fractional differential equations has attracted much attention by using variational methods, for example, [7, 8, 13–19]. We also mention that in the recent works [20, 21], the authors have developed a general approach concerting the existence of solutions.

Fractional differential equations containing left and right fractional differential operators have received attention from scientists due to their applications in physical phenomena exhibiting anomalous diffusion. In [7], appropriate fractional derivative spaces were defined and existence and uniqueness results for a fractional boundary value problem were proven using the Lax-Milgram theorem.

Jiao and Zhou [8] showed the variational structure of a fractional boundary value problem under an appropriate functional space; they used the least action principle and the Mountain Pass theorem to obtain the existence of at least one solution. Sun and Zhang [22] obtained the existence result for a fractional boundary value problem by using the Mountain Pass method and an iterative technique. In [23], the authors discussed the existence of a fractional boundary value problem with linear growth about the fractional derivative in a nonlinearity term. Compared with some integral-order partial differential equations such as [6, 24–31], the fractional derivatives have hereditary and nonlocal properties so that they are much more suitable for describing long-memory processes than the classical integer-order derivatives.

Motivated by the above papers, in this paper, we first investigated the existence of solutions for the following fractional boundary value problems:

$$\begin{cases} {}_tD_T^{\alpha}{}_0D_t^{\alpha}u(t) + {}_0D_t^{\alpha}{}_tD_T^{\alpha}u(t) = f(t, u(t), {}_0D_t^{\alpha}u(t)), \text{a.e.} t \in [0, T], \\ u(0) = u(T) = 0, \end{cases} \tag{1}$$

where ${}_0D_t^\alpha$ and ${}_tD_T^\alpha$ are the left and right fractional Riemann-Liouville derivatives of order $1/2 < \alpha < 1$, respectively, $f : [0, T] \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$.

Note that problem (1) is not variational due to the fractional derivative contained in nonlinearity, so we cannot find a functional such that its critical point is the weak solution corresponding to (1). In order to overcome this difficulty, we consider the following fractional boundary value problem which is independent on the fractional derivative of the solution

$$\begin{cases} {}_tD_{T}^{\alpha}{}_0D_t^\alpha u(t) + {}_0D_t^\alpha{}_tD_T^\alpha u(t) = f(t, u(t), {}_0D_t^\alpha w(t)), \text{a.e.t} \in [0, T], \\ u(0) = u(T) = 0, \end{cases} \tag{2}$$

where $w$ is an element of fractional Sobolev space $E^\alpha$. First, by using variational methods, we obtain the existence of solutions for (2). Then, under the assumption that $f$ is linear growth about the fractional derivative and based on iterative methods, we show there exists a solution for (1). Our conditions are weaker than that in [23].

We also discuss the following fractional boundary value problem:

$$\begin{cases} {}_tD_{T}^{\alpha}{}_0D_t^\alpha u(t) + {}_0D_t^\alpha{}_tD_T^\alpha u(t) = a(t)g(u(t)) + \lambda|{}_0D_t^\alpha u(t)|^2, \text{a.e.t} \in [0, T], \\ u(0) = u(T) = 0, \end{cases} \tag{3}$$

where $a \in L^1(0, T; \mathbb{R}^+)$, $\lambda$ is a parameter, and $g \in C(\mathbb{R}; \mathbb{R})$. Compared with (1), the nonlinearity of (3) is quadratic growth about a fractional derivative. By using variational methods and an iterative technique, we obtain that there exists solutions for (3) when $\lambda$ and $a$ satisfy suitable conditions. To the best of the authors' knowledge, there are no results on the solutions to the fractional boundary problem which have quadratic growth about the fractional derivative in the nonlinearity term .

The paper is organized as follows. In Section 2, we will list some important properties of the basic functional space. We show the existence results for (1) and (3) in Section 3 and Section 4, respectively.

## 2. Preliminary

Let us briefly recall the property of a fractional derivative which will be used to construct the variational functional.

**Lemma 1** [1]. *For $0 < \beta \le 1$, if $f \in L^p([a, b]\mathbb{R})$ with $p(1-\beta) > 1$, ${}_0J_t^{1-\beta} f(t)$ is absolutely continuous, and $g : ([a, b], \mathbb{R})$ is absolutely continuous with $g(b) = 0$, then*

$$\int_a^b \left[{}_aD_t^\beta f(t)\right] g(t)dt = \int_a^b \left[{}_tD_b^\beta g(t)\right] f(t)dt. \tag{4}$$

Now, we recall some properties of the basic function space which have been studied in [32].

Throughout this paper, let $1/2 < \alpha < 1$.

The fractional derivative space $E^\alpha$ is defined by the completion of $C_0^\infty(0, T)$ with respect to the norm $\|u\| = \left(\int_0^T |u(t)|^2 dt + \int_0^T |{}_0D_t^\alpha u(t)|^2 dt\right)^{1/2}$, where ${}_0D_t^\alpha$ is the $\alpha$-order left Riemann-Liouville fractional derivative. Then, $E^\alpha$ is a reflexive and separable Hilbert space. And the Riemann-Liouville fractional derivative exists for the elements in $E^\alpha$ [22].

**Lemma 2** [32]. *For all $u \in E^\alpha$, we have*

$$\|u\|_{L^2} \le \frac{T^\alpha}{\Gamma(\alpha+1)} \|{}_0D_t^\alpha u\|_{L^2}, \tag{5}$$

$$\|u\|_\infty \le \frac{T^{\alpha-(1/2)}}{\Gamma(\alpha)(2\alpha-1)^{1/2}} \|{}_0D_t^\alpha u\|_{L^2}. \tag{6}$$

According to (5), one can consider $E^\alpha$ with respect to the equivalent norm

$$\|u\|_\alpha = \|{}_0D_t^\alpha u\|_{L^2}. \tag{7}$$

**Lemma 3** [32]. *If the sequence $\{u_k\}$ converges weakly to $u$ in $E^\alpha$, i.e., $u_k \rightharpoonup u$, then $u_k \longrightarrow u$ in $C([0, T], \mathbb{R}^N)$, i.e., $\|u - u_k\|_\infty \longrightarrow 0$ as $k \longrightarrow \infty$.*

*By the proof of Proposition 4.1 in [8], we have the following property.*

**Lemma 4.** *For any $u \in E^\alpha$,*

$$|\cos(\pi\alpha)|^2 \|u\|_\alpha^2 \le \|{}_tD_T^\alpha u\|_{L^2}^2 \le \frac{1}{|\cos(\pi\alpha)|^2} \|u\|_\alpha^2. \tag{8}$$

## 3. Existence Result for (1)

We assume that $f(t, x, y)$ satisfies the following conditions:

$(H_0)$ $f : [0, T] \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is measurable in $t \in [0, T]$ for every $(x, y) \in \mathbb{R} \times \mathbb{R}$ and continuous in $(x, y) \in \mathbb{R} \times \mathbb{R}$ for a.e.t $\in [0, T]$, and there exist $a \in C(\mathbb{R}^+, \mathbb{R}^+)$, $b \in L^1(0, T; \mathbb{R}^+)$, such that

$$|f(t, x, y)| \le a(|x|)b(t) + y^2, \tag{9}$$

for all $t \in [0, T]$, $x, y \in \mathbb{R}$.

$(H_1)$ There are constants $\mu > 2$ and $R > 0$ such that, for $|x| \ge R$, $y \in \mathbb{R}$,

$$0 < \mu F(t, x, y) \le f(t, x, y)x, \tag{10}$$

where $F(t, x, y) = \int_0^x f(t, s, y)ds$.

$(H_2)$ $\lim_{x\to 0}(f(t, x, y)/x) = 0$ uniformly for $t \in [0, T]$ and $y \in \mathbb{R}$.

In order to derive a weak solution of (1), we suppose that $u$ is a solution of (1), and multiplying (1) by an arbitrary $v \in C_0^\infty(0,T)$ and by Lemma 1, we have

$$\int_0^T {}_0D_t^\alpha u(t)\,{}_0D_t^\alpha v(t) + {}_tD_T^\alpha u(t)\,{}_tD_T^\alpha v(t)\,dt - \int_0^T f(t,u(t),{}_0D_t^\alpha u(t))v(t)\,dt = 0. \quad (11)$$

Since (11) is well defined for $u, v \in E^a$, the weak solution of (1) may be defined as follows.

*Definition 5.* A weak solution of (1) is a function $u \in E^\alpha$ such that

$$\int_0^T {}_0D_t^\alpha u(t)\,{}_0D_t^\alpha v(t) + {}_tD_T^\alpha u(t)\,{}_tD_T^\alpha v(t)\,dt - \int_0^T f(t,u(t),{}_0D_t^\alpha u(t))v(t)\,dt = 0, \quad (12)$$

for every $v \in E^\alpha$.

*Definition 6.* A function $u$ is called a solution of (1), if ${}_0D_t^\alpha u(t)$ and ${}_tD_T^\alpha u(t)$ exist, $({}_{-t}J_T^{1-\alpha}\,{}_0D_t^\alpha u(t) + {}_0J_t^{1-\alpha}\,{}_tD_T^\alpha u(t))$ is derivable for almost every $t \in [0,T]$, and $u$ satisfies (1).

For a given $w \in E^\alpha$, we consider the functional $\varphi_w : E^\alpha \longrightarrow \mathbb{R}$, defined by

$$\varphi_w(u) = \frac{1}{2}\int_0^T |{}_0D_t^\alpha u(t)|^2 + |{}_tD_T^\alpha u(t)|^2\,dt - \int_0^T F(t,u(t),{}_0D_t^\alpha w(t))\,dt. \quad (13)$$

In view of assumption $(H_0)$, we know that $\varphi$ is continuously differentiable and

$$\varphi_w'(u)v = \int_0^T {}_0D_t^\alpha u(t)\,{}_0D_t^\alpha v(t) + {}_tD_T^\alpha u(t)\,{}_tD_T^\alpha v(t)\,dt - \int_0^T f(t,u(t),{}_0D_t^\alpha w(t))v(t)\,dt, \quad (14)$$

for $u$, $v \in E^\alpha$. Hence, a critical point of $\varphi_w$ gives us a weak solution of (2).

**Lemma 7.** *If $u$ is a weak solution of (1), then $u$ is also a solution of (1).*

*Proof.* Let $u$ be a weak solution of (1), then $u \in E^\alpha$, so ${}_0D_t{}^\alpha u(t)$ and ${}_tD_T{}^\alpha u(t)$ exist and $u(0) = u(T) = 0$. For every $v \in C_0^\infty$, we have

$$\int_0^T {}_0D_t^\alpha u(t)\,{}_0D_t^\alpha v(t) + {}_tD_T^\alpha u(t)\,{}_tD_T^\alpha v(t)\,dt - \int_0^T f(t,u(t),{}_0D_t^\alpha u(t))v(t)\,dt = 0. \quad (15)$$

Since $v \in C_0^\infty$, we have ${}_0D_t^\alpha v(t) = {}_0J_t^\alpha v'(t)$ and ${}_tD_T^\alpha v(t) = {}_{-t}J_T^\alpha v'(t)$, so

$$\int_0^T \left( {}_{-t}J_T^{1-\alpha}\,{}_0D_t^\alpha u(t) + {}_0J_t^{1-\alpha}\,{}_tD_t^\alpha u(t) - \int_0^t f(s,u(s),{}_0D_s^\alpha u(s))\,ds \right) v'(t)\,dt = 0. \quad (16)$$

Then, there exists a constant $C$ such that

$${}_{-t}J_T^{1-\alpha}\,{}_0D_t^\alpha u(t) + {}_0J_t^{1-\alpha}\,{}_tD_T^\alpha u(t) - \int_0^t f(s,u(s),{}_0D_s^\alpha u(s))\,ds = C, \quad (17)$$

thus,

$${}_tD_T^\alpha\,{}_0D_t^\alpha u(t) + {}_0D_t^\alpha\,{}_tD_T^\alpha u(t) = f(t,u(t),{}_0D_t^\alpha u(t)),\ t \in [0,T]. \quad (18)$$

**Lemma 8.** *Suppose $(H_0)$ and $(H_1)$ hold, then $\varphi_w$ satisfies the (PS) condition.*

*Proof.* Let $\{u_n\} \subset E^\alpha$, $\{\varphi_w(u_n)\}$ is bounded, and $\varphi_w'(u_n) \longrightarrow 0$; we first show that $\{u_n\}$ is bounded.

It follows from $(H_1)$ that

$$\begin{aligned}
\mu\varphi_w(u_n) - \varphi_w'(u_n)u_n &= \left(\frac{\mu}{2} - 1\right)\int_0^T |{}_0D_t^\alpha u_n(t)|^2 + |{}_tD_T^\alpha u_n(t)|^2\,dt \\
&\quad + \int_{\{t\in[0,T] \,\|\, u_n(t)| \ge R\}} f(t,u_n(t),{}_0D_t^\alpha w(t))u_n(t) \\
&\quad - \mu F(t,u_n(t),{}_0D_t^\alpha w(t))\,dt \\
&\quad + \int_{\{t\in[0,T] \,\|\, u_n(t)| < R\}} f(t,u_n(t),{}_0D_t^\alpha w(t))u_n(t) \\
&\quad - \mu F(t,u_n(t),{}_0D_t^\alpha w(t))\,dt \\
&\ge \left(\frac{\mu}{2} - 1\right)\int_0^T |{}_0D_t^\alpha u_n(t)|^2 + |{}_tD_T^\alpha u(t)|^2\,dt \\
&\quad + \int_{\{t\in[0,T] \,\|\, u_n(t)| < R\}} (f(t,u_n(t),{}_0D_t^\alpha w(t))u_n(t) \\
&\quad - \mu F(t,u_n(t),{}_0D_t^\alpha w(t))),
\end{aligned} \quad (19)$$

which implies that $\{u_n\}$ is bounded.

From the reflexivity of $E^\alpha$, we may extract a weakly convergent subsequence that, for simplicity, we call $\{u_n\}$, $u_n \rightharpoonup u$ then $\|u_n - u\|_\infty \longrightarrow 0$. Next, we will prove that $\{u_n\}$ strongly converges to $u$. By $(H_0)$, we know that

$$\int_0^T (f(t, u_n(t), {}_0D_t^\alpha w(t)) - f(t, u(t), {}_0D_t^\alpha w(t))(u_n(t) - u(t))dt \longrightarrow 0 \text{ as } n \longrightarrow \infty. \quad (20)$$

Note that

$$\begin{aligned}\left(\varphi_w'(u_n) - \varphi_w'(u)\right)(u_n - u) &= \int_0^T |{}_0D_t^\alpha (u_n(t) - u(t))|^2 \\ &+ |{}_tD_T^\alpha (u_n(t) - u(t))|^2 dt - \int_0^T (f(t, u_n(t), {}_0D_t^\alpha w(t)) \\ &- f(t, u(t), {}_0D_t^\alpha w(t))(u_n(t) - u(t))dt \geq \left(1 + |\cos \pi\alpha|^2\right) \|u_n - u\|_\alpha^2 \\ &- \int_0^T (f(t, u_n(t), {}_0D_t^\alpha w(t)) - f(t, u(t), {}_0D_t^\alpha w(t))(u_n(t) - u(t))dt.\end{aligned} \quad (21)$$

By $\varphi'(u_n) \longrightarrow 0$ and $u_n \rightharpoonup u$, we obtain that

$$\left(\varphi'(u_n) - \varphi'(u)\right)(u_n - u) \longrightarrow 0 \text{ as } n \longrightarrow \infty. \quad (22)$$

Thus, $\|u_n - u\|_\alpha \longrightarrow 0$ as $n \longrightarrow \infty$. Therefore, $\varphi_w$ satisfies the (PS) condition.

**Lemma 9.** *Let $w \in E^\alpha$ and suppose $(H_0)$, $(H_1)$, and $(H_2)$ hold, then (2) has at least one nontrivial solution.*

*Proof.* The proof relies on the Mountain Pass theorem [33, 34]. It is clear that $\varphi_w \in C^1(E^\alpha, R)$, $\varphi_w(0) = 0$, and $\varphi_w$ satisfies the (PS) condition from Lemma 8. By $(H_2)$, for all $\varepsilon > 0$, there is a $\delta > 0$ such that

$$F(t, x, y) \leq \varepsilon |x|^2, \ |x| < \delta, \ t \in [0, T], \ y \in \mathbb{R}. \quad (23)$$

Let $\rho = \delta \Gamma(\alpha)(2\alpha - 1)^{1/2} / 2T^{\alpha-(1/2)}$ and choose $\|u\|_\alpha = \rho$; by the above inequality and (6), we have

$$\begin{aligned}\varphi_w(u) &= \frac{1}{2}\int_0^T |{}_0D_t^\alpha u(t)|^2 + |{}_tD_T^\alpha u(t)|^2 dt \\ &- \int_0^T F(t, u(t), {}_0D_t^\alpha w(t))dt \geq \frac{1 + |\cos \pi\alpha|^2}{2} \|u\|_\alpha^2 \\ &- \varepsilon T \|u\|_\infty^2 \geq \left(\frac{1 + |\cos \pi\alpha|^2}{2} - \frac{\varepsilon T^{2\alpha}}{\Gamma(\alpha)^2 (2\alpha - 1)}\right) \|u\|_\alpha^2.\end{aligned} \quad (24)$$

It suffices to choose $\varepsilon = (1 + |\cos \pi\alpha|^2)\Gamma(\alpha)^2(2\alpha - 1)/4 T^{2\alpha}$ and $\sigma = ((1 + |\cos \pi\alpha|^2)/4)\rho$ to get $\varphi_w(u) \geq \sigma$. Thus, there exist positive numbers $\rho$ and $\sigma$ which are independent of $w$ such that $\varphi_w(u) \geq \sigma$ for $u \in E^\alpha$ satisfies $\|u\|_\alpha = \rho$.

It follows from $(H_1)$ that there exist $c_1, c_2 > 0$ such that

$$F(t, x, y) \geq c_1 |x|^\mu - c_2, \quad \forall t \in [0, T], \ x, y \in \mathbb{R}. \quad (25)$$

Choosing $\tilde{u} \in E^\alpha$ satisfies $\|\tilde{u}\|_\alpha = 1$, and we obtain

$$\begin{aligned}\varphi_w(r\tilde{u}) &= \frac{r^2}{2}\int_0^T |{}_0D_t^\alpha \tilde{u}(t)|^2 + |{}_tD_T^\alpha \tilde{u}(t)|^2 dt \\ &- \int_0^T F(t, r\tilde{u}(t), {}_0D_t^\alpha w(t))dt \leq \frac{r^2}{2}\left(1 + \frac{1}{|\cos \pi\alpha|^2}\right) \\ &- c_1 r^\mu \int_0^T |\tilde{u}|^\mu dt + c_2 T,\end{aligned} \quad (26)$$

which implies that $\varphi(r\tilde{u}) \longrightarrow -\infty$ as $r \longrightarrow \infty$. Hence, we obtain that there exists a $\beta > 0$ independent of $u_1$ and $w$ such that $\varphi_w(u) \leq 0$ for all $\|u\|_\alpha > \beta$.

The above discussions combined with the Mountain Pass theorem show that (2) has at least one nontrivial solution $u_w$ which can be characterized as

$$\varphi_w'(u_w) = 0, \varphi_w(u_w) = \inf_{g \in \Gamma} \max_{u \in g([0,1])} \varphi_w(u), \quad (27)$$

where $\Gamma = \{g \in C([0, 1], E^\alpha) | g(0) = 0, g(1) = \beta\tilde{u}\}$.

In order to obtain the existence of solutions for (1), we need the following Lipschitz condition.

$(H_3)$ There exist $L_1, L_2 > 0$ such that

$$\begin{aligned}&|f(t, x_1, y) - f(t, x_2, y)| \leq L_1 |x_1 - x_2|, \\ &\quad \forall t \in [0, T], \ x_1, x_2 \in [0, r_1], \ y \in \mathbb{R}, \\ &|f(t, x, y_1) - f(t, x, y_2)| \leq L_2 |y_1 - y_2|, \\ &\quad \forall t \in [0, T], \ x \in [0, r_1], \ y_1, y_2 \in \mathbb{R},\end{aligned} \quad (28)$$

where $r_1 = c_2 T^{\alpha-(1/2)} / \Gamma(\alpha)(2\alpha - 1)^{1/2}$; $C_2$ will be determined later.

**Theorem 10.** *Let $\gamma = L_2 T^\alpha \Gamma(\alpha + 1) / ((1 + |\cos \pi\alpha|^2)\Gamma(\alpha + 1)^2 - L_1 T^{2\alpha})$, suppose $(H_0)$–$(H_3)$ hold, $2R/(1 + |\cos \pi\alpha|^2) < 1$, and $0 < \gamma < 1$, then problem (1) has a nontrivial solution.*

*Proof.* For $n = 1, 2, \cdots$ and $u_1 = 0$, we construct a sequence $\{u_n\}$, where $u_n$ is the solution of the following problem:

$$\begin{cases} {}_tD_T^\alpha {}_0D_t^\alpha u_n(t) + {}_0D_t^\alpha {}_tD_T^\alpha u_n(t) = F(t, u_n(t), {}_0D_t^\alpha u_{n-1}(t)), \text{ a.e. } t \in [0, T], \\ u_n(0) = u_n(T) = 0. \end{cases} \quad (29)$$

Now, we assume $\|u_{n-1}\|_\alpha \leq C_2$; by the mathematical induction, we will prove that $\|u_n\|_\alpha \leq C_2$. It follows from $u_n$ satisfying (29) that

$$\varphi_{u_{n-1}}(u_n) = \inf_{g\in\Gamma u\in g([0,1])} \max \varphi_{u_{n-1}}((u) \le \max_{r\in[0,\beta]} \varphi_{u_{n-1}}((r\tilde{u}) \le \max_{r\in[0,\beta]} \left(\frac{r^2}{2}\left(1+\frac{1}{|\cos\pi\alpha|^2}\right) - c_1 r^\mu \int_0^T |\tilde{u}|^\mu dt + c_2 T\right). \quad (30)$$

Let

$$h(r) = \frac{r^2}{2}\left(1+\frac{1}{|\cos\pi\alpha|^2}\right) - c_1 r^\mu \int_0^T |\tilde{u}|^\mu dt + c_2 T, \quad (31)$$

then $h(r)$ can achieve its maximum at $\tilde{r} = ((|\cos\pi\alpha|^2+1)/\cos\pi\alpha|^2 c_1\mu\int_0^T |u_1|^\mu dt)^{1/(\mu-2)}$, so $\varphi_{u_{n-1}}(u_n) \le h(\tilde{r})$. By $(H_1)$ and $(H_0)$, we have

$$\begin{aligned} h(\tilde{r}) \ge \varphi_{u_{n-1}}(u_n) &= \frac{1}{2}\int_0^T |{}_0D_t^\alpha u_n(t)|^2 + |{}_tD_T^\alpha u_n(t)|^2 dt \\ &\quad - \int_0^T F(t,u_n(t),{}_0D_t^\alpha u_{n-1}(t))dt \\ &\ge \frac{1}{2}\left(1+|\cos\pi\alpha|^2\right)\|u_n\|_\alpha^2 - \int_{\{t\in[0,T]\,|\,|u_n(t)|\le R\}} F(t,u_n(t),{}_0D_t^\alpha u_{n-1}(t))dt \\ &\ge \frac{1}{2}\left(1+|\cos\pi\alpha|^2\right)\|u_n\|_\alpha^2 \int_{\{t\in[0,T]\,|\,|u_n(t)|\le R\}}\int_0^{u_n(t)} f(t,s,{}_0D_t^\alpha u_{n-1}(t))dsdt \\ &\ge \frac{1}{2}\left(1+|\cos\pi\alpha|^2\right)\|u_n\|_\alpha^2 - \int_{\{t\in[0,T]\,|\,|u_n(t)|\le R\}}\int_0^R a(|s|)b(t)dsdt \\ &\quad - R\int_0^t |{}_0D_t^\alpha u_{n-1}(t)|^2 dt \ge \frac{1}{2}\left(1+|\cos\pi\alpha|^2\right)\|u_n\|_\alpha^2 \\ &\quad - \int_0^T b(t)dt\int_0^R a(|s|)ds - R\|u_{n-1}\|_\alpha^2. \end{aligned} \quad (32)$$

Then,

$$\begin{aligned} \|u_n\|_\alpha^2 &\le \frac{2R}{1+|\cos\pi\alpha|^2}\|u_{n-1}\|_\alpha^2 + \frac{2\left(h(\tilde{r}) + \int_0^T b(t)dt\int_0^R a(|s|)ds\right)}{1+|\cos\pi\alpha|^2} \\ &\le \left(\frac{2R}{1+|\cos\pi\alpha|^2}\right)^{n-1}\|u_1\|_\alpha^2 \\ &\quad + \frac{2\left(h(\tilde{r}) + \int_0^T b(t)dt\int_0^R a(|s|)ds\right)}{1+|\cos\pi\alpha|^2}\sum_{k=0}^{n-2}\left(\frac{2R}{1+|\cos\pi\alpha|^2}\right)^k \\ &= \left(\frac{2R}{1+|\cos\pi\alpha|^2}\right)^{n-1}\|u_1\|_\alpha^2 \\ &\quad + \frac{2\left(1+|\cos\pi\alpha|^2\right)\left(h(\tilde{r}) + \int_0^T b(t)dt\int_0^R a(|s|)ds\right)}{\left(1+|\cos\pi\alpha|^2\right)\left(1+|\cos\pi\alpha|^2-2R\right)}. \end{aligned} \quad (33)$$

Thus, we can choose $C_2 = 2(1+|\cos\pi\alpha|^2)(h(\tilde{r}) + \int_0^T b(t)dt\int_0^R a(|s|)ds)/(1+|\cos\pi\alpha|^2)(1+|\cos\pi\alpha|^2-2R)$.

Since $(\varphi'(u_{n+1}) - \varphi'(u_n))(u_{n+1}-u_n) = 0$ and $\|u_n\|\alpha \le C_2$, then $\|u_n\|\infty\le r_1$, where $r_1$ is given by $(H_3)$. From $(H_3)$, we have

$$\begin{aligned} &\int_0^T |{}_0D_t^\alpha(u_{n+1}(t)-u_n(t))|^2 + |{}_tD_T^\alpha(u_{n+1}(t)-u_n(t))|^2 dt \\ &= \int_0^T (f(t,u_{n+1}(t),{}_0D_t^\alpha u_n(t)) \\ &\quad - f(t,u_n(t),{}_0D_t^\alpha u_{n-1}(t)))(u_{n+1}(t)-u_n(t))dt \\ &= \int_0^T (f(t,u_{n+1}(t){}_0D_t^\alpha u_n(t)) - f(t,u_n(t),{}_0D_t^\alpha u_n(t)))(u_{n+1}(t) \\ &\quad - u_n(t))dt + \int_0^T (f(t,u_n(t),{}_0D_t^\alpha u_n(t)) \\ &\quad - f(t,u_n(t),{}_0D_t^\alpha u_{n-1}(t)))(u_{n+1}(t)-u_n(t))dt \\ &\le L_1\|u_{n+1}-u_n\|_{L^2}^2 + L_2\int_0^T |{}_0D_t^\alpha u_n(t) - {}_0D_t^\alpha u_{n-1}(t)|(u_{n+1}(t)-u_n(t))dt \\ &\le L_1\|u_{n+1}-u_n\|_{L^2}^2 + L_2\|u_n-u_{n-1}\|_\alpha\|u_{n+1}-u_n\|_{L^2}. \end{aligned} \quad (34)$$

Combining the above estimates with (8) and (5), we obtain

$$\begin{aligned} &\left(1+|\cos\pi\alpha|^2 - \frac{L_1T^{2\alpha}}{\Gamma(\alpha+1)^2}\right)\|u_{n+1}-u_n\|_\alpha \\ &\quad \le \frac{L_2T^\alpha}{\Gamma(\alpha+1)}\|u_n-u_{n-1}\|_\alpha, \end{aligned} \quad (35)$$

that is $\|u_{n+1}-u_n\|_\alpha \le \gamma\|u_n-u_{n-1}\|_\alpha$. Since $0<\gamma<1$, the above inequality implies that $\{u_n\}$ is a Cauchy sequence in $E^\alpha$. Thus, there is a $u$ such that $\{u_n\}$ converges strongly to $u$ in $E^\alpha$.

Now, we show that $u\equiv 0$. By the proof of Lemma 9, we know $\varphi(u_n)\ge\sigma$.

If $\varphi(u_n)\longrightarrow\varphi(u)$, then$\varphi(u)>\sigma$, and since $\varphi(0)=0$, we obtain $u\equiv 0$. In order to show $\varphi(u_n)\longrightarrow\varphi(u)$, we only need to show $\int_0^T F(t,u_n,{}_0D_t^\alpha u_{n-1}(t))dt \longrightarrow \int_0^T F(t,u,{}_0D_t^\alpha u(t))dt$. In fact,

$$\begin{aligned} &\int_0^T F(t,u_n,{}_0D_t^\alpha u_{n-1}(t))dt - \int_0^T F(t,u,{}_0D_t^\alpha u(t))dt \\ &= \int_0^T\int_0^{u_n} f(t,x,{}_0D_t^\alpha u_{n-1}(t))dt - \int_0^T\int_0^u f(t,x,{}_0D_t^\alpha u(t))dt \\ &= \int_0^T\int_u^{u_n} f(t,x,{}_0D_t^\alpha u_{n-1}(t)) - f(t,x,{}_0D_t^\alpha u(t))dt \\ &\le L_2\int_0^T |{}_0D_t^\alpha u_{n-1}(t) - {}_0D_t^\alpha u(t))\|u_n-u|dt \\ &\le L_2\left\|{}_0D_t^\alpha u_{n-1}(t) - {}_0D_t^\alpha u(t))\right\|L^2\left\|u_n-u\right\|L^2 \\ &\le \frac{L_2T^\alpha}{\Gamma(\alpha+1)}\|u_n-u\|_\alpha^2 \longrightarrow 0. \end{aligned} \quad (36)$$

Next, we show that for any $v\in E^\alpha$,

$$\int_0^T {}_0D_t^\alpha u(t)_0D_t^\alpha v(t) + {}_tD_T^\alpha u(t)_tD_T^\alpha v(t)dt = \int_0^T f(t, u(t), {}_0D_t^\alpha u(t))v(t)dt. \tag{37}$$

It remains only to show

$$\int_0^T f(t, u_n(t), {}_0D_t^\alpha u_{n-1}(t))v(t)dt \longrightarrow \int_0^T f(t, u(t), {}_0D_t^\alpha u(t))\, v(t)dt. \tag{38}$$

Note that

$$\begin{aligned}
&\int_0^T (f(t, u_n(t), {}_0D_t^\alpha u_{n-1}(t)) - f(t, u(t), {}_0D_t^\alpha u(t)))v(t)dt \\
&= \int_0^T (f(t, u_n(t), {}_0D_t^\alpha u_{n-1}(t)) - (f(t, u(t), {}_0D_t^\alpha u_{n-1}(t))v(t)dt \\
&\quad + \int_0^T (f(t, u(t), {}_0D_t^\alpha u_{n-1}(t)) - f(t, u(t), {}_0D_t^\alpha u(t)))v(t)dt \\
&\le L_1\|u_n - u\|_{L^2}\|v\|_{L^2} + L_2\|u_{n-1} - u\|_\alpha\|v\|_{L^2} \\
&\le C\left(\|u_n - u\|_\alpha + \|u_{n-1} - u\|_\alpha\right)\|v\|_\alpha,
\end{aligned} \tag{39}$$

where $C$ is a constant. Thus, we obtain a nontrivial solution of problem (1).

## 4. Existence Result for (3)

In Section 3, condition $(H_3)$ implies that the nonlinearity is linear growth about a fractional derivative of solutions; this section will consider the fractional boundary value problem (3) in which the nonlinearity is quadratic growth about the fractional derivative of solutions.

Assume that $g$ satisfies the following conditions:

$(H_1')$ There are constants $\mu > 2$ and $R > 0$ such that, for $|x| \ge R$, $0 < \mu G(x) \le g(x)x$, where $G(x) = \int_0^x g(s)ds$.

$(H_2')$ $\lim_{x\to 0}(g(x)/x) = 0$.

Similar to Section 3, since (3) is not variational, given $w \in E^\alpha$, we consider the following problem which is independent on the fractional derivative of the solution:

$$\begin{cases} {}_tD_T^\alpha {}_0D_t^\alpha u(t) + {}_0D_t^\alpha {}_tD_T^\alpha u(t) = a(t)g(u(t)) + \lambda|{}_0D_t^\alpha w(t)|^2, \text{ a.e.} t \in [0, T], \\ u(0) = u(T) = 0. \end{cases} \tag{40}$$

Then, the corresponding functional $\phi : E^\alpha \longrightarrow E^\alpha$ is given by

$$\begin{aligned}
\phi_w(u) &= \frac{1}{2}\int_0^T |{}_0D_t^\alpha u(t)|^2 + |{}_tD_T^\alpha u(t)|^2 dt - \int_0^T a(t)G(u(t))dt \\
&\quad - \lambda\int_0^T |{}_0D_t^\alpha w(t)|^2 u(t)dt, \\
\phi_w'(u)v &= \int_0^T {}_0D_t^\alpha(t)_0D_t^\alpha v(t) + {}_tD_T^\alpha u(t)_tD_T^\alpha v(t)dt \\
&\quad - \int_0^T a(t)g(u(t))v(t)dt - \lambda\int_0^T |{}_0D_t^\alpha w(t)|^2 v(t)dt,
\end{aligned} \tag{41}$$

where $v \in E^\alpha$.

$(H_1')$ implies that there are constants $c_1, c_2 > 0$ such that

$$G(x) \ge c_1|x|^\mu - c_2. \tag{42}$$

By $(H_2')$, there is a $\delta > 0$ such that $|x| \le \delta$ implies

$$g(x) \le \frac{(\Gamma(\alpha))^2(2\alpha - 1)\left(1 + |\cos\ \pi\alpha|^2\right)}{4\int_0^T a(t)dtT^{2\alpha-1}}|x|^2. \tag{43}$$

For convenience of our statement, let us give some notations and denote

$$\begin{aligned}
a &= \frac{(\mu - 2)\left(1 + |\cos\pi\alpha|^2\right)}{2\mu}, \\
b &= \frac{\lambda(\mu - 1)T^{\alpha-(1/2)}}{\Gamma(\alpha)(2\alpha - 1)^{1/2}\mu}, \\
d &= \frac{(\mu - 1)\Gamma(\alpha)(2\alpha - 1)^{1/2}\delta\left(1 + |\cos\pi\alpha|^2\right)}{32T^{\alpha-(1/2)}\mu},
\end{aligned} \tag{44}$$

where $\mu$ is given in $(H_1')$ and $\delta$ is given in (43). Assuming $a^2 > 4bd$, let

$$\varepsilon_1 = \frac{a - \sqrt{a^2 - 4bd}}{2b},\ \varepsilon_2 = \frac{a + \sqrt{a^2 - 4bd}}{2b}. \tag{45}$$

For a fixed function $v \in E^\alpha$ with $\|v\| = 1$, we denote

$$\begin{aligned}
m(t) &= \left(1 + \frac{1}{|\cos\pi\alpha|^2}\right)r^2 - c_1 r^\mu\int_0^T a(t)|v|^\mu dt \\
&\quad + \frac{r\Gamma(\alpha)(2\alpha - 1)^{1/2}\delta\left(1 + |\cos\pi\alpha|^2\right)}{4T^{\alpha-(1/2)}} + c_2\int_0^T a(t),
\end{aligned} \tag{46}$$

$$M = \max_{r\in(0,\infty)} m(t),\ M_1 = M + \int_0^T a(t)dt \max_{|x|\le R}\left(G(x) - \frac{1}{\mu}g(x)\right). \tag{47}$$

For $\varepsilon \in (\varepsilon_1, \varepsilon_2)$, denote

$$R_1 = R_1(\varepsilon) = \frac{M_1 \varepsilon}{a\varepsilon - b\varepsilon^2 - d},\ R_2 = R_2(\varepsilon) = \frac{R_1 T^{\alpha-(1/2)}}{\Gamma(\alpha)(2\alpha-1)^{1/2}}. \tag{48}$$

We also need the following Lipschitz condition:

$(H_3')$There exists $L_{R2} > 0$ such that $|g(x_1) - g(x_2)| \le L_R| x_1 - x_2|$ for $0 < x_1, x_2 < R_2$. Then we have the following result.

**Theorem 11.** *Suppose $(H_1')$–$(H_3')$ hold, $a^2 > 4bd$; if there exists $\varepsilon \in (\varepsilon_1, \varepsilon_2)$, such that*

$$\lambda R_1^2 \le \frac{\Gamma(\alpha)^2 (2\alpha-1)\delta\left(1 + |\cos \pi\alpha|^2\right)}{8T^{2\alpha-1}}, \tag{49}$$

$$\frac{2\lambda R_1 T^{\alpha-(1/2)}}{\Gamma(\alpha)(2\alpha-1)^{1/2}} < 1 + |\cos \pi\alpha|^2 - \frac{T^{2\alpha-1} L_{R_2} \int_0^T a(t)dt}{(\Gamma(\alpha))^2 (2\alpha-1)}. \tag{50}$$

*Then, (3) has a nontrivial weak solution.*

*Proof.* We first verify that for a given $w \in E^\alpha$ with $\|w\|_\alpha \le R_1$, (40) has at least a nontrivial weak solution.

In order to use the Mountain Pass theorem, we first show that $\phi$ satisfies the (PS) condition. Let $\{u_n\} \subset E^\alpha$, $\{\phi_w(u_n)\}$ is bounded, and $\phi_w'(u_n) \longrightarrow 0$, we show that $\{u_n\}$ is bounded.

In fact,

$$\begin{aligned}
\mu\phi_\omega(u_n) - \phi_\omega'(u_n)u_n &= \left(\frac{\mu}{2} - 1\right)\int_0^T |{}_0D_t^\alpha u_n(t)|^2 + |{}_tD_T^\alpha u_n(t)|^2 dt \\
&\quad + \int_0^T a(t)(g(u_n)u_n - \mu G(u_n))dt + \lambda(1-\mu)\int_0^T |{}_0D_t^\alpha \omega(t)|^2 u_n(t)dt \\
&\ge \left(\frac{\mu}{2} - 1\right)\left(1 + |\cos \pi\alpha|^2\right)\|u_n\|_\alpha^2 + \lambda(1-\mu)\|\omega\|_\alpha^2 \|u_n\|_\infty \\
&\quad + \int_{\{t\in[0,T]||u_n(t)|\le R\}} a(t)(u_n(t)g(u_n(t)) - \mu G(u_n(t)))dt \\
&\ge \left(\frac{\mu}{2} - 1\right)\left(1 + |\cos \pi\alpha|^2\right)\|u_n\|_\alpha^2 + \frac{\lambda R_1^2 (1-\mu)T^{\alpha-(1/2)}}{\Gamma(\alpha)(2\alpha-1)^{1/2}}\|u_n\|_\alpha \\
&\quad + \int_{\{t\in[0,T]||u_n(t)|\le R\}} a(t)(u_n(t)g(u_n(t)) - \mu G(u_n(t)))dt,
\end{aligned} \tag{51}$$

which implies that $\{u_n\}$ is bounded, and similar to the last part of Lemma 8, we get that $\phi_w(u)$ satisfies the (PS) condition.

Let $\rho = \Gamma(\alpha)(2\alpha-1)^{1/2}\delta/T^{a-(1/2)}$ and choose $u \in E^\alpha$ with $\|u\| = \rho$, then $\|u\|_\infty \le \delta$. From (43), (6), and (49), we have

$$\begin{aligned}
\phi_w(u) &= \frac{1}{2}\int_0^T |{}_0D_t^\alpha u(t)|^2 + |{}_tD_T^\alpha u(t)|^2 dt - \int_0^T a(t)G(u(t))dt \\
&\quad - \lambda\int_0^T |{}_0D_t^\alpha w(t)|^2 u(t)dt \ge \frac{1}{2}\left(1 + |\cos \pi\alpha|^2\right)\|u\|_\alpha^2 \\
&\quad - \frac{\Gamma(\alpha)^2(2\alpha-1)\left(1 + |\cos \pi\alpha|^2\right)}{4T^{2\alpha-1}}\|u\|_\infty^2 - \lambda\|w\|_\alpha^2\|u\|_\infty \\
&\ge \frac{1}{4}\left(1 + |\cos \pi\alpha|^2\right)\|u\|_\alpha^2 - \frac{\lambda T^{\alpha-(1/2)} R_1^2}{\Gamma(\alpha)(2\alpha-1)^{1/2}}\|u\|_\alpha \\
&= \left(\frac{1}{4}\left(1 + |\cos \pi\alpha|^2\right)\rho - \frac{\lambda T^{\alpha-(1/2)} R_1^2}{\Gamma(\alpha)(2\alpha-1)^{1/2}}\right)\rho \\
&\ge \frac{1}{8}\left(1 + |\cos \pi\alpha|^2\right)\rho^2.
\end{aligned} \tag{52}$$

Then, we obtain that there exists $\beta > 0$ such that for $\|u\|_\alpha = \rho$, $\phi_w(u) \ge \beta$ uniformly for $w \in E^\alpha$ with $\|w\|_\alpha \le R1$.

Let $v \in E^\alpha$ with $\|v\| = 1$, then

$$\begin{aligned}
\phi_w(rv) &\le \frac{r^2}{2}\left(1 + \frac{1}{|\cos \pi\alpha|^2}\right) - c_1 r^\mu \int_0^T a(t)|v|^\mu dt \\
&\quad + \lambda r\int_0^T |{}_0D_t^\alpha w(t)|^2 v dt + c_2\int_0^T a(t)dt \\
&\le \frac{r^2}{2}\left(1 + \frac{1}{|\cos \pi\alpha|^2}\right) - c_1 r^\mu \int_0^T a(t)|v|^\mu dt \\
&\quad + \frac{r\lambda R_1^2 T^{\alpha-(1/2)}}{\Gamma(\alpha)(2\alpha-1)^{1/2}}\|v\|_\alpha \\
&\quad + c_2\int_0^T a(t)dt \longrightarrow -\infty \text{ as } r \longrightarrow \infty,
\end{aligned} \tag{53}$$

which implies there exists $r_0 > \rho$ such that $\phi_w(r_0 v) < 0$. Then, from the Mountain Pass theorem, we get that $\phi_w$ has a nontrivial weak solution $u_w$ which can be characterized as

$$\phi_w'(u_w) = 0,\ \phi_w(u_w) = \inf_{\eta\in\Gamma} \max_{u\in\eta([0,1])} \phi_w(u), \tag{54}$$

where $\Gamma = \{\eta \in C([0,1], E^\alpha) | \eta(0) = 0, \eta(1) = r_0 v\}$.

Let $u_1 = 0$, we can obtain that $\phi_{u1}$ has a nontrivial critical point $u_2$. For $n = 1, 2, \cdots$, we construct a sequence $\{u_n\}$, where $u_n$ is the critical point of $\phi_{u_{n-1}}$. Now, we assume that $\|u_{n-1}\|_\alpha \le R_1$; by the mathematical induction, we will prove that $\|u_n\|_\alpha \le R_1$. In fact, by (54), we have

$$\begin{aligned}
\phi_{u_{n-1}}(u_n) &\le \max_{r\in(0,r_0)} \phi_{u_{n-1}}(rv) \le \left(1+\frac{1}{|\cos\pi\alpha|^2}\right) r^2 \\
&\quad - \int_0^T a(t)(c_1 r^\mu |v|^\mu - c_2)\,dt + \lambda r \int_0^T |{}_0D_t^\alpha u_{n-1}(t)|^2 v(t)\,dt \\
&\le \left(1+\frac{1}{|\cos\pi\alpha|^2}\right) r^2 - c_1 r^\mu \int_0^T a(t)|v|^\mu dt \\
&\quad + \frac{\lambda r T^{\alpha-(1/2)}}{\Gamma(\alpha)(2\alpha-1)^{1/2}} \|u_{n-1}\|_\alpha^2 + c_2\int_0^T a(t)\,dt \\
&\le \left(1+\frac{1}{|\cos\pi\alpha|^2}\right) r^2 - c_1 r^\mu \int_0^T a(t)|v|^\mu dt \\
&\quad + \frac{\lambda r T^{\alpha-(1/2)} R_1^2}{\Gamma(\alpha)(2\alpha-1)^{1/2}} + c_2\int_0^T a(t)\,dt \\
&\le \left(1+\frac{1}{|\cos\pi\alpha|^2}\right) r^2 - c_1 r^\mu \int_0^T a(t)|v|^\mu dt \\
&\quad + \frac{r\Gamma(\alpha)(2\alpha-1)^{1/2}\delta\left(1+|\cos\pi\alpha|^2\right)}{4T^{\alpha-(1/2)}} \\
&\quad + c_2\int_0^T a(t)\,dt \le M,
\end{aligned} \tag{55}$$

where $M$ is given in (47).

Hence,

$$\begin{aligned}
M &\ge \phi_{u_{n-1}}(u_n) = \frac{1}{2}\int_0^T |{}_0D_t^\alpha u_n(t)|^2 + |{}_tD_T^\alpha u_n(t)|^2 dt \\
&\quad - \int_0^T a(t)G(u_n(t))\,dt - \lambda\int_0^T |{}_0D_t^\alpha u_{n-1}(t)|^2 u_n(t)\,dt \\
&\ge \frac{1}{2}\int_0^T |{}_0D_t^\alpha u_n(t)|^2 + |{}_tD_T^\alpha u_n(t)|^2 dt \\
&\quad - \frac{1}{\mu}\int_0^T a(t)g(u_n(t))u_n(t)\,dt - \lambda\int_0^T |{}_0D_t^\alpha u_{n-1}(t)|^2 u_n(t)\,dt \\
&\quad - \int_0^T a(t)\,dt \max_{|x|\le R}\left(G(x) - \frac{1}{\mu}g(x)\right) \\
&= \left(\frac{1}{2}-\frac{1}{\mu}\right)\int_0^T |{}_0D_t^\alpha u_n(t)|^2 + |{}_tD_T^\alpha u_n(t)|^2 dt \\
&\quad + \frac{1}{\mu}\phi_{u_{n-1}}(u_n)u_n - \lambda\left(1-\frac{1}{\mu}\right)\int_0^T |{}_0D_t^\alpha u_{n-1}(t)|^2 u_n(t)\,dt \\
&\quad - \int_0^T a(t)\,dt \max_{|x|\le R}\left(G(x) - \frac{1}{\mu}g(x)\right) \\
&\ge \left(\frac{1}{2}-\frac{1}{\mu}\right)\left(1+|\cos\pi\alpha|^2\right)\|u_n\|_\alpha^2 \\
&\quad - \lambda\left(1-\frac{1}{\mu}\right)\int_0^T |{}_0D_t^\alpha u_{n-1}(t)|^2 u_n(t)\,dt \\
&\quad - \int_0^T a(t)\,dt \max_{|x|\le R}\left(G(x) - \frac{1}{\mu}g(x)\right).
\end{aligned} \tag{56}$$

So,

$$\begin{aligned}
&\left(\frac{1}{2}-\frac{1}{\mu}\right)\left(1+|\cos\pi\alpha|^2\right)\|u_n\|_\alpha^2 \le M \\
&\quad + \lambda\left(1-\frac{1}{\mu}\right)\int_0^T |{}_0D_t^\alpha u_{n-1}(t)|^2 u_n(t)\,dt \\
&\quad + \int_0^T a(t)\,dt \max_{|x|\le R}\left(G(x) - \frac{1}{\mu}g(x)\right) \le M_1 \\
&\quad + \left(1-\frac{1}{\mu}\right)\frac{\lambda T^{\alpha-(1/2)}}{\Gamma(\alpha)(2\alpha-1)^{1/2}}\|u_{n-1}\|_\alpha^2\|u_n\|_\alpha \le M_1 \\
&\quad + \frac{\lambda(\mu-1)T^{\alpha-(1/2)}}{\Gamma(\alpha)(2\alpha-1)^{1/2}\mu}\left(\varepsilon\|u_n\|_\alpha^2 + \frac{R_1^2}{4\varepsilon}\|u_{n-1}\|_\alpha^2\right).
\end{aligned} \tag{57}$$

By (49), we obtain

$$\begin{aligned}
&\left(\frac{(\mu-2)\left(1+|\cos\pi\alpha|^2\right)}{2\mu} - \frac{\lambda\varepsilon(\mu-1)T^{\alpha-(1/2)}}{\Gamma(\alpha)(2\alpha-1)^{1/2}\mu}\right)\|u_n\|_\alpha^2 \\
&\quad \le \frac{\lambda R_1^2(\mu-1)T^{\alpha-(1/2)}}{4\varepsilon\Gamma(\alpha)(2\alpha-1)^{1/2}\mu}\|u_{n-1}\|_\alpha^2 + M_1 \\
&\quad \le \frac{(\mu-1)\Gamma(\alpha)(2\alpha-1)^{1/2}\delta\left(1+|\cos\pi\alpha|^2\right)}{32T^{\alpha-(1/2)}\mu\varepsilon}\|u_{n-1}\|_\alpha^2 + M_1.
\end{aligned} \tag{58}$$

That is

$$(a-b\varepsilon)\|u_n\|_\alpha^2 \le \frac{d}{\varepsilon}\|u_{n-1}\|_\alpha^2 + M_1. \tag{59}$$

When $\varepsilon \in (\varepsilon_1, \varepsilon_2)$, where $\varepsilon_1$ and $\varepsilon_2$ are given in (45), we have

$$\frac{d}{\varepsilon} < a - b\varepsilon. \tag{60}$$

Then,

$$\begin{aligned}
\|u_n\|_\alpha^2 &\le \frac{d/\varepsilon}{a-b\varepsilon}\|u_{n-1}\|_\alpha^2 + \frac{M_1}{a-b\varepsilon} \\
&\le \left(\frac{d/\varepsilon}{a-b\varepsilon}\right)^{n-1}\|u_1\|_\alpha^2 + \frac{M_1}{a-b\varepsilon}\sum_{k=0}^{n-2}\left(\frac{d/\varepsilon}{a-b\varepsilon}\right)^k \\
&\le \|u_1\|_\alpha^2 + \frac{M_1\varepsilon}{a\varepsilon - b\varepsilon^2 - d}.
\end{aligned} \tag{61}$$

Therefore, the above argument implies that $\|u_n\|_\alpha \le R_1$.

Finally, we show that $\{u_n\}$ is convergent to a nontrivial solution of (3).

Since $\|u_n\|_\alpha \le R_1$, we have $\|u_n\|_\infty \le R_2$, then by $(H_3')$, we obtain

$$\begin{aligned}
\left(1+|\cos \pi\alpha|^2\right)\|u_{n+1}-u_n\|_\alpha^2 \le{}& \int_0^T \left|{}_0D_t^\alpha(u_{n+1}-u_n(t))\right|^2 \\
&+ \left|{}_tD_T^\alpha(u_{n+1}-u_n(t))\right|^2 dt \\
={}& \left(\phi'_{u_n}(u_{n+1})-\phi_{u_{n-1}}(u_n)\right)(u_{n+1}-u_n(t)) \\
&+ \int_0^T a(t)(f(u_n)-f(u_{n+1}))(u_{n+1}-u_n)dt \\
&+ \lambda\int_0^T \left(\left|{}_0D_t^\alpha u_n(t)\right|^2-\left|{}_0D_t^\alpha u_{n-1}\right|^2\right)(u_{n+1}-u_n(t))dt \\
\le{}& L_{R_2}\|u_{n+1}-u_n\|_\infty^2\int_0^T a(t)dt \\
&+ \lambda\int_0^T {}_0D_t^\alpha u_n(t)\,{}_0D_t^\alpha(u_n-u_{n-1})dt\|u_{n+1}-u_n\|_\infty \\
&+ \lambda\int_0^T {}_0D_t^\alpha u_{n-1}(t)\,{}_0D_t^\alpha(u_n-u_{n-1})dt\|u_{n+1}-u_n\|_\infty \\
\le{}& L_{R_2}\|u_{n+1}-u_n\|_\infty^2\int_0^T a(t)dt+\lambda\|u_n\|_\alpha\|u_n-u_{n-1}\|_\alpha\|u_{n+1}-u_n\|_\infty \\
&+ \lambda\|u_{n-1}\|_\alpha\|u_n-u_{n-1}\|_\alpha\|u_{n+1}-u_n\|_\infty \\
\le{}& \frac{T^{2\alpha-1}L_{R_2}\int_0^T a(t)dt}{(\Gamma(\alpha))^2(2\alpha-1)}\|u_{n+1}-u_n\|_\alpha^2 \\
&+ \frac{2\lambda R_1 T^{\alpha-(1/2)}}{\Gamma(\alpha)(2\alpha-1)^{1/2}}\|u_n-u_{n-1}\|_\alpha\|u_{n+1}-u_n\|_\alpha.
\end{aligned} \tag{62}$$

So

$$\begin{aligned}
&\left(1+|\cos \pi\alpha|^2-\frac{T^{2\alpha-1}L_{R_2}\int_0^T a(t)dt}{(\Gamma(\alpha))^2(2\alpha-1)}\right)\|u_{n+1}-u_n\|_\alpha \\
&\quad\le \frac{2\lambda R_1 T^{\alpha-(1/2)}}{\Gamma(\alpha)(2\alpha-1)^{1/2}}\|u_n-u_{n-1}\|_\alpha.
\end{aligned} \tag{63}$$

From (50), we know that $\{u_n\}$ is a Cauchy sequence in $E^\alpha$ and $u$ is a weak solution of (3). Since $\phi_{u_{n-1}}(u_n)>\beta>0$ for $n=1,2,\cdots$ and $\beta$ does not depend on $n$, we obtain that $u$ is a nontrivial weak solution of (3).

**Corollary 12.** *Suppose $(H_1')$-$(H_3')$ hold; if the right-hand side of (50) is greater than 0, then there exists a constant $\lambda_0>0$, such that (1) has a solution when $\lambda\in(0,\lambda_0)$.*

## Data Availability

No data were used to support this study.

## Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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