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Research Article

Analysis on Existence of Positive Solutions for a Class Second Order Semipositone Differential Equations

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In this paper, we study the existence of positive solutions of the following second-order semipositone system (see equation 1). By applying a well-known fixed-point theorem, we prove that the problem admits at least one positive solution, if f is bounded below.

1. Introduction

This paper is focused on the existence of positive solutions of a second-order semipositone system

$$\begin{cases} -u'' + \rho u = \phi u + f(t, u, \phi), & t \in (0, 1), \\ -\phi'' = \mu u, & t \in (0, 1), \\ u(0) = u(1) = \phi(0) = \phi(1) = 0, \end{cases} \tag{1}$$

where μ is a positive constant and f satisfies the following assumption: $(F_0) f : [0, 1] \times \mathbb{R}_+^2 \longrightarrow \mathbb{R}$ is continuous, and

$$f(t, u, \phi) \geq -e(t), \quad \text{for } (t, u, \phi) \in [0, 1] \times \mathbb{R}_+^2, \tag{2}$$

where $e : [0, 1] \longrightarrow \mathbb{R}_+$ is continuous and $e(t) \not\equiv 0$ on [0,1].

The second-order elliptic systems

$$\begin{cases} -\Delta u + \rho u = \phi u + f(u), & x \in \Omega, \\ -\Delta \phi = \mu u, & x \in \Omega, \\ u = \phi = 0, & x \in \partial\Omega, \end{cases} \tag{3}$$

have a strong physical meaning in quantum mechanics models [1, 2], in semiconductor theory [3], or in a time- and space-dependent mathematical model of nuclear reactors in a closed container [4]. To the best of our knowledge, existence and multiplicity of nontrivial solutions of BVP(1) have been widely studied by using the variational method [5], bifurcation techniques [6, 7], or fixed-point theorems [8–11]. In general, in order to ensure the positivity of the solutions of Equation (1), one of the crucial assumptions is that the nonlinearity f is nonnegative. Of course, the natural question is whether Equation (1) has a positive solution or not if f satisfies the assumption (F_0).

On the other hand, many authors have been interested in finding the relations between the positivity of solutions and the changing sign of the nonlinearity in order to prove the existence of the positive solutions. We refer the readers to [12–16] and the references.

Inspired by these references, the purpose of this paper is to find some new conditions, which are used to study the existence and multiplicity of positive solutions of the semipositone Equation (1). The main tool is the following well-known fixed-point theorem.

Lemma 1 [17]. *Let E be a Banach space and K be a cone in E. Assume Ω_r and Ω_R are open bounded subsets of E with $\Omega_r \cap K \neq \phi, \overline{\Omega_r \cap K} \subset \Omega_R \cap K$. Let $T : \overline{\Omega_R \cap K} \longrightarrow K$ be a completely continuous operator such that*

(a) $\|Tu\| \leq \|u\|$, for $u \in \partial(\Omega_r \cap K)$, and

(b) there exists a $\eta(t) \in K\backslash\{0\}$ such that

$$u \neq Tu + \lambda\eta(t), \quad \text{for } u \in \partial(\Omega_R \cap K), \lambda > 0. \tag{4}$$

Then, T has a fixed point in $\overline{\Omega_R \cap K}\backslash\Omega_r \cap K$. The same conclusion remains valid if (a) holds on $\partial(\Omega_R \cap K)$ and (b) holds on $\partial(\Omega_r \cap K)$.

The paper is organized as follows: in Section 2, we give some preliminaries, which are about the properties of the Green functions, the notations of some sets, etc.; in Section 3, we give the main results and the corresponding proof. In Section 4, some examples are given to illustrate the main results.

2. Preliminary

Let $G(t, s)$ be the Green function of linear boundary value problem

$$-u'' + \rho u = 0, \quad u(0) = u(1) = 0, \tag{5}$$

where $\rho > -\pi^2$.

Lemma 2 [18]. *Let $\omega = \sqrt{|\rho|}$, then $G(t, s)$ can be expressed by*

(i) $G(t, s) = \left\{ \begin{array}{l} \sinh \omega t \sinh \omega(1-s)/\omega \sinh \omega, \quad 0 \leq t \leq s \leq 1, \\ \sinh \omega s \sinh \omega(1-t)/\omega \sinh \omega, 0 \leq s \leq t \leq 1, \text{ if } \rho > 0 \end{array} \right.$

(ii) $G(t, s) = \begin{cases} t(1-s), & 0 \leq t \leq s \leq 1, \\ s(1-t), & 0 \leq s \leq t \leq 1, \text{ if } \rho = 0 \end{cases}$

(iii) $G(t, s) = \{ \sin \omega t \sin \omega(1-s)/\omega \sin \omega, \quad 0 \leq t \leq s \leq 1,$ $\sin \omega s \sin \omega(1-t)/\omega \sin \omega, 0 \leq s \leq t \leq 1, \text{ if } -\pi^2 < \rho > 0.$

Lemma 3 [18]. *The function $G(t, s)$ has the following properties:*

(i) $G(t, s) > 0$, $\forall t, s \in (0, 1)$

(ii) $G(t, s) \leq CG(s, s)$, $\forall t, s \in [0, 1]$

(iii) $G(t, s) \geq \delta G(t, t)G(s, s)$, $\forall t, s \in [0, 1]$

where $C = 1, \delta = \omega/\sinh \omega$, if $\rho > 0$; $C = 1, \delta = 1$, if $\rho = 0$; and $C = 1/\sin \omega, \delta = \omega \sin \omega$, if $-\pi^2 < \rho < 0$.

Lemma 4. *For the function $G(t, s)$, there exists a $\xi > 0$ such that*

$$G(t, t) \geq \xi \int_0^1 G(t, s) ds. \tag{6}$$

Proof.

(i) For $\rho > 0$, we have

$$\int_0^1 G(t, s) ds = \frac{1}{\omega^2 \sinh w} \{\sinh \omega t[\cosh \omega(1-t) - 1] + \sinh \omega(1-t)[\cosh \omega t - 1]\}. \tag{7}$$

Let

$$J_1(t) = \begin{cases} \dfrac{\cosh \omega(1-t) - 1}{\sinh \omega(1-t)}, & 0 \leq t < 1, \\ 0, & t = 1, \end{cases}$$
$$J_2(t) = \begin{cases} \dfrac{\cosh \omega t - 1}{\sinh \omega t}, & 0 < t \leq 1, \\ 0, & t = 0. \end{cases} \tag{8}$$

Since $J_1(t)$ is positive and continuous on [0,1] and $J_1(1) = 0$, we have

$$J_1^* = \max_{t \in [0,1]} J_1(t) > 0. \tag{9}$$

In the similar way, we also have

$$J_2^* = \max_{t \in [0,1]} J_2(t) > 0. \tag{10}$$

Choosing $\xi < \omega/(J_1^* + J_2^*)$. Then, for any $t \in (0, 1)$, we have

$$\begin{aligned}
\xi \int_0^1 G(t, s) ds &= \xi \frac{1}{\omega^2 \sinh w} \{\sinh \omega t[\cosh \omega(1-t) - 1] \\
&\quad + \sinh \omega(1-t)[\cosh \omega t - 1]\} \\
&= \xi \frac{1}{\omega^2 \sinh w} \left\{ \sinh \omega t \sinh \omega(1-t) \frac{\cosh \omega(1-t) - 1}{\sinh \omega(1-t)} \right. \\
&\quad \left. + \sinh \omega t \sinh \omega(1-t) \frac{\cosh \omega t - 1}{\sinh \omega t} \right\} \\
&= \frac{\xi}{\omega} \frac{1}{\omega \sinh w} \{\sinh \omega t \sinh \omega(1-t) J_1^* \\
&\quad + \sinh \omega t \sinh \omega(1-t) J_2^*\} \\
&= \frac{\xi(J_1^* + J_2^*)}{\omega} G(t, t) \leq G(t, t).
\end{aligned} \tag{11}$$

Since $G(0,0)=\int_0^1 G(0,s)ds = G(1,1) = \int_0^1 G(1,s)ds = 0$, then for any $\xi > 0$, we have

$$G(t,t) = \xi \int_0^1 G(t,s)ds, \quad \text{for } t = 0, 1. \tag{12}$$

Therefore, there exists a $\xi > 0$ such that

$$G(t,t) \geq \xi \int_0^1 G(t,s)ds. \tag{13}$$

(ii) For $\rho = 0$, it is obvious that

$$\int_0^1 G(t,s)ds = \frac{1}{2}t^2(1-t) + \frac{1}{2}t(1-t)^2 \leq t(1-t)^2 = G(t,t). \tag{14}$$

(iii) For $-\pi^2 < \rho < 0$, we have

$$\begin{aligned}\int_0^1 G(t,s)ds = \frac{1}{\omega^2 \sin w} &\{\sin \omega t[1 - \cos \omega(1-t)] \\ &+ \sinh \omega(1-t)[1 - \cos \omega t]\}.\end{aligned} \tag{15}$$

Using the similar discussion of Case (i), it follows that there exists a $\xi > 0$ such that

$$G(t,t) \geq \xi \int_0^1 G(t,s)ds. \tag{16}$$

For convenience, let $K(t,s)$ denote the Green function for $\rho = 0$. Then, Equation (1) can be rewritten as

$$\begin{cases} -u'' + \rho u = \mu u \displaystyle\int_0^1 K(t,s)u(s)ds + f\left(t, u\displaystyle\int_0^1 K(t,s)u(s)ds\right), \\ u(0) = u(1) = 0. \end{cases} \tag{17}$$

Furthermore, let $x = u + \omega$, where $\omega(t) = \int_0^1 G(t,s)e(s)ds$ is the unique solution of the linear boundary value problem

$$\begin{cases} -u'' + \rho u = e(t), \\ u(0) = u(1) = 0. \end{cases} \tag{18}$$

Then, we rewrite (17) as

$$\begin{cases} -x'' + \rho x = F(t, x - \omega), \\ x(0) = x(1) = 0, \end{cases} \tag{19}$$

where

$$\begin{aligned} F(t, x - \lambda\omega) = \mu(x-\omega)&\int_0^1 K(t,s)[x(s) - \omega(s)]ds \\ &+ \left\{ f\left(t, x - \omega, \int_0^1 K(t,s)[x(s) - \omega(s)]ds\right) + e(t)\right\}. \end{aligned} \tag{20}$$

From the above discussion, then we have the following lemma.

Lemma 5. *Assume that (F_0) holds. Then, $u(t)$ is a positive solution of (1) if only if $x(t)$ is a positive solution of the following problem:*

$$-x'' + \rho x = F(t, H(x-\omega)(x-\omega)), \tag{21}$$

with $x(t) \geq \omega(t)$. Here, $H(t)$ denotes the Heaviside function of a single real variable:

$$H(t) = \begin{cases} 1, & t \geq 0, \\ 0, & t < 0. \end{cases} \tag{22}$$

Let E denote the Banach space $C[0,1]$ with the norm $\|x\| = \max_{t\in[0,1]}|x(t)|$.

Define a cone $K \subset E$ by

$$K = \left\{ x(t) \in C[0,1]: \min_{t\in[\theta,1-\theta]} x(t) \geq \sigma\|x\| \right\}, \tag{23}$$

where $\theta \in (0, 1/2)$, $\sigma = \min_{t\in[\theta,1-\theta]}(\sigma/C)G(t,t) \in (0,1)$. Define an operator T by

$$T(x)(t) = \int_0^1 G(t,s)F(s, H(x-\omega)(x-\omega))ds. \tag{24}$$

Lemma 6. *Assume that (F_0) holds. Then, $T(K) \subseteq K$, and $T: K \longrightarrow K$ is completely continuous.*

Proof. For any $x(t) \in K$, from Lemma 3, it follows that

$$\begin{aligned} T(x)(t) &= \int_0^1 G(t,s)F(s, H(x-\omega)(x-\omega))ds \\ &\geq \delta \int_0^1 G(t,t)G(s,s)F(s, H(x-\omega)(x-\omega))ds \\ &= \frac{\delta}{C}G(t,t)\int_0^1 CG(s,s)F(s, H(x-\omega)(x-\omega))ds \\ &\geq \frac{\delta}{C}G(t,t)\int_0^1 G(t,s)F(s, H(x-\omega)(x-\omega))ds, \end{aligned} \tag{25}$$

which implies that $T(K) \subseteq K$.

Now, we show that $T: K \longrightarrow K$ is completely continuous.

First, we show that T maps the bounded set into itself. Since e and f are continuous, for any given $c > 0$, let

$$L = \max \{F(t, H(x-\omega)(x-\omega)) \colon 0 \le t \le 1, 0 \le x \le c\}. \quad (26)$$

Then, for $x \in \bar{K}_c$, we have

$$|T(x)(t)|_\infty = \left|\int_0^1 G(t,s)F(s, H(x-\omega)(x-\omega))ds\right|_\infty \le L \max_{0\le t,s\le 1} G(t,s), \quad (27)$$

which implies that $T(\bar{K}_c)$ is uniformly bounded.

Second, for $t_1, t_2 \in [0,1]$, we have

$$\begin{aligned} |Tx(t_2) - Tx(t_1)| &= \left|\int_0^1 [G(t_2,s) - G(t_1,s)]F(s, H(x-\omega)(x-\omega))ds\right| \\ &= \left|\int_0^1 [G_i(t_2,s) - G_i(t_1,s)]\tilde{f}_i(s, u(s) - \omega(s))ds\right| \\ &\le L \max_{0\le t,s\le 1}\left|\frac{\partial G(t,s)}{\partial t}\right| \|t_2 - t_1\|, \end{aligned} \quad (28)$$

which implies that the operator T is equicontinuous.

Thus, by applying the Arzela-Ascoli theorem [17], we obtain that $T(\overline{K_c})$ is relatively compact, namely, the operator T is compact.

Finally, we claim that $T : \overline{K_c} \longrightarrow K$ is continuous. Assume that $\{x_n\}_{n=1}^\infty \subset \overline{K_c}$ which converges to $x(t)$ uniformly on [0,1]. By Lebesgue's dominated convergence theorem and letting $n \longrightarrow \infty$, we have

$$\begin{aligned} &\|Tx_n(t) - Tx(t)\| \\ &\quad = \left\|\int_0^1 G(t,s)[F(s, H(x_n-\omega)(x_n-\omega)) \right. \\ &\quad\quad \left. - F(s, H(x_n-\omega)(x_n-\omega))]ds\right\| \\ &\quad \le C\int_0^1 G(s,s)[F(s, H(x_n-\omega)(x_n-\omega)) \\ &\quad\quad - F(s, H(x-\omega)(x-\omega))]ds \longrightarrow 0, \quad \text{as } n \longrightarrow +\infty. \end{aligned} \quad (29)$$

So, T is continuous on K_c. The proof is completed.

At the end of this section, let

$$e^* = \max_{t\in[0,1]} e(t) > 0, \omega^* = \max_{t\in[0,1]} \omega(t). \quad (30)$$

Define the height functions

$$\begin{aligned} \Phi_*(r) &= \min\left\{f(t,u,\phi) \colon (t,u,\phi) \in [0,1] \times [0,r] \times \left[0, \frac{r}{6}\right]\right\}, \\ \Phi_*(t,r) &= \max\left\{f(t,u,\phi) \colon (u,\phi) \in [0,r] \times \left[0, \frac{r}{6}\right]\right\}. \end{aligned} \quad (31)$$

In addition, we need to select some suitable open bounded sets. For any $\gamma > 0$, let

$$\begin{aligned} &\Omega^\gamma = \left\{x \in E : \min_{t\in[\theta,1-\theta]} x(t) < \sigma\gamma\right\}, B^\gamma = \{x \in E : \|x\| < \gamma\}, \\ &\Omega_K^\gamma = \Omega_\gamma \cap K, \partial\Omega_K^\gamma = \partial\Omega_\gamma \cap K, \\ &B_K^\gamma = B_\gamma \cap K, \partial B_K^\gamma = \partial B_\gamma \cap K. \end{aligned} \quad (32)$$

From [19, 20], we can conclude the lemma below.

Lemma 7.

(i) Ω_K^γ, B_K^γ *are open relative to* K

(ii) $B_K^{\sigma\gamma} \subset \Omega_K^\gamma \subset B_K^\gamma$

(iii) $x \in \partial\Omega_K^\gamma$ *if and only if* $\min_{[\theta,1-\theta]} x(t) = \sigma\gamma$

(iv) If $x \in \partial\Omega_K^\gamma$, *then* $\sigma\gamma \le x(t) \le \gamma$, *for* $t \in [\theta, 1-\theta]$

3. Main Results

Theorem 8. *Assume that* (F_0) *holds. In addition, the function* f *satisfies the following assumption:*

(F_1) *There exists a* $\alpha > 0$ *such that* $\Phi_*(\alpha) \ge 0$ *and*

$$\mu\frac{\alpha^2}{6}C\int_0^1 G(s,s)ds + C\int_0^1 G(s,s)e(s)ds + C\int_0^1 G(s,s)\Phi^*(s,\alpha)ds < \alpha. \quad (33)$$

Then, we have

(i) If $\sigma\alpha > \omega^*$, *then (1) has at least one partly positive solution* (u, ϕ), *namely,*

$$u(t) > 0, \quad \text{for } t \in [\theta, 1-\theta] \quad (34)$$

(ii) If $\alpha\delta\xi > Ce^*$, *then (1) has at least one positive solution* (u, ϕ), *which satisfies*

$$u(t) > 0, \quad \text{for } t \in [0,1] \quad (35)$$

Proof. For any $x \in \partial B_K^\alpha$, it is obvious that

$$\begin{aligned} &H(x-\omega)(x-\omega) \le \|x\| = \alpha, \\ &\int_0^1 K(t,s)H(x-\omega)[x(s) - \omega(s)]ds \\ &\quad \le \int_0^1 K(s,s)H(x-\omega)[x(s) - \omega(s)]ds \le \frac{1}{6}\alpha. \end{aligned} \quad (36)$$

Then, from (F_1) it follows that

$$\begin{aligned}
T(x)(t) &= \int_0^1 G(t,s)F(s,H(x-\omega)(x-\omega))ds \\
&= \mu\int_0^1 G(t,s)\left[H(x-\omega)(x-\omega)\int_0^1 K(s,\tau)H(x-\omega)\right. \\
&\quad \left.\cdot [x(\tau)-\omega(\tau)]d\tau\right]ds + \int_0^1 G(t,s)[f(s,H(x-\omega) \\
&\quad \cdot (x-\omega)\int_0^1 K(s,\tau)H(x-\omega)[x(\tau)-\omega(\tau)]d\tau) \\
&\quad \left.+ e(s)\right]ds \le \mu\frac{\alpha^2}{6}C\int_0^1 G(s,s)ds + C\int_0^1 G(s,s)e(s)ds \\
&\quad + C\int_0^1 G(s,s)f(s,H(x-\omega)(x-\omega), \\
&\quad \int_0^1 K(s,\tau)H(x-\omega)[x(\tau)-\omega(\tau)]d\tau)ds \\
&\le \mu\frac{\alpha^2}{6}C\int_0^1 G(s,s)ds + C\int_0^1 G(s,s)e(s)ds \\
&\quad + C\int_0^1 G(s,s)\Phi^*(s,\alpha)ds < \alpha = \|x\|,
\end{aligned} \tag{37}$$

which implies that (a) of Lemma 1 holds.

Let

$$\Psi(\rho) = \max_{0\le x\le\rho}\left\{H(x-\omega)(x-\omega)\cdot\int_\theta^{1-\theta} K(\tau,\tau)H(x-\omega)[x(\tau)-\omega(\tau)]d\tau\right\}. \tag{38}$$

From [21], we have that

$$\lim_{\rho\to+\infty}\frac{\Psi(\rho)}{\rho} = +\infty. \tag{39}$$

Then, there exists a $\beta >$ with $\sigma\beta > \alpha$ such that

$$H(\sigma\beta-\omega)(\sigma\beta-\omega)\cdot\int_\theta^{1-\theta} K(\tau,\tau)H(\sigma\beta-\omega)[\sigma\beta-\omega(\tau)]d\tau > \Lambda\beta, \tag{40}$$

where Λ satisfies

$$\Lambda\mu\sigma C\int_\theta^{1-\theta} G(s,s)K(s,s)ds > 1. \tag{41}$$

Let $\eta(t) = 1$; now we prove that $x \ne Tx + \lambda$, for $x \in \partial\Omega_K^\beta$ and $\lambda > 0$. On the contrary, if there exists a pair of $x_0 \in \partial\Omega_K^\beta$ and $\lambda_0 > 0$ such that $x_0(t) = T(x_0)(t) + \lambda_0$, then from (iv) of Lemma 7, it follows that

$$\sigma\beta = \sigma\|x_0\| \le x_0(t) \le \beta, \quad \text{for } t \in [\theta, 1-\theta]. \tag{42}$$

Furthermore, for $t \in [\theta, 1-\theta]$, we have

$$\begin{aligned}
\|x_0\| &\ge \min_{\theta\le t\le 1-\theta} x_0(t) = \min_{\theta\le t\le 1-\theta} Tx_0(t) + \lambda_0 \\
&= \min_{\theta\le t\le 1-\theta}\mu\int_0^1 G(t,s)H(x_0-\omega)(x_0-\omega)\int_0^1 K(s,\tau)H(x-\omega) \\
&\quad \cdot [x(\tau)-\omega(\tau)]d\tau ds + \min_{\theta\le t\le 1-\theta}\int_0^1 G(t,s) \\
&\quad \cdot \left[f\left(s, H(x_0-\omega)(x-\omega), \int_0^1 K(s,\tau)H(x_0-\omega)\right.\right. \\
&\quad \left.\left.\cdot [x(\tau)-\omega(\tau)]d\tau\right) + e(s)\right]ds + \lambda_0 \\
&\ge \min_{\theta\le t\le 1-\theta}\mu\int_0^1 G(t,s)\left[H(x_0-\omega)(x_0-\omega)\int_0^1 K(s,\tau)H\right. \\
&\quad \left.\cdot (x-\omega)[x(\tau)-\omega(\tau)]d\tau\right]ds \ge \min_{\theta\le t\le 1-\theta}\mu\delta G(t,t) \\
&\quad \cdot \int_0^1 G(s,s)\left[H(x_0-\omega)(x_0-\omega)\int_0^1 K(s,\tau)H(x-\omega)\right. \\
&\quad \left.\cdot [x(\tau)-\omega(\tau)]d\tau\right]ds \ge \mu\sigma C\int_0^1 G(s,s) \\
&\quad \cdot \left[H(x_0-\omega)(x_0-\omega)\int_0^1 K(s,s)K(\tau,\tau)H(x-\omega)\right. \\
&\quad \left.\cdot [x(\tau)-\omega(\tau)]d\tau\right]ds = \mu\sigma C\int_0^1 G(s,s)K(s,s) \\
&\quad \cdot \left[H(x_0-\omega)(x_0-\omega)\int_0^1 K(\tau,\tau)H(x-\omega)[x(\tau)-\omega(\tau)]d\tau\right]ds \\
&= \mu\sigma C\int_0^1 G(s,s)K(s,s)\left[H(x_0-\omega)(x_0-\omega)\int_0^{1-\theta} K(\tau,\tau)H(x-\omega)\right. \\
&\quad \cdot [x(\tau)-\omega(\tau)]d\tau]ds = \mu\sigma C\int_0^1 G(s,s)K(s,s) \\
&\quad \cdot \left[H(\sigma\|x_0\|-\omega)(\sigma\|x_0\|-\omega)\int_\theta^{1-\theta} K(\tau,\tau)H(\sigma\|x_0\|-\omega)\right. \\
&\quad \left.\cdot [\sigma\|x_0\|-\omega(\tau)]d\tau\right]ds \ge \mu\sigma C\int_0^1 G(s,s)K(s,s)ds\cdot\Lambda\beta > \beta = \|x_0\|,
\end{aligned} \tag{43}$$

which contradicts with the statement (iii) of Lemma 7. So (b) holds.

Since $\alpha < \sigma\beta$, from Lemma 7, we have $\overline{B_K^\alpha} \subset B_K^{\sigma\beta} \subset \Omega_K^\beta$. Therefore, by Lemma 1, we can get that T has at least one positive fixed-point $x(t) \in \overline{\Omega_K^\beta} \setminus B_K^\alpha$. Hence, the inequalities hold,

$$\|x\| \ge \alpha, \sigma\alpha \le \min_{t\in[\theta,1-\theta]} x(t) \le \sigma\beta. \tag{44}$$

On the other hand, since $\sigma\|x\| \le \min_{t\in[\theta,1-\theta]} x(t) \le \sigma\beta$, we have $\|x\| \le \beta$.

(i) Since

$$\min_{t\in[\theta,1-\theta]} x(t) \ge \sigma\alpha > \omega^* > \omega(t) = \int_0^1 G(t,s)e(s)ds, \tag{45}$$

we have

$$u(t)=x(t)-\omega(t)>0, \quad t\in[\theta,1-\theta] \tag{46}$$

(ii) From Lemmas 3 and 4, we have

$$\begin{aligned} x(t) &\geq \frac{\delta}{C}G(t,t)\|t\| = \frac{\delta}{C}G(t,t)\alpha \geq \alpha\frac{\delta}{Ce^*}\xi e^*\int_0^1 G(t,s)ds \\ &\geq \alpha\frac{\delta}{Ce^*}\xi\int_0^1 G(t,s)e(s)ds = \alpha\frac{\delta}{Ce^*}\xi\cdot\omega(t) > \omega(t), \end{aligned} \tag{47}$$

which implies that $u(t)=x(t)-\omega(t)>0$

Therefore, (1) has one positive solution $(u,\phi)=(u,\int_0^1 K(t,s)u(s)ds)$.

Theorem 9. *Assume that (F_0) holds. In addition, the function f satisfies the following assumptions:*

(F_2) There exists a $\alpha>\max\{\omega^*/\sigma, Ce^*/\delta\xi\}$ such that $\Phi_*(\alpha)\leq 0$ and

$$\mu\frac{\alpha^2}{6}C\int_0^1 G(s,s)ds + C\int_0^1 G(s,s)e(s)ds + C\int_0^1 G(s,s)\Phi^*(s,\alpha)ds < \alpha. \tag{48}$$

(F_3) There exists a $r^*\in(0,\sigma\alpha-\omega^*)$ such that

$$\begin{aligned} &\mu\frac{r^{*2}}{6}C\int_0^1 G(s,s)ds + C\int_0^1 G(s,s)e(s)ds \\ &\quad + C\int_0^1 G(s,s)\Phi^*(s,r^*)ds < r^*. \end{aligned} \tag{49}$$

(F_4) $\lim_{u+\phi\to 0^+}(f(t,u,\phi)/(u+\phi))=+\infty$, uniformly for $t\in[0,1]$.

Then, (1) has at least two positive solution(u_i,ϕ_i) $(i=1,2)$, which satisfies

$$0\leq u_1(t)<r^*, \quad \min_{t\in[\theta,1-\theta]} u_2(t) > \sigma\alpha-\omega^*. \tag{50}$$

Proof. From (F_2) and Theorem 8, it follows that there exists a solution $u_2(t)\geq 0$ and

$$\min_{t\in[\theta,1-\theta]} u_2(t) > \sigma\alpha-\omega^*. \tag{51}$$

Now, we apply Lemma 1 to prove the existence of another solution $u_1(t)$.

Since $r^*<\sigma\alpha-\omega^*<\alpha$ and $\Phi_*(\alpha)$, then we can define the operator

$$\bar{T}(u)(t)=\int_0^1 G(t,s)F(s,u(s))ds. \tag{52}$$

For any $u\in\partial B_K^{r^*}$, it is obvious that

$$\int_0^1 K(t,s)u(s)ds \leq \int_0^1 K(s,s)u(s)ds \leq \frac{1}{6}r^*. \tag{53}$$

Then, we have

$$\begin{aligned} \bar{T}(u)(t) &= \int_0^1 G(t,s)F(s,u(s))ds \\ &\leq \mu\frac{r^{*2}}{6}C\int_0^1 G(s,s)ds + C\int_0^1 G(s,s)e(s)ds + C\int_0^1 G(s,s)f \\ &\quad \cdot\left(s, H(x-\omega)(x-\omega), \int_0^1 K(s,\tau)H(x-\omega)[x(\tau)-\omega(\tau)]d\tau\right)ds \\ &\leq \mu\frac{r^{*2}}{6}C\int_0^1 G(s,s)ds + C\int_0^1 G(s,s)e(s)ds \\ &\quad + C\int_0^1 G(s,s)\Phi^*(s,r^*)ds < r^* = \|u\|, \end{aligned} \tag{54}$$

which implies that (a) of Lemma 1 holds.

Since $\lim_{u+\phi\to 0^+}(f(t,u,\phi)/(u+\phi))=+\infty$, uniformly for $t\in[0,1]$, there exists a $r_*<r^*$ such that

$$f(t,u,\phi) > M(u+\phi), \quad \text{for } 0<u+\phi\leq r_*, \tag{55}$$

where M satisfies

$$M\delta\sigma\cdot\max_{0\leq t\leq 1} G(t,t)\cdot\int_\theta^{1-\theta} G(s,s)\left[1+\int_\theta^{1-\theta} K(s,\tau)d\tau\right]ds > 1. \tag{56}$$

Let $\eta(t)=1$, now we prove that $u\neq\bar{T}u+\lambda$, for $u\in\partial\Omega_K^{r_*/2}$ and $\lambda>0$. On the contrary, if there exists a pair of $u_0\in\partial\Omega_K^{r_*/2}$ and $\lambda_0>0$ such that $u_0(t)=\bar{T}(u_0)(t)+\lambda_0$, then from (iv) of Lemma 7, it follows that

$$u_0(t)\leq\frac{r_*}{2}, \phi_0(t)=\int_0^1 K(t,s)u_0(s)ds < \frac{r_*}{12}. \tag{57}$$

Furthermore, we have

$$\begin{aligned} \|u_0\| &= \left\|\tilde{T}(u_0)(t)\right\| + \lambda_0 \geq \left\|\int_0^1 G(t,s)f\left(s,u(s),\int_0^1 K(s,\tau)u(\tau)d\tau\right)ds\right\| \\ &\geq \delta\max_{0\leq t\leq 1} G(t,t)\int_0^1 G(s,s)f\left(s,u(s),\int_0^1 K(s,\tau)u(\tau)d\tau\right)ds \\ &\geq \delta\max_{0\leq t\leq 1} G(t,t)\int_0^1 G(s,s)M\left[u,(s)+\int_0^1 K(s,\tau)u(\tau)d\tau\right]ds \\ &\geq \delta\max_{0\leq t\leq 1} G(t,t)\int_0^{1-\theta} G(s,s)M\left[u,(s)+\int_0^{1-\theta} K(s,\tau)u(\tau)d\tau\right]ds \\ &\geq M\delta\sigma\cdot\max_{0\leq t\leq 1} G(t,t)\cdot\int_0^{1-\theta} G(s,s)\left[1+\int_0^{1-\theta} K(s,\tau)d\tau\right]ds \\ &\quad \cdot\|u_0\| > \|u_0\|, \end{aligned} \tag{58}$$

which contradicts with the statement (iii) of Lemma 7. So (b) holds.

Therefore, from Lemma 1, we can get that $\tilde{\mathbf{T}}$ has at least one positive fixed-point $u_1(t) \in \overline{\Omega_K^{r^*}} \backslash B_K^{r_*/2}$. Hence, the inequalities hold

$$\|u_1\| \geq \frac{r_*}{2}, \sigma \frac{r_*}{2} \leq \min_{t\in[\theta,1-\theta]} u_1(t) \leq \sigma r^*. \tag{59}$$

On the other hand, since $\sigma\|u_1\| \leq \min_{t\in[\theta,1-\theta]} u_1(t) \leq \sigma r^*$, we have $\|u_1\| \leq r^*$.

Finally, since

$$\min_{t\in[\theta,1-\theta]} u_2(t) \leq \sigma\alpha - \omega^* > r^*, \|u_1\| \leq r^*, \tag{60}$$

(1) has at least two positive solutions.

4. Examples

Example 10. Let us consider the following system:

$$\begin{cases} -u'' + u = \phi u + \frac{1}{4}e^u + \frac{1}{4}\cos \pi\phi - \frac{1}{4}t^2 0 < t < 1, \\ -\phi'' = \frac{1}{4}u, \\ \phi(0) = \phi(1) = 0, \phi''(0) = \phi''(1) = 0, \end{cases} \tag{61}$$

where $\mu = 1, \rho = 1, f : [0,1] \times \mathbb{R}_+^2 \longrightarrow \mathbb{R}$ is continuous, and

$$f = \frac{1}{4}\left(e^u + \cos \pi\phi - t^2\right) \geq -\frac{1}{4}t^2 = -e(t) \quad \text{for } (t,u,\phi) \in [0,1] \times \mathbb{R}_+^2. \tag{62}$$

It is obvious that (F_0) holds. Via some computations, we have

$$e^* = \frac{1}{4}, C = 1, \delta = \frac{1}{\sinh 1}, J_1^* = J_2^* = \frac{e^{1/2} - e^{-(1/2)}}{e^{1/2} + e^{-(1/2)}} < 1. \tag{63}$$

Choosing $\theta = 1/4 \in (0, 1/2), \xi = 1/2$. Then, we have

$$\begin{aligned} \sigma &= \min_{t\in[1/4,3/4]} \frac{\delta}{C} G(t,t) = \frac{\sinh(3/4)\sinh(1/4)}{(\sinh 1)^2}, \\ \omega^* &= \frac{\sinh 1 - 2\sinh(1/2)}{\sinh 1}. \end{aligned} \tag{64}$$

Furthermore, we have

$$\begin{aligned} &\frac{\omega^*}{\sigma} = \frac{(\sinh 1 - 2\sinh(1/2))\sinh 1}{\sinh(3/4)\sinh(1/4)} < 3, \\ &\frac{Ce^*}{\delta\xi} < \sin 1(J_1^* + J_2^*) < 3. \end{aligned} \tag{65}$$

Choosing $\alpha = 3 > \max\{\omega^*/\sigma, Ce^*/\delta\xi\}, r^* = 3/10 \in (0, \sigma\alpha - \omega^*)$. Then,

$$\begin{aligned} &\Phi_*(3) \geq 0, \Phi^*(t,3) = \frac{1}{4}\left(e^3 + 1\right), \\ &\Phi_*\left(\frac{3}{10}\right) \geq 0, \Phi^*\left(t, \frac{3}{10}\right) = \frac{1}{4}\left(e^{3/10} + 1\right). \end{aligned} \tag{66}$$

It is easy to get

$$\begin{aligned} &\mu\frac{\alpha^2}{6}C\int_0^1 G(s,s)ds + C\int_0^1 G(s,s)e(s)ds + C\int_0^1 G(s,s)\Phi^*(s,\alpha)ds \\ &\leq \frac{3}{2}\frac{e^{-1}}{2\sinh 1} + \frac{e^{-1}}{2\sinh 1} + \frac{e^{-1}}{2\sinh 1}\left[\frac{1}{4}\left(e^3 + 1\right)\right] \leq 3, \end{aligned} \tag{67}$$

$$\begin{aligned} &\mu\frac{(r^*)^2}{6}C\int_0^1 G(s,s)ds + C\int_0^1 G(s,s)e(s)ds + C\int_0^1 G(s,s)\Phi^*(s,r^*)ds \\ &\leq \frac{3}{200}\frac{e^{-1}}{2\sinh 1} + \frac{e^{-1}}{2\sinh 1} + \frac{e^{-1}}{2\sinh 1}\left[\frac{1}{4}\left(e^{3/10} + 1\right)\right] \leq \frac{3}{10}, \end{aligned} \tag{68}$$

which implies that (F_2) and (F_3) hold.

Finally, it is obvious that

$$\lim_{u+\phi\to 0+} \frac{f(t,u,\phi)}{u+\phi} = +\infty, \quad \text{uniformly for } t \in [0,1]. \tag{69}$$

So (F_4) holds.

Therefore, by Theorem 9, (61) has two positive solutions.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed equally and significantly in writing this article. All the authors read and approved the final manuscript.

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