Research Article

Positive Solutions for a Singular Superlinear Fourth-Order Equation with Nonlinear Boundary Conditions

Dongliang Yan

Department of Mathematics, Northwest Normal University, Lanzhou 730070, China

Correspondence should be addressed to Dongliang Yan; yuhulu@163.com

Received 7 November 2019; Revised 10 February 2020; Accepted 2 March 2020; Published 28 March 2020

Academic Editor: Gennaro Infante

Copyright © 2020 Dongliang Yan. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We show the existence of positive solutions for a singular superlinear fourth-order equation with nonlinear boundary conditions.

$$\begin{cases} u''''(x) = \lambda h(x)f(u(x)), & x \in (0, 1), \\ u(0) = u'(0) = 0, \\ u''(1) = 0, & u''''(1) + c(u(1))u(1) = 0, \end{cases}$$
 where $\lambda > 0$ is a small positive parameter, $f: (0, \infty) \rightarrow \mathbb{R}$ is continuous, superlinear at ∞ , and is allowed to be singular at 0, and $h: [0, 1] \rightarrow [0, \infty)$ is continuous. Our approach is based on the fixed-point result of Krasnoselskii type in a Banach space.

1. Introduction

Consider the singular superlinear fourth-order nonlinear boundary value problem

$$\begin{cases} u''''(x) = \lambda h(x)f(u(x)), & x \in (0, 1), \\ u(0) = u'(0) = 0, \\ u''(1) = 0, & u''''(1) + c(u(1))u(1) = 0, \end{cases} \quad (1)$$

where $\lambda > 0$ is a small positive parameter, $f: (0, \infty) \rightarrow \mathbb{R}$ is continuous, $h: [0, 1] \rightarrow [0, \infty)$ is continuous. If $f(0) > 0$, then (1) is called a positone problem; if $f(0) < 0$, then (1) is called a semipositone problem.

The study of semipositone problems was formally introduced by Castro and Shivaji [1]. From an application viewpoint, one is usually interested in the existence of positive solutions for semipositone problems. Significant processes on second-order semipositone problems have taken place in the last 10 years, see [1–5] and the references therein.

Fourth-order boundary value problems modeling bending equilibria of elastic beams have been considered in several papers [6–9]. Most of them are concerned with nonlinear equations with null boundary conditions. When the boundary conditions are nonzero or nonlinear, fourth-

order equations can model beams resting on elastic bearings located in their extremities. See for instance, [10–12] and the references therein.

For instance, Cabrera et al. [10] studied the existence of positive solutions for the fourth-order positone problem

$$\begin{cases} u''''(x) = f(t, u(x), (Hu)(x)), & x \in (0, 1), \\ u(0) = u'(0) = 0, \\ u''(1) = 0, & u''''(1) + \tilde{c}(u(1)) = 0, \end{cases} \quad (2)$$

where $f(t, x, y): [0, 1] \times [0, +\infty) \times [0, +\infty) \rightarrow [0, +\infty)$. $f(t, x, y)$ is increasing in x and decreasing in y , for fixed $t \in [0, +\infty]$. For convenience, we denote $\tilde{c}(s) = c(s)s$ in this article. In [10–12], the authors studied positive solutions of fourth-order nonlinear boundary value problems in the positone case based on a mixed monotone operator method and a well-known fixed-point theorem in cones.

It should be noted that nonlinear part f is either bounded or positive states in [1, 10–12]. In this paper, we prove the existence of positive solution to (1) by assuming that $f: (0, \infty) \rightarrow \mathbb{R}$ is continuous and is allowed to be singular at 0; in other words, f may be unbounded from below and satisfies the superlinear condition. Moreover, we prove a useful lemma (Lemma 3) in this paper which plays a key role

to guarantee the positivity of solution. It can be obtained by concavity and convexity of solution or calculation for the second-order boundary value problem, but for the fourth-order boundary value problem, this becomes complicated.

In addition, we will replace $h(x)f(u(x))$ by $m(x)$ and γ by $c(s)$, and we perform a study of the sign of Green's function of the corresponding linear problems:

$$\begin{cases} w''''(x) = m(x), & x \in (0, 1), \\ w(0) = w'(0) = 0, \\ w''(1) = 0, & w'''(1) + \gamma w(1) = 0. \end{cases} \quad (3)$$

In detail, we construct Green's function $G(x, s)$ by disconjugacy theory and give a sufficient condition to make ensure that $G(x, s)$ is either positive or negative. This fact is crucial for our arguments.

Motivated by the above facts, in this paper, we shall obtain the existence of positive solutions for a singular superlinear fourth-order equation with nonlinear boundary conditions via the fixed-point result of Krasnoselskii type in a Banach space.

We shall make the following assumptions:

- (H1): $h: [0, 1] \rightarrow [0, \infty)$ is continuous.
- (H2): $c: [0, \infty) \rightarrow [0, \infty)$ is continuous.
- (H3): there exists a constant β with $\beta < (1/2)$ such that

$$\limsup_{t \rightarrow 0^+} t^\beta |f(t)| < \infty. \quad (4)$$

- (H4): $f: (0, \infty) \rightarrow \mathbb{R}$ is continuous and $\lim_{t \rightarrow \infty} (f(t)/t) = \infty$.

By a positive solution of (1), we mean function $u \in C^3[0, 1] \cap C^4(0, 1)$ with $u > 0$ on $(0, 1)$ and satisfying (1).

Our main result is as follows.

Theorem 1. *Let (H1)–(H4) hold. Then, there exists a constant $\lambda_0 > 0$ such that (1) has a positive solution u_λ for $\lambda < \lambda_0$ with $u_\lambda \rightarrow \infty$ as $\lambda \rightarrow 0^+$ uniformly on compact subsets of $(0, 1)$. The paper is organized as follows. In Section 2, we present some preliminaries and lemmas that will be used to prove our main results. In Section 3, we prove the main result that the existence of a large positive solution to (1) for λ small.*

2. Notation and Preliminaries

Suppose that E is a real Banach space which is partially ordered by a cone $P \subset E$, i.e., $x \leq y$ if and only if $y - x \in P$. If $x \leq y$ and $x \neq y$, then we denote $x < y$ or $y > x$. By θ , we denote the zero element of E . Recall that a nonempty closed convex set $P \subset E$ is a cone if it satisfies (i) $x \in P, \lambda \geq 0 \implies \lambda x \in P$ and (ii) $x \in P, -x \in P \implies x = \theta$.

We first recall the following fixed-point result of Krasnoselskii type in a Banach space (see e.g., [1], Theorem 12.3).

Lemma 1. *Let E be a Banach space and $T: P \rightarrow P$ be a completely continuous operator. Suppose there exist $h \in E, h_0 \neq 0$ and positive constants r, R with $r \neq R$ such that*

- (a) *If $y \in P$ satisfies $y = \zeta Ty, \theta \in [0, 1]$, then $\|y\| \neq r$*
- (b) *If $y \in P$ satisfies $y = Ty + \xi h, \xi \geq 0$, then $\|y\| \neq R$*

Then, T has a fixed point $y \in P$ with $\min\{r, R\} < \|y\| < \max\{r, R\}$. In the sequel, we describe the Banach space where the fixed points are found, as well as some notations which are used along the paper. Let us consider the Banach space $\mathcal{B} = C(I), I = [0, 1]$ coupled with the norm

$$\|u\|_\infty = \max_{t \in I} |u(t)|, \quad (5)$$

and the Banach space $X = L^1(0, 1)$ equipped with the norm $\|\phi\|_X = \int_0^1 |\phi(t)| dt$.

Lemma 2. *Let w satisfy*

$$\begin{cases} w''''(x) = m(x), & x \in (0, 1), \\ w(0) = w'(0) = 0, \\ w''(1) = 0, & w'''(1) + \gamma w(1) = 0, \end{cases} \quad (6)$$

where $m \in L^1(0, 1), m(t) \geq 0$, for a.e. $t \in (0, 1), \gamma \in [0, 2]$. Then,

$$w(x) \geq \frac{3-\gamma}{9-\gamma} \|w\|_\infty q(x), \quad x \in (0, 1), \quad (7)$$

where

$$q(x) = \min(x^2, x) = x^2. \quad (8)$$

Proof of Lemma 2. It is easy to verify that

$$w(x) = \int_0^1 G(x, s) m(s) ds := \mathcal{L}w(x). \quad (9)$$

Green's function $G(x, s)$ is computed as follows:

$$G(x, s) = \frac{1}{6} \begin{cases} s^2(3x-s) + \frac{\gamma}{6-2\gamma} x^2 s^2 (x-3)(s-3), & 0 \leq s \leq x \leq 1, \\ x^2(3s-x) + \frac{\gamma}{6-2\gamma} x^2 s^2 (x-3)(s-3), & 0 \leq x \leq s \leq 1. \end{cases} \quad (10)$$

Obviously, $G(x, s)$ is positive for any $\gamma \in [0, 2]$, while $G(x, s)$ changes sign for γ is large enough, and we can compute the value of $G(x, s)$ when $\gamma = 10000$ by Mathematic 9.0:

$$\begin{aligned} G(0.587022479, 0.999) &\approx -0.0000059476, \\ G(0.5852750296, 0.5852750296) &\approx 0.00979528106. \end{aligned} \quad (11)$$

Let $\|w\|_\infty = w(x_0)$ for some $x_0 \in (0, 1]$. We verify that

$$\frac{G(x, s)}{G(x_0, s)} \geq q(x), \quad s, x, x_0 \in (0, 1]. \quad (12)$$

Indeed, if $x, x_0 \leq s$,

$$\begin{aligned} \frac{G(x, s)}{G(x_0, s)} &= \frac{x^2(3s-x) + (\gamma/(6-2\gamma))x^2s^2(x-3)(s-3)}{x_0^2(3s-x_0) + (\gamma/(6-2\gamma))x_0^2s^2(x_0-3)(s-3)} \geq \frac{x^2(3-(x/s))}{x_0^2(3-(x_0/s)) + (\gamma/(6-2\gamma))x_0^2s(x_0-3)(s-3)} \\ &\geq \frac{x^2(3-(x/s))}{3s^2 + (\gamma/(6-2\gamma))s^3(s-3)^2} \geq \frac{2x^2}{3 + (\gamma/(6-2\gamma))4} \geq \frac{3-\gamma}{9-\gamma}x^2. \end{aligned} \quad (13)$$

If $x \leq s \leq x_0$,

$$\begin{aligned} \frac{G(x, s)}{G(x_0, s)} &= \frac{x^2(3s-x) + (\gamma/(6-2\gamma))x^2s^2(x-3)(s-3)}{s^2(3x_0-s) + (\gamma/(6-2\gamma))x_0^2s^2(x_0-3)(s-3)} \geq \frac{x^2(3-(x/s))}{s(3x_0) + (\gamma/(6-2\gamma))x_0^2s(x_0-3)(s-3)} \\ &\geq \frac{x^2(3-(x/s))}{3x_0 + (\gamma/(6-2\gamma))(s-3)^2} \geq \frac{2x^2}{3 + (\gamma/(6-2\gamma))9} \geq \frac{3-\gamma}{9-\gamma}x^2. \end{aligned} \quad (14)$$

If $s \leq x \leq x_0$,

$$\begin{aligned} \frac{G(x, s)}{G(x_0, s)} &= \frac{s^2(3x-s) + (\gamma/(6-2\gamma))x^2s^2(x-3)(s-3)}{s^2(3x_0-s) + (\gamma/(6-2\gamma))x_0^2s^2(x_0-3)(s-3)} \geq \frac{(3x-s)}{(3x_0-s) + (\gamma/(6-2\gamma))x_0^2(x_0-3)(s-3)} \\ &\geq \frac{(3x-s)}{3x_0 + (\gamma/(6-2\gamma))x_0^2(s-3)^2} \geq \frac{2x}{3 + (\gamma/(6-2\gamma))9} \geq \frac{3-\gamma}{9-\gamma}x. \end{aligned} \quad (15)$$

If $x_0 \leq s \leq x$,

$$\begin{aligned} \frac{G(x, s)}{G(x_0, s)} &= \frac{s^2(3x-s) + (\gamma/(6-2\gamma))x^2s^2(x-3)(s-3)}{x_0^2(3s-x_0) + (\gamma/(6-2\gamma))x_0^2s^2(x_0-3)(s-3)} \geq \frac{3x-s}{(x_0/s)^2(3s-x_0) + (\gamma/(6-2\gamma))x_0^2(x_0-3)(s-3)} \\ &\geq \frac{3x-s}{(s/s)^23s + (\gamma/(6-2\gamma))x_0^2(x_0-3)^2} \geq \frac{2x}{3 + (\gamma/(6-2\gamma))4} \geq \frac{3-\gamma}{9-\gamma}x. \end{aligned} \quad (16)$$

Thus,

$$w(x) = \int_0^1 \frac{G(x, s)}{G(x_0, s)} G(x_0, s) w(s) ds \geq \frac{3-\gamma}{9-\gamma} \|w\|_\infty q(x). \quad (17)$$

Lemma 3. Let $k \in L^1(0, 1)$ with $k \geq 0$, and let $u \in C^3[0, 1] \cap C^4(0, 1]$ satisfy

$$\begin{cases} u''''(x) \geq -k, & x \in (0, 1), \\ u(0) \geq 0, & u'(0) \geq 0, \\ u''(1) \geq 0, & u'''(1) + \gamma u(1) \geq 0. \end{cases} \quad (18)$$

Suppose $\|u\|_\infty > 2((9-\gamma)/(3-\gamma))^2 \|k\|_X$. Then, $u(x) \geq 0$ and

$$u(x) \geq \frac{3-\gamma}{2(9-\gamma)} \|u\|_\infty q(x), \quad x \in (\tau, 1). \quad (19)$$

Proof. Let $w_0(x) \in \mathcal{B}$ be the unique solution of the problem

$$\begin{cases} w''''(x) = -k, & x \in (0, 1), \\ w(0) = 0, & w'(0) = 0, \\ w''(1) = 0, & w'''(1) + \gamma w(1) = 0. \end{cases} \quad (20)$$

Then,

$$\begin{aligned} -w_0(x) &= \frac{1}{6} \left(\int_0^x (s^2(3x-s) + (\gamma/(6-2\gamma))x^2s^2(x-3)(s-3)) k(s) ds \right. \\ &\quad \left. + \int_x^1 (x^2(3s-x) + (\gamma/(6-2\gamma))x^2s^2(x-3)(s-3)) k(s) ds \right), \end{aligned} \quad (21)$$

and accordingly,

$$\begin{aligned}
& -w_0(x) \\
& \leq \int_0^x \left(3x^2s + \frac{\gamma}{6-2\gamma} x^2 s^2 (x-3)(s-3) \right) k(s) ds \\
& \quad + \int_x^1 \left(3x^2s + \frac{\gamma}{6-2\gamma} x^2 s^2 (x-3)(s-3) \right) k(s) ds \\
& \leq \int_0^1 \left(3x^2s + \frac{\gamma}{6-2\gamma} x^2 s^2 (x-3)(s-3) \right) k(s) ds \\
& \leq x^2 \int_0^1 \left(3s + \frac{\gamma}{6-2\gamma} (1-3)^2 s \right) k(s) ds \leq \frac{9-\gamma}{3-\gamma} \|k\|_X q(x).
\end{aligned} \tag{22}$$

Set $y(x) = u(x) - w_0(x)$, $(y'''(1) + \gamma y(1)) - \gamma(y(0) + y'(0))/(6+\gamma) \leq y'''(1) \leq (y'''(1) + \gamma y(1)) - \gamma(y(0) + y'(0))$, $y(\tau) = \|y\|_\infty$. Then,

$$\begin{cases} y'''(x) \geq 0, & x \in (0, 1), \\ y(0) \geq 0, & y'(0) \geq 0, \\ y''(1) \geq 0, & y'''(1) + \gamma y(1) \geq 0, \end{cases} \tag{23}$$

and by Lemma 2, we can get

$$y(x) \geq \frac{3-\gamma}{9-\gamma} \|y\|_\infty q(x), \quad x \in (\tau, 1). \tag{24}$$

It follows from (22) and (24) and the fact that $y(x) = u(x) - w_0(x)$ that

$$\begin{aligned}
u(x) &= y(x) + w_0(x) \\
&\geq \frac{3-\gamma}{9-\gamma} \|y\|_\infty q(x) - \frac{9-\gamma}{3-\gamma} \|k\|_X q(x) \\
&= \frac{3-\gamma}{9-\gamma} \|u - w_0\|_\infty q(x) - \frac{9-\gamma}{3-\gamma} \|k\|_X q(x) \\
&\geq \frac{3-\gamma}{9-\gamma} \|u\|_\infty q(x) - \frac{9-\gamma}{3-\gamma} \|w_0\|_\infty q(x) - \frac{9-\gamma}{3-\gamma} \|k\|_X q(x) \\
&\geq \frac{3-\gamma}{9-\gamma} \|u\|_\infty q(x) + \frac{9-\gamma}{3-\gamma} \|k\|_X (q(x))^2 - \frac{9-\gamma}{3-\gamma} \|k\|_X q(x) \\
&\geq \frac{3-\gamma}{9-\gamma} \|u\|_\infty q(x) - \frac{9-\gamma}{3-\gamma} \|k\|_X q(x) \\
&= \left(\frac{3-\gamma}{9-\gamma} \|u\|_\infty - \frac{9-\gamma}{3-\gamma} \|k\|_X \right) q(x),
\end{aligned} \tag{25}$$

for $x \in (\tau, 1)$. This completes the Proof of Lemma 3.

3. Proof of the Main Results

Proof of Theorem 1. Let $\lambda > 0$. For $v \in \mathcal{B}$, define $\mathcal{L}_\lambda v = u$, where u is the solution of

$$\begin{cases} u''''(x) = \lambda h(x) f(\bar{v}), & x \in (0, 1), \\ u(0) = 0, & u'(0) = 0, \\ u''(1) = 0, & u'''(1) + \gamma_v u(1) = 0, \end{cases} \tag{26}$$

where

$$\bar{v}(x) = \max\{v(x), q(x)\}, \tag{27}$$

$\gamma_v = c(|v(1)|)$, and $q(x)$ is defined in Lemma 2. Then,

$$u(x) = \lambda \int_0^1 G_v(x, s) h(s) f(\bar{v}(s)) ds, \tag{28}$$

where Green's function $G_v(x, s)$ is given by

$$G_v(x, s) = \frac{1}{6} \begin{cases} s^2(3x-s) + \frac{\gamma_v}{6-2\gamma_v} x^2 s^2 (x-3)(s-3), & 0 \leq s \leq x \leq 1, \\ x^2(3s-x) + \frac{\gamma_v}{6-2\gamma_v} x^2 s^2 (x-3)(s-3), & 0 \leq x \leq s \leq 1. \end{cases} \tag{29}$$

Note that $G_v(x, s) \leq 1$ for all x, s . By (H3), there exists a constant $M > 0$ depending on $\|v\|_\infty$ such that

$$|f(\bar{v}(x))| \leq \frac{M}{\bar{v}^\beta} \leq \frac{M}{q^\beta} \in L^1(0, 1). \tag{30}$$

It follows from the Lebesgue dominated convergence theorem that

$$u \in \mathcal{B} \text{ i.e. } \mathcal{L}_\lambda: \mathcal{B} \longrightarrow \mathcal{B}. \tag{31}$$

Define the cone P in $\mathcal{B}$ by

$$P = \left\{ u \in \mathcal{B} \mid u(x) \geq 0, \quad u(x) \geq \frac{3-\gamma}{2(9-\gamma)} \|u\|_\infty q(x), \quad x \in (\tau, 1) \right\}. \tag{32}$$

For $u \in P$, it follows that, from (H3) and Lemma 3, we can get $\mathcal{L}_\lambda(P) \subset P$.

We next show that $\mathcal{L}_\lambda: P \longrightarrow P$ is a completely continuous operator.

Now, we show $\mathcal{L}_\lambda: P \longrightarrow P$ is continuous. To this end, let $(v_n) \in C[0, 1]$ be such that $v_n \longrightarrow v$ in P and let

$$u_n = \mathcal{L}_\lambda v_n, \quad u = \mathcal{L}_\lambda v. \tag{33}$$

Fix $x, s \in (0, 1)$, and define

$$H(z) = x^2(3s-x) + \frac{z}{6-2z} x^2 s^2 (x-3)(s-3), \quad \text{for } 0 \leq z \leq 2. \tag{34}$$

Then, $H'(z) = (1/(6-2z)^2) x^2 s^2 (x-3)(s-3)$, and there exists a constant N such that $|H'(z)| \leq N$ for $0 \leq z \leq 2$, and the mean value theorem gives

$$|G_{v_n}(x, s) - G_v(x, s)| \leq N |v_n - v|. \tag{35}$$

Notice that

$$\tilde{\sigma}_n(s) = |f(\tilde{v}_n(s)) - f(\tilde{v}(s))| \longrightarrow 0, \quad n \longrightarrow \infty \text{ for } s \in (0, 1), \quad (36)$$

and the fact

$$|f(\tilde{v}(x))| \leq \frac{M}{q^\beta} \in L^1(0, 1). \quad (37)$$

Now, these together with the Lebesgue dominated convergence theorem guarantee that

$$\begin{aligned} & \|\mathcal{L}_\lambda v_n - \mathcal{L}_\lambda v\| \\ & \leq \lambda \left(\int_0^1 |G_{v_n}(x, s) - G_v(x, s)| h(s) |f(\tilde{v}_n) - f(\tilde{v})| ds + \int_0^1 G_v(x, s) h(s) |\tilde{\sigma}_n(s)| ds \right) \\ & \leq \lambda \left(\int_0^1 N |\gamma_{v_n} - \gamma_v| h(s) |f(\tilde{v}_n)| ds + \int_0^1 G_v(x, s) h(s) |\tilde{\sigma}_n(s)| ds \right) \longrightarrow 0 \text{ as } n \longrightarrow \infty. \end{aligned} \quad (38)$$

Hence, $\mathcal{L}_\lambda: P \longrightarrow P$ is continuous.

Finally, we prove that $\mathcal{L}_\lambda: P \longrightarrow P$ is completely continuous. In fact, for $x \in P$,

$$\|\mathcal{L}_\lambda v\| \leq \int_0^1 G_{\gamma_v}(x, s) h(s) \frac{M}{q^\beta(s)} ds, \quad (39)$$

and for $x, x' \in [0, 1]$, we have

$$\|\mathcal{L}_\lambda v(x) - \mathcal{L}_\lambda v(x')\| \leq \int_0^1 |G_{\gamma_v}(x, s) - G_{\gamma_v}(x', s)| h(s) \frac{M}{q^\beta(s)} ds. \quad (40)$$

Now, the Arzela–Ascoli theorem guarantees that $\mathcal{L}_\lambda: P \longrightarrow P$ is compact.

Let $a > 1$ be such that $f(z) > 0$ for $z \geq a$. Since $\limsup_{x \rightarrow 0^+} x^\beta |f(x)| < \infty$, there exists a constant $b > 0$ such that

$$|f(z)| \leq \frac{b}{z^\beta}, \quad z \in (0, a). \quad (41)$$

Hence,

$$f(z) \geq -\frac{b}{z^\beta}, \quad (42)$$

$$|f(z)| \leq \frac{b}{z^\beta} + \hat{f}(\max(z, a)) \quad (43)$$

for all $z > 0$, where $\hat{f}(x) = \sup_{a \leq z \leq x} f(z)$ for $x \geq a$. Note that $\hat{f}$ is nondecreasing.

Suppose $\lambda < (a/(2(c_1 + c_2 \hat{f}(a))))$, where $c_1 = b \int_0^1 (h(s)/(q^\beta(s))) ds$ and $c_2 = \int_0^1 h(s) ds$. Next, we shall verify the assumptions of Lemma 1 hold for $\mathcal{L}_\lambda$.

(a) There exists $r_\lambda > 0$ such that if $u \in P$ satisfy $u = \theta \mathcal{L}_\lambda u$, $\theta \in I$, then $\|u\|_\infty \neq r_\lambda$.

Indeed, let $u \in P$ satisfy $u = \theta \mathcal{L}_\lambda u$ for some $\theta \in (0, 1]$. Then, $u/\theta = \mathcal{L}_\lambda u$, and therefore, u satisfies

$$u(x) = \lambda \theta \int_0^1 G_u(x, s) h(s) f(\tilde{u}) ds, \quad (44)$$

for $x \in I$. Since $q(s) \leq 1$ for all $s \in I$ and $a > 1$, it follows from (43) that

$$|f(\tilde{u})| \leq \frac{b}{\tilde{u}^\beta} + \hat{f}(\max\{\tilde{u}(s), a\}) \leq \frac{b}{q^\beta} + \hat{f}(\max\{\tilde{u}(s), a\}). \quad (45)$$

Hence,

$$\begin{aligned} u(x) & \leq \lambda \left(b \int_0^1 \frac{h(s)}{q^\beta} ds + \hat{f}(\max\{\|u\|_\infty, a\}) \int_0^1 h(s) ds \right) \\ & \leq \lambda (c_1 + c_2 \hat{f}(\max\{\|u\|_\infty, a\})), \end{aligned} \quad (46)$$

for $x \in I$, which implies

$$\frac{\|u\|_\infty}{c_1 + c_2 \hat{f}(\max\{\|u\|_\infty, a\})} \leq \lambda. \quad (47)$$

Since $(a/(c_1 + c_2 \hat{f}(a))) > 2\lambda$ and $\lim_{z \rightarrow \infty} (z/(c_1 + c_2 \hat{f}(z))) = 0$ by (H4), there exists a constant $r_\lambda > a$ such that

$$\frac{r_\lambda}{c_1 + c_2 \hat{f}(r_\lambda)} = 2\lambda. \quad (48)$$

Combining (47) and (48), we deduce that $\|u\|_\infty \neq r_\lambda$. Note that $r_\lambda \longrightarrow \infty$ as $\lambda \longrightarrow 0$.

(b) There exists $R_\lambda > r_\lambda$ such that if $u = \mathcal{L}_\lambda u + \xi$, $\xi \geq 0$, then $\|u\|_\infty \neq R_\lambda$.

Let $u \in P$ satisfy $u = \mathcal{L}_\lambda u + \xi$ for some $\xi \geq 0$. Then, $u - \xi = \mathcal{L}_\lambda u$, and so,

$$u(x) - \xi = \lambda \int_0^1 G_u(x, s) h(s) f(\tilde{u}) ds. \quad (49)$$

Let

$$k(x) = \frac{bh(x)}{q^\beta(x)}, \quad x \in (0, 1]. \quad (50)$$

Then, $k \in X$. Since u satisfies

$$\begin{cases} u''' = \lambda h(x) f(\tilde{u}), & 0 < x < 1, \\ u(0) = \xi \geq 0, & u'(0) = u''(1) = 0, \\ u'''(1) + \lambda_v u(1) = \lambda_v \xi \geq 0, \end{cases} \quad (51)$$

and by (41) and (42),

$$h(x) f(\tilde{u}(x)) \geq -\frac{bh(x)}{\tilde{u}^\beta(x)} \geq -\frac{bh(x)}{q^\beta(x)} = -k(x), \quad (52)$$

it follows from Lemma 3 that

$$u(x) \geq \left(\frac{3-\gamma}{9-\gamma} \|u\|_\infty - \frac{9-\gamma}{3-\gamma} \|k\|_X \right) q(x), \quad x \in (\tau, 1). \quad (53)$$

Suppose $\|u\|_\infty > \max\{((9-\gamma)/(3-\gamma))^2 \|k\|_X, 2/q(\sigma)\}$. Then,

$$u(s) \geq \frac{2(9-\gamma)^2 - (9-\gamma)^2}{2(9-\gamma)^2} \|u\|_\infty q(\sigma) \geq \frac{q(\sigma)}{2} \|u\|_\infty, \quad (54)$$

$$s \in [\tau + \sigma, 1 - \sigma] =: \Lambda.$$

Since

$$G_u(x, s) \geq \frac{1}{6} \begin{cases} 2x^2 s^2, & s \leq x, \\ 2x^2 s^2, & x \leq s. \end{cases} \geq \frac{1}{256}, \quad (55)$$

for $s, x \in \Lambda$, it follows from (52), (54), and (55) that

$$\begin{aligned} u(x) &\geq \lambda \left(\int_\Lambda G_u(x, s) h(s) f(\tilde{u}) ds + \int_{\Lambda^c} G_u(x, s) h(s) f(\tilde{u}) ds \right) \\ &\geq \lambda \left((1/256) \tilde{f}((q(\sigma)\|u\|_\infty)/2) \int_\Lambda h(s) ds - \|k\|_X \right), \end{aligned} \quad (56)$$

for $x \in \Lambda$, where $\tilde{f}(x) = \inf_{z \geq x} f(z)$. Consequently,

$$\frac{(1/256) \tilde{f}((q(\sigma)\|u\|_\infty)/2) \int_\Lambda h(s) ds - \|k\|_X}{\|u\|_\infty} \leq \frac{1}{\lambda}. \quad (57)$$

Since the left side of this inequality goes to ∞ as $\|u\|_\infty \rightarrow \infty$, it follows that $\|u\|_\infty < R_\lambda$ for $R_\lambda \gg 1$, which establishes (b).

By Lemma 1, $\mathcal{L}_\lambda$ has a fixed point u_λ with $\|u_\lambda\|_\infty > r_\lambda$. Since (19) holds (with $\xi = 0$) and $r_\lambda \rightarrow \infty$ as $\lambda \rightarrow 0$, and it follows that u_λ is a positive solution of (1) if λ is sufficiently small and $u_\lambda(x) \rightarrow \infty$ as $\lambda \rightarrow 0$ uniformly for x in compact subsets of $(0, 1)$. This completes the Proof of Theorem 1. $\square$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors' Contributions

The authors contributed equally to this paper. All authors read and approved the final manuscript.

Acknowledgments

This work was supported by the NSFC (No. 11671322).

References

- [1] A. Castro and R. Shivaji, "Non-negative solutions for a class of non-positone problems," *Proceedings of the Royal Society of Edinburgh: Section A Mathematics*, vol. 108, no. 3-4, pp. 291-302, 1988.
- [2] V. Anuradha, D. D. Hai, and R. Shivaji, "Existence results for superlinear semipositone BVP'S," *Proceedings of the American Mathematical Society*, vol. 124, no. 3, pp. 757-764, 1996.
- [3] D. D. Hai and R. Shivaji, "Positive radial solutions for a class of singular superlinear problems on the exterior of a ball with nonlinear boundary conditions," *Journal of Mathematical Analysis and Applications*, vol. 456, no. 2, pp. 872-881, 2017.
- [4] R. Ma, R. Wang, and L. Ren, "Existence results for semipositone boundary value problems," *Acta Mathematica Scientia*, vol. 21, no. 2, pp. 189-195, 2001.
- [5] X. Xu and D. O'Regan, "Existence of positive solutions for operator equations and applications to semipositone problems," *Positivity*, vol. 10, no. 2, pp. 315-328, 2006.
- [6] Z. Bai and H. Wang, "On positive solutions of some nonlinear fourth-order beam equations," *Journal of Mathematical Analysis and Applications*, vol. 270, no. 2, pp. 357-368, 2002.
- [7] A. Cabada and S. Tersian, "Multiplicity of solutions of a two point boundary value problem for a fourth-order equation," *Applied Mathematics and Computation*, vol. 219, no. 10, pp. 5261-5267, 2013.
- [8] R. Ma, "Multiple positive solutions for a semipositone fourth-order boundary value problem," *Hiroshima Mathematical Journal*, vol. 33, no. 2, pp. 217-227, 2003.
- [9] R. Ma and L. Xu, "Existence of positive solutions of a nonlinear fourth-order boundary value problem," *Applied Mathematics Letters*, vol. 23, no. 5, pp. 537-543, 2010.
- [10] I. J. Cabrera, B. López, and K. Sadarangani, "Existence of positive solutions for the nonlinear elastic beam equation via a mixed monotone operator," *Journal of Computational and Applied Mathematics*, vol. 327, pp. 306-313, 2018.
- [11] S. Li and C. Zhai, "New existence and uniqueness results for an elastic beam equation with nonlinear boundary conditions," *Boundary Value Problems*, vol. 2015, no. 1, p. 104, 2015.
- [12] T. F. Ma and A. L. M. Martinez, "Positive solutions for a fourth order equation with nonlinear boundary conditions," *Mathematics and Computers in Simulation*, vol. 80, no. 11, pp. 2177-2184, 2010.