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Research Article

Equivalent Parameter Conditions for the Validity of Half-Discrete Hilbert-Type Multiple Integral Inequality with Generalized Homogeneous Kernel

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Let $G(u,v)$ be a homogeneous nonnegative function of order λ, $K(n,\|x\|_{m,\rho})=G(n^{\lambda_1},\|x\|_{m,\rho}^{\lambda_2})$. By using the weight coefficient method, the equivalent parameter conditions and best constant factors for the validity of the following half-discrete Hilbert-type multiple integral inequality $\int_{\mathbb{R}_+^m}\sum_{n=1}^{\infty}K(n,\|x\|_{m,\rho})a_n f(x)\mathrm{d}x\le M\|\tilde{a}\|_{p,\alpha}\|f\|_{q,\beta}$ are discussed. Finally, its applications in operator theory are discussed.

1. Preliminaries

Suppose that $m\in\mathbb{N}_+$, $\mathbb{R}_+^m=\{x=(x_1,x_2,\cdots,x_m): x_i\ge 0, i=1,2,\cdots,m\}$. In this paper, it is always assumed that $x\in\mathbb{R}_+^m$, $\|x\|_{m,\rho}=(x_1^\rho+x_2^\rho+\cdots+x_m^\rho)^{1/\rho}(\rho>0)$. For any given $r>0$, $\alpha\in\mathbb{R}$, we write

$$l_r^\alpha=\left\{\tilde{a}=\{a_n\}: \|\tilde{a}\|_{r,\alpha}=\left(\sum_{n=1}^{\infty}n^\alpha a_n^r\right)^{1/r}<+\infty, a_n\ge 0\right\},$$
$$L_r^\alpha(\mathbb{R}_+^m)=\left\{f(x)\ge 0: \|f\|_{r,\alpha}=\left(\int_{\mathbb{R}_+^m}\|x\|_{m,\rho}^\alpha f^r(x)\mathrm{d}x\right)^{1/r}<+\infty\right\}. \quad (1)$$

Let $G(u,v)$ be a homogeneous nonnegative function of order λ. We call $K(u,v)=G(u^{\lambda_1},v^{\lambda_2})$ a generalized homogeneous function. Obviously, $K(u,v)$ satisfies

$$K(u,v)=u^{\lambda\lambda_1}K\left(1,u^{-\lambda_1/\lambda_2}v\right)=v^{\lambda\lambda_2}K\left(v^{-\lambda_2/\lambda_1}u,1\right). \quad (2)$$

We will discuss the equivalent parameter conditions for the validity of the half-discrete Hilbert-type multiple integral inequality with generalized homogeneous kernel and the optimal constant factors of the inequality under certain special conditions. Finally, the boundedness and norm of the corresponding series operator and integral operator are discussed. The relevant literature can be found in [1–16].

Lemma 1 (see [17]). *Let $m \in \mathbb{N}_+, \rho > 0, r > 0, \psi(u)$ be measurable. $\|x\|_{m,\rho} \le r$ represents the region $\Omega_r = \{x = (x_1, x_2, \cdots, x_m) : \|x\|_{m,\rho} \le r, x_i \ge 0\}$. Then,*

$$\int_{\|x\|_{m,\rho} \le r} \psi\left(\|x\|_{m,\rho}\right) dx = \frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)} \int_0^r \psi(u) u^{m-1} du,$$
$$\int_{\|x\|_{m,\rho} \ge r} \psi\left(\|x\|_{m,\rho}\right) dx = \frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)} \int_r^{+\infty} \psi(u) u^{m-1} du, \quad (3)$$

where $\Gamma(t)$ is the Gamma function.

In view of Lemma 1, one has

$$\int_{\mathbb{R}_+^m} \psi\left(\|x\|_{m,\rho}\right) dx = \frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)} \int_0^{+\infty} \psi(u) u^{m-1} du. \quad (4)$$

Lemma 2. *Assume that $(1/p) + (1/q) = 1 (p > 1)$, $m \in \mathbb{N}_+$, $\rho > 0$, $\lambda_1 \lambda_2 > 0$, $G(u, v)$ is a homogeneous nonnegative measurable function of order λ, $K(n, \|x\|_{m,\rho}) = G(n^{\lambda_1}, \|x\|_{m,\rho}^{\lambda_2})$, $(1/\lambda_2)(((\alpha\lambda_2 - m\lambda_1)/p) + ((\beta\lambda_1 - \lambda_2)/q) - \lambda\lambda_1\lambda_2) = c$, $K(t, 1)t^{-(a+1/p)+c}$ is monotonically decreasing on $(0, +\infty)$. Denote*

$$W_1 = \int_0^{+\infty} K(1, t) t^{-((b+m)/q)+m-1} dt, \; W_2 = \int_0^{+\infty} K(t, 1) t^{-((a+1)/p)+c} dt. \quad (5)$$

Then,

$$\lambda_1 W_2 = \lambda_2 W_1, \omega_1(n) = \int_{\mathbb{R}_+^m} K\left(n, \|x\|_{m,\rho}\right) \|x\|_{m,\rho}^{-((\beta+m)/q)} dx$$
$$= \frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)} n^{((\alpha+1)/p)-1-c} W_1, \omega_2(x)$$
$$= \sum_{n=1}^{\infty} K\left(n, \|x\|_{m,\rho}\right) n^{-((\alpha+1)/p)+c} \le \|x\|_{m,\rho}^{((\beta+m)/q)-m} W_2. \quad (6)$$

Proof. According to $(1/\lambda_2)(((\alpha\lambda_2 - m\lambda_1)/p) + ((\beta\lambda_1 - \lambda_2)/q) - \lambda\lambda_1\lambda_2) = c$, after a simple calculation, we have $-(\lambda_1/\lambda_2)(\lambda\lambda_2 - ((\beta + m)/q) + m - 1) - (\lambda_1/\lambda_2) - 1 = -((\alpha + 1)/p) + c$. Then,

$$W_1 = \int_0^{+\infty} K\left(t^{-(\lambda_2/\lambda_1)}, 1\right) t^{\lambda\lambda_2 - ((\beta+m)/q)+m-1} dt$$
$$= \frac{\lambda_1}{\lambda_2} \int_0^{+\infty} K(u, 1) u^{-(\lambda_1/\lambda_2)(\lambda\lambda_2 - ((\beta+m)/q)+m-1)-(\lambda_1/\lambda_2)-1} du$$
$$= \frac{\lambda_1}{\lambda_2} \int_0^{+\infty} K(u, 1) u^{-((\alpha+1)/p)+c} du = \frac{\lambda_1}{\lambda_2} W_2. \quad (7)$$

Hence, $\lambda_1 W_2 = \lambda_2 W_1$.

It follows from Lemma 1 that

$$\omega_1(n) = \frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)} \int_0^{+\infty} K(n, t) t^{-((\beta+m)/q)+m-1} dt$$
$$= \frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)} \int_0^{+\infty} n^{\lambda\lambda_1} K\left(1, t \cdot n^{-(\lambda_1/\lambda_2)}\right) t^{-((\beta+m)/q)+m-1} dt$$
$$= \frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)} n^{\lambda\lambda_1 - (\lambda_1/\lambda_2)(((\beta+m)/q)-m+1)}$$
$$+ (\lambda_1/\lambda_2) \int_0^{+\infty} K(1, u) u^{-((\beta+m)/q)+m-1} du$$
$$= \frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)} n^{((\alpha+m)/p)-1-c} W_1. \quad (8)$$

Notice that $K(t, 1)t^{-((\alpha+1)/p)+c}$ is monotonically decreasing on $(0, +\infty)$;we have

$$\omega_2(x) = \sum_{n=1}^{\infty} \|x\|_{m,\rho}^{\lambda\lambda_2} K\left(n \cdot \|x\|_{m,\rho}^{-\lambda_2/\lambda_1}, 1\right) n^{-((\alpha+1)/p)+c}$$
$$= \|x\|_{m,\rho}^{\lambda\lambda_2 - (\lambda_2/\lambda_1)(((\alpha+1)/p)-c)} \sum_{n=1}^{\infty} K$$
$$\cdot \left(n \cdot \|x\|_{m,\rho}^{-\lambda_2/\lambda_1}, 1\right) \left(n \cdot \|x\|_{m,\rho}^{-\lambda_2/\lambda_1}\right)^{-((\alpha+1)/p)+c}$$
$$\le \|x\|_{m,\rho}^{\lambda\lambda_2 - (\lambda_2/\lambda_1)(((\alpha+1)/p)-c)} \int_0^{+\infty} K$$
$$\cdot \left(t \cdot \|x\|_{m,\rho}^{-\lambda_2/\lambda_1}, 1\right) \left(t \cdot \|x\|_{m,\rho}^{-\lambda_2/\lambda_1}\right)^{-((\alpha+1)/p)+c} dt$$
$$= \|x\|_{m,\rho}^{\lambda\lambda_2 - (\lambda_2/\lambda_1)(((\alpha+1)/p)-1-c)} \int_0^{+\infty} K(u, 1) u^{-((\alpha+1)/p)+c} du$$
$$= \|x\|_{m,\rho}^{((\beta+m)/q)-m} W_2. \quad (9)$$

2. Main Results

Theorem 3. *Suppose that $(1/p) + (1/q) = 1 (p > 1)$, $m \in \mathbb{N}_+$, $\rho > 0$, $\lambda_1 \lambda_2 > 0$, $G(u, v)$ is a homogeneous nonnegative measurable function of order λ, $(1/\lambda_2)(((\alpha\lambda_2 - m\lambda_1)/p) + ((\beta\lambda_1 - m\lambda_2)/q) - \lambda\lambda_1\lambda_2) = c$, $G(t^{\lambda_1}, 1)t^{-(\alpha+1)/p}$ and $G(t^{\lambda_1}, 1)t^{-((\alpha+1)/p)+c}$ are monotonically decreasing on $(0, +\infty)$, and*

$$W_0 = |\lambda_2| \int_0^{+\infty} G\left(1, t^{\lambda_2}\right) t^{-((\beta+m)/q)+m-1} dt \quad (10)$$

is convergent. Then,

(i) The necessary and sufficient condition for the validity of inequality

$$\int_{\mathbb{R}_+^m} \sum_{n=1}^{\infty} G\left(n^{\lambda_1}, \|x\|_{m,\rho}^{\lambda_2}\right) a_n f(x) dx \le M \|\tilde{a}\|_{p,\alpha} \|f\|_{q,\beta}, \quad (11)$$

for some constant $M > 0$ is $c \geq 0$, where $\tilde{a} = \{a_n\} \in l_p^\alpha, f(x) \in L_q^\beta(\mathbb{R}_+^m)$

(ii) When $c = 0$, that is, $((\alpha\lambda_2 - m\lambda_1)/p) + ((\beta\lambda_1 - m\lambda_2)/q) - \lambda\lambda_1\lambda_2 = 0$, the best constant factor of (11) is

$$\inf M = \frac{W_0}{|\lambda_1|^{1/q}|\lambda_2|^{1/p}}\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/p} \quad (12)$$

.

Proof. Denote $K(n, \|x\|_{m,\rho}) = G(n^{\lambda_1}, \|x\|_{m,\rho}^{\lambda_2})$. (i) Suppose that (11) holds. We prove that $c \geq 0$ by using reduction to absurdity. If $c < 0$, for $\varepsilon = (-c/(2|\lambda_1|)) > 0$, take

$$a_n = n^{(-\alpha-1-|\lambda_1|\varepsilon)/p}, \quad n = 1, 2, \cdots, n,$$
$$f(x) = \begin{cases} \|x\|_{m,\rho}^{(-\beta-m-|\lambda_2|\varepsilon)/q}, & \|x\|_{m,\rho} \geq 1, \\ 0, & 0 < \|x\|_{m,\rho} < 1. \end{cases} \quad (13)$$

It follows from Lemma 1 that

$$\begin{aligned} M\|\tilde{a}\|_{p,\alpha}\|f\|_{q,\beta} &= M\left(\sum_{n=1}^{\infty} n^{-1-|\lambda_1|\varepsilon}\right)^{1/p}\left(\int_{\|x\|_{m,\rho}\geq 1}\|x\|_{m,\rho}^{-m-|\lambda_2|\varepsilon}\mathrm{d}x\right)^{1/q} \\ &= M\left(1 + \sum_{k=2}^{\infty} n^{-1-|\lambda_1|\varepsilon}\right)^{1/p}\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\int_1^{+\infty} t^{-1-|\lambda_2|\varepsilon}\mathrm{d}t\right)^{1/q} \\ &\leq M\left(1 + \int_1^{+\infty} t^{-1-|\lambda_1|\varepsilon}\mathrm{d}t\right)^{1/p}\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\cdot\frac{1}{|\lambda_2|\varepsilon}\right)^{1/q} \\ &= \frac{M}{\varepsilon\cdot|\lambda_1|^{1/p}|\lambda_2|^{1/q}}\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/q}(1 + |\lambda_1|\varepsilon)^{1/p} \\ &= \frac{2M}{-c}\left(\frac{\lambda_1}{\lambda_2}\right)^{1/q}\left(1 - \frac{c}{2}\right)^{1/p}\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/q}. \end{aligned} \quad (14)$$

Since $G(t^{\lambda_1}, 1)t^{-((\alpha+1)/p)} = K(t, 1)t^{-((\alpha+1)/p)}$ is monotonically decreasing on $(0, +\infty)$, we have

$$\begin{aligned} &\int_{\mathbb{R}_+^m}\sum_{n=1}^{\infty} G\left(n^{\lambda_1}, \|x\|_{m,\rho}^{\lambda_2}\right)a_n f(x)\mathrm{d}x \\ &= \int_{\|x\|_{m,\rho}\geq 1}\|x\|_{m,\rho}^{-(\beta+m+|\lambda_2|\varepsilon)/q}\left(\sum_{n=1}^{\infty} K\left(n, \|x\|_{m,\rho}\right)n^{-(\alpha+1+|\lambda_1|\varepsilon)/p}\right)\mathrm{d}x \\ &= \int_{\|x\|_{m,\rho}\geq 1}\|x\|_{m,\rho}^{\lambda\lambda_2-((\beta+m+|\lambda_2|\varepsilon)/q)-(\lambda_2/\lambda_1)((\alpha+1+|\lambda_1|\varepsilon)/p)} \\ &\left[\sum_{n=1}^{\infty} K\left(n\cdot\|x\|_{m,\rho}^{-\lambda_2/\lambda_1}, 1\right)\left(n\cdot\|x\|_{m,\rho}^{-\lambda_2/\lambda_1}\right)^{-(\alpha+1+|\lambda_1|\varepsilon)/p}\right]\mathrm{d}x \\ &\geq \int_{\|x\|_{m,\rho}\geq 1}\|x\|_{m,\rho}^{\lambda\lambda_2-((\beta+m+|\lambda_2|\varepsilon)/q)-(\lambda_2/\lambda_1)((\alpha+1+|\lambda_1|\varepsilon)/p)} \\ &\times\left[\int_1^{+\infty} K\left(u\cdot\|x\|_{m,\rho}^{-\lambda_2/\lambda_1}, 1\right)\left(u\cdot\|x\|_{m,\rho}^{-\lambda_2/\lambda_1}\right)^{-(\alpha+1+|\lambda_1|\varepsilon)/p}\mathrm{d}u\right]\mathrm{d}x \\ &= \int_{\|x\|_{m,\rho}\geq 1}\|x\|_{m,\rho}^{\lambda\lambda_2-((\beta+m+|\lambda_2|\varepsilon)/q)-(\lambda_2/\lambda_1)((\alpha+1+|\lambda_1|\varepsilon)/p)+(\lambda_2/\lambda_1)} \\ &\cdot\left(\int_{\|x\|_{m,\rho}^{-(\lambda_2/\lambda_1)}}^{+\infty} K(t, 1)t^{-(\alpha+1+|\lambda_1|\varepsilon)/p}\mathrm{d}t\right)\mathrm{d}x \\ &= \int_{\|x\|_{m,\rho}\geq 1}\|x\|_{m,\rho}^{-m-(\lambda_2/\lambda_1)c-|\lambda_2|\varepsilon}\left(\int_{\|x\|_{m,\rho}^{-(\lambda_2/\lambda_1)}}^{+\infty} K(t, 1)t^{-(\alpha+1+|\lambda_1|\varepsilon)/p}\mathrm{d}t\right)\mathrm{d}x \\ &\geq \int_{\|x\|_{m,\rho}\geq 1}\|x\|_{m,\rho}^{-m-(\lambda_2/\lambda_1)c-|\lambda_2|\varepsilon}\mathrm{d}x\int_1^{+\infty} K(t, 1)t^{-(\alpha+1+|\lambda_1|\varepsilon)/p}\mathrm{d}t \\ &= \frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\int_1^{+\infty} t^{-1-(\lambda_2/\lambda_1)c-|\lambda_2|\varepsilon}\mathrm{d}t\int_1^{+\infty} K(t, 1)t^{-(\alpha+1+|\lambda_1|\varepsilon)/p}\mathrm{d}t \\ &= \frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\int_1^{+\infty} t^{-1-(\lambda_2/2\lambda_1)c}\mathrm{d}t\int_1^{+\infty} K(t, 1)t^{-(\alpha+1+|\lambda_1|\varepsilon)/p}\mathrm{d}t. \end{aligned} \quad (15)$$

It follows from (11), (14), and (15) that

$$\begin{aligned} &\int_1^{+\infty} t^{-1-(\lambda_2/2\lambda_1)c}\mathrm{d}t\int_1^{+\infty} K(t, 1)t^{-(\alpha+1+|\lambda_1|\varepsilon)/p}\mathrm{d}t \\ &\leq \frac{2M}{-c}\left(\frac{\lambda_1}{\lambda_2}\right)^{1/q}\left(1 - \frac{c}{2}\right)^{1/p}\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{-(1/\rho)} < +\infty. \end{aligned} \quad (16)$$

Since $(\lambda_2/2\lambda_1)c < 0$, then $\int_1^{+\infty} t^{-1-(\lambda_2/2\lambda_1)c}\mathrm{d}t = +\infty$, which contradicts (16). Therefore, $c \geq 0$.

Suppose that $c \geq 0$. It follows from Hölder's inequality and Lemma 2 that

$$\begin{aligned} &\int_{\mathbb{R}_+^m}\sum_{n=1}^{\infty} G\left(n^{\lambda_1}, \|x\|_{m,\rho}^{\lambda_2}\right)a_n f(x)\mathrm{d}x \\ &= \int_{\mathbb{R}_+^m}\sum_{n=1}^{\infty}\left(\frac{n^{(\alpha+1-cp)/(pq)}}{\|x\|_{m,\rho}^{(\beta+m)/(pq)}}\cdot a_n\right)\left(\frac{\|x\|_{m,\rho}^{(\beta+m)/(pq)}}{n^{(\alpha+1-cp)/(pq)}}\cdot f(x)\right)K\left(n, \|x\|_{m,\rho}\right)\mathrm{d}x \\ &\leq \left(\int_{\mathbb{R}_+^m}\sum_{n=1}^{\infty} n^{(\alpha+1-cp)/q}\|x\|_{m,\rho}^{-(\beta+m)/q}a_n^p K\left(n, \|x\|_{m,\rho}\right)\mathrm{d}x\right)^{1/p} \\ &\times\left(\int_{\mathbb{R}_+^m}\sum_{n=1}^{\infty}\|x\|_{m,\rho}^{(\beta+m)/p}n^{-(\alpha+1-cp)/p}f^q(x)K\left(n, \|x\|_{m,\rho}\right)\mathrm{d}x\right)^{1/q} \\ &= \left(\sum_{n=1}^{\infty} n^{(\alpha+1-cp)/q}a_n^p\omega_1(n)\right)^{1/p}\left(\int_{\mathbb{R}_+^m}\|x\|_{m,\rho}^{(\beta+m)/p}f^q(x)\omega_2(x)\mathrm{d}x\right)^{1/q} \\ &\leq W_1^{1/p}W_2^{1/q}\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/p}\left(\sum_{n=1}^{\infty} n^{((\alpha+1-cp)/q)+((\alpha+1)/p)-1-c}a_n^p\right)^{1/p} \\ &\times\left(\int_{\mathbb{R}_+^m}\|x\|_{m,\rho}^{((\beta+m)/p)+((\beta+m)/q)-m}f^q(x)\mathrm{d}x\right)^{1/q} \\ &= \frac{W_0}{|\lambda_1|^{1/q}|\lambda_2|^{1/p}}\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/p}\left(\sum_{n=1}^{\infty} n^{\alpha-pc}a_n^p\right)^{1/p}\left(\int_{\mathbb{R}_+^m}\|x\|_{m,\rho}^{\beta}f^q(x)\mathrm{d}x\right)^{1/q} \\ &\leq \frac{W_0}{|\lambda_1|^{1/q}|\lambda_2|^{1/p}}\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/p}\left(\sum_{n=1}^{\infty} n^{\alpha}a_n^p\right)^{1/p}\left(\int_{\mathbb{R}_+^m}\|x\|_{m,\rho}^{\beta}f^q(x)\mathrm{d}x\right)^{1/q} \\ &= \frac{W_0}{|\lambda_1|^{1/q}|\lambda_2|^{1/p}}\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/p}\|\tilde{a}\|_{p,\alpha}\|f\|_{q,\beta}. \end{aligned} \quad (17)$$

Take $M \geq (W_0/(|\lambda_1|^{1/q}|\lambda_2|^{1/p}))((\Gamma^m(1/\rho))/(\rho^{m-1}\Gamma(m/\rho)))^{1/p}$ arbitrarily; one can get (11).

(ii) When $c = 0$, if (12) does not hold, then, there exists constant $M_0 > 0$, such that

$$M_0 < \frac{W_0}{|\lambda_1|^{1/q}|\lambda_2|^{1/p}}\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/p}, \tag{18}$$

$$\int_{\mathbb{R}_+^m}\sum_{n=1}^{\infty} G\left(n^{\lambda_1}, \|x\|_{m,\rho}^{\lambda_2}\right) a_n f(x)\mathrm{d}x \leq M_0\|\tilde{a}\|_{p,\alpha}\|f\|_{q,\beta}. \tag{19}$$

For $\varepsilon > 0$ and δ are sufficiently small, take

$$a_n = n^{-(\alpha+1+|\lambda_1|\varepsilon)/p}, \quad n = 1, 2, \cdots, n,$$
$$f(x) = \begin{cases} \|x\|_{m,\rho}^{-(\beta+m+|\lambda_2|\varepsilon)/q}, & \|x\|_{m,\rho} \geq \delta, \\ 0, & 0 < \|x\|_{m,\rho} < \delta. \end{cases} \tag{20}$$

Then,

$$M_0\|\tilde{a}\|_{p,\alpha}\|f\|_{q,\beta} \leq \frac{M_0}{\varepsilon|\lambda_1|^{1/p}|\lambda_2|^{1/q}}\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/q} \cdot (1+|\lambda_1|\varepsilon)^{1/p}\delta^{-|\lambda_2|\varepsilon/q}, \tag{21}$$

$$\begin{aligned}
&\int_{\mathbb{R}_+^m}\sum_{n=1}^{\infty} G\left(n^{\lambda_1}, \|x\|_{m,\rho}^{\lambda_2}\right) a_n f(x)\mathrm{d}x \\
&= \sum_{n=1}^{\infty} n^{-(\alpha+1+|\lambda_1|\varepsilon)/p}\left(\int_{\|x\|_{m,\rho}\geq\delta}\|x\|_{m,\rho}^{-(\beta+m+|\lambda_2|\varepsilon)/q}K\left(n, \|x\|_{m,\rho}\right)\mathrm{d}x\right) \\
&= \frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\sum_{n=1}^{\infty} n^{-(\alpha+1+|\lambda_1|\varepsilon)/p}\left(\int_{\delta}^{+\infty}K(n,u)u^{-((\beta+m+|\lambda_2|\varepsilon)/q)+m-1}\mathrm{d}u\right) \\
&= \frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\sum_{n=1}^{\infty} n^{\lambda\lambda_1-(\alpha+1+|\lambda_1|\varepsilon)/p} \\
&\cdot\left(\int_{\delta}^{+\infty}K\left(1, u\cdot n^{-\lambda_1/\lambda_2}\right)u^{-((\beta+m+|\lambda_2|\varepsilon)/q)+m-1}\mathrm{d}u\right) \\
&= \frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\sum_{n=1}^{\infty} n^{\lambda\lambda_1-((\alpha+1+|\lambda_1|\varepsilon)/p)-(\lambda_1/\lambda_2)(((\beta+m+|\lambda_2|\varepsilon)/q)-m+1)+(\lambda_1/\lambda_2)} \\
&\times\int_{\delta\cdot n^{-\lambda_1/\lambda_2}}^{+\infty}K(1,t)t^{-((\beta+m+|\lambda_2|\varepsilon)/q)+m-1}\mathrm{d}t \\
&\geq \frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\sum_{n=1}^{\infty} n^{-1-|\lambda_1|\varepsilon}\int_{\delta}^{+\infty}K(1,t)t^{-((\beta+m+|\lambda_2|\varepsilon)/q)+m-1}\mathrm{d}t \\
&\geq \frac{\Gamma^m(1/\rho)}{|\lambda_1|\varepsilon\rho^{m-1}\Gamma(m/\rho)}\int_{\delta}^{+\infty}K(1,t)t^{-((\beta+m+|\lambda_2|\varepsilon)/q)+m-1}\mathrm{d}t.
\end{aligned} \tag{22}$$

It follows from (19), (21), and (22) that

$$\frac{|\lambda_1|^{1/p}|\lambda_2|^{1/q}}{|\lambda_1|}\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/p}\int_{\delta}^{+\infty}K(1,t)t^{-((\beta+m+|\lambda_2|\varepsilon)/q)+m-1}\mathrm{d}t \leq M_0(1+|\lambda_1|\varepsilon)^{1/p}\delta^{-|\lambda_2|\varepsilon/q}. \tag{23}$$

Let $\varepsilon \to 0^+$; then,

$$\frac{|\lambda_1|^{1/p}|\lambda_2|^{1/q}}{|\lambda_1|}\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/p}\int_{\delta}^{+\infty}K(1,t)t^{-((\beta+m)/q)+m-1}\mathrm{d}t \leq M_0. \tag{24}$$

In addition, let $\delta \to 0^+$; we have

$$\frac{W_0}{|\lambda_1|^{1/q}|\lambda_2|^{1/p}}\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/p} \leq M_0. \tag{25}$$

This contradicts (18). Hence, (12) holds.

3. Applications

Define series operator T_1 and singular integral operator T_2 with kernel $G(n^{\lambda_1}, \|x\|_{m,\rho}^{\lambda_2})$ by, respectively,

$$T_1(\tilde{a})(x) = \sum_{n=1}^{\infty} G\left(n^{\lambda_1}, \|x\|_{m,\rho}^{\lambda_2}\right)a_n,$$
$$T_2(f)_n = \int_{\mathbb{R}_+^m} G\left(n^{\lambda_1}, \|x\|_{m,\rho}^{\lambda_2}\right)f(x)\mathrm{d}x. \tag{26}$$

According to the basic theory of Hilbert-type inequality (11) can be equivalently written as the following two expressions:

$$\|T_1(\tilde{a})\|_{p,\beta(1-p)} \leq M\|\tilde{a}\|_{p,\alpha},$$
$$\|T_2(f)\|_{q,\alpha(1-q)} \leq M\|f\|_{q,\beta}. \tag{27}$$

Thus, Theorem 3 is equivalent to the following theorem.

Theorem 4. *Assume that* $(1/p) + (1/q) = 1 (p > 1)$, $m \in \mathbb{N}_+$, $\rho > 0$, $\lambda_1\lambda_2 > 0$, $G(u, v)$ *is a homogeneous nonnegative measurable function of order* λ, $(1/\lambda_1)(((\alpha\lambda_2 - m\lambda_1)/p) + ((\beta\lambda_1 - \lambda_2)/q) - \lambda\lambda_1\lambda_2) = c, G(t^{\lambda_1}, 1)t^{-(\alpha+1)/p}$ *and* $G(t^{\lambda_1}, 1)$ $t^{-((\alpha+1)/p)+c}$ *are monotonically decreasing on* $(0, +\infty)$, *operators* T_1 *and* T_2 *are as defined in (26), and*

$$W_0 = |\lambda_2|\int_0^{+\infty} G\left(1, t^{\lambda_2}\right)t^{-((\beta+m)/q)+m-1}dt \tag{28}$$

is convergent. Then,

(i) $T_1 : l_p^{\alpha} \to L_p^{\beta(1-p)}(\mathbb{R}_+^m)$ *and* $T_2 : L_q^{\beta}(\mathbb{R}_+^m) \to l_q^{\alpha(1-q)}$ *are bounded operators if and only if* $c \geq 0$

(ii) When $c = 0$, *the operator norms of* T_1 *and* T_2 *are*

$$\|T_1\| = \|T_2\| = \frac{W_0}{|\lambda_1|^{1/q}|\lambda_2|^{1/p}}\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/p} \quad (29)$$

.

Corollary 5. *Suppose that* $m \in \mathbb{N}_+$, $\rho > 0$, $(1/p) + (1/q) = 1$ $(p > 1)$, $a > 0$, $\lambda_1 > 0$, $\lambda_2 > 0$, $(1/\lambda_2)(((\alpha\lambda_2 - m\lambda_1)/p) + ((\beta\lambda_1 - \lambda_2)/q) + a\lambda_1\lambda_2) = c$, $\alpha \geq \max\{\lambda_1 pb - 1, -1, \lambda_1 pb + pc - 1, pc - 1\}$, $m(q-1) - \lambda_2 q(a+b) < \beta < m(q-1)$. *Denote*

$$W_0 = \int_0^1 \frac{1}{(1+t)^a}\left[t^{(1/\lambda_2)(m-((\beta+m)/q))-1} + t^{a+b-(1/\lambda_2)(m-((\beta+m)/q))-1}\right]dt. \quad (30)$$

Define operators T_1 *and* T_2 *by, respectively,*

$$T_1(\tilde{a})(x) = \sum_{n=1}^{\infty} \frac{\left(\min\left\{1, n^{\lambda_1}/\|x\|_{m,\rho}^{\lambda_2}\right\}\right)^b}{\left(n^{\lambda_1} + \|x\|_{m,\rho}^{\lambda_2}\right)^a} \cdot a_n, \quad \tilde{a} = \{a_n\} \in l_p^{\alpha},$$
$$T_2(f)_n = \int_{\mathbb{R}_+^m} \frac{\left(\min\left\{1, n^{\lambda_1}/\|x\|_{m,\rho}^{\lambda_2}\right\}\right)^b}{\left(n^{\lambda_1} + \|x\|_{m,\rho}^{\lambda_2}\right)^a} \cdot f(x)dx, \quad f(x) \in L_q^{\beta}(\mathbb{R}_+^m). \quad (31)$$

Then,

(i) T_1 *is a bounded operator from* l_p^{α} *to* $L_p^{\beta(1-p)}(\mathbb{R}_+^m)$*, and* T_2 *is a bounded operator from* $L_q^{\beta}(\mathbb{R}_+^m)$ *to* $l_q^{\alpha(1-q)}$ *if and only if* $c \geq 0$

(ii) When $c = 0$*, the operator norms of* T_1 *and* T_2 *are*

$$\|T_1\| = \|T_2\| = \frac{W_0}{\lambda_1^{1/q}\lambda_2^{1/p}}\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/p} \quad (32)$$

Proof. Denote

$$G(u,v) = \frac{(\min\{1, u/v\})^b}{(u+v)^a}. \quad (33)$$

Then, $G(u,v)$ is a homogeneous nonnegative function of $\lambda = -a$ order.

It follows from $m(q-1) - \lambda_2 q(a+b) < \beta < m(q-1)$ that $(1/\lambda_2)(m - ((\beta+m)/q)) > 0$ and $a + b - (1/\lambda_2)(m - ((\beta+m)/q)) > 0$. Hence, W_0 is convergent and

$$\begin{aligned}|\lambda_2|\int_0^{+\infty} G\left(1, t^{\lambda_2}\right)t^{-((\beta+m)/q)+m-1}dt &= \int_0^{+\infty} G(1,u)u^{(1/\lambda_2)(m-((\beta+m)/q))-1}du \\ &= \int_0^{+\infty} \frac{\left(\min\{1, u^{-1}\}\right)^b}{(1+u)^a} \cdot u^{(1/\lambda_2)(m-((\beta+m)/q))-1}du \\ &= \int_0^1 \frac{u^{(1/\lambda_2)(m-((\beta+m)/q))-1}}{(1+u)^a}du + \int_1^{+\infty} \frac{u^{-b+(1/\lambda_2)(m-((\beta+m)/q))-1}}{(1+u)^a}du \\ &= \int_0^1 \frac{1}{(1+t)^a}\left[t^{(1/\lambda_2)(m-((\beta+m)/q))-1} + t^{a+b-(1/\lambda_2)(m-((\beta+m)/q))-1}\right]dt.\end{aligned} \quad (34)$$

In view of $\alpha \geq \max\{\lambda_1 pb + pc - 1, pc - 1\}$, we obtain $\lambda_1 b - ((\alpha+1)/p) + c < 0$ and $-((\alpha+1)/p) + c < 0$. Therefore,

$$G\left(t^{\lambda_1}, 1\right)t^{-((\alpha+1)/p)+c} = \begin{cases} \dfrac{1}{\left(1+t^{\lambda_1}\right)^a} \cdot t^{\lambda_1 b-((\alpha+1)/p)+c}, & 0 < t \leq 1, \\ \dfrac{1}{\left(1+t^{\lambda_1}\right)^a} \cdot t^{-((\alpha+1)/p)+c}, & t > 1 \end{cases} \quad (35)$$

is monotonically decreasing on $(0, +\infty)$.

According to $\alpha \geq \max\{\lambda_1 pb - 1, -1\}$, it is also known that $G(t^{\lambda_1}, 1)t^{-(\alpha+1)/p}$ is monotonically decreasing on $(0, +\infty)$.

In summary, it follows from Theorem 4 that Corollary 5 holds.

Take $b = 0$ in Corollary 5, by virtue of the properties of Beta function; we get

$$\begin{aligned}W_0 &= \int_0^1 \frac{1}{(1+t)^a}\left[t^{(1/\lambda_2)(m-((\beta+m)/q))-1} + t^{a-(1/\lambda_2)(m-((\beta+m)/q))-1}\right]dt \\ &= B\left(\frac{1}{\lambda_2}\left(m - \frac{\beta+m}{q}\right), a - \frac{1}{\lambda_2}\left(m - \frac{\beta+m}{q}\right)\right)\end{aligned} \quad (36)$$

and the following result.

Corollary 6. *Assume that* $m \in \mathbb{N}_+$, $\rho > 0$, $(1/p) + (1/q) = 1$ $(p > 1)$, $a > 0$, $\lambda_1 > 0$, $\lambda_2 > 0$, $(1/\lambda_2)(((\alpha\lambda_2 - m\lambda_1)/p) + ((\beta\lambda_1 - \lambda_2)/q) + a\lambda_1\lambda_2) = c$, $\alpha \geq \max\{-1, pc - 1\}$, $m(q-1) - \lambda_2 qa < \beta < m(q-1)$. *Define operators* T_1 *and* T_2 *by, respectively,*

$$T_1(\tilde{a})(x) = \sum_{n=1}^{\infty} \frac{a_n}{\left(n^{\lambda_1} + \|x\|_{m,\rho}^{\lambda_2}\right)^a}, \quad \tilde{a} = \{a_n\} \in l_p^{\alpha},$$
$$T_2(f)_n = \int_{\mathbb{R}_+^m} \frac{f(x)}{\left(n^{\lambda_1} + \|x\|_{m,\rho}^{\lambda_2}\right)^a}dx, \quad f(x) \in L_q^{\beta}(\mathbb{R}_+^m). \quad (37)$$

Then,

(i) $T_1 : l_p^{\alpha} \to L_p^{\beta(1-p)}(\mathbb{R}_+^m)$ *and* $T_2 : L_q^{\beta}(\mathbb{R}_+^m) \to l_q^{\alpha(1-q)}$ *are bounded operators if and only if* $c \geq 0$

(ii) *When* $c = 0$*, the operator norms of* T_1 *and* T_2 *are*

$$\|T_1\| = \|T_2\| = \frac{1}{\lambda_1^{1/q}\lambda_2^{1/p}} B\left(\frac{1}{\lambda_2}\left(m - \frac{\beta+m}{q}\right), a - \frac{1}{\lambda_2}\left(m - \frac{\beta+m}{q}\right)\right)\left(\frac{\Gamma^m(1/\rho)}{\rho^{m-1}\Gamma(m/\rho)}\right)^{1/p} \quad (38)$$

In Corollary 6, let $\alpha = (\lambda_1/\lambda_2)(1-(p/r)a\lambda_2)$, $\beta = (\lambda_2/\lambda_1)(1-(q/s)a\lambda_1)$, $m = 1$, $(1/r)+(1/s)(r>1)$*; then,*

$$\frac{\alpha\lambda_2 - \lambda_1}{p} + \frac{\beta\lambda_1 - \lambda_2}{q} + a\lambda_1\lambda_2 = 0. \quad (39)$$

The following results can be obtained.

Corollary 7. *Assume that* $(1/p)+(1/q)=1(p>1)$, $(1/r)+(1/s)(r>1)$, $a>0, \lambda_1>0, \lambda_2>0$, $\alpha \geq -1$, $q(1-\lambda_2 a)-1<\beta<q-1$, $\alpha = (\lambda_1/\lambda_2)(1-(p/r)a\lambda_2)$, $\beta = (\lambda_2/\lambda_1)(1-(q/s)(q/s)a\lambda_1)$*. Define operators* T_1 *and* T_2 *by, respectively,*

$$T_1(\tilde{a})(x) = \sum_{n=1}^{\infty} \frac{a_n}{\left(n^{\lambda_1}+x^{\lambda_2}\right)^a}, \quad \tilde{a} = \{a_n\} \in l_p^{\alpha},$$
$$T_2(f)_n = \int_0^{+\infty} \frac{f(x)}{\left(n^{\lambda_1}+x^{\lambda_2}\right)^a} dx, \quad f(x) \in L_q^{\beta}(0,+\infty). \quad (40)$$

Then, $T_1 : l_p^{\alpha} \to L_p^{\beta(1-p)}(0,+\infty)$ *and* $T_2 : L_q^{\beta}(0,+\infty) \to l_q^{\alpha(1-q)}$ *are bounded operators, and the operator norms of* T_1 *and* T_2 *are*

$$\|T_1\| = \|T_2\| = \frac{1}{\lambda_1^{1/q}\lambda_2^{1/p}} B\left(\frac{a}{r} + \frac{1}{\lambda_1 q} - \frac{1}{\lambda_2 p}, \frac{a}{s} + \frac{1}{\lambda_2 p} - \frac{1}{\lambda_1 q}\right). \quad (41)$$

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflict of interests.

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