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Research Article

Multiplicity of Weak Positive Solutions for Fractional p & q Laplacian Problem with Singular Nonlinearity

Dandan Yang and Chuanzhi Bai

Department of Mathematics, Huaiyin Normal University, Huaian, Jiangsu 223300, China

Correspondence should be addressed to Chuanzhi Bai; czbai@hytc.edu.cn

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In this paper, we prove the existence and multiplicity of positive solutions for a class of fractional p & q Laplacian problem with singular nonlinearity. Our approach relies on the variational method, some analysis techniques, and the method of Nehari manifold.

1. Introduction

In this paper, we consider the following fractional p & q Laplacian problem

$$\begin{aligned} &(-\Delta)_p^s u + (-\Delta)_q^s u = a(x)u^{-\gamma} + \lambda b(x)u^{\beta}, \quad x \in \Omega, \\ &u > 0, \quad x \in \Omega, \\ &u = 0, \quad x \in \partial\Omega, \end{aligned} \tag{1}$$

where $\Omega \subset \mathbb{R}^N (N \geq 3)$ is an open bounded domain with smooth boundary $\partial\Omega$, $N > 2s(0 < s < 1)$, $1 \leq q < p < N/s$, $0 < \gamma < 1 < \beta < p_s^* - 1$, λ is a positive parameter. The weight functions $a : \Omega \to \mathbb{R}$ is in $L^{p_s^*/(p_s^*+\gamma-1)}$ with $a(x) > 0$ for almost every $x \in \Omega$, $b : \Omega \to \mathbb{R}$ is bounded with $b(x) > 0$ for almost every $x \in \Omega$, and $p_s^* = pN/(N - ps)$ denotes the critical Sobolev exponent. $(-\Delta)_r^s$, with $r \in \{p, q\}$, is the fractional r-Laplacian operator defined for any $u \in \mathscr{C}_c^\infty(\mathbb{R}^N)$ by

$$(-\Delta)_r^s u(x) = 2 \lim_{\varepsilon \searrow 0} \int_{B_\varepsilon(x)^c} \frac{|u(x) - u(y)|^{r-2}(u(x) - u(y))}{|x - y|^{N+sr}} dy, \quad x \in \mathbb{R}^N. \tag{2}$$

Recently, Wang and Zhang [1] investigated the following singular elliptic boundary value problem involving the fractional Laplacian

$$\begin{aligned} &(-\Delta)^s u = \lambda u^{\beta} + a(x)u^{-\gamma}, \quad x \in \Omega, \\ &u > 0, \quad x \in \Omega, \\ &u = 0, \quad x \in \partial\Omega, \end{aligned} \tag{3}$$

the existence and multiplicity of weak positive solutions of (3) have been obtained in [1] by using the variational method, Nehari Maniford method, and Fibering Map analysis.

In [2], Crandall et al. firstly studied the semilinear problem with singular nonlinearity, since then, the local setting ($s = 1$) for (3) and some other versions of the problem have been extensively studied during the past decades, see for example [3–10] and the references therein.

In recent years, the fractional Laplacian problems have been extensively investigated. For more details, we cite the reader to [11–15]. There are many different definitions of weak solutions for the fractional Laplacian equation (3). In [16], Fang say that $u \in H_0^s(\Omega)$ is a weak solution of (3) with $\lambda = 0$ and $a \equiv 1$ if the identity

$$\int_\Omega (-\Delta)^{s/2} u (-\Delta)^{s/2} \varphi dx = \int_\Omega u^{-\gamma} \varphi dx, \quad \forall \varphi \in H_0^s(\Omega) \tag{4}$$

holds. In virtue of the method of sub-supersolution, the author gives the sufficient conditions for the existence and uniqueness of positive solution.

Very recently, great attention has been devoted to the study of fractional p-Laplacian problems, see for instance [17–20]. However, in literature, there are only a few papers [21–23] dealing with fractional p & q problems. Motivated by the works [1, 23, 24], in this paper, we investigate the existence and multiplicity of solutions for the fractional p & q Laplacian problem (1) and extend the main results of Wang and Zhang [1].

This article is organized as follows. In Section 2, we give some notations and preliminaries. Section 3 is devoted to proving that problem (1) has at least two positive solutions for λ sufficiently small.

2. Preliminaries

For any $s \in (0,1), 1 < p < \infty$, we define

$$W^{s,p}(\mathbb{R}^N) = \Big\{u \big| u : \mathbb{R}^N \to \mathbb{R} \text{ is measurable}, u\big|_\Omega \in L^p(\Omega), \text{ and } \int_Q \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} dxdy < \infty \Big\}, \tag{5}$$

where $Q = \mathbb{R}^{2N} \backslash (\mathscr{C}\Omega \times \mathscr{C}\Omega)$ with $\mathscr{C}\Omega = \mathbb{R}^N \backslash \Omega$.

The space $W_0^{s,p}(\Omega) = \{u \in W^{s,p}(\mathbb{R}^N) : u = 0 \text{ in } \mathbb{R}^N \backslash \Omega\}$ endowed with the norm

$$\|u\| = [u]_{s,p} = \left(\int_Q \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy \right)^{1/p}. \tag{6}$$

From [23], we know that $W_0^{s,p}(\Omega) \subset W_0^{s,q}(\Omega)$ $(q \le p)$, which allows us to study (1) in $W_0^{s,p}(\Omega)$.

Throughout this section, we denote the best Sobolev constant S_p for the embedding of $W_0^{s,p}(\Omega)$ into $L^{p_s^*}(\Omega)$, defined as

$$S_p = \inf_{u \in W_0^{s,p}(\Omega)\backslash\{0\}} \frac{\int_{\mathbb{R}^{2N}} |u(x)-u(y)|^p / |x-y|^{N+sp} dxdy}{\left(\int_{\mathbb{R}^N} |u|^{p_s^*} dx\right)^{p/p_s^*}} < 0. \tag{7}$$

By a weak solution u of (1), we mean $u \in W_0^{s,p}(\Omega)$ satisfies (1) weakly, that is, we are looking for a function $u \in W_0^{s,p}(\Omega)$, $u(x) > 0$ a.e. in Ω such that

$$\begin{aligned} &\int_Q \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{N+ps}} dxdy \\ &+ \int_Q \frac{|u(x)-u(y)|^{q-2}(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{N+qs}} dxdy \\ &= \int_\Omega \left(a(x)u^{-\gamma}\phi + \lambda b(x) u^\beta \phi\right) dx, \quad \forall \phi \in W_0^{s,p}(\Omega). \end{aligned} \tag{8}$$

To study weak solutions to (1), we consider the following energy functional

$$\mathscr{J}_\lambda(u) = \frac{1}{p}[u]_{s,p}^p + \frac{1}{q}[u]_{s,q}^q - \frac{1}{1-\gamma}\int_\Omega a(x)|u|^{1-\gamma} dx - \frac{\lambda}{1+\beta}\int_\Omega b(x)|u|^{1+\beta} dx. \tag{9}$$

The following Lemmas will be useful in the study of our problem (1).

Lemma 1 [23]. *If $q \le p$, then $W_0^{s,p}(\Omega) \subset W_0^{s,q}(\Omega)$ with continuous embedding.*

Lemma 2 [23]. *If $\{u_n\}_{n\in\mathbb{N}}$ is a sequence weakly convergent to some u in $W_0^{s,p}(\Omega)$, then*

$$[u_n - u]_{s,p}^p = [u_n]_{s,p}^p - [u]_{s,p}^p + o(1). \tag{10}$$

The Nehari manifold $\mathscr{N}_\lambda(\Omega)$ is closed linked to the behavior of functions of the form $h_\lambda : t \to \mathscr{J}_\lambda(tu)$ for $t > 0$ that named fibering maps [25]. If $u \in W_0^{s,p}(\Omega)$, we have

$$h_\lambda(t) = \frac{t^p}{p}[u]_{s,p}^p + \frac{t^q}{q}[u]_{s,q}^q - \frac{t^{1-\gamma}}{1-\gamma}\int_\Omega a(x)|u|^{1-\gamma}dx - \lambda\frac{t^{1+\beta}}{1+\beta}\int_\Omega b(x)|u|^{1+\beta}dx, \tag{11}$$

$$h_\lambda'(t) = t^{p-1}[u]_{s,p}^p + t^{q-1}[u]_{s,q}^q - t^{-\gamma}\int_\Omega a(x)|u|^{1-\gamma}dx - \lambda t^\beta \int_\Omega b(x)|u|^{1+\beta}dx, \tag{12}$$

and

$$h_\lambda''(t) = (p-1)t^{p-2}[u]_{s,p}^p + (q-1)t^{q-2}[u]_{s,q}^q + \gamma t^{-\gamma-1}\int_\Omega a(x)|u|^{1-\gamma}dx - \lambda\beta t^{\beta-1}\int_\Omega b(x)|u|^{1+\beta}dx. \tag{13}$$

Obviously,

$$\begin{aligned} t h_\lambda'(t) &= t^p[u]_{s,p}^p + t^q[u]_{s,q}^q - t^{1-\gamma}\int_\Omega a(x)|u|^{1-\gamma}dx - \lambda t^{1+\beta}\int_\Omega b(x)|u|^{1+\beta}dx \\ &= \langle \mathscr{J}_\lambda'(tu), tu \rangle, \end{aligned} \tag{14}$$

which implies that for $u \in W_0^{s,p}(\Omega)\backslash\{0\}$ and $t > 0$, $h_\lambda'(t) = 0$ if and only if $tu \in \mathscr{N}_\lambda(\Omega)$, i.e., positive critical points of h_λ' correspond to points on the Nehari manifold. In particular, $h_\lambda'(1) = 0$ if and only if $u \in \mathscr{N}_\lambda(\Omega)$. Hence, we define

$$\begin{aligned} \mathscr{N}_\lambda^+(\Omega) &= \{u \in \mathscr{N}_\lambda(\Omega) : h_\lambda''(1) > 0\} \\ &= \Big\{u \in \mathscr{N}_\lambda(\Omega) : (p+\gamma-1)[u]_{s,p}^p + (q+\gamma-1)[u]_{s,q}^q > \lambda(\beta+\gamma)\int_\Omega b(x)|u|^{1+\beta}dx\Big\}, \\ \mathscr{N}_\lambda^0(\Omega) &= \{u \in \mathscr{N}_\lambda(\Omega) : h_\lambda''(1) = 0\} \\ &= \Big\{u \in \mathscr{N}_\lambda(\Omega) : (p+\gamma-1)[u]_{s,p}^p + (q+\gamma-1)[u]_{s,q}^q = \lambda(\beta+\gamma)\int_\Omega b(x)|u|^{1+\beta}dx\Big\}, \\ \mathscr{N}_\lambda^-(\Omega) &= \{u \in \mathscr{N}_\lambda(\Omega) : h_\lambda''(1) < 0\} \\ &= \Big\{u \in \mathscr{N}_\lambda(\Omega) : (p+\gamma-1)[u]_{s,p}^p + (q+\gamma-1)[u]_{s,q}^q < \lambda(\beta+\gamma)\int_\Omega b(x)|u|^{1+\beta}dx\Big\}. \end{aligned} \tag{15}$$

Throughout this paper, we make the following assumptions:

(H1) $a \in L^{p/(p-1)}(\Omega)$ with $a(x) > 0$ for all almost every $x \in \Omega$.

(H2) $b(x) > 0$ is bounded for all almost every $x \in \Omega$.

By (H2), we know that there exists $\bar{b} > 0$ such that $0 < b(x) \le \bar{b}$ for all almost every $x \in \Omega$.

For $p_s^* > p$, one has that $p_s^*/(p_s^*-1+\gamma) < p/(p-1+\gamma) < p/(p-1)$. Thus, there exists a constant C such that

$$\|a\|_{p_s^*/(p_s^*-1+\gamma)} \le C\|a\|_{p/(p-1)}. \quad (16)$$

Lemma 3. *Assume that condition (H1) holds with $p < 1+\beta$. Then there exists $M > 0$ such that $\|u\| \le M$, for each $u \in \mathcal{N}_\lambda^+(\Omega)$.*

Proof. If $u \in \mathcal{N}_\lambda^+(\Omega)$, by the definition of $\mathcal{N}_\lambda(\Omega)$ and $\mathcal{N}_\lambda^+(\Omega)$, we get that

$$\begin{aligned}&(p+\gamma-1)[u]_{s,p}^p + (q+\gamma-1)[u]_{s,q}^q \\ &\quad > (\beta+\gamma)\left([u]_{s,p}^p + [u]_{s,q}^q - \int_\Omega a(x)|u|^{1-\gamma}dx\right). \quad (17)\end{aligned}$$

By (H1), (16), and the Hölder inequality and fractional Sobolev inequalities (7), we obtain

$$\begin{aligned}\int_\Omega a(x)|u|^{1-\gamma}dx &\le \left(\int_\Omega |a(x)|^{p_s^*/(p_s^*-1+\gamma)}dx\right)^{(p_s^*-1+\gamma)/p_s^*}\left(\int_\Omega |u|^{p_s^*}dx\right)^{(1-\gamma)/p_s^*} \\ &= \|a\|_{p_s^*/(p_s^*-1+\gamma)}\|u\|_{p_s^*}^{1-\gamma} \le \|a\|_{p_s^*/(p_s^*-1+\gamma)} S_p^{-(1-\gamma)/p}\|u\|^{1-\gamma} \\ &\le C\|a\|_{p/(p-1)} S_p^{-(1-\gamma)/p}\|u\|^{1-\gamma}. \quad (18)\end{aligned}$$

It follows from (17) and (18) that

$$\begin{aligned}(\beta+1-p)\|u\|^p &< (\beta+1-p)[u]_{s,p}^p + (\beta+1-q)[u]_{s,q}^q \\ &< (\beta+\gamma)\int_\Omega a(x)[u]^{1-\gamma}dx \\ &\le (\beta+\gamma)CS_p^{-(1-\gamma)/p}\|a\|_{p/(p-1)}\|u\|^{1-\gamma}. \quad (19)\end{aligned}$$

Thus, one has

$$\|u\| \le M := \left(\frac{\beta+\gamma}{\beta+1-p}CS_p^{-(1-\gamma)/p}\|a\|_{p/(p-1)}\right)^{1/(p+\gamma-1)}, \quad \forall u \in \mathcal{N}_\lambda^+(\Omega). \quad (20)$$

The proof is complete. □

Lemma 4. *If (H1) holds with $p < 1+\beta$, then the functional $\mathcal{J}_\lambda$ is coercive and bounded below on $\mathcal{N}_\lambda(\Omega)$.*

Proof. By using the definition of $\mathcal{N}_\lambda(\Omega)$ and (16), we have

$$\begin{aligned}\mathcal{J}_\lambda(u) &= \left(\frac{1}{p}-\frac{1}{1+\beta}\right)[u]_{s,p}^p + \left(\frac{1}{q}-\frac{1}{1+\beta}\right)[u]_{s,q}^q \\ &\quad - \left(\frac{1}{(1-\gamma)}-\frac{1}{1+\beta}\right)\int_\Omega a(x)|u|^{1-\gamma}dx \\ &\ge \left(\frac{1}{p}-\frac{1}{1+\beta}\right)\|u\|^p - \left(\frac{1}{1-\gamma}-\frac{1}{1+\beta}\right)CS_p^{-(1-\gamma)/p}\|a\|_{p/(p-1)}\|u\|^{1-\gamma}. \quad (21)\end{aligned}$$

This implies that $\mathcal{J}_\lambda(\Omega)$ is coercive and bounded from below on $\mathcal{N}_\lambda(\Omega)$. The proof is complete. □

Lemma 5. *Assume that conditions (H1) and (H2) hold with $p < 1+\beta$. Then the minimal value $m_\lambda = \inf_{u\in\mathcal{N}_\lambda^+(\Omega)}\mathcal{J}_\lambda(u) < 0$.*

Proof. For $1 < \beta < p_s^*-1$, applying (H2), the Hölder inequality and fractional Sobolev inequalities (7), we have

$$\begin{aligned}\int_\Omega b(x)|u|^{1+\beta}dx &\le \bar{b}\left(\int_\Omega dx\right)^{(p_s^*-1-\beta)/p_s^*}\left(\int_\Omega |u|^{p_s^*}dx\right)^{(1+\beta)/p_s^*} \\ &= \bar{b}\|\Omega\|^{(p_s^*-1-\beta)/p_s^*}\|u\|_{p_s^*}^{1+\beta} \le \bar{b}|\Omega|^{(p_s^*-1-\beta)/p_s^*}S_p^{-(1+\beta)/p}\|u\|^{1+\beta}. \quad (22)\end{aligned}$$

Using inequality (18) and (22), we obtain

$$\begin{aligned}\mathcal{J}_\lambda(u) &\ge \frac{1}{p}[u]_{s,p}^p - \frac{1}{1-\gamma}\int_\Omega a(x)|u|^{1-\gamma}dx - \frac{\lambda}{1+\beta}\int_\Omega b(x)|u|^{1+\beta}dx \\ &\ge \frac{1}{p}\|u\|^p - \frac{C}{1-\gamma}S_p^{-(1-\gamma)/p}\|a\|_{p/(p-1)}\|u\|^{1-\gamma} \\ &\quad - \frac{\lambda}{1+\beta}\bar{b}S_p^{-(1+\beta)/p}|\Omega|^{p_s^*/(p_s^*-\beta-1)}\|u\|^{1+\beta}. \quad (23)\end{aligned}$$

It follows from $0 < \gamma < 1 \le q < p < 1+\beta < p_s^*$ that there exist $\rho > 0$, $\varepsilon > 0$, and $\lambda_1 > 0$ small enough such that for any fixed $\lambda \in (0, \lambda_1)$

$$\mathcal{J}_\lambda(u) \ge \frac{\varepsilon}{2} > 0, \quad \forall u \in \partial B_\rho, \quad (24)$$

where $B_\rho = \{u \in \mathcal{N}_\lambda^+(\Omega) : \|u\| \le \rho\}$. On the other hand, for any fixed $v \in \mathcal{N}_\lambda^+(\Omega)$, since $0 < \gamma < 1 \le q < p < 1+\beta$, we can derive that

$$\begin{aligned}\mathcal{J}_\lambda(tv) &= \frac{t^p}{p}[v]_{s,p}^p + \frac{t^q}{q}[v]_{s,q}^q - \frac{t^{1-\gamma}}{1-\gamma}\int_\Omega a(x)|v|^{1-\gamma}dx \\ &\quad - \frac{\lambda}{1+\beta}t^{1+\beta}\int_\Omega b(x)|v|^{1+\beta}dx \\ &= -At^{1-\gamma} + o(t^q)\ (t \to 0), \quad (25)\end{aligned}$$

where $A = (1)/(1-\gamma)\int_\Omega a(x)|v|^{1-\gamma}dx$. Thus, $\mathcal{J}_\lambda(tv) < 0$ for all $v \ne 0$, provided $t > 0$ is sufficiently small. Hence, $m_\lambda = \inf_{u\in\mathcal{N}_\lambda^+(\Omega)}\mathcal{J}_\lambda(u) < 0$. This completes the proof of Lemma 5. □

Lemma 6. *For each $\lambda \in (0, \lambda_1)$, there exists $u_\lambda \in \mathcal{N}_\lambda^+(\Omega)$ such that $\mathcal{J}_\lambda(u_\lambda) = m_\lambda = \inf_{\mathcal{N}_\lambda(\Omega)}\mathcal{J}_\lambda$.*

Proof. Let $\{u_n\} \subset B_\rho$ be a minimizing sequence such that $\mathcal{J}_\lambda(u_n) \to m_\lambda$ as $n \to \infty$. From Lemma 3 we know that the sequence $\{u_n\}$ is bounded in $W_0^{s,p}(\Omega)$. Hence, we can obtain that there exists a sub-sequence of $\{u_n\}$ (still denoted by $\{u_n\}$) such that $u_n \rightharpoonup u_\lambda$ weakly in $W_0^{s,p}(\Omega)$, strongly in $L^r(\Omega)$ $(1 \le r < p_s^*)$ and pointwise a.e. in Ω. By using Hölder inequality, we derive that as $n \to \infty$,

$$\begin{aligned}\int_\Omega a(x)u_n^{1-\gamma}dx &\le \int_\Omega a(x)u_\lambda^{1-\gamma}dx + \int_\Omega a(x)|u_n-u_\lambda|^{1-\gamma}dx \\ &\le \int_\Omega a(x)u_\lambda^{1-\gamma}dx + \left(\int_\Omega a(x)^{p/(p-1)}dx\right)^{(p-1)/p} \\ &\quad \cdot \left(\int_\Omega |u_n-u_\lambda|^p dx\right)^{(1-\gamma)/p} \\ &= \int_\Omega a(x)u_\lambda^{1-\gamma}dx + \|a\|_{p/(p-1)}\|u_n-u_\lambda\|_p^{1-\gamma} \\ &= \int_\Omega a(x)u_\lambda^{1-\gamma}dx + o(1) \quad (26)\end{aligned}$$

and

$$\int_\Omega a(x)u_\lambda^{1-\gamma}dx \le \int_\Omega a(x)u_n^{1-\gamma}dx + \int_\Omega a(x)|u_n - u_\lambda|^{1-\gamma}dx \le \int_\Omega a(x)u_n^{1-\gamma}dx + \|a\|_{p/(p-1)}\|u_n - u_\lambda\|_p^{1-\gamma} = \int_\Omega a(x)u_n^{1-\gamma}dx + o(1). \quad (27)$$

Thus, we obtain that

$$\int_\Omega a(x)u_n^{1-\gamma}dx = \int_\Omega a(x)u_\lambda^{1-\gamma}dx + o(1). \quad (28)$$

Since $b(x) > 0$ is bounded in Ω, we have that $b(x)^{1/(1+\beta)}u_n(x)$ is also bounded in $W_0^{s,p}(\Omega)$. If $r \in (1,\infty)$ and $\{g_n\}_{n\in\mathbb{N}} \subset L^r(\mathbb{R}^k)$ is a bounded sequence such that $g_n \to g$ a.e. in $\mathbb{R}^k$, then we have from the Brezis–Lieb Lemma that

$$|g_n - g|_{L^r(\mathbb{R}^k)}^r = |g_n|_{L^r(\mathbb{R}^k)}^r - |g|_{L^r(\mathbb{R}^k)}^r + o(1). \quad (29)$$

Taking

$$g_n = b(x)^{1/(1+\beta)}u_n(x), \quad g = b(x)^{1/(1+\beta)}u(x), \quad r = 1+\beta \text{ and } k = N \quad (30)$$

in (29), we get

$$\int_\Omega b(x)|u_n(x) - u(x)|^{1+\beta}dx = \int_\Omega b(x)|u_n(x)|^{1+\beta}dx - \int_\Omega b(x)|u(x)|^{1+\beta}dx + o(1). \quad (31)$$

Since

$$0 \le \int_\Omega b(x)|u_n(x) - u(x)|^{1+\beta}dx \le b\|u_n - u\|_{1+\beta}^{1+\beta} \to 0, \quad n \to \infty, \quad (32)$$

we can deduce that as $n \to \infty$

$$\int_\Omega b(x)|u_n(x)|^{1+\beta}dx = \int_\Omega b(x)|u(x)|^{1+\beta}dx + o(1). \quad (33)$$

By Lemmas 1 and 2, we have

$$[u_n - u]_{s,q}^q = [u_n]_{s,q}^q - [u]_{s,q}^q + o(1) \quad (34)$$

and

$$[u_n - u]_{s,p}^p = [u_n]_{s,p}^p - [u]_{s,p}^p + o(1). \quad (35)$$

From $m_\lambda < 0$, we have $\|u_n\| \le r_0 < r$ for some positive constant r independent of n. Thus, by (35) and $u_\lambda \in B_r$ we obtain $u_n - u_\lambda \in B_r$ if n is sufficiently large. Combining above arguments with (28), (33), (34), and (35), we can obtain

$$\begin{aligned} m_\lambda &= \mathcal{J}_\lambda(u_n) + o(1) \\ &= \mathcal{J}_\lambda(u_\lambda) + \frac{1}{p}[u_n - u_\lambda]_{s,p}^p + \frac{1}{q}[u_n - u_\lambda]_{s,q}^q + o(1) \\ &\ge \mathcal{J}_\lambda(u_\lambda) + o(1) \\ &\ge m_\lambda + o(1)(n \to \infty), \end{aligned} \quad (36)$$

which yields that

$$0 \ge \mathcal{J}_\lambda(u_\lambda) - m_\lambda + o(1) \ge o(1). \quad (37)$$

Passing to the limit as $n \to \infty$, we get $\mathcal{J}_\lambda(u_\lambda) = m_\lambda$.

In the following, we will show that $u_\lambda \in \mathcal{N}_\lambda^+(\Omega)$. It suffices to prove that $u_n \to u_\lambda$ strongly in $W_0^{s,p}(\Omega)$. By $\mathcal{J}_\lambda(u_\lambda) = m_\lambda$ and

$$m_\lambda = \mathcal{J}_\lambda(u_\lambda) + \frac{1}{p}[u_n - u_\lambda]_{s,p}^p + \frac{1}{q}[u_n - u_\lambda]_{s,q}^q + o(1), \quad (38)$$

we have

$$0 = \frac{1}{p}[u_n - u_\lambda]_{s,p}^p + \frac{1}{q}[u_n - u_\lambda]_{s,q}^q + o(1) \ge \frac{1}{p}\|u_n - u_\lambda\|^p + o(1), \quad (39)$$

which implies that $\|u_n - u_\lambda\|^p \to 0$ as $n \to \infty$, i.e., $u_n \to u_\lambda$ strongly in $W_0^{s,p}(\Omega)$. Thus, $u_\lambda \in \mathcal{N}_\lambda^+(\Omega)$ is a minimizer of $\mathcal{J}_\lambda$ in $\mathcal{N}_\lambda^+(\Omega)$. The proof is complete. □

3. Main Result

Similar to the proof of Lemma 3.7 in [1], we can easily obtain the following Lemma.

Lemma 7. *For any $u \in \mathcal{N}_\lambda^+(\Omega)$, then there exists $\varepsilon > 0$ and a continuous function $f = f(\omega) > 0$, $\omega \in W_0^{s,p}(\Omega)$ with $\|\omega\| < \varepsilon$ satisfying that $f(0) = 1$, $f(\omega)(u+\omega) \in \mathcal{N}_\lambda^+(\Omega)$.*

Lemma 8. *For each given $\phi \in W_0^{s,p}(\Omega)$, $\phi \ge 0$, there exists $T > 0$ such that $\mathcal{J}_\lambda(u_\lambda + t\phi) \ge \mathcal{J}_\lambda(u_\lambda)$ for all $t \in [0, T]$ and $u_\lambda \in \mathcal{N}_\lambda^+(\Omega)$.*

Proof. Let

$$\eta(t) = (p-1)[u_\lambda + t\phi]_{s,p}^p + (q-1)[u_\lambda + t\phi]_{s,q}^q + \gamma\int_\Omega a(x)|u_\lambda + t\phi|^{1-\gamma}dx - \lambda\beta\int_\Omega b(x)|u_\lambda + t\phi|^{1+\beta}dx, \quad \forall t > 0. \quad (40)$$

By using the continuity of η, $\eta(0) > 0$ and $u_\lambda \in \mathcal{N}_\lambda^+(\Omega)$, we can infer that there exists $T > 0$, such that $\eta(t) > 0$ for all $t \in [0, T]$. On the other hand, applying Lemma 7 we get that for any $t > 0$ there exists $t' > 0$ such that $t'(u_\lambda + t\phi) \in \mathcal{N}_\lambda^+(\Omega)$. Hence, $t' \to 1$ as $t \to 0$ and for each $t \in [0, T]$ we obtain

$$\mathcal{J}_\lambda(u_\lambda + t\phi) \ge \mathcal{J}_\lambda[t'(u_\lambda + t\phi)] \ge \inf_{\mathcal{N}_\lambda^+(\Omega)} \mathcal{J}_\lambda = \mathcal{J}_\lambda(u_\lambda). \quad (41)$$

The proof of this Lemma is completed. □

Lemma 9. *The minimizer $u_\lambda \in \mathcal{N}_\lambda^+(\Omega)$ is a weak solution of problem (1). Moreover, $u_\lambda(x) > 0$ for each $x \in \Omega$.*

Proof. Firstly, we prove that $u_\lambda \in \mathcal{N}_\lambda^+(\Omega)$ is a weak solution of (1). From Lemma 8, we derive that for any $\phi \in W_0^{s,p}(\Omega)$, $\phi \geq 0$ and $t \in (0, T]$.

$$\begin{aligned} 0 &\leq \frac{\mathcal{J}_\lambda(u_\lambda + t\phi) - \mathcal{J}_\lambda(u_\lambda)}{t} \\ &= \frac{1}{pt}\int_Q \frac{|u_\lambda(x) - u_\lambda(y) + t(\phi(x) - \phi(y))|^p - |u_\lambda(x) - u_\lambda(y)|^p}{|x-y|^{N+qs}}dxdy \\ &\quad + \frac{1}{qt}\int_Q \frac{|u_\lambda(x) - u_\lambda(y) + t(\phi(x) - \phi(y))|^q - |u_\lambda(x) - u_\lambda(y)|^q}{|x-y|^{N+qs}}dxdy \\ &\quad - \frac{1}{1-\gamma}\int_\Omega a(x)\frac{|u_\lambda + t\phi|^{1-\gamma} - |u_\lambda|^{1-\gamma}}{t}dx \\ &\quad - \frac{\lambda}{1+\beta}\int_\Omega b(x)\frac{|u_\lambda + t\phi|^{1+\beta} - |u_\lambda|^{1+\beta}}{t}dx. \end{aligned} \tag{42}$$

Letting $t \to 0^+$, we infer that

$$\begin{aligned} &\frac{1}{1-\gamma}\liminf_{t\to 0^+}\int_\Omega a(x)\frac{|u_\lambda + t\phi| - |u_\lambda|^{1-\gamma}}{t}dx \\ &\leq \int_Q \frac{|u_\lambda(x) - u_\lambda(y)|^{p-2}(u_\lambda(x) - u_\lambda(y))(\phi(x) - \phi(y))}{|x-y|^{n+ps}}dxdy \\ &\quad + \int_Q \frac{|u_\lambda(x) - u_\lambda(y)|^{q-2}(u_\lambda(x) - u_\lambda(y))(\phi(x) - \phi(y))}{|x-y|^{n+qs}}dxdy \\ &\quad - \lambda\int_\Omega b(x)u_\lambda^\beta\phi dx. \end{aligned} \tag{43}$$

From Fatou's Lemma, we have

$$\int_\Omega a(x)u_\lambda^{-\gamma}\phi dx \leq \frac{1}{1-\gamma}\liminf_{t\to 0^+}\int_\Omega a(x)\frac{|u_\lambda + t\phi|^{1-\gamma} - |u_\lambda|^{1-\gamma}}{t}dx. \tag{44}$$

Together with (42) with (43), it yields

$$\begin{aligned} &\int_Q \frac{|u_\lambda(x) - u_\lambda(y)|^{p-2}(u_\lambda(x) - u_\lambda(y))(\phi(x) - \phi(y))}{|x-y|^{n+ps}}dxdy \\ &\quad + \int_Q \frac{|u_\lambda(x) - u_\lambda(y)|^{q-2}(u_\lambda(x) - u_\lambda(y))(\phi(x) - \phi(y))}{|x-y|^{n+qs}}dxdy \\ &\quad - \lambda\int_\Omega b(x)u_\lambda^\beta\phi dx - \int_\Omega a(x)u_\lambda^{-\gamma}\phi dx \geq 0. \end{aligned} \tag{45}$$

For any given $\psi \in W_0^{s,p}(\Omega)$, taking

$$\phi = (u_\lambda + t\psi)^+ \in W_0^{s,p}(\Omega), \quad \phi \geq 0 \tag{46}$$

into (44), we obtain

$$\begin{aligned} 0 &\leq \int_{Q'} \frac{|u_\lambda(x) - u_\lambda(y)|^{p-2}(u_\lambda(x) - u_\lambda(y))((u_\lambda + t\psi)(x) - (u_\lambda + t\psi)(y))}{|x-y|^{n+ps}}dxdy \\ &\quad + \int_{Q'} \frac{|u_\lambda(x) - u_\lambda(y)|^{q-2}(u_\lambda(x) - u_\lambda(y))((u_\lambda + t\psi)(x) - (u_\lambda + t\psi)(y))}{|x-y|^{n+qs}}dxdy \\ &\quad - \lambda\int_{\{x|(u_\lambda + t\psi)\geq 0\}} b(x)u_\lambda^\beta(u_\lambda + t\psi)dx - \int_{\{x|(u_\lambda + t\psi)\geq 0\}} a(x)u_\lambda^{-\gamma}(u_\lambda + t\psi)dx \\ &= \|u_\lambda\|^p + \|u_\lambda\|^q - \lambda\int_\Omega b(x)u_\lambda^{\beta+1}dx - \int_\Omega a(x)u_\lambda^{1-\gamma}dx \\ &\quad + t\bigg[\int_Q \frac{|u_\lambda(x) - u_\lambda(y)|^{p-2}(u_\lambda(x) - u_\lambda(y))(\psi(x) - \psi(y))}{|x-y|^{n+ps}}dxdy \\ &\quad + \int_Q \frac{|u_\lambda(x) - u_\lambda(y)|^{q-2}(u_\lambda(x) - u_\lambda(y))(\psi(x) - \psi(y))}{|x-y|^{n+qs}}dxdy \\ &\quad - \lambda\int_\Omega b(x)u_\lambda^\beta\psi dx - \int_\Omega a(x)u_\lambda^{-\gamma}\psi dx\bigg] \\ &\quad - \int_{Q''} \frac{|u_\lambda(x) - u_\lambda(y)|^{p-2}(u_\lambda(x) - u_\lambda(y))((u_\lambda + t\psi)(x) - (u_\lambda + t\psi)(y))}{|x-y|^{n+ps}}dxdy \\ &\quad - \int_{Q''} \frac{|u_\lambda(x) - u_\lambda(y)|^{q-2}(u_\lambda(x) - u_\lambda(y))((u_\lambda + t\psi)(x) - (u_\lambda + t\psi)(y))}{|x-y|^{n+qs}}dxdy \\ &\quad - \lambda\int_{\{x|(u_\lambda + t\psi)< 0\}} b(x)u_\lambda^\beta(u_\lambda + t\psi)dx - \int_{\{x|(u_\lambda + t\psi)< 0\}} a(x)u_\lambda^{-\gamma}(u_\lambda + t\psi)dx \\ &\leq t\bigg[\int_Q \frac{|u_\lambda(x) - u_\lambda(y)|^{p-2}(u_\lambda(x) - u_\lambda(y))(\psi(x) - \psi(y))}{|x-y|^{n+ps}}dxdy \\ &\quad + \int_Q \frac{|u_\lambda(x) - u_\lambda(y)|^{q-2}(u_\lambda(x) - u_\lambda(y))(\psi(x) - \psi(y))}{|x-y|^{n+qs}}dxdy \\ &\quad - \lambda\int_\Omega b(x)u_\lambda^\beta\psi dx - \int_\Omega a(x)u_\lambda^{-\gamma}\psi dx\bigg] \\ &\quad - t\int_{Q''} \frac{|u_\lambda(x) - u_\lambda(y)|^{p-2}(u_\lambda(x) - u_\lambda(y))(\psi(x) - \psi(y))}{|x-y|^{n+ps}}dxdy \\ &\quad - t\int_{Q''} \frac{|u_\lambda(x) - u_\lambda(y)|^{q-2}(u_\lambda(x) - u_\lambda(y))(\psi(x) - \psi(y))}{|x-y|^{n+qs}}dxdy, \end{aligned} \tag{47}$$

where $Q' = \mathbb{R}^{2n} \backslash (\mathscr{C}\Omega' \times \mathscr{C}\Omega')$, $\mathscr{C}\Omega' = \mathbb{R}^n \backslash \{x|(u_\lambda + t\psi) \geq 0\}$ and $Q'' = \mathbb{R}^{2n} \backslash (\mathscr{C}\Omega'' \times \mathscr{C}\Omega'')$, $\mathscr{C}\Omega'' = \mathbb{R}^n \backslash \{x|(u_\lambda + t\psi) < 0\}$. Since the measure of the set $\{x|(u_\lambda + t\psi) < 0\}$ tends to 0 as $t \to 0^+$, one has

$$\int_{Q''} \frac{|u_\lambda(x) - u_\lambda(y)|^{p-2}(u_\lambda(x) - u_\lambda(y))(\psi(x) - \psi(y))}{|x-y|^{n+ps}}dxdy \to 0, t \to 0^+, \tag{48}$$

$$\int_{Q''} \frac{|u_\lambda(x) - u_\lambda(y)|^{q-2}(u_\lambda(x) - u_\lambda(y))(\psi(x) - \psi(y))}{|x-y|^{n+qs}}dxdy \to 0, t \to 0^+. \tag{49}$$

Hence, dividing by $t > 0$ we infer that

$$\begin{aligned} &\int_Q \frac{|u_\lambda(x) - u_\lambda(y)|^{p-2}(u_\lambda(x) - u_\lambda(y))(\psi(x) - \psi(y))}{|x-y|^{n+ps}}dxdy \\ &\quad + \int_Q \frac{|u_\lambda(x) - u_\lambda(y)|^{q-2}(u_\lambda(x) - u_\lambda(y))(\psi(x) - \psi(y))}{|x-y|^{n+qs}}dxdy \\ &\quad - \lambda\int_\Omega b(x)u_\lambda^\beta\psi dx - \int_\Omega a(x)u_\lambda^{-\gamma}\psi dx \geq 0, \quad \forall\psi \in W_0^{s,p}(\Omega). \end{aligned} \tag{50}$$

Since $\psi \in W_0^{s,p}(\Omega)$ is arbitrary, replacing ψ by $-\psi$ in above inequality, one obtains

$$\int_Q \frac{|u_\lambda(x)-u_\lambda(y)|^{p-2}(u_\lambda(x)-u_\lambda(y))(\psi(x)-\psi(y))}{|x-y|^{n+ps}}dxdy + \int_Q \frac{|u_\lambda(x)-u_\lambda(y)|^{q-2}(u_\lambda(x)-u_\lambda(y))(\psi(x)-\psi(y))}{|x-y|^{n+qs}}dxdy - \lambda\int_\Omega b(x)u_\lambda^\beta \psi dx - \int_\Omega a(x)u_\lambda^{-\gamma}\psi dx \le 0, \quad \forall \psi \in W_0^{s,p}(\Omega). \tag{51}$$

Thus

$$\int_Q \frac{|u_\lambda(x)-u_\lambda(y)|^{p-2}(u_\lambda(x)-u_\lambda(y))(\psi(x)-\psi(y))}{|x-y|^{n+ps}}dxdy + \int_Q \frac{|u_\lambda(x)-u_\lambda(y)|^{q-2}(u_\lambda(x)-u_\lambda(y))(\psi(x)-\psi(y))}{|x-y|^{n+qs}}dxdy - \lambda\int_\Omega b(x)u_\lambda^\beta \psi dx - \int_\Omega a(x)u_\lambda^{-\gamma}\psi dx = 0, \quad \forall \psi \in W_0^{s,p}(\Omega). \tag{52}$$

Hence, u_λ is a weak solution of (1).

Secondly, we prove that $u_\lambda > 0$ for almost each $x \in \Omega$. Since $\mathcal{J}_\lambda(u_\lambda) = c < 0$, we get $u_\lambda \ge 0$ and $u_\lambda \ne 0$. Let $e_1 \in W_0^{s,p}(\Omega)$ be the first eigenfunction of the operator $(-\Delta)_p^s$ with $e_1 > 0$ and $\|e_1\| = 1$. Taking $\psi = e_1$ in (51), one gets

$$\begin{aligned}
\int_\Omega a(x)u_\lambda^{-\gamma}e_1 dx &\le \left(\int_Q \frac{|u_\lambda(x)-u_\lambda(y)|^{p-1}}{|x-y|^{n+ps}}dxdy\right)^{(p-1)/p} \cdot \left(\int_Q \frac{|e_1(x)-e_1(y)|^{p}}{|x-y|^{n+ps}}dxdy\right)^{1/p} \\
&\quad + \left(\int_Q \frac{|u_\lambda(x)-u_\lambda(y)|^{q-1}}{|x-y|^{n+qs}}dxdy\right)^{(q-1)/q} \cdot \left(\int_Q \frac{|e_1(x)-e_1(y)|^{q}}{|x-y|^{n+qs}}dxdy\right)^{1/q} \\
&\quad + \lambda\bar{b}\left(\int_\Omega |u_\lambda|^{p_s^*}dx\right)^{\beta/p_s^*} \cdot \left(\int_\Omega |e_1|^{p_s^*/(p_s^*-\beta)}dx\right)^{(p_s^*-\beta)/p_s^*} \\
&= \|u_\lambda\|^{(p-1)/p}\|e_1\|^{1/p} + \|u_\lambda\|^{(q-1)/q}\|e_1\|^{1/q} + \lambda\bar{b}\|u_\lambda\|_{p_s^*}^\beta \left(\int_\Omega |e_1|^{p_s^*/(p_s^*-\beta)}dx\right)^{(p_s^*-\beta)/p_s^*} \\
&\le \|u_\lambda\|^{(p-1)/p}\|e_1\|^{1/p} + \|u_\lambda\|^{(q-1)/q}\|e_1\|^{1/q} + \lambda\bar{b}S_p^{-\beta/p}\|u_\lambda\|^\beta \left[\left(\int_\Omega dx\right)^{(p_s^*-\beta-1)/(p_s^*-\beta)} \cdot \left(\int_\Omega |e_1|^{p_s^*}dx\right)^{1/(p_s^*-\beta)}\right]^{(p_s^*-\beta)/p_s^*} \\
&= \|u_\lambda\|^{(p-1)/p}\|e_1\|^{1/p} + \|u_\lambda\|^{(q-1)/q}\|e_1\|^{1/q} + \lambda\bar{b}S_p^{-\beta/p}\|\Omega\|^{(p_s^*-\beta-1)/p_s^*}\|u_\lambda\|^\beta\|e_1\|_{p_s^*} \\
&\le \|u_\lambda\|^{(p-1)/p}\|e_1\|^{1/p} + \|u_\lambda\|^{(q-1)/q}\|e_1\|^{1/q} + \lambda\bar{b}S_p^{-(1+\beta)/p}|\Omega|^{(p_s^*-\beta-1)/p_s^*}\|u_\lambda\|^\beta\|e_1\| < \infty,
\end{aligned} \tag{53}$$

which implies that $u_\lambda > 0$ for almost every $x \in \Omega$. This completes the proof of Lemma 9. ☐

Lemma 10. *Assume that (H1) and (H2) hold with $p < 1 + \beta$. Then there exists $\lambda_2 > 0$ such that for any $\lambda \in (0, \lambda_2)$, one has $\mathcal{N}_\lambda^0(\Omega) = \{0\}$. Moreover, $\mathcal{N}_\lambda^-(\Omega)$ is closed in $W_0^{s,p}(\Omega)$for all $\lambda \in (0, \lambda_2)$.*

Proof. If not, there is a $u_0 \in \mathcal{N}_\lambda^0(\Omega)$ with $\|u_0\| \ne 0$. Then by the definition of $\mathcal{N}_\lambda^0(\Omega)$, we get

$$\begin{aligned}
(p+\gamma-1)[u_0]_{s,p}^p &\le (p+\gamma-1)[u_0]_{s,p}^p + (q+\gamma-1)[u_0]_{s,q}^q \\
&= \lambda(\beta+\gamma)\int_\Omega b(x)|u_0|^{1+\beta}dx,
\end{aligned} \tag{54}$$

which implies that

$$\begin{aligned}
\|u_0\|^p = [u_0]_{s,p}^p &\le \frac{\lambda(\beta+\gamma)}{p+\gamma-1}\int_\Omega b(x)|u_0|^{1+\beta}dx \\
&\le \frac{\lambda b(\beta+\gamma)}{p+\gamma-1}|\Omega|^{(p_s^*-1-p)/p_s^*}S_p^{-(1+\beta)/p}\|u_0\|^{1+\beta},
\end{aligned} \tag{55}$$

and so

$$\|u_0\| \ge \left(\frac{p+\gamma-1}{\lambda b(\beta+\gamma)}|\Omega|^{(p+1-p_s^*)/p_s^*}S_p^{(1+\beta)/p}\right)^{1/(1+\beta-p)}. \tag{56}$$

On the other hand, from the definition of $\mathcal{N}_\lambda^0(\Omega)$, we can obtain

$$\begin{aligned}
&(\beta+1-p)[u_0]_{s,p}^p + (\beta+1-q)[u_0]_{s,q}^q \\
&\quad = (\beta+\gamma)\int_\Omega a(x)|u_0|^{1-\gamma}dx, \quad u_0 \in \mathcal{N}_\lambda^0(\Omega).
\end{aligned} \tag{57}$$

By using the Hölder inequalities and (7), we have by (57) that

$$\begin{aligned}
(\beta+1-p)[u_0]_{s,p}^p &\le (\beta+1-p)[u_0]_{s,p}^p \\
&\quad + (\beta+1-q)[u_0]_{s,q}^q = (\beta+\gamma)\int_\Omega a(x)|u_0|^{1-\gamma}dx \\
&\le (\beta+\gamma)\|a\|_{p_s^*/(p_s^*-1+\gamma)}S_p^{-(1-\gamma))/p}\|u_0\|^{1-\gamma},
\end{aligned} \tag{58}$$

which implies that

$$\|u_0\| \leq \left(\frac{\beta+\gamma}{\beta+1-p} \|a\|_{p_s^*/(p_s^*-1+\gamma)} S_p^{-(1-\gamma)/p} \right)^{1/(p+\gamma-1)}. \tag{59}$$

If $\lambda > 0$ is sufficiently small, however, (55) contradicts to (58). Hence, we conclude that there exists $\lambda_2 > 0$ such that $\mathcal{N}_\lambda^0(\Omega) = \{0\}$ for $\lambda \in (0, \lambda_2)$.

Let $\{u_n\} \subset \mathcal{N}_\lambda^-(\Omega)$ be a sequence satisfying $u_n \to u_*$ in the $W_0^{s,p}(\Omega)$. By using the Sobolev inequalities and continuous compact embedding, we have $u_n \to u_*$ in $L^{1+\beta}(\Omega)$ and $u_* \in \mathcal{N}_\lambda^-(\Omega) \cup \mathcal{N}_\lambda^0(\Omega)$. From the definition of $\mathcal{N}_\lambda^-(\Omega)$, we obtain that

$$\begin{aligned}\bar{b}\lambda(\beta+\gamma)|\Omega|^{(p_s^*-1-\beta)/p_s^*} \|u_*\|_{p_s^*}^{1+\beta} &\geq \bar{b}\lambda(\beta+\gamma)\|u_*\|_{1+\beta}^{1+\beta} \\ &\geq \lambda(\beta+\gamma)\int_\Omega b(x)|u_*|^{1+\beta}dx \\ &> (p+\gamma-1)[u_*]_{s,p}^p \\ &\quad + (q+\gamma-1)[u_*]_{s,q}^q \geq (p+\gamma-1)\|u_*\|^p \\ &\geq (p+\gamma-1)S_p\|u_*\|_{p_s^*}^p\end{aligned} \tag{60}$$

that is,

$$\|u_*\|_{p_s^*} \geq \left(\frac{p+\gamma-1}{\bar{b}\lambda(\beta+\gamma)} S_p |\Omega|^{(1+\beta-p_s^*)/p_s^*} \right)^{1/(1+\beta-p)} > 0 \tag{61}$$

which yields $u_* \neq 0$, i.e., $u_* \in \mathcal{N}_\lambda^-(\Omega)$. The proof of this Lemma is completed. □

Lemma 11. *Assume that (H1) and (H2) hold with $p < 1+\beta$. Then there exists $\lambda_3 > 0$, such that $\mathcal{J}_\lambda(u) \geq 0$, for each $u \in \mathcal{N}_\lambda^-(\Omega)$ while $\lambda \in (0, \lambda_3)$.*

Proof. If not, there is a $v \in \mathcal{N}_\lambda^-(\Omega)$ such that $\mathcal{J}_\lambda(v) < 0$. Then

$$\begin{aligned}&\frac{1}{p}\left([v]_{s,p}^p + [v]_{s,q}^q\right) - \frac{1}{1-\gamma}\int_\Omega a(x)|v|^{1-\gamma}dx \\ &\quad - \frac{\lambda}{1+\beta}\int_\Omega b(x)|v|^{1+\beta}dx \\ &\quad \leq \frac{1}{p}[v]_{s,p}^p + \frac{1}{q}[v]_{s,q}^q - \frac{1}{1-\gamma}\int_\Omega a(x)|v|^{1-\gamma}dx \\ &\quad - \frac{\lambda}{1+\beta}\int_\Omega b(x)|v|^{1+\beta}dx < 0.\end{aligned} \tag{62}$$

By (62) and the definition of $\mathcal{N}_\lambda^-(\Omega)$, it follows that

$$\lambda\left(\frac{1}{p} - \frac{1}{1+\beta}\right)\int_\Omega b(x)|v|^{1+\beta}dx - \left(\frac{1}{1-\gamma} - \frac{1}{p}\right)\int_\Omega a(x)|v|^{1-\gamma}dx < 0. \tag{63}$$

Combining with (18) and (22), we get

$$\begin{aligned}&\lambda\bar{b}\left(\frac{1}{p} - \frac{1}{1+\beta}\right)|\Omega|^{(p_s^*-1-\beta)/p_s^*}\|v\|_{p_s^*}^{1+\beta} \\ &\quad < \left(\frac{1}{1-\gamma} - \frac{1}{p}\right)\|a\|_{p_s^*/(p_s^*-1+\gamma)}\|v\|_{p_s^*}^{1-\gamma},\end{aligned} \tag{64}$$

which implies that

$$\|v\|_{p_s^*} < \left(\frac{(1+\beta)(p+\gamma-1)}{\lambda\bar{b}(1-\gamma)(\beta+1-p)} |\Omega|^{(1+\beta-p_s^*)/p_s^*} \|a\|_{p_s^*/(p_s^*-1+\gamma)} \right)^{1/(\beta+\gamma)}. \tag{65}$$

With the help of inequalities (61) and (65), for any $u \in \mathcal{N}_\lambda^-(\Omega)$, we get that

$$\begin{aligned}&\left(\frac{p+\gamma-1}{\bar{b}\lambda(\beta+\gamma)} S_p |\Omega|^{(1+\beta-p_s^*)/p_s^*} \right)^{1/(1+\beta-p)} \leq \|u\|_{p_s^*} \\ &\quad < \left(\frac{(1+\beta)(p+\gamma-1)}{\lambda\bar{b}(1-\gamma)(\beta+1-p)} |\Omega|^{(1+\beta-p_s^*)/p_s^*} \|a\|_{p_s^*/(p_s^*-1+\gamma)} \right)^{1/(\beta+\gamma)}.\end{aligned} \tag{66}$$

Direct calculations show that

$$\begin{aligned}&\frac{\left(p+\gamma-1/\bar{b}(\beta+\gamma)\cdot S_p|\Omega|^{(1+\beta-p_s^*)/p_s^*}\right)^{1/(1+\beta-p)}}{\left((1+\beta)(p+\gamma-1)/\bar{b}(1-\gamma)(\beta+1-p)\cdot|\Omega|^{(1+\beta-p_s^*)/p_s^*}\|a\|_{p_s^*/(p_s^*-1+\gamma)}\right)^{1/(\beta+\gamma)}} \\ &\quad < \lambda^{(p+\gamma-1)/((1+\beta-p)(\beta+\gamma))},\end{aligned} \tag{67}$$

which contradicts the fact $\lambda^{(p+\gamma-1)/(1+\beta-p)(\beta+\gamma)} \to 0$ as $\lambda \to 0^+$. We complete the proof of Lemma 11. □

Lemma 12. *Assume that (H1) and (H2) hold with $p < 1+\beta$. Then there exists $\lambda_4 > 0$ small enough such that for each $\lambda \in (0, \lambda_4)$, there exists $v_\lambda \in \mathcal{N}_\lambda^-(\Omega)$ satisfying $\mathcal{J}_\lambda(v_\lambda) = \overline{m}_\lambda = \inf_{u \in \mathcal{N}_\lambda^-(\Omega)} \mathcal{J}_\lambda(u)$. Furthermore, v_λ is a weak positive solution of problem (1).*

Proof. From Lemma 4, $\mathcal{J}_\lambda$ is coercive on $\mathcal{N}_\lambda(\Omega)$, and it is also true for $\mathcal{N}_\lambda^-(\Omega)$. If sequence $\{v_n\} \subset \mathcal{N}_\lambda^-(\Omega)$ satisfying $\mathcal{J}_\lambda(v_n) \to \overline{m}_\lambda = \inf_{u \in \mathcal{N}_\lambda^-(\Omega)} \mathcal{J}_\lambda(u)$ as $n \to \infty$. By using the coercive of $\mathcal{J}_\lambda$, we can get $\{v_n\}$ is bounded in $\mathcal{N}_\lambda^-(\Omega)$. Hence, we may assume that $v_n \rightharpoonup v_\lambda$ weakly as $n \to \infty$ in $\mathcal{N}_\lambda^-(\Omega)$. Since $\mathcal{N}_\lambda^-(\Omega)$ is completed in $W_0^{s,p}(\Omega)$ (Lemma 10), it follows from the same arguments as in proving the existence of minimizer u_λ (Lemma 6) and the compactness of the embedding $W_0^{s,p}(\Omega) \to L^{1+\beta}(\Omega)$ ($\beta < p_s^* - 1$) we get $v_\lambda \in \mathcal{N}_\lambda^-(\Omega)$ is the minimizer of $\mathcal{J}_\lambda$. Furtherover, similar to the proof of weak positive solution u_λ as in Lemma 9, one can prove that $v_\lambda \in W_0^{s,p}(\Omega)$ is also a weak positive solution for problem (1). The proof is complete. □

Theorem 13. *Let $0 < \gamma < 1 \leq q < p < 1+\beta < p_s^*$. Assume that (H1) and (H2) hold. Then there exists a positive number λ^* such that for each $\lambda \in (0, \lambda^*)$ problem (1) possesses at least two weak positive solution $u_\lambda, v_\lambda \in W_0^{s,p}(\Omega)$.*

Proof. Let $\lambda^* = \min\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$. Obviously, Lemma 3 and Lemma 12 are true for all $\lambda \in (0, \lambda^*)$. Thus, it follows from Lemma 9 and Lemma 12 that u_λ and v_λ are the weak positive solutions of problem (1). The proof is complete. □

4. Conclusions

In this paper, the existence and multiplicity of positive solutions for a class of fractional p & q Laplacian problem with singular nonlinearity have been investigated. It is worthy to point out that few studies have been done on this issue. By means of the variational method, Nehari manifold method and some analysis techniques, the sufficient conditions of existence and multiplicity of positive solutions to this problem have been presented in Theorem 13. Our results generalize the main conclusions of Wang and Zhang in [1].

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

Authors' Contributions

The authors contributed equally to this paper. All authors read and approved the final manuscript.

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