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Research Article

Multiplicity Results for Variable-Order Nonlinear Fractional Magnetic Schrödinger Equation with Variable Growth

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In this paper, we prove the multiplicity of nontrivial solutions for a class of fractional-order elliptic equation with magnetic field. Under appropriate assumptions, firstly, we prove that the system has at least two different solutions by applying the mountain pass theorem and Ekeland's variational principle. Secondly, we prove that these two solutions converge to the two solutions of the limit problem. Finally, we prove the existence of infinitely many solutions for the system and its limit problems, respectively.

1. Introduction

In this paper, we consider the multiplicity of nontrivial solutions of the following concave-convex elliptic equation involving variable-order nonlinear fractional magnetic Schrödinger equation:

$$\begin{cases} (-\Delta)_A^{s(\cdot)} u + V_\lambda(x) u = f(x)|u|^{p(x)-2}u + g(x)|u|^{q(x)-2}u \text{ in } \Omega, \\ u = 0 \text{ in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1}$$

where $N \geq 1$; $s(\cdot)$: $\mathbb{R}^N \times \mathbb{R}^N \longrightarrow (0,1)$ is a continuous function; Ω is a bounded subset in $\mathbb{R}^N$ with $N > 2s(x,y)$ for all $(x,y) \in \Omega \times \Omega$; $(-\Delta)_A^{s(\cdot)}$ is the variable-order fractional magnetic Laplace operator; the potential $V_\lambda(x) = \lambda V^+(x) - V^-(x)$ with $V^\pm = \max\{\pm V, 0\}$; $\lambda > 0$ is a parameter; magnetic field $A \in C^{0,\alpha}(\mathbb{R}^N, \mathbb{R}^N)$ with $\alpha \in (0,1]$; $f, g > 0$ are two bounded nonnegative measurable function; $p, q \in C(\Omega)$; and $u : \mathbb{R}^N \longrightarrow \mathbb{C}$. In [1], the fractional magnetic Laplacian has been defined as

$$(-\Delta)_A^s u(x) = \lim_{r\to 0} \int_{B_r^c(x)} \frac{u(x) - e^{i(x-y)\cdot A((x+y)/2)} u(y)}{|x-y|^{N+2s}} dy, \tag{2}$$

for $x \in \mathbb{R}^N$. In [2], the variable-order fractional magnetic Laplace $(-\Delta)^{s(\cdot)}$ is defined as follows: for each $x \in \mathbb{R}^N$,

$$(-\Delta)^{s(\cdot)} \varphi(x) = 2P.V \int_{\mathbb{R}^N} \frac{\varphi(x) - \varphi(y)}{|x-y|^{N+2s(x,y)}} dy, \tag{3}$$

along any $\varphi \in C_0^\infty(\Omega)$. Inspired by them, we define the variable-order fractional magnetic Laplacian $(-\Delta)_A^{s(\cdot)}$ as follows: for each $x \in \mathbb{R}^N$,

$$(-\Delta)_A^{s(\cdot)} u(x) = \lim_{r\to 0} \int_{B_r^c(x)} \frac{u(x) - e^{i(x-y)\cdot A((x+y)/2)} u(y)}{|x-y|^{N+2s(x,y)}} dy. \tag{4}$$

Since $s(\cdot)$ is a function, magnetic field $A \in C^{0,\alpha}(\mathbb{R}^N, \mathbb{R}^N)$ with $\alpha \in (0,1]$, we see that operator $(-\Delta)_A^{s(\cdot)}$ is variable order fractional magnetic Laplace operator. Especially, when $s(\cdot) \equiv$ constant, $(-\Delta)_A^{s(\cdot)}$ reduce to the usual fractional magnetic Laplace operator. When $s(\cdot) \equiv$ constant, $A = 0$, $(-\Delta)_A^{s(\cdot)}$ reduce to the usual fractional Laplace operator. Very recently, for $s(\cdot) = 1$, $p(x), q(x) \equiv$ constant, and $A = 0$; in [3], under appropriate assumptions, the authors obtained the multiplicity and concentration of the positive solution of the following indefinite semilinear elliptic equations involving concave-convex nonlinearities by the variational method:

$$\begin{cases} -\Delta u + V_\lambda(x)u = f(x)|u|^{q-2}u + g(x)|u|^{p-2}u \text{ in } \mathbb{R}^N, \\ u \geq 0 \text{ in } \mathbb{R}^N. \end{cases} \tag{5}$$

For $s(\cdot) = \alpha, p(x), q(x) \equiv$ constant, and $A = 0$; in [4], the authors obtained the existence, multiplicity, and concentration of nontrivial solutions for the following indefinite fractional elliptic equation by using Nehari manifold decomposition:

$$\begin{cases} (-\Delta)^\alpha u + V_\lambda(x)u = a(x)|u|^{q-2}u + b(x)|u|^{p-2}u \text{ in } \mathbb{R}^N, \\ u \geq 0 \text{ in } \mathbb{R}^N. \end{cases} \tag{6}$$

When $A = 0$, $V^-(x) = 0$, and $f(x), g(x) \equiv$ constant, the authors in [2] give some sufficient conditions to ensure the existence of two different weak solutions and use the variational method and the mountain pass theorem to obtain the two weak solutions of problem (12) which converge to two solutions of its limit problems and the existence of infinitely many solutions to its limit problem:

$$\begin{cases} (-\Delta)^{s(\cdot)}u + \lambda V(x)u = \alpha|u|^{p(x)-2}u + \beta|u|^{q(x)-2}u \text{ in } \Omega, \\ u = 0 \text{ in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{7}$$

For $s(\cdot) = s, p(x), q(x) \equiv$ constant, in [1], the authors study the existence of solutions for the following equation on the whole space by using the method of Nehari manifold decomposition,and obtain some sufficient conditions for the existence of nontrivial solutions of the following equation:

$$(-\Delta)_A^s u + V_\lambda(x)u = f(x)|u|^{q-2}u + g(x)|u|^{p-2}u \text{ in } \mathbb{R}^N. \tag{8}$$

In recent years, with the continuous deepening of research, the fractional magnetic problem has attracted extensive attention of researchers. More and more researchers have studied the solvability of the fractional magnetic problem (see [5–8]). We know that the fractional magnetic Laplacian operator $(-\Delta)_A^s$ is introduced in literature [9]; $(-\Delta)_A^s$ comes from magnetic Laplacian $(\nabla - iA)^{2s}$. However, as far as we know, up to now, few papers have studied the existence and multiplicity of solutions for the variable-order fractional magnetic Schrödinger equation. Therefore, motivated by the above literature, we are interested in the existence and multiplicity of solutions to problem (1) with variable growth. As far as we know, this is the first time to study the existence and multiplicity of nontrivial solutions for the variable-order nonlinear fractional magnetic Schrödinger equation with variable exponents.

It is worth noting that in this paper, we not only prove that there exist two different nontrivial solutions to problem (1), but we also show that the two nontrivial solutions of problem (1) converge to two solutions of the limit problem for problem (1). The novelty of this paper is that, compared with [1], we write the original fractional magnetic Schrödinger equation without variable growth to a variable-order fractional magnetic Schrödinger equation with variable growth. In addition, compared with [2], we write the variable order fractional Schrödinger equation to a variable order fractional magnetic Schrödinger equation.

Inspired by the above works, we assume that $s : \mathbb{R}^N \times \mathbb{R}^N \longrightarrow (0, 1)$ and V are continuous functions satisfying the following:

(S_1): $0 < s^- := \min_{(x,y)\in\mathbb{R}^N\times\mathbb{R}^N} s(x, y) \leq s^+ := \max_{(x,y)\in\mathbb{R}^N\times\mathbb{R}^N} s(x, y) < 1$.

(S_2): $s(\cdot)$ is symmetric, that is, $s(x, y) = s(y, x)$ for all $(x, y) \in \mathbb{R}^N \times \mathbb{R}^N$.

(V_1): $X = \operatorname{int}\left((V^+)^{-1}(0)\right) \subset \Omega$ is a nonempty bounded domain and $\tilde{X} = (V^+)^{-1}(0)$.

(V_2): there exists a nonempty open domain $\Omega_0 \subset X$ such that $V^+ \equiv 0$, $V^- \equiv 0$ for all $x \in \bar{\Omega}_0$.

(V_3): V^+ is a continuous function on Ω and $V^- \in L^{N/2}(\Omega)$.

(V_4): there exists a constant $\vartheta_0 > 1$ such that

$$\inf_{u\in H_{0,A}^{s(\cdot)}(\Omega,\mathbb{C})\setminus\{0\}} \frac{\int_{\mathbb{R}^{2N}}\left(\left(\left|u(x) - e^{i(x-y)\cdot A(x+y/2)}u(y)\right|^2\right)/|x-y|^{N+2s(x,y)}\right)dxdy + \lambda\int_\Omega V^+u^2dx}{\int_\Omega V^-u^2dx} \geq \vartheta_0, \tag{9}$$

or all $\lambda > 0$, where $H_{0,A}^{s(\cdot)}(\Omega, \mathbb{C})$ is the Hilbert space related to magnetic field A (see Section 2).

For the variable exponents p, q, we assume that $p, q \in C(\bar{\Omega})$ and satisfy the following assumption:

(H_1): $2 < p(x) < 2N/(N - 2s(x, x))$ for all $x \in \bar{\Omega}$.

(H_2): $1 < q(x) < 2$ for all $x \in \bar{\Omega}$.

In addition, we assume that f, g satisfy the following assumption:

(H_3): $f, g : \mathbb{R}^N \longrightarrow [0, \infty)$ are bounded nonnegative measurable function such that $f > 0$, $g > 0$ on open interval $\Omega_f, \Omega_g \subset \Omega$ and

$$\|f\|_\infty = \|f\|_{L^\infty(\mathbb{R}^N)} \leq \frac{p^-(2-q^+)(\vartheta_0 - 1)}{\max\left\{C_p^{p^+}, C_p^{p^-}\right\}2\vartheta_0(p^+ - q^+)},$$

$$\|g\|_\infty = \|g\|_{L^\infty(\mathbb{R}^N)} \leq \frac{q^-(p^+-2)(\vartheta_0 - 1)}{\max\left\{C_q^{q^+}, C_q^{q^-}\right\}2\vartheta_0(p^+ - q^+)} \cdot \left[\frac{(2-q^+)(\vartheta_0 - 1)}{2A_1\vartheta_0(p^+ - q^+)}\right]^{(2-q^+)/(p^+-2)}. \tag{10}$$

Based on the hypothesis (S_2), we can give the following definition of weak solutions for problem (1).

Definition 1. We say that $u \in E_\lambda$ is a weak solution of problem (1), if

$$\begin{aligned} &\Re\int_{\mathbb{R}^{2N}} \frac{\left(u(x) - e^{i(x-y)\cdot A((x+y)/2)}u(y)\right)\overline{\left(v(x)e^{i(xy)\cdot A((x+y)/2)}v(y)\right)}}{|x-y|^{N+2s(x,y)}}dxdy \\ &\quad + \lambda\Re\int_\Omega V^+ u\bar{v}dx - \Re\int_\Omega V^- u\bar{v}dx \\ &\quad - \Re\int_\Omega \left(f(x)|u|^{p(x)-2}u + g(x)|u|^{q(x)-2}u\right)\bar{v}dx = 0, \end{aligned} \tag{11}$$

for any $v \in E_\lambda$, where E_λ will be defined in Section 2.

Now, we will describe the first main result as follows.

Theorem 2. *Assume that $(S_1), (S_2), (V_1) - (V_4)$, and $(H_1) - (H_3)$ hold. Let $N > 2s^+$. Then, the problem (1) allows at least two different solutions for all $\lambda > 0$.*

Theorem 3. *Let u_λ^1 and u_λ^2 be two solutions obtained in Theorem 2. Then, $u_\lambda^1 \longrightarrow u^1$ and $u_\lambda^2 \longrightarrow u^2$ in $H_{0,A}^{s(\cdot)}(\Omega, \mathbb{C})$ as $\lambda \longrightarrow \infty$, where $u^1 \neq u^2$ are two nontrivial solutions of the following problem:*

$$\begin{cases} (-\Delta)_A^{s(\cdot)}u - V^-(x)u = f(x)|u|^{p(x)-2}u + g(x)|u|^{q(x)-2}u \text{ in } \Omega_0, \\ u = 0 \text{ in } \mathbb{R}^N \setminus \Omega_0. \end{cases} \tag{12}$$

Remark 4. In general, if $s(\cdot)$: $\mathbb{R}^N \times \mathbb{R}^N \longrightarrow (0, 1)$ is a continuous function, magnetic field $A \in C^{0,\alpha}(\mathbb{R}^N, \mathbb{R}^N)$ with $\alpha \in (0, 1]$, then the variable-order fractional magnetic Laplacian can be defined as follows: for each $u \in C_0^\infty(\Omega, \mathbb{C})$,

$$\left\langle (-\Delta)_A^{s(\cdot)}u, v \right\rangle = \Re\int_{\mathbb{R}^{2N}} \frac{\left(u(x) - e^{i(x-y)\cdot A((x+y)/2)}u(y)\right)\overline{\left(v(x)e^{i(xy)\cdot A((x+y)/2)}v(y)\right)}}{|x-y|^{N+2s(x,y)}}dxdy, \tag{13}$$

along any $v \in C_0^\infty(\Omega, \mathbb{C})$.

2. Preliminaries and Notations

In this section, we first give the definition of the variable exponential Lebesgue space. Secondly, we define variable-order fractional magnetic Sobolev spaces and prove the compact conditions between them. Finally, we give the variational setting for problem (1) and theorems that will be used later.

In this paper, we use $|\Omega|$ to represent n-dimensional Lebesgue measure of a measurable set $\Omega \subset \mathbb{R}^N$. In addition, for each $a \in \mathbb{C}$, we will use $\Re a$ to represent the real part of a and $\bar{a}$ to represent the complex conjugate of a. Let $N \geq 1$ and $\Omega \subset \mathbb{R}^N$ be a nonempty set. A measurable function $r : \bar{\Omega} \longrightarrow [1, \infty)$ is called a variable exponent, and we define $r^+ = \text{esssup}_{x\in\Omega} r(x)$, $r^- = \text{essinf}_{x\in\Omega} r(x)$. If r^+ is finite, then the exponent r is said to be bounded. The variable exponent Lebesgue space is

$$L^{r(x)}(\Omega, \mathbb{C}) = \left\{ u : \Omega \longrightarrow \mathbb{C} \text{ is a measurable function ; } \int_\Omega |u(x)|^{r(x)}dx < \infty \right\} \tag{14}$$

with the Luxemburg norm

$$\|u\|_{L^{r(x)}(\Omega,\mathbb{C})} = \inf\left\{ \mu > 0 : \int_\Omega \left(\frac{|u(x)|}{\mu}\right)^{r(x)} dx \leq 1 \right\}, \tag{15}$$

then $L^{r(x)}(\Omega, \mathbb{C})$ is a Banach space, and when r is bounded, we have the following relations

$$\begin{aligned} &\min\left\{ \|u\|_{L^{r(x)}(\Omega,\mathbb{C})}^{r^-}, \|u\|_{L^{r(x)}(\Omega,\mathbb{C})}^{r^+} \right\} \\ &\quad \leq \int_\Omega |u(x)|^{r(x)}dx \leq \max\left\{ \|u\|_{L^{r(x)}(\Omega,\mathbb{C})}^{r^-}, \|u\|_{L^{r(x)}(\Omega,\mathbb{C})}^{r^+} \right\}. \end{aligned} \tag{16}$$

For bounded exponent, the dual space $(L^{r(x)}(\Omega, \mathbb{C}))'$ can be identified with $L^{r'(x)}(\Omega, \mathbb{C})$, where the conjugate exponent r' is defined by $r' = r/(r-1)$. If $1 < r^- \leq r^+ < \infty$, then the variable exponent Lebesgue space $L^{r(x)}(\Omega, \mathbb{C})$ is a separable and reflexive. In particular,

$$L^2(\Omega, \mathbb{C}) = \left\{ u : \Omega \longrightarrow \mathbb{C} \text{ is a measurable function ; } \int_\Omega |u(x)|^2 dx < \infty \right\} \tag{17}$$

with the scalar product

$$\langle u, v \rangle_{L^2(\Omega,\mathbb{C})} = \Re\int_\Omega u\bar{v}dx. \tag{18}$$

By Lemma 3.2.20 of [10] and $\|\cdot\|_{L^{r(x)}(\Omega,\mathbb{C})} = \||\cdot|\|_{L^{r(x)}(\Omega,\mathbb{R})}$, we know that in the variable exponent Lebesgue space, Hölder inequality is still valid. For all $u \in L^{r(x)}(\Omega, \mathbb{C})$, $v \in L^{r'(x)}(\Omega, \mathbb{C})$ with $r(x) \in (1, \infty)$, the following inequality holds:

$$\int_\Omega |u||v|dx \le \left(\frac{1}{r^-} + \frac{1}{\left(r'\right)^-}\right)\|u\|_{L^{r(x)}(\Omega,\mathbb{C})}\|v\|_{L^{r'(x)}(\Omega,\mathbb{C})} \le 2\|u\|_{L^{r(x)}(\Omega,\mathbb{C})}\|v\|_{L^{r'(x)}(\Omega,\mathbb{C})}. \tag{19}$$

Let Ω be a nonempty open subset of $\mathbb{R}^N$, and let $s(\cdot)$: $\mathbb{R}^N \times \mathbb{R}^N \longrightarrow (0, 1)$ be a measurable function, and there exist two constants $0 < s_0 < s_1 < 1$ such that $s_0 < s(x, y) < s_1$ for all $(x, y) \in \mathbb{R}^N \times \mathbb{R}^N$. Set

$$H^{s(\cdot)}(\Omega, \mathbb{C}) = \left\{u \in L^2(\Omega, \mathbb{C}): \left(\int_\Omega\int_\Omega \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s(x,y)}} dxdy\right)^{1/2} < \infty\right\}. \tag{20}$$

Equip $H^{s(\cdot)}(\Omega, \mathbb{C})$ with the norm

$$\|u\|^2_{H^{s(\cdot)}(\Omega,\mathbb{C})} = \|u\|^2_{L^2(\Omega,\mathbb{C})} + \int_\Omega\int_\Omega \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s(x,y)}} dxdy. \tag{21}$$

Especially, if $s(\cdot) \equiv$ constant, then the space $H^{s(\cdot)}(\Omega, \mathbb{C})$ is the usual fractional Sobolev space $H^s(\Omega, \mathbb{C})$.

Lemma 8. *Let Ω be a smooth bounded subset of $\mathbb{R}^N$ and let $s(\cdot)$: $\mathbb{R}^N \times \mathbb{R}^N \longrightarrow (0, 1)$ satisfying (S_1) and $r : \bar{\Omega} \longrightarrow (1, \infty)$ satisfying $1 \le r \le 2N/(N - 2s(x, x))$. Then, there exists $\tilde{C}_r = C(N, r^+, s^+, s^-) > 0$ such that $\|u\|_{L^{r(x)}(\Omega,\mathbb{C})} \le \tilde{C}_r\|u\|_{H^{s(\cdot)}(\Omega,\mathbb{C})}$, for any $u \in H^{s(\cdot)}(\Omega, \mathbb{C})$. That is, the embedding $H^{s(\cdot)}(\Omega, \mathbb{C})^\circ L^{r(x)}(\Omega, \mathbb{C})$ is continuous. Moreover, $H^{s(\cdot)}(\Omega, \mathbb{C})^\circ L^{r(x)}(\Omega, \mathbb{C})$ is compact.*

Proof. By Theorem 2.1 of [2], we know that $H^{s(\cdot)}(\Omega, \mathbb{R})^\circ L^{r(x)}(\Omega, \mathbb{R})$ is continuous and compact, there exists $\tilde{C}_r = C(N, r^+, s^+, s^-) > 0$ such that $\|u\|_{L^{r(x)}(\Omega,\mathbb{R})} \le \tilde{C}_r\|u\|_{H^{s(\cdot)}(\Omega,\mathbb{R})}$. then, for any $u \in H^{s(\cdot)}(\Omega, \mathbb{C})$, we have

$$\begin{aligned}\|u\|_{L^{r(x)}(\Omega,\mathbb{C})} &= \||u|\|_{L^{r(x)}(\Omega,\mathbb{R})} \le \tilde{C}_r\||u|\|_{H^{s(\cdot)}(\Omega,\mathbb{R})} \\ &= \tilde{C}_r\left(\||u|\|^2_{L^2(\Omega,\mathbb{R})} + \int_\Omega\int_\Omega \frac{||u(x)| - |u(y)||^2}{|x - y|^{N+2s(x,y)}} dxdy\right)^{1/2} \\ &\le \tilde{C}_r\left(\|u\|^2_{L^2(\Omega,\mathbb{C})} + \int_\Omega\int_\Omega \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s(x,y)}} dxdy\right)^{1/2} \\ &= \tilde{C}_r\|u\|_{H^{s(\cdot)}(\Omega,\mathbb{C})}.\end{aligned} \tag{22}$$

Hence, the embedding $H^{s(\cdot)}(\Omega, \mathbb{C})^\circ L^{r(x)}(\Omega, \mathbb{C})$ is continuous and compact.

Let $A : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ be a continuous function and $A \in L^\infty_{loc}(\mathbb{R}^N, \mathbb{R}^N)$. For a function $u : \mathbb{R}^N \longrightarrow \mathbb{C}$, define

$$[u]^2_{H^{s(\cdot)}_A(\mathbb{R}^N,\mathbb{C})} := \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\left|u(x) - e^{i(x-y)\cdot A((x+y)/2)}u(y)\right|^2}{|x - y|^{N+2s(x,y)}} dxdy \tag{23}$$

and the corresponding norm denoted by $\|u\|^2_{H^{s(\cdot)}_A(\mathbb{R}^N,\mathbb{C})} = \|u\|^2_{L^2(\mathbb{R}^N,\mathbb{C})} + [u]^2_{H^{s(\cdot)}_A(\mathbb{R}^N,\mathbb{C})}$. We consider the space H of measurable functions $u : \mathbb{R}^N \longrightarrow \mathbb{C}$ such that $\|u\|_{H^{s(\cdot)}_A(\mathbb{R}^N,\mathbb{C})} < \infty$; then, $(H, \langle\cdot, \cdot\rangle_{H^{s(\cdot)}_A(\mathbb{R}^N,\mathbb{C})})$ is a Hilbert space. Define $H^{s(\cdot)}_A(\mathbb{R}^N, \mathbb{C})$ as the closure of $C^\infty_c(\mathbb{R}^N, \mathbb{C})$ in H; then, $H^{s(\cdot)}_A(\mathbb{R}^N, \mathbb{C})$ is a Hilbert space. Especially, if $A = 0$, then the space $H^{s(\cdot)}_A(\mathbb{R}^N, \mathbb{C})$ is the variable-order fractional Sobolev space $H^{s(\cdot)}(\mathbb{R}^N, \mathbb{C})$; if $A = 0$ and $s(\cdot) \equiv$ constant, then the space $H^{s(\cdot)}_A(\mathbb{R}^N, \mathbb{C})$ is the usual fractional Sobolev space $H^s(\mathbb{R}^N, \mathbb{C})$.

In order to define weak solutions of problem (1), we introduce the functional space

$$H^{s(\cdot)}_{0,A}(\Omega, \mathbb{C}) = \left\{u \in H^{s(\cdot)}_A\left(\mathbb{R}^N, \mathbb{C}\right): u = 0 \ a.e. \text{in } \mathbb{R}^N \setminus \Omega\right\}, \tag{24}$$

equipping $H^{s(\cdot)}_{0,A}(\Omega, \mathbb{C})$ with the scalar product

$$\langle u, v\rangle_{H^{s(\cdot)}_{0,A}(\Omega,\mathbb{C})} := \Re\int_{\mathbb{R}^{2N}} \frac{\left(u(x) - e^{i(x-y)\cdot A((x+y)/2)}u(y)\right)\overline{\left(v(x)e^{i(xy)\cdot A((x+y)/2)}v(y)\right)}}{|x - y|^{N+2s(x,y)}} dxdy, \tag{25}$$

which induces the following norm $\|u\|_{H^{s(\cdot)}_{0,A}(\Omega,\mathbb{C})} := \langle u, u\rangle^{1/2}_{H^{s(\cdot)}_{0,A}(\Omega,\mathbb{C})}$. Hence, $H^{s(\cdot)}_{0,A}(\Omega, \mathbb{C})$ generalizes to the variable-order fractional Sobolev space (see [2]) and the magnetic framework the space introduced in [9]. Next, we state and prove some properties of space $H^{s(\cdot)}_{0,A}(\Omega, \mathbb{C})$, which will be useful in the sequel.

Lemma 6. *There exists a constant $C_2 > 0$, depending only on N, s_1 and Ω, such that*

$$\|u\|_{H^{s(\cdot)}_{0,A}(\Omega,\mathbb{C})} \le \|u\|_{H^{s(\cdot)}_A\left(\mathbb{R}^N,\mathbb{C}\right)} \le C_2\|u\|_{H^{s(\cdot)}_{0,A}(\Omega,\mathbb{C})}, \tag{26}$$

for any $u \in H^{s(\cdot)}_{0,A}(\Omega, \mathbb{C})$. Thus, $\|u\|_{H^{s(\cdot)}_{0,A}(\Omega,\mathbb{C})}$ is an equivalent norm of $H^{s(\cdot)}_A(\mathbb{R}^N, \mathbb{C})$.

Proof. For any $u \in H^{s(\cdot)}_{0,A}(\Omega, \mathbb{C})$, by Lemma 3.1 in [9], we have the pointwise diamagnetic inequality

$$\left|u(x) - e^{i(x-y)\cdot A((x+y)/2)}u(y)\right| \geq ||u(x)| - |u(y)||, \quad \text{for } a.e. x, y \in \mathbb{R}^N, \tag{27}$$

from which we immediately have

$$\begin{aligned}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\left|u(x) - e^{i(x-y)\cdot A((x+y)/2)}u(y)\right|^2}{|x-y|^{N+2s(x,y)}} dxdy &\geq \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{||u(x)| - |u(y)||^2}{|x-y|^{N+2s(x,y)}} dxdy \\ &\geq \int_{\mathbb{R}^N\backslash\Omega}\left(\int_\Omega \frac{||u(x)| - |u(y)||^2}{|x-y|^{N+2s(x,y)}} dy\right)dx \\ &= \int_{\mathbb{R}^N\Omega}\left(\int_\Omega \frac{|u(y)|^2}{|x-y|^{N+2s(x,y)}} dy\right)dx.\end{aligned} \tag{28}$$

Since Ω is bounded, there exists $r > 1/2$ such that $\Omega \subset B_r$ and $|B_r \setminus \Omega| > 0$; then, we have

$$\begin{aligned}\int_{\mathbb{R}^N\backslash\Omega}\left(\int_\Omega \frac{|u(y)|^2}{|x-y|^{N+2s(x,y)}} dy\right)dx &\geq \int_{B_r\Omega}\left(\int_\Omega \frac{|u(y)|^2}{|2r|^{N+2s(x,y)}} dy\right)dx \\ &\geq \int_{B_r\backslash\Omega}\left(\int_\Omega \frac{|u(y)|^2}{|2r|^{N+2s_1}} dy\right)dx \\ &= \frac{1}{(2r)^{N+2s_1}}\int_{B_r\backslash\Omega}\left(\int_\Omega |u(y)|^2 dy\right)dx \\ &= \frac{|B_r \setminus \Omega|}{(2r)^{N+2s_1}}\|u\|^2_{L^2(\Omega,\mathbb{C})}.\end{aligned} \tag{29}$$

Thus, we obtain

$$\begin{aligned}\|u\|^2_{L^2(\Omega,\mathbb{C})} &\leq \frac{(2r)^{N+2s_1}}{|B_r \setminus \Omega|}\int_{\mathbb{R}^{2N}} \frac{\left|u(x) - e^{i(x-y)\cdot A((x+y)/2)}u(y)\right|^2}{|x-y|^{N+2s(x,y)}} dxdy \\ &= C_1\|u\|^2_{H^{s(\cdot)}_{0,A}(\Omega,\mathbb{C})}, \\ \|u\|^2_{H^{s(\cdot)}_A(\mathbb{R}^N,\mathbb{C})} &= \|u\|^2_{L^2(\Omega,\mathbb{C})} + [u]^2_{H^{s(\cdot)}_A(\mathbb{R}^N,\mathbb{C})} \\ &\leq C_1\|u\|^2_{H^{s(\cdot)}_{0,A}(\Omega,\mathbb{C})} + \|u\|^2_{H^{s(\cdot)}_{0,A}(\Omega,\mathbb{C})} \\ &= C_2^2\|u\|^2_{H^{s(\cdot)}_{0,A}(\Omega,\mathbb{C})},\end{aligned} \tag{30}$$

where $C_1 = (2r)^{N+2s_1}/|B_r \setminus \Omega|$ and $C_2^2 = C_1 + 1$.

In addition,

$$\begin{aligned}\|u\|^2_{H^{s(\cdot)}_{0,A}(\Omega,\mathbb{C})} &\leq \int_\Omega |u(x)|^2 dx + \int_{\mathbb{R}^{2N}} \cdot \frac{\left|u(x) - e^{i(x-y)\cdot A((x+y)/2)}u(y)\right|^2}{|x-y|^{N+2s(x,y)}} dxdy \\ &= \|u\|^2_{H^{s(\cdot)}_A(\mathbb{R}^N,\mathbb{C})}.\end{aligned} \tag{31}$$

Combining the above two aspects, we have

$$\|u\|_{H^{s(\cdot)}_{0,A}(\Omega,\mathbb{C})} \leq \|u\|_{H^{s(\cdot)}_A(\mathbb{R}^N,\mathbb{C})} \leq C_2\|u\|_{H^{s(\cdot)}_{0,A}(\Omega,\mathbb{C})}, \tag{32}$$

which implies that $\|u\|_{H^{s(\cdot)}_{0,A}(\Omega,\mathbb{C})}$ is the equivalent norm of a norm $\|u\|_{H^{s(\cdot)}_A(\mathbb{R}^N,\mathbb{C})}$.

Lemma 2.3. *Let Ω be a bounded subset of $\mathbb{R}^N$. Assume that $s(\cdot)$: $\mathbb{R}^N \times \mathbb{R}^N \longrightarrow (0, 1)$ is a continuous function satisfying (S_1) and $r : \bar{\Omega} \longrightarrow [1, \infty)$ is a continuous function satisfying $1 \leq r \leq 2N/(N - 2s(x, x))$. If $u \in H^{s(\cdot)}_{0,A}(\Omega, \mathbb{C})$, then*

$$H^{s(\cdot)}_{0,A}(\Omega, \mathbb{C})^\circ H^{s(\cdot)}(\Omega, \mathbb{C}) \tag{33}$$

is continuous and

$$H^{s(\cdot)}_{0,A}(\Omega, \mathbb{C})^\circ L^{r(x)}(\Omega, \mathbb{C}) \tag{34}$$

is compact, that is, there exists $C_r = C(N, r^+, s^+, s^-) > 0$ such that

$$\|u\|_{L^{r(x)}(\Omega,\mathbb{C})} \leq C_r\|u\|_{H^{s(\cdot)}_{0,A}(\Omega,\mathbb{C})}. \tag{35}$$

Proof. For any $u \in H^{s(\cdot)}_{0,A}(\Omega, \mathbb{C})$, we have

$$\begin{aligned}\|u\|^2_{H^{s(\cdot)}(\Omega,\mathbb{C})} &= \int_\Omega |u(x)|^2 dx + \int_\Omega\int_\Omega \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s(x,y)}} dxdy \\ &\leq \int_\Omega |u(x)|^2 dx \\ &\quad + 2\int_\Omega\int_\Omega \frac{\left|u(x) - e^{i(x-y)\cdot A((x+y)/2)}u(y)\right|^2}{|x-y|^{N+2s(x,y)}} dxdy \\ &\quad + 2\int_\Omega\int_\Omega \frac{|u(y)|^2\left|e^{i(x-y)\cdot A((x+y)/2)} - 1\right|^2}{|x-y|^{N+2s(x,y)}} dxdy \\ &= 2\|u\|^2_{H^{s(\cdot)}_A(\Omega,\mathbb{C})} + 2J,\end{aligned} \tag{36}$$

where

$$
\begin{aligned}
J &= \int_\Omega\int_\Omega \frac{|u(y)|^2\left|e^{i(x-y)\cdot A((x+y)/2)}-1\right|^2}{|x-y|^{N+2s(x,y)}}dxdy \\
&= \int_\Omega |u(y)|^2\left(\int_{\Omega\cap\{|x-y|>1\}} \frac{\left|e^{i(x-y)\cdot A((x+y)/2)}-1\right|^2}{|x-y|^{N+2s(x,y)}}dx\right)dy \\
&\quad+ \int_\Omega |u(y)|^2\left(\int_{\Omega\cap\{|x-y|\le 1\}} \frac{\left|e^{i(x-y)\cdot A((x+y)/2)}-1\right|^2}{|x-y|^{N+2s(x,y)}}dx\right)dy \\
&= J_1 + J_2.
\end{aligned} \tag{37}
$$

Since $|e^{it}-1|\le 2$, we have

$$
\begin{aligned}
J_1 &\le 4\int_\Omega |u(y)|^2\left(\int_{\Omega\cap\{|x-y|>1\}} \frac{1}{|x-y|^{N+2s(x,y)}}dx\right)dy \\
&\le 4\int_\Omega |u(y)|^2\left(\int_{\Omega\cap\{|x-y|>1\}} \frac{1}{|x-y|^{N+2s_0}}dx\right)dy \\
&= 4\int_\Omega |u(y)|^2\left(\int_{\Omega\cap\{|z|>1\}} \frac{1}{|z|^{N+2s_0}}dz\right)dy \\
&\le C_3\int_\Omega |u(y)|^2dy = C_3\|u\|^2_{L^2(\Omega,\mathbb{C})}.
\end{aligned} \tag{38}
$$

In view of Ω which is bounded, there exists a compact set $K\subset\mathbb{R}^N$ such that $\Omega\subset K$. By Lemma 2.2 of [11], we know that A is locally bounded and $K\subset\mathbb{R}^N$is compact, $\left|e^{i(x-y)\cdot A((x+y)/2)}-1\right|^2\le C_4|x-y|^2$, for $|x-y|\le 1$, $x,y\in K$.

Thus, we obtain

$$
\begin{aligned}
J_2 &\le \int_K |u(y)|^2\left(\int_{K\cap\{|x-y|\le 1\}} \frac{\left|e^{i(x-y)\cdot A((x+y)/2)}-1\right|^2}{|x-y|^{N+2s(x,y)}}dx\right)dy \\
&\le \int_K |u(y)|^2\left(\int_{K\cap\{|x-y|\le 1\}} \frac{C_4}{|x-y|^{N+2s(x,y)-2}}dx\right)dy \\
&\le \int_K |u(y)|^2\left(\int_{K\cap\{|x-y|\le 1\}} \frac{C_4}{|x-y|^{N+2s_1-2}}dx\right)dy \\
&\le C_4\int_K |u(y)|^2\left(\int_{K\cap\{|z|\le 1\}} \frac{1}{|z|^{N+2s_1-2}}dz\right)dy \\
&\le C_5\int_K |u(y)|^2dy = C_5\int_{K\setminus\Omega}|u(y)|^2dy + C_5\int_\Omega |u(y)|^2dy \\
&= C_5\int_\Omega |u(y)|^2dy = C_5\|u\|^2_{L^2(\Omega,\mathbb{C})}.
\end{aligned} \tag{39}
$$

Equations (36)–(39) together with Lemma 6, we have

$$
\begin{aligned}
\|u\|^2_{H^{s(\cdot)}(\Omega,\mathbb{C})} &\le 2\|u\|^2_{H_A^{s(\cdot)}(\Omega,\mathbb{C})} + 2C_3\|u\|^2_{L^2(\Omega,\mathbb{C})} \\
&\quad + 2C_5\|u\|^2_{L^2(\Omega,\mathbb{C})} \le C_6\|u\|^2_{H_A^{s(\cdot)}(\Omega,\mathbb{C})} \\
&\le C_6\|u\|^2_{H_A^{s(\cdot)}(\mathbb{R}^N,\mathbb{C})} \le C_7\|u\|^2_{H_{0,A}^{s(\cdot)}(\Omega,\mathbb{C})},
\end{aligned} \tag{40}
$$

which implies that the embedding $H_{0,A}^{s(\cdot)}(\Omega,\mathbb{C})\hookrightarrow H^{s(\cdot)}(\Omega,\mathbb{C})$ is continuous. In addition, by Lemma 5, we know that $H^{s(\cdot)}(\Omega,\mathbb{C})\hookrightarrow L^{r(x)}(\Omega,\mathbb{C})$ is compact. Therefore, the embedding $H_{0,A}^{s(\cdot)}(\Omega,\mathbb{C})\hookrightarrow L^{r(x)}(\Omega,\mathbb{C})$ is compact.

Next, we give the variational setting for problem (1). For $\lambda>0$, we need the following scalar product and norm:

$$
\langle u,v\rangle_\lambda := \Re\int_{\mathbb{R}^{2N}} \frac{\left(u(x)-e^{i(x-y)\cdot A((x+y)/2)}u(y)\right)\left(\overline{v(x)e^{i(xy)\cdot A((x+y)/2)}v(y)}\right)}{|x-y|^{N+2s(x,y)}}dxdy + \lambda\cdot\Re\int_\Omega V^+u\bar{v}dx,\ \|u\|_\lambda := \langle u,u\rangle_\lambda^{1/2}. \tag{41}
$$

Let $E=\{u\in H_{0,A}^{s(\cdot)}(\Omega,\mathbb{C}): \int_\Omega V^+u^2dx<\infty\}$ be equipped with the norm $\|u\|_E=\|u\|_1$ (that is, $\lambda=1$ in $\|u\|_\lambda$). Obviously, $\|u\|_E\le\|u\|_\lambda$ for $\lambda\ge 1$. Set $E_\lambda=(E,\|\cdot\|_\lambda)$. Moreover, for $r(x)\in(1,2N/(N-2s(x,x)))$, we can get

$$
\begin{aligned}
\int_\Omega |u(x)|^{r(x)}dx &\le \max\left\{\|u\|^{r^+}_{L^{r(\cdot)}(\Omega,\mathbb{C})}, \|u\|^{r^-}_{L^{r(\cdot)}(\Omega,\mathbb{C})}\right\} \\
&\le \max\left\{C_r^{r^+}\|u\|^{r^+}_{H_{0,A}^{s(\cdot)}(\Omega,\mathbb{C})}, C_r^{r^-}\|u\|^{r^-}_{H_{0,A}^{s(\cdot)}(\Omega,\mathbb{C})}\right\} \\
&\le \max\left\{C_r^{r^+}\|u\|^{r^+}_\lambda, C_r^{r^-}\|u\|^{r^-}_\lambda\right\}.
\end{aligned} \tag{42}
$$

For simplicity, we let $\|u\|^2_{\lambda,V} := \|u\|^2_{H_{0,A}^{s(\cdot)}(\Omega,\mathbb{C})} + \int_\Omega V_\lambda u^2dx$. Therefore, by condition (V_4),

$$
\|u\|^2_\lambda \ge \|u\|^2_{\lambda,V} \ge \frac{\vartheta_0-1}{\vartheta_0}\|u\|^2_\lambda, \quad \text{for all } \lambda\ge 0. \tag{43}
$$

Associated with problem (1), we consider the energy functional $\Psi_\lambda: E_\lambda\longrightarrow\mathbb{R}$,

$$
\begin{aligned}
\Psi_\lambda(u) &= \frac{1}{2}\|u\|^2_\lambda - \frac{1}{2}\int_\Omega V^-u^2dx \\
&\quad - \int_\Omega\left(\frac{f(x)}{p(x)}|u|^{p(x)} + \frac{g(x)}{q(x)}|u|^{q(x)}\right)dx \\
&= \frac{1}{2}\|u\|^2_{\lambda,V} - \int_\Omega\left(\frac{f(x)}{p(x)}|u|^{p(x)} + \frac{g(x)}{q(x)}|u|^{q(x)}\right)dx.
\end{aligned} \tag{44}
$$

In fact, one can verify that Ψ_λ is well-defined of class C^1 in E_λ and

$$\Re\int_{\mathbb{R}^{2N}}\frac{\left(u(x)-e^{i(x-y)\cdot A((x+y)/2)}u(y)\right)\overline{\left(v(x)e^{i(xy)\cdot A((x+y)/2)}v(y)\right)}}{|x-y|^{N+2s(x,y)}}dxdy+\lambda\Re\int_{\Omega}V^{+}u\bar{v}dx \\ -\Re\int_{\Omega}V^{-}u\bar{v}dx-\Re\int_{\Omega}\left(f(x)|u|^{p(x)-2}u+g(x)|u|^{q(x)-2}u\right)\bar{v}dx=0, \tag{45}$$

for all $u, v \in E_\lambda$. Therefore, if $u \in E_\lambda$ is a critical point of Ψ_λ, then u is a solution of problem (1).

Now we give the theorems that we need later.

Theorem 8 (see [2, 12]). *Let X be a real infinite dimensional Banach space and $I \in C'(X)$ a functional satisfying the $(PS)_c$ condition as well as the following three properties:*

(1) $I(0) = 0$, and there exists two constants $\rho, \delta > 0$ such that $I(u) \geq \delta$ for all $u \in X$ with $\|u\| = \rho$

(2) I is even

(3) For all finite dimensional subspaces $Y \subset X$, there exists $R = R(Y) > 0$ such that $I(u) \leq 0$ for all $u \in X \setminus B_R(Y)$, where $B_R(Y) = \{u \in Y : \|u\| \leq R\}$. Then, I poses an unbounded sequence of critical values characterized by a minimax argument

Theorem 9 (see [13]). *Let X be a real Banach space and $I \in C^1(X, \mathbb{R})$. Suppose I satisfies the (PS) condition, which is even and bounded from below, and $I(0) = 0$. If for any $k \in N$, there exists a k-dimensional subspace X^k of X and $\rho_k > 0$ such that $\sup_{X^k\cap S_{\rho_k}} I < 0$, where $S_{\rho_k} = \{u \in X : \|u\|_X = \rho_k\}$, then at least one of the following conclusions holds:*

(i) There exists a sequence of critical points $\{u_k\}$ satisfying $I(u_k) < 0$ for all k and $\|u_k\|_X \longrightarrow 0$ as $k \longrightarrow \infty$

(ii) There exists $r > 0$ such that for any $0 < a < r$, there exists a critical point u such that $\|u\|_X = a$ and $I(u) = 0$

It is easy to verify that E_λ is a separable Hilbert space. Let $X_i = span\{e_i\}$, then $E_\lambda = \oplus_{i\geq 1} X_i$. we define

$$Y_k = \oplus_{i=1}^{k} X_i, Z_k = \overline{\oplus_{i=k+1}^{\infty} X_i}. \tag{46}$$

Theorem 10 (see [14], fountain theorem). *Suppose that $I \in C^1(E, \mathbb{R})$ satisfying $I(-u) = I(u)$. Assume that for every $k \in N$, there exist $r_k > \gamma_k > 0$ such that*

(D_1): $a_k = \max\{I(u): u \in Y_k, \|u\| = r_k\} \leq 0$.

(D_2): $b_k = \inf\{I(u): u \in Z_k, \|u\| = \gamma_k\} \longrightarrow \infty$ as $k \longrightarrow \infty$.

(D_3): I satisfies $(PS)_c$ condition for every $c > 0$.

Then, I has an unbounded sequence of critical values which have the form

$$c_k = \inf_{r\in\Gamma_k}\max_{u\in B_k} I(\eta(u)), \tag{47}$$

where $\Gamma_k = \{\eta \in C(B_k, E)$: η is equivariant and $\eta|_{\partial B_k} = id\}$.

Theorem 11 (see [15], dual fountain theorem). *Suppose that $I \in C^1(E, \mathbb{R})$ satisfying $I(-u) = I(u)$. Assume that for every $k \geq k_0$, there exist $r_k > \gamma_k > 0$ such that*

(D_4): $a_k = \inf\{I(u): u \in Z_k, \|u\| = r_k\} \geq 0$.

(D_5): $b_k = \max\{I(u): u \in Y_k, \|u\| = \gamma_k\} < 0$.

(D_6): $d_k = \max\{I(u): u \in Z_k, \|u\| \leq \gamma_k\} \longrightarrow 0$ as $k \longrightarrow \infty$.

(D_7): I satisfies $(PS)_c^*$ condition for every $c \in [d_{k_0}, 0]$.

Then, I has a sequence of negative critical values converging to 0.

3. Proof of Theorem 1

In this part, we first recall that definitions of functional Ψ_λ satisfies the $(\mathrm{PS})_c$ condition and $(\mathrm{PS})_c^*$ condition in E_λ at the level $c \in \mathbb{R}$ and use the usual mountain pass theorem (see [2]) to find a $(\mathrm{PS})_c$ sequence in E_λ. Second, we show that functional Ψ_λ satisfies the $(\mathrm{PS})_c^*$ condition in E_λ at the level $c < c_0$. Finally, we give the proof of problem (1).

Definition 12 (see [2]). Let $I \in C^1(E, \mathbb{R})$ and $c \in \mathbb{R}$. The functional I satisfies the $(\mathrm{PS})_c$ condition if any sequence $\{u_n\} \subset E$ such that $I(u_n) \longrightarrow c$ and $I'(u_n) \longrightarrow 0$ as $n \longrightarrow \infty$ admits a strongly convergent subsequence in E.

Definition 13 (see [16]). Let $I \in C^1(E, \mathbb{R})$ and $c \in \mathbb{R}$. The functional I satisfies the $(\mathrm{PS})_c^*$ condition (with respect to Y_n) if any sequence $\{u_n\} \subset E$ such that $\{u_n\} \in Y_n$, $I(u_n) \longrightarrow c$ and $I'|_{Y_n}(u_n) \longrightarrow 0$ as $n \longrightarrow \infty$ admits a strongly convergent subsequence in E.

Remark 14. From Remark 2.1 in [16], we get that the $(\mathrm{PS})_c^*$ condition means the $(\mathrm{PS})_c$ condition.

Theorem 15 (Theorem 3.1, [2]). *Let E be a real Banach space and $I \in C^1(E, \mathbb{R})$ with $I(0) = 0$. Suppose that*

(i) there exist $\delta > 0$ and $\rho > 0$ such that $I(u) \geq \delta$ for each $u \in E$ subject to $\|u\|_E = \rho$

(ii) there exists $e \in E$ with $\|e\|_E > \rho$ such that $I(e) < 0$

Define $\Gamma = \{\gamma \in C^1([0,1], E): \gamma(0) = 1, \gamma(1) = e\}$. Then,

$$c = \inf_{\gamma \in \Gamma} \max_{0 \le t \le 1} I(\gamma(t)) \ge \delta, \tag{48}$$

and there exists a $(PS)_c$ sequence $\{u_n\}_n \subset E$.

In order to obtain our main results by using the mountain pass theorem, we first prove that Ψ_λ satisfies the mountain pass geometry (i) and (ii).

Lemma 16. *Assume that (S_1), (V_1)–(V_4), and (H_1)–(H_3) hold. Then, for each $\lambda > 0$, there exists $\rho > 0$ and $\tau > 0$ such that*

$$\Psi_\lambda(u) > \tau \quad \text{for all } u \in E_\lambda \text{ with} \|u\|_\lambda = \rho. \tag{49}$$

Proof. In view of (42) and the fractional Sobolev inequality, for each $u \in E_\lambda$, one has

$$\begin{aligned} &\int_\Omega \left(\frac{f(x)}{p(x)} |u|^{p(x)} + \frac{g(x)}{q(x)} |u|^{q(x)} \right) dx \\ &\le \frac{\|f\|_\infty}{p^-} \int_\Omega |u|^{p(x)} dx + \frac{\|g\|_\infty}{q^-} \int_\Omega |u|^{q(x)} dx \\ &\le \frac{\|f\|_\infty}{p^-} \max\left\{ C_p^{p^+} \|u\|_\lambda^{p^+}, C_p^{p^-} \|u\|_\lambda^{p^-} \right\} \\ &\quad + \frac{\|g\|_\infty}{q^-} \max\left\{ C_q^{q^+} \|u\|_\lambda^{q^+}, C_q^{q^-} \|u\|_\lambda^{q^-} \right\}, \end{aligned} \tag{50}$$

where C_p, C_q are two constants of embedding from variable-order fractional Sobolev space $H_{0,A}^{s(\cdot)}(\Omega, \mathbb{C})$ to $L^{p(x)}(\Omega, \mathbb{C})$ and $L^{q(x)}(\Omega, \mathbb{C})$, respectively. Making use of (43) and (50), we obtain that

$$\begin{aligned} \Psi_\lambda(u) &= \frac{1}{2} \|u\|_\lambda^2 - \frac{1}{2} \int_\Omega V^- u^2 dx \\ &\quad - \int_\Omega \left(\frac{f(x)}{p(x)} |u|^{p(x)} + \frac{g(x)}{q(x)} |u|^{q(x)} \right) dx \\ &= \frac{1}{2} \|u\|_{\lambda,V}^2 - \int_\Omega \left(\frac{f(x)}{p(x)} |u|^{p(x)} + \frac{g(x)}{q(x)} |u|^{q(x)} \right) dx \\ &\ge \frac{1}{2} \frac{\vartheta_0 - 1}{\vartheta_0} \|u\|_\lambda^2 - \|f\|_\infty \frac{\max\left\{ C_p^{p^+}, C_p^{p^-} \right\}}{p^-} \|u\|_\lambda^{p^+} \\ &\quad - \|g\|_\infty \frac{\max\left\{ C_q^{q^+}, C_q^{q^-} \right\}}{q^-} \|u\|_\lambda^{q^+} \\ &= \frac{\vartheta_0 - 1}{2\vartheta_0} \|u\|_\lambda^2 - A_1 \|u\|_\lambda^{p^+} - A_2 \|u\|_\lambda^{q^+} \\ &= \|u\|_\lambda^{q^+} \left(\frac{\vartheta_0 - 1}{2\vartheta_0} \|u\|_\lambda^{2-q^+} - A_1 \|u\|_\lambda^{p^+ - q^+} - A_2 \right), \end{aligned} \tag{51}$$

for each $u \in E_\lambda$ with $\|u\|_\lambda \ge 1$.

Set

$$\begin{aligned} A_1 &= \frac{\|f\|_\infty \max\left\{ C_p^{p^+}, C_p^{p^-} \right\}}{p^-}, \\ A_2 &= \frac{\|g\|_\infty \max\left\{ C_q^{q^+}, C_q^{q^-} \right\}}{q^-}. \end{aligned} \tag{52}$$

Define $\Phi_1(t)\colon [0, \infty) \longrightarrow \mathbb{R}$ as follows

$$\Phi_1(t) = \Phi_2(t) t^{q^+} \quad \text{for all } t \ge 0, \tag{53}$$

where

$$\Phi_2(t) = \frac{\vartheta_0 - 1}{2\vartheta_0} t^{2-q^+} - A_1 t^{p^+ - q^+} - A_2. \tag{54}$$

As long as

$$A_2 < \left(\frac{(2 - q^+)(\vartheta_0 - 1)}{2A_1 \vartheta_0 (p^+ - q^+)} \right)^{(2-q^+)/(p^+-2)} \frac{(\vartheta_0 - 1)(p^+ - 2)}{2\vartheta_0 (p^+ - q^+)}, \tag{55}$$

that is,

$$A_1^{2-q^+} A_2^{p^+-2} \le \left(\frac{(2 - q^+)(\vartheta_0 - 1)}{2\vartheta_0 (p^+ - q^+)} \right)^{2-q^+} \left(\frac{(\vartheta_0 - 1)(p^+ - 2)}{2\vartheta_0 (p^+ - q^+)} \right)^{p^+-2}, \tag{56}$$

that is,

$$\begin{aligned} \|g\|_\infty \le &\frac{q^-(p^+ - 2)(\vartheta_0 - 1)}{\max\left\{ C_q^{q^+}, C_q^{q^-} \right\} 2\vartheta_0 (p^+ - q^+)} \\ &\cdot \left[\frac{(2 - q^+)(\vartheta_0 - 1)}{2A_1 \vartheta_0 (p^+ - q^+)} \right]^{(2-q^+)/(p^+-2)}, \end{aligned} \tag{57}$$

we can easily show that for $t = \tilde{t} = [(2 - q^+)(\vartheta_0 - 1)/2A_1\vartheta_0 (p^+ - q^+)]^{1/(p^+-2)}$, we have

$$\Phi_2(\tilde{t}) = \max_{t \ge 0} \Phi_2(t) > 0. \tag{58}$$

By

$$\|f\|_\infty \le \frac{p^-(2 - q^+)(\vartheta_0 - 1)}{\max\left\{ C_p^{p^+}, C_p^{p^-} \right\} 2\vartheta_0 (p^+ - q^+)}, \tag{59}$$

it is easy to derive that

$$\tilde{t} = \left[\frac{(2 - q^+)(\vartheta_0 - 1)}{2A_1 \vartheta_0 (p^+ - q^+)} \right]^{1/(p^+-2)} \ge 1. \tag{60}$$

By letting $\rho = \tilde{t} > 0$ and $\tau = \Phi(\tilde{t}) > 0$, we can easily get $\Psi_\lambda(u) > \tau$ for all $u \in E_\lambda$ with $\|u\|_\lambda = \rho$.

Lemma 17. *Suppose that (S_1), $(V_1)-(V_4)$, and $(H_1)-(H_3)$ hold. Then, there exists $e\in E_\lambda$ with $\|e\|_\lambda>\rho$ such that $\Psi_\lambda(e)<0$ for all $\lambda>0$, where $\rho>0$ is given by Lemma 16.*

Proof. Notice that $f:\mathbb{R}^N\longrightarrow[0,\infty)$ is a bounded nonnegative measurable function such that $f>0$ on open interval $\Omega_f\subset\Omega$; then, we can select $w_0\in E_\lambda$ such that

$$\|w_0\|_\lambda=1 \text{ and} \int_\Omega f(x)|w_0(x)|^{p(x)}dx>0. \tag{61}$$

For all $t\geq 1$, combining (43) with (44), we have

$$\begin{aligned}\Psi_\lambda(tw_0)&=\frac{1}{2}\|tw_0\|_\lambda^2-\frac{1}{2}\int_\Omega V^-(tw_0)^2dx\\&\quad-\int_\Omega\left(\frac{f(x)}{p(x)}|tw_0|^{p(x)}+\frac{g(x)}{q(x)}|tw_0|^{q(x)}\right)dx\\&=\frac{1}{2}\|tw_0\|_{\lambda,V}^2-\int_\Omega\left(\frac{f(x)}{p(x)}|tw_0|^{p(x)}+\frac{g(x)}{q(x)}|tw_0|^{q(x)}\right)dx\\&\leq\frac{1}{2}\|tw_0\|_\lambda^2-\int_\Omega\left(\frac{f(x)}{p(x)}|tw_0|^{p(x)}+\frac{g(x)}{q(x)}|tw_0|^{q(x)}\right)dx\\&\leq\frac{t^2}{2}\|w_0\|_\lambda^2-\frac{t^{p^-}}{p^+}\int_\Omega f(x)|w_0|^{p(x)}dx.\end{aligned}\tag{62}$$

Since $p^->2$, then there exists $t^*\geq 1$ large enough such that $\|t^*w_0\|_\lambda>\rho$ and $\Psi_\lambda(t^*w_0)<0$. By letting $e=t^*w_0$, we can easily reach the conclusion.

Define

$$\begin{aligned}c_\lambda&=\inf_{\gamma\in\Gamma_1}\max_{0\leq t\leq 1}\Psi_\lambda(\gamma(t)),\\c(\Omega_0)&=\inf_{\gamma\in\Gamma_2}\max_{0\leq t\leq 1}\Psi_\lambda|_{H_{0,A}^{s(\cdot)}(\Omega_0,\mathbb{C})}(\gamma(t)),\end{aligned}\tag{63}$$

where $\Psi_\lambda|_{H_{0,A}^{s(\cdot)}(\Omega_0,\mathbb{C})}$ is a restriction of Ψ_λ on $H_{0,A}^{s(\cdot)}(\Omega_0,\mathbb{C})$ and

$$\begin{aligned}\Gamma_1&=\{\gamma\in C([0,1],E_\lambda):\gamma(0)=0,\gamma(1)=e\},\\\Gamma_2&=\left\{\gamma\in C\left([0,1],H_{0,A}^{s(\cdot)}(\Omega_0,\mathbb{C})\right):\gamma(0)=0,\gamma(1)=e\right\}.\end{aligned}\tag{64}$$

Observe that

$$\begin{aligned}\Psi_\lambda|_{H_{0,A}^{s(\cdot)}(\Omega_0,\mathbb{C})}(u)&=\frac{1}{2}\|u\|_{H_{0,A}^{s(\cdot)}(\Omega_0,\mathbb{C})}^2-\frac{1}{2}\int_{\Omega_0}V^-u^2dx\\&\quad-\int_{\Omega_0}\left(\frac{f(x)}{p(x)}|u|^{p(x)}+\frac{g(x)}{q(x)}|u|^{q(x)}\right)dx,\end{aligned}\tag{65}$$

for all $u\in H_{0,A}^{s(\cdot)}(\Omega_0,\mathbb{C})$. Obviously, $c(\Omega_0)$ is independent of λ. From the proofs of Lemma 16 and Lemma 17, we can easily derive that $\Psi_\lambda|_{H_{0,A}^{s(\cdot)}(\Omega_0,\mathbb{C})}$ satisfies the mountain pass geometry. Since $H_{0,A}^{s(\cdot)}(\Omega_0,\mathbb{C})\subset E_\lambda$ for all $\lambda>0$, we have $0<\tau\leq c_\lambda\leq c(\Omega_0)$ for all $\lambda>0$. Evidently, for any $t\in[0,1]$, $te\in\Gamma_2$. Consequently, there exists $c_0>0$ such that

$$c(\Omega_0)\leq\max_{0\leq t\leq 1}\Psi_\lambda(te)\leq c_0<\infty, \tag{66}$$

being $p^->2$. Then,

$$0<\tau\leq c_\lambda\leq c(\Omega_0)<c_0, \tag{67}$$

for all $\lambda>0$. In view of Lemma 16, Lemma 17, and Theorem 15, it is easy to get that for all $\lambda>0$, there exists $\{u_n\}\subset E_\lambda$ such that

$$\begin{aligned}&\Psi_\lambda(u_n)\longrightarrow c_\lambda>0,\\&\Psi_\lambda'(u_n)\longrightarrow 0,\quad \text{as } n\longrightarrow\infty.\end{aligned}\tag{68}$$

Lemma 18. *Assume that (S_1),(V_1)–(V_4) and (H_1)–(H_3) hold. Then, Ψ_λ satisfies the $(PS)_c^*$ condition in E_λ for all $c<c_0$ and $\lambda>0$.*

Proof. Assume that $\{u_n\}$ be a $(PS)_c^*$ sequence in E_λ with $c<c_0$; then, $\{u_n\}\in Y_n$, $\Psi_\lambda(u_n)\longrightarrow c_\lambda$ and $\Psi_\lambda'|_{Y_n}(u_n)\longrightarrow 0$ as $n\longrightarrow\infty$. It follows from (42) and (43) and the Hölder inequality that

$$\begin{aligned}&c_\lambda+o(1)\|u_n\|_\lambda\\&\geq\Psi_\lambda(u_n)-\frac{1}{p^-}\left\langle\Psi_\lambda'(u_n),u_n\right\rangle=\frac{1}{2}\|u_n\|_{\lambda,V}^2\\&\quad-\int_\Omega\left(\frac{f(x)}{p(x)}|u_n|^{p(x)}+\frac{g(x)}{q(x)}|u_n|^{q(x)}\right)dx-\frac{1}{p^-}\|u_n\|_{\lambda,V}^2\\&\quad+\frac{1}{p^-}\int_\Omega\left(f(x)|u_n|^{p(x)}+g(x)|u_n|^{q(x)}\right)dx\\&=\left(\frac{1}{2}-\frac{1}{p^-}\right)\|u_n\|_{\lambda,V}^2-\int_\Omega f(x)\left(\frac{1}{p(x)}-\frac{1}{p^-}\right)\\&\quad\cdot|u_n|^{p(x)}dx-\int_\Omega g(x)\left(\frac{1}{q(x)}-\frac{1}{p^-}\right)|u_n|^{q(x)}dx\\&\geq\frac{p^--2}{2p^-}\frac{\vartheta_0-1}{\vartheta_0}\|u_n\|_\lambda^2-\left(\frac{1}{q^-}-\frac{1}{p^-}\right)\int_\Omega g(x)|u_n|^{q(x)}dx\\&\geq\frac{p^--2}{2p^-}\frac{\vartheta_0-1}{\vartheta_0}\|u_n\|_\lambda^2-\left(\frac{1}{q^-}-\frac{1}{p^-}\right)\\&\quad\cdot\|g\|_\infty\max\left\{C_q^{q^+}\|u_n\|_\lambda^{q^+},C_q^{q^-}\|u_n\|_\lambda^{q^-}\right\}.\end{aligned}\tag{69}$$

On the contrary, we suppose that $\{u_n\}$ is not bounded in E_λ. Then, there exists a subsequence still denoted by

$\{u_n\}$ such that $\|u_n\|_\lambda \longrightarrow \infty$ as $n \longrightarrow \infty$. Then, it follows from (69) that

$$\begin{aligned} &\frac{c_\lambda}{\|u_n\|_\lambda^2} + o(1)\frac{1}{\|u_n\|_\lambda} \\ &\quad \geq \frac{p^- - 2}{2p^-}\frac{\vartheta_0 - 1}{\vartheta_0} - \left(\frac{1}{q^-} - \frac{1}{p^-}\right) \\ &\quad \cdot \|g\|_\infty \max\left\{C_q^{q^+}\|u_n\|_\lambda^{q^+-2}, C_q^{q^-}\|u_n\|_\lambda^{q^--2}\right\}, \end{aligned} \tag{70}$$

which leads to a contradiction. Hence, $\{u_n\}$ is bounded in E_λ for all $\lambda > 0$. Therefore, there exist a subsequence of $\{u_n\}$ still denoted by $\{u_n\}$ and u_0 in E_λ such that

$$\begin{aligned} &u_n \rightharpoonup u_0 \text{ in } E_\lambda, u_n \longrightarrow u_0 \; a.e. \text{in } \Omega, \\ &|u_n|^{r(x)-2}u_n \rightharpoonup |u_0|^{r(x)-2}u_0 \text{ in } L^{r'(x)}(\Omega, \mathbb{C}), \end{aligned} \tag{71}$$

where $r'(x) = r(x)/r(x) - 1$. The next step is to show that $u_n \longrightarrow u_0$ in E_λ. By Lemma 7, we can get $u_n \longrightarrow u_0$ in $L^{r(x)}(\Omega, \mathbb{C})$. Thus,

$$\lim_{n\to\infty}\int_\Omega |u_n - u_0|^{p(x)}dx = 0, \tag{72}$$

$$\lim_{n\to\infty}\int_\Omega |u_n - u_0|^{q(x)}dx = 0. \tag{73}$$

Making use of Hölder inequality, we can obtain

$$\begin{aligned} &\int_\Omega \left| f(x)\left(|u_n|^{p(x)-2}u_n - |u_0|^{p(x)-2}u_0\right)\overline{(u_n u_0)}\right| dx \\ &\quad \leq \|f\|_\infty \int_\Omega \left| \left(|u_n|^{p(x)-2}u_n - |u_0|^{p(x)-2}u_0\right)\overline{(u_n u_0)}\right| dx \\ &\quad = \|f\|_\infty \int_\Omega \left| |u_n|^{p(x)-2}u_n - |u_0|^{p(x)-2}u_0\right| \left|\overline{(u_n u_0)}\right| dx \\ &\quad = \|f\|_\infty \int_\Omega \left| |u_n|^{p(x)-2}u_n - |u_0|^{p(x)-2}u_0\right| |u_n - u_0| dx \\ &\quad \leq \|f\|_\infty \left(\int_\Omega \left| |u_n|^{p(x)-2}u_n - |u_0|^{p(x)-2}u_0\right|^{p(x)/(p(x)-1)} dx\right)^{(p(x)-1)/p(x)} \\ &\qquad \cdot \left(\int_\Omega |u_n - u_0|^{p(x)} dx\right)^{1/p(x)}. \end{aligned} \tag{74}$$

Combining (H_3), (71), and (72), we have

$$\lim_{n\to\infty}\int_\Omega f(x)\left(|u_n|^{p(x)-2}u_n - |u_0|^{p(x)-2}u_0\right)\overline{(u_n u_0)}dx = 0. \tag{75}$$

Similarly, we have

$$\lim_{n\to\infty}\int_\Omega g(x)\left(|u_n|^{q(x)-2}u_n - |u_0|^{q(x)-2}u_0\right)\overline{(u_n u_0)}dx = 0. \tag{76}$$

We notice that $u_n \rightharpoonup u_0$ in E_λ and $\Psi'_\lambda(u_n) \longrightarrow 0$; then, we obtain

$$\lim_{n\to\infty}\left\langle \Psi'_\lambda(u_n) - \Psi'_\lambda(u_0), u_n - u_0 \right\rangle = 0. \tag{77}$$

Therefore,

$$\begin{aligned} o(1) &= \left\langle \Psi'_\lambda(u_n) - \Psi'_\lambda(u_0), u_n - u_0 \right\rangle \\ &= \langle u_n, u_n - u_0\rangle_{\lambda,V} - \mathfrak{R}\int_\Omega \left(f(x)|u_n|^{p(x)-2}u_n \right. \\ &\quad \left. + g(x)|u_n|^{q(x)-2}u_n\right)\overline{(u_n u_0)}dx - \langle u_0, u_n - u_0\rangle_{\lambda,V} \\ &\quad + \mathfrak{R}\int_\Omega \left(f(x)|u_0|^{p(x)-2}u_0 + g(x)|u_0|^{q(x)-2}u_0\right)\overline{(u_n u_0)}dx \\ &= \langle u_n - u_0, u_n - u_0\rangle_{\lambda,V} - \mathfrak{R}\int_\Omega f(x)\left(|u_n|^{p(x)-2}u_n \right. \\ &\quad \left. - |u_0|^{p(x)-2}u_0\right)\overline{(u_n u_0)}dx - \mathfrak{R}\int_\Omega g(x) \\ &\quad \cdot \left(|u_n|^{q(x)-2}u_n - |u_0|^{q(x)-2}u_0\right)\overline{(u_n u_0)}dx, \end{aligned} \tag{78}$$

which means that

$$\lim_{n\to\infty}\|u_n - u_0\|_{\lambda,V} = 0. \tag{79}$$

It follows from (43) that $\lim_{n\to\infty}\|u_n - u_0\|_\lambda = 0$.

Proof of Theorem 2. In view of Lemma 16, Lemma 17, and Theorem 15, we can easily infer that for all $\lambda > 0$, there exists a $(\mathrm{PS})_{c_\lambda}$ sequence $\{u_n\}$ for Ψ_λ on E_λ. It derives from Lemma 18 and $0 < c_\lambda < c(\Omega_0) < c_0$ for all $\lambda > 0$ that there exists a subsequence of $\{u_n\}$ still denoted by $\{u_n\}$ and $u_\lambda^1 \in E_\lambda$ such that $u_n \longrightarrow u_\lambda^1$ in E_λ. Furthermore, $\Psi_\lambda(u_n) \longrightarrow c_\lambda \geq \tau$ and u_λ^1 is a solution of problem (1).

The next step is to prove that system (1) has another solution. Set

$$\bar{c}_\lambda = \inf\left\{\Psi_\lambda(u) \colon u \in \overline{B_\rho}\right\}, \tag{80}$$

where $B_\rho = \{u \in E_\lambda : \|u\|_\lambda < \rho\}$ and $\rho > 0$ is given by Lemma 16. Then, $\bar{c}_\lambda < 0$ for all $\lambda > 0$. For this purpose, we first prove there exists $v_0 \in E_\lambda$ such that $\Psi_\lambda(\sigma v_0) < 0$ for all $\sigma > 0$

sufficiently small. Let $v_0 \in H^{s(\cdot)}_{0,A}(\Omega,\mathbb{C})$ such that $\int_\Omega g(x)|v_0|^{q(x)}dx > 0$. Making use of the hypothesis (H_2) and (43), we obtain that for $\sigma \in (0,1)$ small enough,

$$\begin{aligned}\Psi_\lambda(\sigma v_0) &= \frac{\sigma^2}{2}\|v_0\|^2_\lambda - \frac{1}{2}\int_{\mathbb{R}^N} V^-(\sigma v_0)^2 dx \\ &\quad - \int_\Omega \left(\frac{f(x)}{p(x)}|\sigma v_0|^{p(x)} + \frac{g(x)}{q(x)}|\sigma v_0|^{q(x)}\right)dx \\ &\le \frac{\sigma^2}{2}\|v_0\|^2_{\lambda,V} - \sigma^{p^+}\int_\Omega \frac{f(x)}{p(x)}|v_0|^{p(x)}dx \\ &\quad - \sigma^{q^+}\int_\Omega \frac{g(x)}{q(x)}|v_0|^{q(x)}dx \\ &\le \frac{\sigma^2}{2}\|v_0\|^2_\lambda - \frac{\sigma^{q^+}}{q^+}\int_\Omega g(x)|v_0|^{q(x)}dx < 0.\end{aligned} \tag{81}$$

Consequently, there exists $v_0 \in E_\lambda$ such that $\Psi_\lambda(\sigma v_0) < 0$ for all $\sigma > 0$ sufficiently small.

Applying Lemma 16 and the Ekeland variational principle to $\bar{B}_\rho$, there exists a sequence $\{u_n\}$ such that

$$\begin{aligned}\bar{c}_\lambda &\le \Psi_\lambda(u_n) \le \bar{c}_\lambda + \frac{1}{n}, \\ \Psi_\lambda(v) &\ge \Psi_\lambda(u_n) - \frac{\|u_n - v\|_\lambda}{n},\end{aligned} \tag{82}$$

for all $v \in \overline{B_\rho}$. Now we prove that $\|u_n\|_\lambda < \rho$ for n enough large. On the contrary, we suppose that $\|u_n\|_\lambda = \rho$ for infinitely many n. Without loss of generality, we can suppose that $\|u_n\|_\lambda = \rho$ for $n \in N$. It follows from Lemma 16 that

$$\Psi_\lambda(u_n) \ge \tau > 0. \tag{83}$$

Combining with (82), we obtain $\bar{c}_\lambda \ge \tau > 0$, which is contradictive with $\bar{c}_\lambda < 0$. Next, we prove $\Psi'_\lambda(u_n) \longrightarrow 0$ in E^*_λ as $n \longrightarrow \infty$. Let

$$y_n = u_n + \sigma v, \quad \text{for all } v \in B_1 := \{u \in E_\lambda : \|u\|_\lambda = 1\}, \tag{84}$$

where $\sigma > 0$ small enough such that $2\sigma\rho + \sigma^2 \le \rho^2 - \|u_n\|^2_\lambda$ for fixed n large. Then,

$$\begin{aligned}\|y_n\|^2_\lambda &= \|u_n\|^2_\lambda + 2\sigma\rho\langle u_n, v\rangle_\lambda + \sigma^2 \\ &\le \|u_n\|^2_\lambda + 2\sigma\rho + \sigma^2 = \rho^2,\end{aligned} \tag{85}$$

which implies that $y_n \in \bar{B}_\rho$. Hence, by using (50), we get

$$\Psi_\lambda(y_n) \ge \Psi_\lambda(u_n) - \frac{\|u_n - y_n\|_\lambda}{n}, \tag{86}$$

that is,

$$\frac{\Psi_\lambda(u_n + \sigma v) - \Psi_\lambda(u_n)}{\sigma} \ge -\frac{1}{n}. \tag{87}$$

Set $\sigma \longrightarrow 0^+$; we obtain $\langle \Psi'_\lambda(u_n), v\rangle \ge -1/n$ for each fixed n large. Similarly, choosing $\sigma < 0$ and $|\sigma|$ small enough and repeating the process above, we can easily get that

$$\left\langle \Psi'_\lambda(u_n), v \right\rangle \le \frac{1}{n}, \tag{88}$$

for each fixed n large.

In short, we have

$$\lim_{n\to\infty} \sup_{v\in B_1} \left|\Psi'_\lambda(u_n), v\right| = 0, \tag{89}$$

which immediately concludes that $\Psi'_\lambda(u_n) \longrightarrow 0$ in E^*_λ as $n \longrightarrow \infty$. Therefore, $\{u_n\}$ is a $(PS)_{\bar{c}_\lambda}$ sequence for the functional Ψ_λ. Making use of a similar proof as Lemma 18, there exists u^2_λ such that $u_n \longrightarrow u^2_\lambda$ in E_λ. Therefore, we obtain a nontrivial solution u^2_λ of problem (1) satisfying

$$\begin{aligned}\Psi_\lambda(u^2_\lambda) &\le \xi < 0, \\ \|u^2_\lambda\|_\lambda &< \rho.\end{aligned} \tag{90}$$

Hence, it is easy to conclude that

$$\Psi_\lambda(u^2_\lambda) = \bar{c}_\lambda \le \xi < 0 < \tau < c_\lambda = \Psi_\lambda(u^1_\lambda), \quad \text{for all } \lambda > 0, \tag{91}$$

which completes the proof.

4. Proof of Theorem 3

In this section, we mainly give the proof of Theorem 3. In addition, inspired by [2, 17], we obtain the method to prove Theorem 3.

Proof of Theorem 3. For each sequence $\{\lambda_n\}$ such that $1 \le \lambda_n \longrightarrow \infty$ as $n \longrightarrow \infty$, set $u^{(i)}_n$ to be the critical points of Ψ_λ obtained in Theorem 2, where $i = 1, 2$. Therefore, one has

$$\begin{aligned}&\Psi_{\lambda_n}\left(u^{(2)}_n\right) \le \xi < 0 < \tau < c_{\lambda_n} = \Psi_{\lambda_n}\left(u^{(1)}_n\right) < c_0, \\ &\Psi'_{\lambda_n}\left(u^{(1)}_n\right) = \Psi'_{\lambda_n}\left(u^{(2)}_n\right) = 0\end{aligned} \tag{92}$$

$$\begin{aligned}\Psi_{\lambda_n}\left(u_n^{(i)}\right) &= \frac{1}{2}\left\|u_n^{(i)}\right\|_{\lambda_n}^2 - \frac{1}{2}\int_\Omega V^-\left(u_n^{(i)}\right)^2 dx \\ &\quad - \int_\Omega \frac{f(x)}{p(x)}\left|u_n^{(i)}\right|^{p(x)} dx - \int_\Omega \frac{g(x)}{q(x)}\left|u_n^{(i)}\right|^{q(x)} dx \\ &= \frac{1}{2}\left\|u_n^{(i)}\right\|_{\lambda_n,V}^2 - \int_\Omega \frac{f(x)}{p(x)}\left|u_n^{(i)}\right|^{p(x)} dx \\ &\quad - \int_\Omega \frac{g(x)}{q(x)}\left|u_n^{(i)}\right|^{q(x)} dx \geq \frac{1}{2}\frac{\vartheta_0 - 1}{\vartheta_0}\left\|u_n^{(i)}\right\|_{\lambda_n}^2 \\ &\quad - \int_\Omega \frac{g(x)}{p(x)}\left|u_n^{(i)}\right|^{p(x)} dx - \int_\Omega \frac{g(x)}{q(x)}\left|u_n^{(i)}\right|^{q(x)} dx \\ &\geq \frac{p^- - 2}{2p^-}\frac{\vartheta_0 - 1}{\vartheta_0}\left\|u_n^{(i)}\right\|_{\lambda_n}^2 \\ &\quad + \left(\frac{1}{p^-} - \frac{1}{q^-}\right)\int_\Omega g(x)\left|u_n^{(i)}\right|^{q(x)} dx \\ &\geq \frac{p^- - 2}{2p^-}\frac{\vartheta_0 - 1}{\vartheta_0}\left\|u_n^{(i)}\right\|_{\lambda_n}^2 - \left(\frac{1}{q^-} - \frac{1}{p^-}\right) \\ &\quad \cdot \|g\|_\infty \max\left\{C_q^{q^+}\left\|u_n^{(i)}\right\|_{\lambda_n}^{q^+}, C_q^{q^-}\left\|u_n^{(i)}\right\|_{\lambda_n}^{q^-}\right\}.\end{aligned} \tag{93}$$

In view of (92) and (93), it gains

$$\left\|u_n^{(i)}\right\|_{\lambda_n} \leq c_2, \tag{94}$$

where $c_2 > 0$ is independent of λ_n. So we can suppose that $u_n^{(i)} \rightharpoonup u^{(i)}$ in $H_{0,A}^{s(\cdot)}(\Omega, \mathbb{C})$ and $u_n^{(i)} \longrightarrow u^i$ in $L^{r(x)}(\Omega, \mathbb{C})$. Making use of Fatou's lemma, we can easily obtain

$$\int_\Omega V^+\left|u^{(i)}\right|^2 dx \leq \liminf_{n\to\infty}\int_\Omega V^+\left|u_n^{(i)}\right|^2 dx \leq \liminf_{n\to\infty}\frac{\left\|u_n^{(i)}\right\|_{\lambda_n}^2}{\lambda_n} = 0. \tag{95}$$

Consequently, $u^{(i)} = 0$ a.e. in $\mathbb{R}^N \setminus (V^+)^{-1}(0)$ and $u^{(i)} \in H_{0,A}^{s(\cdot)}(\Omega_0, \mathbb{C})$.

Next, we will prove that $u_n^{(i)} \longrightarrow u^{(i)}$ in $H_{0,A}^{s(\cdot)}(\Omega, \mathbb{C})$. Indeed, combining $u_n^{(i)} \longrightarrow u^{(i)}$ in $L^{r(x)}(\Omega, \mathbb{C})$ and (H_3), one has

$$\begin{aligned}\int_\Omega f(x)\left|u_n^{(i)} - u^{(i)}\right|^{p(x)} dx &\leq \|f\|_\infty \int_\Omega \left|u_n^{(i)} - u^{(i)}\right|^{p(x)} dx, \\ \int_\Omega g(x)\left|u_n^{(i)} - u^{(i)}\right|^{q(x)} dx &\leq \|g\|_\infty \int_\Omega \left|u_n^{(i)} - u^{(i)}\right|^{q(x)} dx.\end{aligned} \tag{96}$$

Then,

$$\lim_{n\to\infty}\int_\Omega f(x)\left|u_n^{(i)} - u^{(i)}\right|^{p(x)} dx = 0, \tag{97}$$

$$\lim_{n\to\infty}\int_\Omega g(x)\left|u_n^{(i)} - u^{(i)}\right|^{q(x)} dx = 0. \tag{98}$$

We notice that

$$\lim_{n\to\infty}\left\langle \Psi_{\lambda_n}'\left(u_n^{(i)}\right), u_n^{(i)}\right\rangle = \lim_{n\to\infty}\left\langle \Psi_{\lambda_n}'\left(u_n^{(i)}\right), u^{(i)}\right\rangle = 0. \tag{99}$$

Therefore, we have

$$\begin{aligned}\left\|u_n^{(i)}\right\|_{\lambda_n}^2 &= \int_\Omega V^-\left(u_n^{(i)}\right)^2 dx \\ &\quad + \int_\Omega \left(f(x)\left|u_n^{(i)}\right|^{p(x)} + g(x)\left|u_n^{(i)}\right|^{q(x)}\right) dx + o(1),\end{aligned} \tag{100}$$

$$\begin{aligned}\left\langle u_n^{(i)}, u^{(i)}\right\rangle_{\lambda_n} &= \Re\int_\Omega V^- u_n^{(i)}\overline{u^{(i)}} dx \\ &\quad + \Re\int_\Omega f(x)\left|u_n^{(i)}\right|^{p(x)-2} u_n^{(i)}\overline{u^{(i)}} dx \\ &\quad + \Re\int_\Omega g(x)\left|u_n^{(i)}\right|^{q(x)-2} u_n^{(i)}\overline{u^{(i)}} dx + o(1).\end{aligned} \tag{101}$$

By (97)–(101), we have

$$\lim_{n\to\infty}\left\|u_n^{(i)}\right\|_{\lambda_n}^2 = \lim_{n\to\infty}\left\langle u_n^{(i)}, u^{(i)}\right\rangle_{\lambda_n} = \left\|u^{(i)}\right\|_{H_{0,A}^{s(\cdot)}(\Omega,\mathbb{C})}^2. \tag{102}$$

On the other hand, the weak lower semicontinuity of norm yields that

$$\begin{aligned}\left\|u^{(i)}\right\|_{H_{0,A}^{s(\cdot)}(\Omega,\mathbb{C})}^2 &\leq \liminf_{n\to\infty}\left\|u_n^{(i)}\right\|_{H_{0,A}^{s(\cdot)}(\Omega,\mathbb{C})}^2 \\ &\leq \limsup_{n\to\infty}\left\|u_n^{(i)}\right\|_{H_{0,A}^{s(\cdot)}(\Omega,\mathbb{C})}^2 \\ &\leq \lim_{n\to\infty}\left\|u_n^{(i)}\right\|_{\lambda_n}^2.\end{aligned} \tag{103}$$

To sum up, we can see that

$$\limsup_{n\to\infty}\left\|u_n^{(i)}\right\|_{H_{0,A}^{s(\cdot)}(\Omega,\mathbb{C})} \leq \left\|u^{(i)}\right\|_{H_{0,A}^{s(\cdot)}(\Omega,\mathbb{C})}. \tag{104}$$

By Proposition 3.32 of [18], we can obtain that $u_n^{(i)} \longrightarrow u^{(i)}$ in $H_{0,A}^{s(\cdot)}(\Omega, \mathbb{C})$. We notice that $\lim_{n\to\infty}\langle \Psi_{\lambda_n}'(u_n^{(i)}), v\rangle = 0$, for any $v \in C_0^\infty(\Omega_0, \mathbb{C})$. Hence,

$$\begin{aligned}&\Re\int_{\mathbb{R}^{2N}} \frac{\left(u^{(i)}(x) - e^{i(x-y)\cdot A((x+y)/2)}u^{(i)}(y)\right)\overline{\left(v(x)e^{i(xy)\cdot A((x+y)/2)}v(y)\right)}}{|x-y|^{N+2s(x,y)}} dxdy \\ &\quad - \Re\int_{\Omega_0} V^- u^{(i)}\bar{v} dx - \Re\int_{\Omega_0}\left(f(x)\left|u^{(i)}\right|^{p(x)-2}u^{(i)}\right. \\ &\quad \left. + g(x)\left|u^{(i)}\right|^{q(x)-2}u^{(i)}\right)\bar{v} dx = 0.\end{aligned} \tag{105}$$

Since the density of $C_0^{\infty}(\Omega_0, \mathbb{C})$ in $H_{0,A}^{s(\cdot)}(\Omega_0, \mathbb{C})$, we can obtain that $u^{(i)}$ is a weak solution of problem (12).

Together with (92), $u^{(i)} = 0$ a.e. in $\mathbb{R}^N \setminus (V^+)^{-1}(0)$ and the constants ξ, τ are independent of λ; we have

$$\begin{aligned} &\frac{1}{2}\left\|u^{(1)}\right\|^2_{H_{0,A}^{s(\cdot)}(\Omega_0,\mathbb{C})} - \frac{1}{2}\int_{\Omega_0} V^-\left(u^{(1)}\right)^2 dx \\ &\quad - \int_{\Omega_0}\left(\frac{f(x)}{p(x)}\left|u^{(1)}\right|^{p(x)} + \frac{g(x)}{q(x)}\left|u^{(1)}\right|^{q(x)}\right)dx \geq \tau > 0, \\ &\frac{1}{2}\left\|u^{(2)}\right\|^2_{H_{0,A}^{s(\cdot)}(\Omega_0,\mathbb{C})} - \frac{1}{2}\int_{\Omega_0} V^-\left(u^{(2)}\right)^2 dx \\ &\quad - \int_{\Omega_0}\left(\frac{f(x)}{p(x)}\left|u^{(2)}\right|^{p(x)} + \frac{g(x)}{q(x)}\left|u^{(2)}\right|^{q(x)}\right)dx \leq \xi < 0, \end{aligned} \tag{106}$$

which implies that $u^{(i)} \neq 0$ and $u^{(1)} \neq u^{(2)}$.

Now we consider the case where $f(x), g(x) \equiv$ constants; that is

$$\begin{cases} (-\Delta)_A^{s(\cdot)} u + V_\lambda(x)u = a|u|^{p(x)-2}u + b|u|^{q(x)-2}u \text{ in } \Omega, \\ u = 0 \text{ in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{107}$$

where a, b are two nonnegative constants. Correspondingly, the energy functional $\Psi_\lambda : E_\lambda \longrightarrow \mathbb{R}$ is

$$\begin{aligned} \Psi_\lambda(u) &= \frac{1}{2}\|u\|_\lambda^2 - \frac{1}{2}\int_\Omega V^- u^2 dx \\ &\quad - \int_\Omega \left(\frac{a}{p(x)}|u|^{p(x)} + \frac{b}{q(x)}|u|^{q(x)}\right)dx \\ &= \frac{1}{2}\|u\|_{\lambda,V}^2 - \int_\Omega \left(\frac{a}{p(x)}|u|^{p(x)} + \frac{b}{q(x)}|u|^{q(x)}\right)dx. \end{aligned} \tag{108}$$

Next, we mainly prove the existence of infinitely many solutions to problem (107) by using four different methods.

Theorem 19. *Assume that* (S_1), (S_2), (V_1)–(V_4), *and* (H_1)–(H_3) *hold. Let* $N > 2s^+$. *Then, problem (107) has infinitely many solutions.*

Proof. Method 1: It is easy to verify that functional Ψ_λ is even and satisfies $\Psi_\lambda(0) = 0$. Furthermore, Lemma 18 shows that functional Ψ_λ is bounded from below in E_λ and satisfies the (PS) condition. For any $k \in N$ and $\rho_k > 0$, let $S_{\rho_k} = \{u \in E_\lambda, \|u\|_\lambda = \rho_k\}$; then, for any $u \in S_{\rho_k}$, one has

$$\begin{aligned} \Psi_\lambda(u) &= \frac{1}{2}\|u\|_{\lambda,V}^2 - \int_\Omega \frac{a}{p(x)}|u|^{p(x)}dx - \int_\Omega \frac{b}{q(x)}|u|^{q(x)}dx \\ &\leq \frac{1}{2}\|u\|_\lambda^2 - \frac{b}{q^+}\int_\Omega |u|^{q(x)}dx \\ &\leq \frac{1}{2}\|u\|_\lambda^2 - \frac{b}{q^+}\min\left\{\|u\|^{q^+}_{L^{q(x)}(\Omega,\mathbb{C})}, \|u\|^{q^-}_{L^{q(x)}(\Omega,\mathbb{C})}\right\}. \end{aligned} \tag{109}$$

We find that there exists $C_{S_{\rho_k}} > 0$ such that $\|u\|_{L^{q(x)}(\Omega)} \geq C_{S_{\rho_k}}\|u\|_\lambda$, since all norms are equivalent on finite dimensional Banach space. Then, by $1 < q(x) < 2$, it gains

$$\sup_{u \in X^k \cap S_{\rho_k}} \Psi_\lambda(u) \leq \frac{1}{2}\|u\|_\lambda^2 - \frac{b}{p^+}\min\left\{C^{q^+}_{S_{\rho_k}}, C^{q^-}_{S_{\rho_k}}\right\}\|u\|_\lambda^{q^+}. \tag{110}$$

Letting $\|u\|_\lambda = \rho_k$ small enough, we can obtain $\sup_{u \in X^k \cap S_{\rho_k}} \Psi_\lambda(u) < 0 = \Psi_\lambda(0)$. Furthermore, we assert that (ii) of Theorem 9 does not work. In fact, (109) means $\Psi_\lambda(u) \neq 0$ since $\Psi_\lambda(tu) < 0$ as t small enough with the given $u \in E_\lambda$. Thus, by Theorem 9, we get that problem (1) has a sequence of solutions $\{u_k\}$ with $\|u_k\|_\lambda \longrightarrow 0$ as $k \longrightarrow \infty$. In short, problem (1) has infinitely many solutions for all $\lambda > 0$.

Method 2. To start with, we assert that for any finite dimensional subspace X of E_λ, there exists $r_1 = r_1(X)$ such that $\Psi_\lambda(u) < 0$ for all $u \in E_\lambda \setminus B_{r_1}(X)$, where $B_{r_1}(X) = \{u \in E_\lambda : \|u\|_\lambda < r_1\}$. Indeed, for each $t \geq 1$, we can easily get that

$$\begin{aligned} \Psi_\lambda(tu) &= \frac{t^2}{2}\|u\|_\lambda^2 - \int_\Omega \frac{a}{p(x)}|tu|^{p(x)}dx - \int_\Omega \frac{b}{q(x)}|tu|^{q(x)}dx \\ &\leq \frac{t^2}{2}\|u\|_\lambda^2 - \frac{at^{p^-}}{p^+}\int_\Omega |u|^{p(x)}dx \\ &\leq \frac{t^2}{2}\|u\|_\lambda^2 - \frac{at^{p^-}}{p^+}\min\left\{\|u\|^{p^+}_{L^{p(x)}(\Omega,\mathbb{C})}, \|u\|^{p^-}_{L^{p(x)}(\Omega,\mathbb{C})}\right\}. \end{aligned} \tag{111}$$

We observe that there exists $C_X > 0$ such that $\|u\|_{L^{p(x)}(\Omega)} \geq C_X\|u\|_\lambda$, since all norms are equivalent on finite dimensional Banach space X. Then, by $p^- > 2$, it gains

$$\begin{aligned} \Psi_\lambda(tu) &\leq \frac{t^2}{2}\|u\|_\lambda^2 - \frac{at^{p^-}}{p^+}\min\left\{C_X^{p^+}, C_X^{p^-}\right\}\|u\|_\lambda^{p^+} \\ &\longrightarrow -\infty \text{as } t \longrightarrow \infty. \end{aligned} \tag{112}$$

Thus, there exists $r_1 > 0$ large enough such that $\Psi_\lambda(u) < 0$ for all $u \in E_\lambda$, with $\|u\|_\lambda = r_2$ and $r_2 \geq r_1$. Consequently, the assertion is valid.

From Lemma18, we know that Ψ_λ satisfies the $(\mathrm{PS})_c$ condition for any $c \in \mathbb{R}$. Obliviously, $\Psi_\lambda(0) = 0$ and Ψ_λ is an even functional. In short, it follows from Theorem 8 that there exists an unbounded sequence of solutions of problem (1) for all $\lambda > 0$.

Lemma 19 (see Lemma 4.1, [10]). *Let $1 < r(x) < 2N/(N-2s(x,x))$ for all $x \in \bar{\Omega}$. For any $k \in N$, define*

$$\beta_k := \left\{ \|u\|_{L^{r(x)}(\Omega,\mathbb{C})} : u \in Z_k, \|u\|_\lambda = 1 \right\}. \quad (113)$$

Then, $\beta_k \longrightarrow 0$, as $k \longrightarrow \infty$.

Method 3. By Remark 14 and Lemma 18, we know that Ψ_λ satisfies the $(PS)_c$ condition for any $c \in \mathbb{R}$. To start with, we will prove (D_1) is satisfied. It follows from (36) and (43) that

$$\begin{aligned} \Psi_\lambda(u) &= \frac{1}{2}\|u\|_{\lambda,V}^2 - \int_\Omega \frac{a}{p(x)}|u|^{p(x)}dx - \int_\Omega \frac{b}{q(x)}|u|^{q(x)}dx \\ &\le \frac{1}{2}\|u\|_\lambda^2 - \frac{a}{p^+}\int_\Omega |u|^{p(x)}dx \\ &\le \frac{1}{2}\|u\|_\lambda^2 - \frac{a}{p^+}\min\left\{ \|u\|_{L^{p(x)}(\Omega,\mathbb{C})}^{p^+}, \|u\|_{L^{p(x)}(\Omega,\mathbb{C})}^{p^-} \right\}. \end{aligned} \quad (114)$$

We find that there exists $C_{Y_k} > 0$ such that $\|u\|_{L^{p(x)}(\Omega)} \ge C_{Y_k}\|u\|_\lambda$, since all norms are equivalent on finite dimensional Banach space Y_k. Then,

$$\Psi_\lambda(u) \le \frac{1}{2}\|u\|_\lambda^2 - \frac{a}{p^+}\min\left\{ C_{Y_k}^{p^+}, C_{Y_k}^{p^-} \right\}\|u\|_\lambda^{p^-}. \quad (115)$$

Then, by $p^- > 2$, we can easily obtain that (D_1) is satisfied for $\|u\|_\lambda = r_k > 1$ big enough. Next, we will show (D_2) is fulfilled. In view of (43) and Lemma 20, for $u \in E_\lambda$, one has

$$\begin{aligned} \Psi_\lambda(u) &= \frac{1}{2}\|u\|_\lambda^2 - \frac{1}{2}\int_{\mathbb{R}^N} V^- u^2 dx \\ &\quad - \int_\Omega \left(\frac{a}{p(x)}|u|^{p(x)} + \frac{b}{q(x)}|u|^{q(x)} \right)dx \\ &= \frac{1}{2}\|u\|_{\lambda,V}^2 - \int_\Omega \left(\frac{a}{p(x)}|u|^{p(x)} + \frac{b}{q(x)}|u|^{q(x)} \right)dx \\ &\ge \frac{\vartheta_0 - 1}{2\vartheta_0}\|u\|_\lambda^2 - \frac{a}{p^-}\int_\Omega |u|^{p(x)}dx - \frac{b}{q^-}\int_\Omega |u|^{q(x)}dx \\ &\ge \frac{\vartheta_0 - 1}{2\vartheta_0}\|u\|_\lambda^2 - \frac{a}{p^-}\beta_k^{p^-}\|u\|_\lambda^{p^+} - \frac{b}{q^-}\beta_k^{q^-}\|u\|_\lambda^{q^+} \\ &= \|u\|_\lambda^{q^+}\left[\frac{\vartheta_0 - 1}{2\vartheta_0}\|u\|_\lambda^{2-q^+} - \frac{a}{p^-}\beta_k^{p^-}\|u\|_\lambda^{p^+-q^+} - \frac{b}{q^-}\beta_k^{q^-} \right] \\ &= \|u\|_\lambda^{q^+}\left[\|u\|_\lambda^{2-q^+}\left(\frac{\vartheta_0 - 1}{2\vartheta_0} - \frac{a}{p^-}\beta_k^{p^-}\|u\|_\lambda^{p^+-2} \right) - \frac{b}{q^-}\beta_k^{q^-} \right] \\ &= \gamma_k^{q^+}\left[\gamma_k^{2-q^+}\left(\frac{\vartheta_0 - 1}{2\vartheta_0} - \frac{a}{p^-}\beta_k^{p^-}\gamma_k^{p^+-2} \right) - \frac{b}{q^-}\beta_k^{q^-} \right]. \end{aligned} \quad (116)$$

Choosing $\gamma_k = ((p^-(\vartheta_0 - 1))/8a\vartheta_0\beta_k^{p^-})^{1/(p^+-2)}$. Combing with Lemma 20, we know that $\beta_k \longrightarrow 0$ as $k \longrightarrow \infty$. Thus, one has $\gamma_k \longrightarrow +\infty$ as $k \longrightarrow \infty$ and

$$\Psi_\lambda(u) \ge \gamma_k^{q^+}\left(\frac{3}{8}\frac{\vartheta_0 - 1}{\vartheta_0}\gamma_k^{2-q^+} - \frac{b}{q^-}\beta_k^{q^-} \right) \longrightarrow +\infty, \quad (117)$$

as $k \longrightarrow \infty$. In conclusion, (D_2) is fulfilled. It is easy to check that satisfying $\Psi_\lambda(-u) = \Psi_\lambda(u)$. Thus, by Theorem 10, we can obtain that problem (1) has infinitely many solutions for all $\lambda > 0$.

Method 4. First, we will show that (D_4) is fulfilled. By (42) and (43), one has

$$\begin{aligned} \Psi_\lambda(u) &\ge \frac{1}{2}\frac{\vartheta_0 - 1}{\vartheta_0}\|u\|_\lambda^2 - \frac{a}{p^-}\max\left\{ C_p^{p^+}\|u\|_\lambda^{p^+}, C_p^{p^-}\|u\|_\lambda^{p^-} \right\} \\ &\quad - \frac{b}{q^-}\max\left\{ \|u\|_{L^{q(x)}(\Omega,\mathbb{C})}^{q^+}, \|u\|_{L^{q(x)}(\Omega,\mathbb{C})}^{q^-} \right\}. \end{aligned} \quad (118)$$

Since $1 < q^- \le q^+ < 2 < p^- \le p^+ < 2N/(N-2s(x,x))$, we can choose $r_3 \in (0,1)$ small enough such that for all $u \in E_\lambda$ with $\|u\|_\lambda \le r_3$,

$$\frac{1}{8}\frac{\vartheta_0 - 1}{\vartheta_0}\|u\|_\lambda^2 \ge \frac{a}{p^-}\max\left\{ C_p^{p^+}\|u\|_\lambda^{p^+}, C_p^{p^-}\|u\|_\lambda^{p^-} \right\} \quad (119)$$

hold. Then,

$$\Psi_\lambda(u) \ge \frac{3}{8}\frac{\vartheta_0 - 1}{\vartheta_0}\|u\|_\lambda^2 - \frac{b}{q^-}\beta_k^{q^-}\|u\|_\lambda^{q^-}. \quad (120)$$

Choosing

$$r_k = \left(\frac{8b\vartheta_0\beta_k^{q^-}}{3q^-(\vartheta_0 - 1)} \right)^{1/(2-q^-)}. \quad (121)$$

Combing with Lemma 20, we know that $\beta_k \longrightarrow 0$ as $k \longrightarrow \infty$. Thus, one has $r_k \longrightarrow 0$ as $k \longrightarrow \infty$. Hence, there exists $\bar{k}$ such that $r_k \le r_3$ as $k \ge \bar{k}$. Consequently, for $k \ge \bar{k}$ and $u \in Z_k$ with $\|u\|_\lambda = r_k$, we can obtain that $\Psi_\lambda(u) \ge 0$. Next, we will show (D_5) is fulfilled. For any $u \in Y_k$, $\|u\|_\lambda = \gamma_k$ with $r_k > \gamma_k > 0$, we get

$$\begin{aligned} \Psi_\lambda(u) &= \frac{1}{2}\|u\|_{\lambda,V}^2 - \int_\Omega \frac{a}{p(x)}|u|^{p(x)}dx - \int_\Omega \frac{b}{q(x)}|u|^{q(x)}dx \\ &\le \frac{1}{2}\|u\|_\lambda^2 - \frac{a}{p^+}\int_\Omega |u|^{p(x)}dx - \frac{b}{q^+}\int_\Omega |u|^{q(x)}dx \\ &\le \frac{1}{2}\|u\|_\lambda^2 - \frac{a}{p^+}\min\left\{ \|u\|_{L^{p(x)}(\Omega,\mathbb{C})}^{p^+}, \|u\|_{L^{p(x)}(\Omega,\mathbb{C})}^{p^-} \right\} \\ &\quad - \frac{b}{q^+}\min\left\{ \|u\|_{L^{q(x)}(\Omega,\mathbb{C})}^{q^+}, \|u\|_{L^{q(x)}(\Omega,\mathbb{C})}^{q^-} \right\}. \end{aligned} \quad (122)$$

We find that there exists $C_{Y_k} > 0$ such that $\|u\|_{L^{q(x)}(\Omega)} \geq C_{Y_k} \|u\|_\lambda$ and $\|u\|_{L^{p(x)}(\Omega)} \geq C_{Y_k} \|u\|_\lambda$, since all norms are equivalent on finite dimensional Banach space Y_k. Then,

$$\begin{aligned}\Psi_\lambda(u) &\leq \frac{1}{2}\|u\|_\lambda^2 - \frac{a}{p^+} \min\left\{C_{Y_k}^{p^+}\|u\|_\lambda^{p^+}, C_{Y_k}^{p^-}\|u\|_\lambda^{p^-}\right\} \\ &\quad - \frac{b}{q^+} \min\left\{C_{Y_k}^{q^+}\|u\|_\lambda^{q^+}, C_{Y_k}^{q^-}\|u\|_\lambda^{q^-}\right\} \\ &\leq \frac{1}{2}\|u\|_\lambda^2 - \frac{a}{p^+} \min\left\{C_{Y_k}^{p^+}, C_{Y_k}^{p^-}\right\}\|u\|_\lambda^{p^+} \\ &\quad - \frac{b}{q^+} \min\left\{C_{Y_k}^{q^+}, C_{Y_k}^{q^-}\right\}\|u\|_\lambda^{q^+} < 0,\end{aligned} \tag{123}$$

as $\gamma_k > 0$ small enough. Now we check (D_6) is fulfilled. It follows from (D_4) that for $k \geq \bar{k}$ and $u \in Z_k$ with $\|u\|_\lambda \leq r_k$,

$$\begin{aligned}\Psi_\lambda(u) &\geq \frac{3}{8}\frac{\vartheta_0 - 1}{\vartheta_0}\|u\|_\lambda^2 - \frac{b}{q^-}\beta_k^{q^-}\|u\|_\lambda^{q^-} \\ &\geq -\frac{b}{q^-}\beta_k^{q^-}\|u\|_\lambda^{q^-} \geq -\frac{b}{q^-}\beta_k^{q^-} r_k^{q^-},\end{aligned} \tag{124}$$

thanks to $\beta_k \longrightarrow 0$ as $k \longrightarrow \infty$. Thus, one has $r_k \longrightarrow 0$ as $k \longrightarrow \infty$. Thus, (D_6) is also satisfied. In conclusion, by Theorem 11, we can obtain that problem (1) has infinitely many solutions for all $\lambda > 0$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

Authors' Contributions

All authors contributed equally to the manuscript and typed, read, and approved the final manuscript.

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References

[1] S. Mao and A. Xia, "Multiplicity results of nonlinear fractional magnetic Schrödinger equation with steep potential," *Applied Mathematics Letters*, vol. 97, pp. 73–80, 2019.

[2] M. Xiang, B. Zhang, and D. Yang, "Multiplicity results for variable-order fractional Laplacian equations with variable growth," *Nonlinear Analysis*, vol. 178, pp. 190–204, 2019.

[3] T.-F. Wu and Y. Cheng, "Multiplicity and concentration of positive solutions for semilinear elliptic equations with steep potential," *Communications on Pure and Applied Analysis*, vol. 15, no. 6, pp. 2457–2473, 2016.

[4] S. Peng and A. Xia, "Multiplicity and concentration of solutions for nonlinear fractional elliptic equations with steep potential," *Communications on Pure & Applied Analysis*, vol. 17, no. 3, pp. 1201–1217, 2018.

[5] A. Fiscella, A. Pinamonti, and E. Vecchi, "Multiplicity results for magnetic fractional problems," *Journal of Differential Equations*, vol. 263, no. 8, pp. 4617–4633, 2017.

[6] F. Colasuonno and P. Pucci, "Multiplicity of solutions for p(x)-polyharmonic elliptic Kirchhoff equations," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 74, no. 17, pp. 5962–5974, 2011.

[7] Z. Liu and Z. Q. Wang, "On Clark's theorem and its applications to partially sublinear problems," *Annales de l'Institut Henri Poincare (C) Non Linear Analysis*, vol. 32, no. 5, pp. 1015–1037, 2015.

[8] T. Bartsch, "Infinitely many solutions of a symmetric Dirichlet problem," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 20, no. 10, pp. 1205–1216, 1993.

[9] T. Bartsch and M. Willem, "On an elliptic equation with concave and convex nonlinearities," *Proceedings of the American Mathematical Society*, vol. 123, no. 11, pp. 3555–3561, 1995.

[10] L. Wang and B. Zhang, "Infinitely many solutions for Kirchhoff-type variable-order fractional Laplacian problems involving variable exponents," *Applicable Analysis*, pp. 1–18, 2019.

[11] T. Bartsch, A. Pankov, and Z. Q. Wang, "Nonlinear Schrödinger equations with steep potential well," *Communications in Contemporary Mathematics*, vol. 3, no. 4, pp. 549–569, 2011.

[12] G. Arioli and A. Szulkin, "A semilinear Schrödinger equation in the presence of a magnetic field," *Archive for Rational Mechanics and Analysis*, vol. 170, no. 4, pp. 277–295, 2003.

[13] S. Cingolani, "Semiclassical stationary states of nonlinear Schrödinger equations with an external magnetic field," *Journal of Differential Equations*, vol. 188, no. 1, pp. 52–79, 2003.

[14] M. J. Esteban and P. L. Lions, "Stationary solutions of nonlinear Schrödinger equations with anexternal magnetic field," in *Partial Differential Equations and the Calculus of Variations. Progress in Nonlinear Differential Equations and Their Applications, vol 1*, F. Colombini, A. Marino, L. Modica, and S. Spagnolo, Eds., pp. 401–449, Birkhäuser, Boston, 1989.

[15] K. Kurata, "Existence and semi-classical limit of the least energy solution to a nonlinear Schrödinger equation with electromagnetic fields," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 41, no. 5-6, pp. 763–778, 2000.

[16] P. d'Avenia and M. Squassina, "Ground states for fractional magnetic operators," *ESAIM: Control, Optimisation and Calculus of Variations*, vol. 24, no. 1, pp. 1–24, 2018.

[17] V. Ambrosio and P. d'Avenia, "Nonlinear fractional magnetic Schrödinger equation: existence and multiplicity," *Journal of Differential Equations*, vol. 264, no. 5, pp. 3336–3368, 2018.

[18] R. Servadei and E. Valdinoci, "Mountain pass solutions for non-local elliptic operators," *Journal of Mathematical Analysis and Applications*, vol. 389, no. 2, pp. 887–898, 2012.