Research Article

Several Fixed-Point Theorems for F -Contractions in Complete Branciari b -Metric Spaces and Applications

Lili Chen,^{1,2} Shuai Huang,¹ Chaobo Li,¹ and Yanfeng Zhao ²

¹Department of Mathematics, Harbin University of Science and Technology, Harbin 150080, China

²College of Mathematics and Systems Science, Shandong University of Science and Technology, Qingdao 266590, China

Correspondence should be addressed to Yanfeng Zhao; 53447556@qq.com

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In this paper, we prove the existence and uniqueness of fixed points for F -contractions in complete Branciari b -metric spaces. Furthermore, an example for supporting the related result is shown. We also present the concept of the weak well-posedness of the fixed-point problem of the mapping T and discuss the weak well-posedness of the fixed-point problem of an F -contraction in complete Branciari b -metric spaces. Besides, we investigate the problem of common fixed points for F -contractions in above spaces. As an application, we apply our main results to solving the existence and uniqueness of solutions for a class of the integral equation and the dynamic programming problem, respectively.

1. Introduction

In the last decades, one can observe a huge amount of interest for the development of fixed-point theory, since it has a lot of applications, especially in metric spaces, see [1–6]. In 1993, Czerwik [7] introduced the concepts of b -metric spaces by weakening the coefficient of the triangle inequality and generalized Banach's Contraction Principle to these spaces. Subsequently, Boriceanu [8] obtained some concrete examples of b -metric spaces and studied the fixed-point properties of set-valued operators in b -metric spaces. In 2000, Branciari [9] suggested a new distance function to improve the notion of metric via substituting the triangle inequality with the quadrilateral inequality and extended Banach's Contraction Principle to Branciari metric spaces. In 2015, George et al. [10] combined Branciari metric spaces with b -metric spaces to introduce a generalization of metric spaces-rectangular b -metric spaces (or Branciari b -metric spaces). They also proved Banach's Contraction Principle and presented the generalized form of Kannan's fixed-point theorem in above spaces. In 2018, Karapinar [11] revisited Kannan's fixed-point theorem under the aspect of interpolation and proposed a new Kannan-type contraction to maximize the rate

of convergence. For more details about fixed-point theorems in metric spaces, please see [12–17].

In fact, fixed-point properties of Branciari b -metric spaces have aroused great interest of many researchers who have extended the results in Branciari metric spaces and b -metric spaces to Branciari b -metric spaces. However, sometimes it is difficult to achieve the desired results in Branciari b -spaces; there are still a large number of problems to be solved. Based on the above-mentioned discussions, we discuss the problem of the existence and uniqueness of fixed points for F -contractions in complete Branciari b -metric spaces which extend the results in [18]. What is more, an example for supporting the result is presented. The concept of the weak well-posedness of the fixed-point problem of the mapping T is also introduced, and the weak well-posedness of the fixed-point problem of an F -contraction satisfying some conditions in complete Branciari b -metric spaces is investigated. Besides, the problem of common fixed points for two F -contractions is studied in above spaces. As an application, by using main results, the existence and uniqueness of solutions for a class of the integral equation and the dynamic programming problem are solved, respectively.

2. Preliminaries

Firstly, we recall some basic notations and facts that we are going to use later.

Definition 1 (see [19]). Let U be a nonempty set and $s \geq 1$ be a real number. A function $d : U \times U \rightarrow [0, \infty)$ is called a b -metric if the following conditions are satisfied, for every $x, y, z \in U$:

- (A1) $d(x, y) = 0$ if and only if $x = y$
- (A2) $d(x, y) = d(y, x)$
- (A3) $d(x, y) \leq s[d(x, z) + d(z, y)]$

In this case, the pair (U, d) is called a b -metric space with constant $s \geq 1$.

Definition 2 (see [10]). Let U be a nonempty set and $s \geq 1$ be a real number. A function $d : U \times U \rightarrow [0, \infty)$ is called a Branciari b -metric if the following conditions are satisfied, for every $x, y \in U$:

- (B1) $d(x, y) = 0$ if and only if $x = y$
- (B2) $d(x, y) = d(y, x)$
- (B3) $d(x, y) \leq s[d(x, u) + d(u, v) + d(v, y)]$ for all distinct points $u, v \in U \setminus \{x, y\}$

In this case, the pair (U, d) is called a Branciari b -metric space with constant $s \geq 1$.

Definition 3 (see [18]). A function $F : (0, \infty) \rightarrow \mathbb{R}$ belongs to $\mathcal{F}$ if it satisfies the following conditions:

- (F1) F is strictly increasing
- (F2) There exists $k \in (0, 1)$ such that $\lim_{x \rightarrow 0^+} x^k F(x) = 0$

In [18], they omitted Wardowski's (F2) condition from the above definition. Explicitly, they do not require that (WF2) if $\{\alpha_n\}_{n \in \mathbb{N}}$ is a sequence of positive real numbers, then $\lim_{n \rightarrow \infty} \alpha_n = 0$ if and only if $\lim_{n \rightarrow \infty} F(\alpha_n) = -\infty$

The reason for this is the following lemma.

Lemma 4 (see [18]). If $F : (0, \infty) \rightarrow \mathbb{R}$ is an increasing function and $\{\alpha_n\}_{n \in \mathbb{N}} \subset (0, +\infty)$ is a decreasing sequence such that $\lim_{n \rightarrow \infty} F(\alpha_n) = -\infty$, then $\lim_{n \rightarrow \infty} \alpha_n = 0$.

We can also see some properties concerning $F_{s,\tau}$ and $F'_{s,\tau}$.

Definition 5 (see [19]). Let $s \geq 1$ and $\tau > 0$. We say that $F \in \mathcal{F}$ belongs to $\mathcal{F}_{s,\tau}$ if it also satisfies $(F_{s,\tau})$ if $\inf F = -\infty$ and $x, y, z \in (0, \infty)$ are such that $\tau + F(sx) \leq F(y)$ and $\tau + F(sy) \leq F(z)$, then

$$\tau + F(s^2x) \leq F(sy). \quad (1)$$

In [20], the authors introduced the following condition (F4) in Definition 3.1:

$(F'_{s,\tau})$ if $\{\alpha_n\}_{n \in \mathbb{N}} \subset (0, \infty)$ is a sequence such that $\tau + F(s\alpha_n) \leq F(\alpha_{n-1})$, for all $n \in \mathbb{N}$ and for some $\tau \geq 0$, then $\tau + F(s^n\alpha_n) \leq F(s^{n-1}\alpha_{n-1})$, for all $n \in \mathbb{N}^*$.

Proposition 6 (see [18]). If F is increasing, then $(\mathcal{F}_{s,\tau})$ is equivalent to $(\mathcal{F}'_{s,\tau})$.

Definition 7 (see [18]). Let (U, d) be a b -metric space with constant $s \geq 1$ and $T : U \rightarrow U$ be an operator. If there exist $\tau > 0$ and $F \in \mathcal{F}_{s,\tau}$, such that for all $x, y \in U$, the inequality $d(Tx, Ty) > 0$ implies

$$(F)\tau + F(s \cdot d(Tx, Ty)) \leq F(d(x, y)), \quad (2)$$

then, T is called an F -contraction.

3. Main Results

In this section, we mainly focus on the existence and uniqueness of fixed points for F -contractions, the weak well-posedness of the fixed-point problem of an F -contraction, and criteria for common fixed points of two F -contractions in complete Branciari b -metric space. Moreover, the existence of fixed points for Hardy-Rogers-type set-valued F -contractions in above spaces is shown. In the sequel, we denote by $\mathbb{R}$ the set of real numbers and by $\mathbb{N}$ the set of positive integer numbers.

3.1. Existence and Uniqueness of Fixed Points for F -Contractions

Theorem 8. Let (U, d) be a complete Branciari b -metric space with constant $s \geq 1$ and $T : U \rightarrow U$ be an F -contraction for $F \in \mathcal{F}_{s,\tau}$. Then, T has a unique fixed point x^* . Furthermore, for any $x_0 \in U$, the sequence $\{x_n\}$ satisfying $x_n = Tx_{n-1}$ is convergent.

Proof. Firstly, we show that T has at most one fixed point. Suppose that x^* and y^* are two different fixed points of T . That is, $Tx^* = x^* \neq y^* = Ty^*$. It follows that

$$d(Tx^*, Ty^*) = d(x^*, y^*) > 0. \quad (3)$$

Therefore, we can apply (F) to get

$$\begin{aligned} \tau + F(s \cdot d(Tx^*, Ty^*)) &\leq F(d(x^*, y^*)) \leq F(s \cdot d(x^*, y^*)) \\ &= F(s \cdot d(Tx^*, Ty^*)), \end{aligned} \quad (4)$$

which implies $\tau \leq 0$, a contradiction. Hence, T has at most one fixed point.

Next, we prove the existence of fixed points of T . For any $x_0 \in U$, set $x_n = Tx_{n-1}$, $\gamma_n = d(x_{n+1}, x_n)$, and $\beta_n = d(x_{n+2}, x_n)$ with $\gamma_0 = d(x_1, x_0)$ and $\beta_0 = d(x_2, x_0)$. Now, we consider the following two cases:

- (i) If there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0-1}$, then we have $Tx_{n_0-1} = x_{n_0-1}$. This shows $x^* = x_{n_0-1}$, a fixed point of T . Therefore, the proof is finished

- (ii) If $x_n \neq x_{n-1}$, for any $n \in \mathbb{N}$, then we have $\gamma_n > 0$, for each $n \in \mathbb{N}$. Hence, (F) implies $F(s\gamma_{n+1}) \leq F(\gamma_n) - \tau$, for every $n \in \mathbb{N}$

By Proposition 6, we obtain

$$F(s^{n+1}\gamma_{n+1}) \leq F(s^n\gamma_n) - \tau, \forall n \in \mathbb{N}. \quad (5)$$

Furthermore, for any $n \in \mathbb{N}$, we have

$$F(s^n\gamma_n) \leq F(s^{n-1}\gamma_{n-1}) - \tau \leq F(s^{n-2}\gamma_{n-2}) - 2\tau \leq \dots \leq F(\gamma_0) - n\tau. \quad (6)$$

Since $\lim_{n \rightarrow \infty} [F(\gamma_0) - n\tau] = -\infty$, then the inequality (6) implies

$$\lim_{n \rightarrow \infty} F(s^n\gamma_n) = -\infty. \quad (7)$$

On the other hand, from (5) and (F1), it follows that the sequence $\{s^n\gamma_n\}_{n=1}^{\infty}$ is decreasing. Furthermore, we can apply Lemma 4 to get

$$\lim_{n \rightarrow \infty} s^n\gamma_n = 0. \quad (8)$$

According to (F2), there exists $k \in (0, 1)$ such that

$$\lim_{n \rightarrow \infty} (s^n\gamma_n)^k F(s^n\gamma_n) = 0. \quad (9)$$

Multiplying (6) by $(s^n\gamma_n)^k$ results

$$0 \leq n(s^n\gamma_n)^k \tau + (s^n\gamma_n)^k F(s^n\gamma_n) \leq (s^n\gamma_n)^k F(\gamma_0), \forall n \in \mathbb{N}, \quad (10)$$

which implies $\lim_{n \rightarrow \infty} n(s^n\gamma_n)^k = 0$. It follows that there exists $n_1 \in \mathbb{N}$ such that $n(s^n\gamma_n)^k \leq 1$, for all $n \geq n_1$. Thus,

$$s^n\gamma_n \leq \frac{1}{n^{1/k}}, \forall n \geq n_1, \quad (11)$$

which shows that the series $\sum_{i=1}^{\infty} s^i\gamma_i$ is convergent.

Now, we prove that $\{x_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence. For all $n, p \in \mathbb{N}$, we drive the proof into two cases.

- (a) If $p > 2$ is odd, we can obtain

$$d(x_{n+3}, x_n) \leq sd(x_{n+3}, x_{n+2}) + sd(x_{n+2}, x_{n+1}) + sd(x_{n+1}, x_n),$$

$$\begin{aligned} d(x_{n+5}, x_n) &\leq sd(x_{n+5}, x_{n+2}) + sd(x_{n+2}, x_{n+1}) + sd(x_{n+1}, x_n) \\ &\leq s^2d(x_{n+5}, x_{n+4}) + s^2d(x_{n+4}, x_{n+3}) \\ &\quad + s^2d(x_{n+3}, x_{n+2}) + s\gamma_{n+1} + s\gamma_n. \end{aligned} \quad (12)$$

Hence, we deduce that

$$\begin{aligned} d(x_{n+p}, x_n) &\leq sd(x_{n+p}, x_{n+2}) + sd(x_{n+2}, x_{n+1}) + sd(x_{n+1}, x_n) \\ &\leq s^2d(x_{n+p}, x_{n+4}) + s^2d(x_{n+4}, x_{n+3}) + s^2d(x_{n+3}, x_{n+2}) \\ &\quad + s\gamma_{n+1} + s\gamma_n \leq s^3d(x_{n+p}, x_{n+6}) + s^3\gamma_{n+5} + s^3\gamma_{n+4} \\ &\quad + s^2\gamma_{n+3} + s^2\gamma_{n+2} + s\gamma_{n+1} + s\gamma_n \leq s^{(p-1)/2}\gamma_{n+p-1} \\ &\quad + s^{(p-1)/2}\gamma_{n+p-2} + s^{(p-1)/2}\gamma_{n+p-3} + s^{(p-3)/2}\gamma_{n+p-4} \\ &\quad + s^{(p-3)/2}\gamma_{n+p-5} + \dots + s^2\gamma_{n+2} + s\gamma_{n+1} + s\gamma_n \\ &\leq s^{(p+1)/2}\gamma_{n+p-1} + s^{p/2}\gamma_{n+p-2} + s^{(p-1)/2}\gamma_{n+p-3} \\ &\quad + s^{(p-2)/2}\gamma_{n+p-4} + s^{(p-3)/2}\gamma_{n+p-5} + \dots + s^{3/2}\gamma_{n+1} \\ &\quad + s^{2/2}\gamma_n \leq s^{p+1}\gamma_{n+p-1} + s^p\gamma_{n+p-2} + s^{p-1}\gamma_{n+p-3} \\ &\quad + s^{p-2}\gamma_{n+p-4} + s^{p-3}\gamma_{n+p-5} + \dots + s^3\gamma_{n+1} + s^2\gamma_n \\ &\leq \frac{1}{s^{n-2}} \left(s^{n+p-1}\gamma_{n+p-1} + s^{n+p-2}\gamma_{n+p-2} + s^{n+p-3}\gamma_{n+p-3} \right. \\ &\quad \left. + \dots + s^{n+1}\gamma_{n+1} + s^n\gamma_n \right) = \frac{1}{s^{n-2}} \sum_{i=n}^{n+p-1} s^i\gamma_i \leq \frac{1}{s^{n-2}} \sum_{i=n}^{\infty} s^i\gamma_i. \end{aligned} \quad (13)$$

- (b) If $p > 2$ is even, we can get

$$d(x_{n+4}, x_n) \leq sd(x_{n+4}, x_{n+2}) + sd(x_{n+2}, x_{n+1}) + sd(x_{n+1}, x_n). \quad (14)$$

Furthermore, we conclude that

$$\begin{aligned} d(x_{n+p}, x_n) &\leq sd(x_{n+p}, x_{n+2}) + sd(x_{n+2}, x_{n+1}) + sd(x_{n+1}, x_n) \\ &\leq s^2d(x_{n+p}, x_{n+4}) + s^2d(x_{n+4}, x_{n+3}) + s^2d(x_{n+3}, x_{n+2}) \\ &\leq s^2d(x_{n+p}, x_{n+4}) + s^2d(x_{n+4}, x_{n+3}) + s^2d(x_{n+3}, x_{n+2}) \\ &\leq s^3d(x_{n+p}, x_{n+6}) + s^3\gamma_{n+5} + s^3\gamma_{n+4} + s^2\gamma_{n+3} + s^2\gamma_{n+2} \leq \\ &\leq s^{(p-2)/2}d(x_{n+p}, x_{n+p-2}) + s^{(p-2)/2}\gamma_{n+p-3} + s^{(p-2)/2}\gamma_{n+p-4} \\ &\quad + s^{(p-4)/2}\gamma_{n+p-5} + s^{(p-4)/2}\gamma_{n+p-6} + \dots + s\gamma_{n+1} + s\gamma_n \\ &\leq s^{(p-2)/2}d(x_{n+p}, x_{n+p-2}) + s^{(p-1)/2}\gamma_{n+p-3} + s^{(p-2)/2}\gamma_{n+p-4} \\ &\quad + s^{(p-3)/2}\gamma_{n+p-5} + s^{(p-4)/2}\gamma_{n+p-6} + \dots + s^{3/2}\gamma_{n+1} + s^{2/2}\gamma_n \\ &\leq s^{(p-2)/2}d(x_{n+p}, x_{n+p-2}) + \frac{1}{s^{n-2}} \left(s^{n+p-3}\gamma_{n+p-3} + s^{n+p-4}\gamma_{n+p-4} \right. \\ &\quad \left. + \dots + s^{n+1}\gamma_{n+1} + s^n\gamma_n \right) \leq s^{(p-2)/2}d(x_{n+p}, x_{n+p-2}) \\ &\quad + \frac{1}{s^{n-2}} \sum_{i=n}^{n+p-3} s^i\gamma_i \leq s^{(p-2)/2}\beta_{n+p-2} + \frac{1}{s^{n-2}} \sum_{i=n}^{\infty} s^i\gamma_i. \end{aligned} \quad (15)$$

Since $\beta_n \geq 0$, we can assume that $\beta_n > 0, \forall n \in \mathbb{N}$. By a similar method, replacing γ_n with β_n in (11), there exists $n_2 \in \mathbb{N}$ such that

$$s^n\beta_n \leq \frac{1}{n^{1/k}}, \forall n \geq n_2, \quad (16)$$

which implies $\lim_{n \rightarrow \infty} s^n \beta_n = 0$ and $\lim_{n \rightarrow \infty} \beta_n = 0$. Together (a) with (b), for every $p \in \mathbb{N}$, letting $n \rightarrow \infty$, we deduce that

$$d(x_{n+p}, x_n) \rightarrow 0. \quad (17)$$

Thus, $\{x_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence. Since (U, d) is complete, there exists $x^* \in U$ such that $\lim_{n \rightarrow \infty} x_n = x^*$. On the other hand, we notice that

$$F(d(Tx, Ty)) \leq \tau + F(d(Tx, Ty)) \leq \tau + F(s \cdot d(Tx, Ty)) \leq F(d(x, y)) \quad (18)$$

holds for all $x, y \in U$ with $d(Tx, Ty) > 0$. Since F is increasing, then

$$d(Tx, Ty) \leq d(x, y). \quad (19)$$

It follows that

$$0 \leq d(x_{n+1}, Tx^*) \leq d(x_n, x^*) \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (20)$$

Hence, we get $x^* = Tx^*$, which completes the proof.

Next, we will give an example to support Theorem 8.

Example 1. Let $U = A \cup B$, where $A = \{1/n : n \in \{2, 3, 4, 5\}\}$ and $B = [1, 2]$. For any $x, y \in U$, we define $d : U \times U \rightarrow [0, \infty)$ by

$$d(x, y) = |x - y|^2. \quad (21)$$

Then, it is not difficult to see that (U, d) is a Branciari b -metric space with constant $s = 12/5 > 1$.

Let $T : U \rightarrow U$ be a mapping satisfying

$$Tx = \begin{cases} \frac{1}{4}, & x \in A, \\ \frac{1}{5}, & x \in B. \end{cases} \quad (22)$$

Now, we verify that T is an F -contraction satisfying the condition (F). We take $x \in A$, $y \in B$, and $\tau = 0.1$. Then, $d(Tx, Ty) = d(1/4, 1/5) = 1/400 > 0$ and

$$0.1 + \frac{12}{5} \cdot d(Tx, Ty) = 0.106 < 0.5 = d(x, y). \quad (23)$$

Let $F : (0, \infty) \rightarrow \mathbb{R}$ be a mapping defined by $F(x) = x$, then it is easy to see that $F \in \mathcal{F}_{s, \tau}$. In addition, we can get

$$\tau + F(s \cdot d(Tx, Ty)) \leq F(d(x, y)). \quad (24)$$

Hence, T satisfies the conditions of Theorem 8 and $x = 1/4$ is the unique fixed point of T .

3.2. Weak Well-Posedness for the Fixed-Point Problem of an F -Contraction. The notion of well-posedness of a fixed-point problem has evoked much interest to several researchers

[21–23]. Now, we introduce the concept of weak well-posedness for the fixed-point problem of the mapping T .

Definition 9. Let (U, d) be a metric space and $T : U \rightarrow U$ be a mapping. The fixed-point problem of T is said to be weakly well-posed if

- (1) T has a unique fixed point x^* in U
- (2) For any sequence $\{x_n\}_{n \in \mathbb{N}}$ in U with $\lim_{n \rightarrow \infty} d(x_n, Tx_n) = 0$ and $\lim_{n, m \rightarrow \infty} d(Tx_n, Tx_m) = 0$, we have $\lim_{n \rightarrow \infty} d(x_n, x^*) = 0$

Theorem 10. Let (U, d) be a complete Branciari b -metric space with constant $s \geq 1$ and $T : U \rightarrow U$ be an F -contraction for $F \in \mathcal{F}_{s, \tau}$ with the continuous differentiable function F' and $\lim_{x \rightarrow \infty} F'(x) < \infty$. Then, the fixed-point problem of T is weakly well-posed.

Proof. From Theorem 8, it follows that T satisfies (5). Let x^* be the unique fixed point of T . For any sequence $\{x_n\}_{n \in \mathbb{N}}$ in U with $\lim_{n \rightarrow \infty} d(x_n, Tx_n) = 0$ and $\lim_{n, m \rightarrow \infty} d(Tx_n, Tx_m) = 0$, in general, the sequence $\{x_n\}_{n \in \mathbb{N}}$ can be decomposed into two subsequences $\{x_n\} = \{y_{n_k}\}_{k \in \mathbb{N}} \cup \{z_{n_k}\}_{k \in \mathbb{N}}$, where $Ty_{n_k} = x^*$ and $Tz_{n_k} \neq x^*$ for any $k \in \mathbb{N}$. Consequently, we can prove the following two assertions: $F(d(x_n, x^*)) - F(sd(Tx_n, Tx^*)) \geq \tau$.

(W1) if $\{x_n\}_{n \in \mathbb{N}}$ is such that $Tx_n = x^*$ for all $n \in \mathbb{N}$, then $\lim_{n \rightarrow \infty} d(x_n, x^*) = 0$

(W2) if $\{x_n\}_{n \in \mathbb{N}}$ is such that $Tx_n \neq x^*$ for all $n \in \mathbb{N}$, then $\lim_{n \rightarrow \infty} d(x_n, x^*) = 0$

In fact, (W1) is trivial, since $d(x_n, x^*) = d(x_n, Tx_n) \rightarrow 0$ as $n \rightarrow \infty$. Now, we prove (W2). Because of the uniqueness of the fixed point of T , we can assume that there exists $n_0 \in \mathbb{N}$ such that $Tx_n \neq x_n$ for $n \geq n_0$. If $Tx_n \neq x^*$ for any $n \in \mathbb{N}$, then we can apply (F) to get

$$F(d(x_n, x^*)) - F(sd(Tx_n, Tx^*)) \geq \tau. \quad (25)$$

Since F is strictly increasing, we have $d(x_n, x^*) > sd(Tx_n, Tx^*)$. Then,

$$\frac{F(d(x_n, x^*)) - F(sd(Tx_n, Tx^*))}{d(x_n, x^*) - sd(Tx_n, Tx^*)} \geq \frac{\tau}{d(x_n, x^*) - sd(Tx_n, Tx^*)}. \quad (26)$$

Then there exists a constant c_n between $d(x_n, x^*)$ and $sd(Tx_n, Tx^*)$ such that

$$\begin{aligned} F'(c_n) &= \frac{F(d(x_n, x^*)) - F(sd(Tx_n, Tx^*))}{d(x_n, x^*) - sd(Tx_n, Tx^*)} \\ &\geq \frac{\tau}{d(x_n, x^*) - sd(Tx_n, Tx^*)}, \end{aligned} \quad (27)$$

which implies

$$0 < \frac{\tau}{F'(c_n)} \leq d(x_n, x^*) - sd(Tx_n, Tx^*). \quad (28)$$

Actually, we can assume that there exists a distinct subsequence $\{Tx_{n_k}\}$ of $\{Tx_n\}$. Otherwise, there exists $x_0 \in U$ and $n_1 \in \mathbb{N}$ such that $Tx_n = x_0$ for $n \geq n_1$. Since $\lim_{n \rightarrow \infty} d(x_n, Tx_n) = 0$, we get $\lim_{n \rightarrow \infty} d(x_n, x_0) = 0$. If $x_0 \neq x^*$, then $x_0 \neq Tx_0$ because of the uniqueness of the fixed point of T . For $n \geq n_1$, we obtain $x_0 = Tx_n \neq Tx_0$. Since T is an F -contraction, together with (19), we have

$$0 < d(Tx_n, Tx_0) \leq d(x_n, x_0) \longrightarrow 0 \text{ as } n \longrightarrow \infty, \quad (29)$$

which is a contradiction. Hence, there exist $m, q, n > n_0$ ($m > q > n$) such that $Tx_m \neq Tx_q \neq Tx_n \neq x_n$. Then,

$$d(x_n, x^*) \leq s[d(x_n, Tx_m) + d(Tx_m, Tx_n) + d(Tx_n, x^*)]. \quad (30)$$

Combining with (28), we deduce that

$$0 < \frac{\tau}{F'(c_n)} \leq s[d(x_n, Tx_m) + d(Tx_m, Tx_n)]. \quad (31)$$

Since

$$0 \leq d(x_n, Tx_m) \leq s[d(x_n, Tx_n) + d(Tx_n, Tx_q) + d(Tx_q, Tx_m)] \longrightarrow 0 \text{ as } n \longrightarrow \infty. \quad (32)$$

Then,

$$0 < \frac{\tau}{F'(c_n)} \leq s[d(x_n, Tx_m) + d(Tx_m, Tx_n)] \longrightarrow 0 \text{ as } n \longrightarrow \infty. \quad (33)$$

Therefore, we get $\lim_{n \rightarrow \infty} F'(c_n) = \infty$ which implies $\lim_{n \rightarrow \infty} c_n = 0$ since F' is continuous and $\lim_{x \rightarrow \infty} F'(x) < \infty$. Hence, we deduce that $\lim_{n \rightarrow \infty} sd(Tx_n, Tx^*) = 0$. From (30), it follows that $\lim_{n \rightarrow \infty} d(x_n, x^*) = 0$.

3.3. The Problem of Common Fixed Points for Two F -Contractions. Now, we continue to solve the problem of common fixed points for two F -contraction mappings in a complete Branciari b -metric space.

Theorem 11. Let (U, d) be a complete Branciari b -metric space with constant $s > 1$. If there exist $\tau > 0$ and $F \in \mathcal{F}_{s, \tau}$ such that $S, T : U \longrightarrow U$ are two F -contraction mappings on U and satisfy

$$\tau + F(d(Sx, Ty)) \leq F(ad(x, y) + bd(x, Sx) + cd(y, Ty)), \quad (34)$$

for any $a, b, c \in [0, 1]$ with $a + b + c < 1$, $sb < 1$, and $\min \{d(Sx, Ty), d(x, y), d(x, Sx), d(y, Ty)\} > 0$ for any $(x, y) \in U \times U$.

Then, there exists a unique common fixed point of the pair mappings (S, T) .

Proof. Let $x_0 \in U$, we define $\{x_n\} \subseteq U$ by

$$\begin{aligned} Sx_{2j} &= x_{2j+1}, \\ Tx_{2j+1} &= x_{2j+2}, \\ j &= 0, 1, 2, \dots \end{aligned} \quad (35)$$

Combining with (34) and (35), we have

$$\begin{aligned} \tau + F(d(x_{2j+1}, x_{2j+2})) &= \tau + F(d(Sx_{2j}, Tx_{2j+1})) \\ &\leq F(ad(x_{2j}, x_{2j+1}) + bd(x_{2j}, Sx_{2j}) + cd(x_{2j+1}, Tx_{2j+1})) \\ &= F(ad(x_{2j}, x_{2j+1}) + bd(x_{2j}, x_{2j+1}) + cd(x_{2j+1}, x_{2j+2})). \end{aligned} \quad (36)$$

Let $\lambda = (a + b)/(1 - c)$, it is not difficult to see $0 < \lambda < 1$ since $a + b + c < 1$. Thanks to the strictly monotone increasing property of F , we deduce

$$d(x_{2j+1}, x_{2j+2}) < \frac{a + b}{1 - c} d(x_{2j}, x_{2j+1}) = \lambda d(x_{2j}, x_{2j+1}). \quad (37)$$

Similarly,

$$d(x_{2j+2}, x_{2j+3}) < \frac{a + b}{1 - c} d(x_{2j+1}, x_{2j+2}) = \lambda d(x_{2j+1}, x_{2j+2}). \quad (38)$$

Hence,

$$d(x_n, x_{n+1}) < \lambda d(x_{n-1}, x_n), \quad n \in \mathbb{N}. \quad (39)$$

For any $n \in \mathbb{N}$, we obtain

$$d(x_n, x_{n+1}) < \lambda d(x_{n-1}, x_n) < \lambda^2 d(x_{n-2}, x_{n-1}) < \dots < \lambda^n d(x_0, x_1). \quad (40)$$

Notice that

$$\begin{aligned} \tau + F(sd(x_1, x_3)) &= \tau + F(sd(Sx_0, Sx_2)) \leq F(d(x_0, x_2)), \\ \tau + F(sd(x_2, x_4)) &= \tau + F(sd(Tx_1, Tx_3)) \leq F(d(x_1, x_3)). \end{aligned} \quad (41)$$

Since F is strictly monotone increasing, we have

$$\begin{aligned} d(x_1, x_3) &\leq \frac{1}{s} d(x_0, x_2), \\ d(x_2, x_4) &\leq \frac{1}{s} d(x_1, x_3) \leq \frac{1}{s^2} d(x_0, x_2). \end{aligned} \quad (42)$$

By induction, we get

$$d(x_n, x_{n+2}) < \frac{1}{s^n} d(x_0, x_2), \quad n \in \mathbb{N}. \quad (43)$$

Next, we will show that $\{x_n\}$ is a Cauchy sequence in U . We consider the following two cases.

(i) Let $m = n + p$, if p is odd and $p > 2$, we have

$$\begin{aligned}
 d(x_n, x_m) &\leq s(d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_m)) \\
 &\leq sd(x_n, x_{n+1}) + sd(x_{n+1}, x_{n+2}) + s^2d(x_{n+2}, x_{n+3}) \\
 &\quad + s^2d(x_{n+3}, x_{n+4}) + s^2d(x_{n+4}, x_m) \leq sd(x_n, x_{n+1}) \\
 &\quad + sd(x_{n+1}, x_{n+2}) + s^2d(x_{n+2}, x_{n+3}) + s^2d(x_{n+3}, x_{n+4}) \\
 &\quad + s^3d(x_{n+4}, x_{n+5}) + s^3d(x_{n+5}, x_{n+6}) + \dots \\
 &\quad + s^{(m-n)/2}d(x_{m-2}, x_{m-1}) + s^{(m-n)/2}d(x_{m-1}, x_m) \\
 &\leq s\lambda^n d(x_0, x_1) + s\lambda^{n+1}d(x_0, x_1) + s^2\lambda^{n+2}d(x_0, x_1) \\
 &\quad + s^2\lambda^{n+3}d(x_0, x_1) + s^3\lambda^{n+3}d(x_0, x_1) + s^3\lambda^{n+4}d(x_0, x_1) \\
 &\quad + \dots + s^{(m-n)/2}\lambda^{m-2}d(x_0, x_1) + s^{(m-n)/2}\lambda^{m-1}d(x_0, x_1) \\
 &\leq (s\lambda^n + s^2\lambda^{n+2} + s^3\lambda^{n+4} + \dots + s^{(m-n)/2}\lambda^{m-2})d(x_0, x_1) \\
 &\quad + (s\lambda^{n+1} + s^2\lambda^{n+3} + s^3\lambda^{n+5} + \dots + s^{(m-n)/2}\lambda^{m-1})d(x_0, x_1) \\
 &\leq (s\lambda^n + s^2\lambda^{n+2} + s^3\lambda^{n+4} + \dots + s^{(m-n)/2}\lambda^{m-2})(1 + \lambda)d(x_0, x_1) \\
 &\leq s\lambda^n (1 + s\lambda^2 + s^2\lambda^4 + s^3\lambda^6 + \dots + s^{(m-n-2)/2}\lambda^{m-n-2}) \\
 &\quad \cdot (1 + \lambda)d(x_0, x_1) \leq s\lambda^n \cdot \frac{1 - s^{(m-n)/2}\lambda^{m-n}}{1 - s\lambda^2} (1 + \lambda)d(x_0, x_1) \\
 &\leq s\lambda^n \cdot \frac{1 - s^{p/2}\lambda^p}{1 - s\lambda^2} (1 + \lambda)d(x_0, x_1).
 \end{aligned} \tag{44}$$

(ii) Let $m = n + p$, if p is even and $p > 2$, we have

$$\begin{aligned}
 d(x_n, x_m) &\leq s(d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_m)) \\
 &\leq sd(x_n, x_{n+1}) + sd(x_{n+1}, x_{n+2}) + s^2d(x_{n+2}, x_{n+3}) \\
 &\quad + s^2d(x_{n+3}, x_{n+4}) + s^2d(x_{n+4}, x_m) \leq sd(x_n, x_{n+1}) \\
 &\quad + sd(x_{n+1}, x_{n+2}) + s^2d(x_{n+2}, x_{n+3}) + s^2d(x_{n+3}, x_{n+4}) \\
 &\quad + s^3d(x_{n+4}, x_{n+5}) + s^3d(x_{n+5}, x_{n+6}) \\
 &\quad + \dots + s^{(m-n-2)/2}d(x_{m-4}, x_{m-3}) + s^{(m-n-2)/2}d(x_{m-3}, x_{m-2}) \\
 &\quad + s^{(m-n-2)/2}d(x_{m-2}, x_m) \leq s\lambda^n d(x_0, x_1) + s\lambda^{n+1}d(x_0, x_1) \\
 &\quad + s^2\lambda^{n+2}d(x_0, x_1) + s^2\lambda^{n+3}d(x_0, x_1) + s^3\lambda^{n+3}d(x_0, x_1) \\
 &\quad + s^3\lambda^{n+4}d(x_0, x_1) + \dots + s^{(m-n-2)/2}\lambda^{m-4}d(x_0, x_1) \\
 &\quad + s^{(m-n-2)/2}\lambda^{m-3}d(x_0, x_1) + s^{(m-n-2)/2}d(x_{m-2}, x_m) \\
 &\leq (s\lambda^n + s^2\lambda^{n+2} + s^3\lambda^{n+4} + \dots + s^{(m-n-2)/2}\lambda^{m-4})d(x_0, x_1) \\
 &\quad + (s\lambda^{n+1} + s^2\lambda^{n+3} + s^3\lambda^{n+5} + \dots + s^{(m-n-2)/2}\lambda^{m-3})d(x_0, x_1) \\
 &\quad + s^{(m-n-2)/2}d(x_{m-2}, x_m) \leq (s\lambda^n + s^2\lambda^{n+2} + s^3\lambda^{n+4} \\
 &\quad + \dots + s^{(m-n-2)/2}\lambda^{m-4})(1 + \lambda)d(x_0, x_1) + s^{(m-n-2)/2}d(x_{m-2}, x_m) \\
 &\leq s\lambda^n (1 + s\lambda^2 + s^2\lambda^4 + s^3\lambda^6 + \dots + s^{(m-n-4)/2}\lambda^{m-n-4}) \\
 &\quad \cdot (1 + \lambda)d(x_0, x_1) + s^{m-n-2}d(x_{m-2}, x_m) \\
 &\leq s\lambda^n \cdot \frac{1 - s^{(m-n-2)/2}\lambda^{m-n-2}}{1 - s\lambda^2} (1 + \lambda)d(x_0, x_1) + s^{-n}d(x_0, x_2) \\
 &\leq s\lambda^n \cdot \frac{1 - s^{(p-2)/2}\lambda^{p-2}}{1 - s\lambda^2} (1 + \lambda)d(x_0, x_1) + s^{-n}d(x_0, x_2).
 \end{aligned} \tag{45}$$

Letting $m, n \rightarrow \infty$, we obtain that $d(x_n, x_m) \rightarrow 0$ for all $p > 2$. Hence, $\{x_n\}$ is a Cauchy sequence in U . From the completeness of U , it follows that there exists $z^* \in U$ such that

$$\lim_{n \rightarrow \infty} d(x_n, z^*) = 0. \tag{46}$$

Next, we verify that z^* is a fixed point of the mapping S . Indeed, if $d(Sz^*, z^*) > 0$, then

$$\begin{aligned}
 \tau + F(d(Sz^*, x_{2j+2})) &\leq F(ad(z^*, x_{2j+1}) + bd(z^*, Sz^*) \\
 &\quad + cd(x_{2j+1}, x_{2j+2})).
 \end{aligned} \tag{47}$$

By using the strictly monotone increasing property of F , we get

$$d(Sz^*, x_{2j+2}) < ad(z^*, x_{2j+1}) + bd(z^*, Sz^*) + cd(x_{2j+1}, x_{2j+2}). \tag{48}$$

We can also see that

$$d(Sz^*, z^*) < s[d(Sz^*, x_{2j+2}) + d(x_{2j+2}, x_{2j+1}) + d(x_{2j+1}, z^*)]. \tag{49}$$

It follows that

$$\begin{aligned}
 \frac{1}{s}d(Sz^*, z^*) &\leq \liminf_{j \rightarrow \infty} d(Sz^*, x_{2j+2}) \\
 &\leq \limsup_{j \rightarrow \infty} d(Sz^*, x_{2j+2}) \leq bd(z^*, Sz^*).
 \end{aligned} \tag{50}$$

Hence, we deduce that $1/s \leq b$ which contradicts the fact $sb < 1$. Therefore, we obtain $d(Sz^*, z^*) = 0$, i.e., $Sz^* = z^*$. Similarly, we can get $Tz^* = z^*$. Therefore, we have

$$Tz^* = Sz^* = z^*. \tag{51}$$

Next, we will show that S and T have a unique common fixed point. Suppose that z^*, z^{**} are two distinct common fixed points of S and T . Then,

$$\begin{aligned}
 \tau + F(d(z^*, z^{**})) &= \tau + F(d(Sz^*, Tz^{**})) \\
 &\leq F(ad(z^*, z^{**}) + bd(z^*, Sz^*) + cd(z^{**}, Tz^{**})) \\
 &= F(ad(z^*, z^{**}) + bd(z^*, z^*) + cd(z^{**}, z^{**})).
 \end{aligned} \tag{52}$$

In view of the strictly monotone increasing property of F , we obtain $(1 - a)d(z^*, z^{**}) < 0$, which is a contradiction. It means $z^* = z^{**}$.

Next, we discuss the existence of fixed points for Hardy-Rogers-type set-valued F -contractions in a complete Branciari b -metric space.

Lemma 12 (see [24]). *If F is right continuous and satisfies (F1), then*

$$F(\inf A) = \inf F(A), \quad (53)$$

for all $A \in (0, \infty)$ with $\inf A > 0$.

Definition 13 (see [24]). Let (U, d) be a Branciari b -metric space and $CB(U)$ be the family of all nonempty closed and bounded subsets of U . Let $H : CB(U) \times CB(U) \rightarrow [0, \infty)$ be a function defined by $H(A, B) = \max_{x \in A} \{\sup_{y \in B} d(x, y)\}$ for all $A, B \in CB(U)$, where $d(x, B) = \inf_{y \in B} \{d(x, y)\}$. Then, H defines a metric on $CB(U)$ called the Hausdorff metric induced by d .

Theorem 14. *Let (U, d) be a complete Branciari b -metric space with constant $s \geq 1$. Assume that there exist $\tau > 0$ and $F \in \mathcal{F}_{s, \tau}$ such that $T : U \rightarrow CB(U)$ satisfies*

$$2\tau + F(H(Tx, Ty)) \leq F(ad(x, y) + bd(x, Tx) + cd(y, Ty)), \quad (54)$$

for all $x, y \in U$ and $a, b, c \geq 0$ with $a + b + c < 1$ and $sc < 1$. If F is right continuous, then T has a fixed point in U .

Proof. Let x_0 be an arbitrary point of U and choose $x_1 \in Tx_0$. If $x_1 \in Tx_1$, then x_1 is a fixed point of T and the proof is finished. Assume that $x_1 \notin Tx_1$, then $Tx_0 \neq Tx_1$. Since F is right continuous and Tx_1 is closed, it follows from Lemma 12 that

$$F(d(x_1, Tx_1)) = \inf_{y \in Tx_1} F(d(x_1, y)). \quad (55)$$

Noticing that $d(x_1, Tx_1) \leq H(Tx_0, Tx_1)$, then there exists $x_2 \in Tx_1$ such that

$$F(d(x_1, x_2)) - \tau < F(d(x_1, Tx_1)) \leq F(H(Tx_0, Tx_1)). \quad (56)$$

Consequently, we get

$$F(d(x_1, x_2)) < F(H(Tx_0, Tx_1)) + \tau, \quad (57)$$

which implies

$$\begin{aligned} 2\tau + F(d(x_1, x_2)) &< 2\tau + F(H(Tx_0, Tx_1)) + \tau \\ &\leq F(ad(x_0, x_1) + bd(x_0, Tx_0) + cd(x_1, Tx_1)) + \tau \\ &\leq F(ad(x_0, x_1) + bd(x_0, x_1) + cd(x_1, x_2)) + \tau \\ &= F((a + b)d(x_0, x_1) + cd(x_1, x_2)) + \tau. \end{aligned} \quad (58)$$

Since F is strictly increasing, we deduce

$$d(x_1, x_2) < (a + b)d(x_0, x_1) + cd(x_1, x_2), \quad (59)$$

and hence,

$$(1 - c)d(x_1, x_2) < (a + b)d(x_0, x_1). \quad (60)$$

Since $a + b + c < 1$, we obtain

$$d(x_1, x_2) < \frac{a + b}{1 - c} d(x_0, x_1) < d(x_0, x_1). \quad (61)$$

Combining with (58), we get

$$\tau + F(d(x_1, x_2)) < F(d(x_0, x_1)). \quad (62)$$

By induction, we can get a sequence $\{x_n\} \subset U$ satisfying $\{d(x_n, x_{n+1})\}$ is decreasing, $x_n \notin Tx_n$, $x_{n+1} \in Tx_n$, and $\tau + F(d(x_n, x_{n+1})) < F(d(x_{n-1}, x_n))$ for all $n \in \mathbb{N}$.

Let $\gamma_n = d(x_n, x_{n+1})$ and $\beta_n = d(x_n, x_{n+2})$. Then, we have

$$F(\gamma_{n+1}) < F(\gamma_n) - \tau, \quad (63)$$

for all $n \in \mathbb{N}$. Similarly to the proof of Theorem 8 (while $s = 1$), we deduce that $\{x_n\}$ is a Cauchy sequence. Since U is complete, there exists $x^* \in U$ such that $x_n \rightarrow x^*$ as $n \rightarrow \infty$. Now, we prove that x^* is a fixed point of T . Since

$$\begin{aligned} 2\tau + F(d(x_n, Tx^*)) &\leq 2\tau + F(H(Tx_{n-1}, Tx^*)) \\ &\leq F(ad(x_{n-1}, x^*) + bd(x_{n-1}, Tx_{n-1}) + cd(x^*, Tx^*)), \\ d(x_{n-1}, Tx_{n-1}) &\leq d(x_{n-1}, x_n), \end{aligned} \quad (64)$$

we deduce that

$$d(x_n, Tx^*) < ad(x_{n-1}, x^*) + bd(x_{n-1}, x_n) + cd(x^*, Tx^*). \quad (65)$$

It follows that

$$\limsup_{n \rightarrow \infty} d(x_n, Tx^*) \leq cd(x^*, Tx^*). \quad (66)$$

Noting that

$$\begin{aligned} d(x^*, Tx^*) &= \inf_{y \in Tx^*} d(x^*, y) \leq s \inf_{y \in Tx^*} [d(x^*, x_{n+1}) + d(x_{n+1}, x_n) \\ &\quad + d(x_n, y)] \leq s[d(x^*, x_{n+1}) + d(x_{n+1}, x_n)] \\ &\quad + sd(x_n, Tx^*), \end{aligned} \quad (67)$$

we obtain

$$d(x^*, Tx^*) \leq \limsup_{n \rightarrow \infty} d(x_n, Tx^*). \quad (68)$$

Combining with (66) and (68), we get

$$\limsup_{n \rightarrow \infty} d(x_n, Tx^*) \leq \limsup_{n \rightarrow \infty} d(x_n, Tx^*). \quad (69)$$

Since $sc < 1$, then $\limsup_{n \rightarrow \infty} d(x_n, Tx^*) = 0$ which implies $d(x^*, Tx^*) = 0$. Hence, x^* is a fixed point of T since Tx^* is closed.

4. Applications

4.1. Existence of Solutions of Integral Equations. Firstly, we will apply Theorem 8 to solving the existence and uniqueness of the solution of the following integral equation:

$$x(t) = g(t) + \int_0^a m(t, u) f(u, x(u)) du, t \in [0, a], a > 0, \quad (70)$$

where $m : [0, a] \times [0, a] \rightarrow [0, \infty)$ is a function such that $m(t, \cdot) \in L^1([0, a])$, $f : [0, a] \times \mathbb{R} \rightarrow \mathbb{R}$, and $g : [0, a] \rightarrow \mathbb{R}$ are two continuous functions.

Let $U = C([0, a], \mathbb{R})$ be the set of real continuous functions defined on $[0, a]$ and the operator $T : U \rightarrow U$ be defined by

$$T(x)(t) = g(t) + \int_0^a m(t, u) f(u, x(u)) du, t \in [0, a]. \quad (71)$$

We will prove the following result.

Theorem 15. Let $U = C([0, a], \mathbb{R})$ and $d : U \times U \rightarrow [0, \infty)$ be given by $d(x, y) = \sup_{t \in [0, a]} |x(t) - y(t)|^2$ for all $x, y \in U$.

Suppose there exists $\zeta : U \times U \rightarrow [0, \sqrt{3}/3]$ such that

$$|f(x, x(u)) - f(y, y(u))| \leq \zeta(x, y) |x(u) - y(u)| \quad (72)$$

and $\int_0^a m(t, u) du \leq \sqrt{e^{-\tau}}$ for $u \in [0, a]$ and $\tau > 0$. Then, the integral equation (70) has a unique solution in X .

Proof. Clearly, each fixed point of T in (71) is a solution of (70). Meanwhile, it is not difficult to see that (U, d) is a complete Branciari b -metric space with $s = 3$. Then, we obtain

$$\begin{aligned} |Tx(t) - Ty(t)|^2 &= \left| \int_0^a m(t, u) [f(u, x(u)) - f(u, y(u))] du \right|^2 \\ &\leq \left(\int_0^a m(t, u) \zeta(x, y) |x(u) - y(u)| du \right)^2 \\ &\leq \frac{1}{3} \left(\int_0^a m(t, u) du \right)^2 \sup_{u \in [0, a]} |x(u) - y(u)|^2 \\ &\leq \frac{e^{-\tau}}{3} d(x, y). \end{aligned} \quad (73)$$

Consequently, for each $x, y \in U$, we have

$$3d(Tx, Ty) \leq e^{-\tau} d(x, y), \quad (74)$$

which implies

$$\tau + \ln 3d(Tx, Ty) \leq \ln d(x, y), \quad (75)$$

for each $x, y \in U$. From Definition 7, the mapping $T : U \rightarrow U$ is an F -contraction with $F(x) = \ln x$. By applying

Theorem 8, the operator T has a unique fixed point, that is, the integral equation (70) has a unique solution.

4.2. Existence of the Common Solution of Functional Equations. Finally, we will apply Theorem 11 to solving the existence and uniqueness of the solution of the dynamic programming problem. In particular, the problem of dynamic programming related to multistage process reduces to solving the existence and uniqueness of the solution of the following functional equations:

$$p(u) = \sup_{v \in B} \{f(u, v) + G_1(u, v, p(\varphi(u, v)))\}, \text{ for all } u \in A, \quad (76)$$

$$q(u) = \sup_{v \in B} \{f(u, v) + G_2(u, v, p(\varphi(u, v)))\}, \text{ for all } u \in A, \quad (77)$$

where $f : A \times B \rightarrow \mathbb{R}$, $\varphi : A \times B \rightarrow A$, and $G_1, G_2 : A \times B \times \mathbb{R} \rightarrow \mathbb{R}$. We assume that C and D are Banach spaces, $A \subseteq C$ is a state space, and $B \subseteq D$ is a decision space. For details, see also [25].

In this section, we will show that the equations (76) and (77) have one common solution. Let U be the set of all bounded real-valued functions defined on a nonempty set A . We define the norm $\|\cdot\|$ by

$$\|x\| = \sup_{u \in A} |x(u)|, \quad (78)$$

for all $x \in U$, it is not difficult to see that $(U, \|\cdot\|)$ is a Banach space. Moreover, we define

$$d(x, y) = \sup_{u \in A} |x(u) - y(u)|^2, \quad (79)$$

for all $x, y \in U$. Then, (U, d) is a complete Branciari b -metric space with $s = 3$. Consider the operators $S, T : U \rightarrow \mathbb{R}$ given by

$$S(x)(u) = \sup_{v \in B} \{f(u, v) + G_1(u, v, x(\varphi(u, v)))\}, \text{ for all } u \in A, \quad (80)$$

$$T(y)(u) = \sup_{v \in B} \{f(u, v) + G_2(u, v, y(\varphi(u, v)))\}, \text{ for all } u \in A, \quad (81)$$

Now, we prove the following theorem.

Theorem 16. Let $S, T : U \rightarrow \mathbb{R}$ be defined as in (80) and (81), respectively. Suppose that the following hypotheses hold:

- (i) $f : A \times B \rightarrow \mathbb{R}$ and $G_1, G_2 : A \times B \times \mathbb{R} \rightarrow \mathbb{R}$ are bounded functions
- (ii) There exists $\lambda > 0$ such that, for all $u \in A$, $v \in B$, and $t, r \in \mathbb{R}$

$$|G_i(u, v, t) - G_i(u, v, r)| \leq \sqrt{\frac{e^{-\lambda}}{3}} |t - r|, i = 1, 2 \quad (82)$$

(iii) There exists $\tau > 0$ such that, for all $u \in A$, $v \in B$, and $x, y \in U$

$$|G_1(u, v, x(u)) - G_2(u, v, y(u))|^2 \leq e^{-\tau} M(x, y), \quad (83)$$

where

$$M(x, y) = ad(x, y) + bd(x, Sx) + cd(y, Ty), \quad (84)$$

for all $a, b, c \in [0, 1]$ with $a + b + c < 1$, $3b < 1$, and $\min \{d(Sx, Ty), d(x, y), d(x, Sx), d(y, Ty)\} > 0$

Then, there exists a unique common fixed point of the pair mappings (S, T) .

Proof. Indeed, let $x \in U$, then x is bounded, there exists $\lambda_1 > 0$ such that $|x(u)| \leq \lambda_1$ for all $u \in A$. By hypothesis (i), there exist λ_2, λ_3 , and λ_4 such that

$$|f(u, v)| \leq \lambda_2, |G_1(u, v, 0)| \leq \lambda_3, |G_2(u, v, 0)| \leq \lambda_4, \quad (85)$$

for all $u \in A$ and $v \in B$. Furthermore, by (ii), for all $u \in A$ and $v \in B$, we have

$$\begin{aligned} & |f(u, v) + G_1(u, v, x(\varphi(u, v)))| \\ & \leq |f(u, v) + |G_1(u, v, x(\varphi(u, v)))|| \\ & \leq |f(u, v)| + |G_1(u, v, x(\varphi(u, v))) - G_1(u, v, 0)| \\ & \quad + |G_1(u, v, 0)| \leq \lambda_2 + \sqrt{\frac{e^{-\lambda}}{3}} |x(\varphi(u, v))| + \lambda_3 \\ & \leq \lambda_2 + \sqrt{\frac{e^{-\lambda}}{3}} \lambda_1 + \lambda_3. \end{aligned} \quad (86)$$

Consequently, for all $u \in A$, $v \in B$, and $x, y \in U$, we get

$$S(x)(u) = \sup_{v \in B} \{f(u, v) + G_1(u, v, x(\varphi(u, v)))\} \leq \lambda_2 + \sqrt{\frac{e^{-\lambda}}{3}} \lambda_1 + \lambda_3. \quad (87)$$

Similarly,

$$T(x)(u) = \sup_{v \in B} \{f(u, v) + G_2(u, v, x(\varphi(u, v)))\} \leq \lambda_2 + \sqrt{\frac{e^{-\lambda}}{3}} \lambda_1 + \lambda_4, \quad (88)$$

which show that S and T are bounded. Let $\varepsilon > 0$, $x_1, x_2 \in U$, $u \in A$, there exist $v_1, v_2 \in B$ such that

$$S(x_1)(u) < f(u, v_1) + G_1(u, v_1, x_1(\varphi(u, v_1))) + \varepsilon, \quad (89)$$

$$T(x_2)(u) < f(u, v_2) + G_2(u, v_2, x_2(\varphi(u, v_2))) + \varepsilon, \quad (90)$$

$$S(x_1)(u) < f(u, v_2) + G_1(u, v_2, x_1(\varphi(u, v_2))), \quad (91)$$

$$T(x_2)(u) < f(u, v_1) + G_2(u, v_1, x_2(\varphi(u, v_1))). \quad (92)$$

Then by (89) and (92), we obtain

$$\begin{aligned} S(x_1)(u) - T(x_2)(u) & < G_1(u, v_1, x_1(\varphi(u, v_1))) \\ & \quad - G_2(u, v_1, x_2(\varphi(u, v_1))) + \varepsilon, \\ & \leq |G_1(u, v_1, x_1(\varphi(u, v_1))) - G_2(u, v_1, x_2(\varphi(u, v_1)))| + \varepsilon. \end{aligned} \quad (93)$$

Similarly, by (90) and (91), we deduce

$$\begin{aligned} T(x_2)(u) - S(x_1)(u) & < |G_2(u, v_1, x_2(\varphi(u, v_1))) \\ & \quad - G_1(u, v_1, x_1(\varphi(u, v_1)))| + \varepsilon. \end{aligned} \quad (94)$$

Besides, we also find

$$\begin{aligned} S(x_1)(u) - S(x_2)(u) & < |G_1(u, v_1, x_1(\varphi(u, v_1))) \\ & \quad - G_1(u, v_1, x_2(\varphi(u, v_1)))| + \varepsilon \leq \sqrt{\frac{e^{-\lambda}}{3}} |x_1(\varphi(u, v_1)) \\ & \quad - x_2(\varphi(u, v_1))| + \varepsilon. \end{aligned} \quad (95)$$

Since ε is arbitrary, we conclude

$$\begin{aligned} & |S(x_1)(u) - T(x_2)(u)|^2 \\ & \leq |G_1(u, v_1, x_1(\varphi(u, v_1))) - G_1(u, v_1, x_2(\varphi(u, v_1)))|^2 \\ & \leq e^{-\tau} M(x_1, x_2), |S(x_1)(u) - S(x_2)(u)|^2 \\ & \leq \frac{e^{-\lambda}}{3} |x_1(\varphi(u, v_1)) - x_2(\varphi(u, v_1))|^2. \end{aligned} \quad (96)$$

Hence, we deduce

$$\begin{aligned} d(Sx_1, Tx_2) & \leq e^{-\tau} M(x_1, x_2), \\ 3d(Sx_1, Sx_2) & \leq e^{-\lambda} d(x_1, x_2). \end{aligned} \quad (97)$$

Similarly, we can get

$$3d(Tx_1, Tx_2) \leq e^{-\lambda} d(x_1, x_2). \quad (98)$$

Then,

$$\begin{aligned} \tau + \ln(d(Sx_1, Tx_2)) & \leq \ln(M(x_1, x_2)), \\ \lambda + \ln(3d(Sx_1, Sx_2)) & \leq \ln(d(x_1, x_2)), \\ \lambda + \ln(3d(Tx_1, Tx_2)) & \leq \ln(d(x_1, x_2)), \end{aligned} \quad (99)$$

which show that S and T are two F -contraction mappings on U with $F(x) = \ln x$. By Theorem 11, there exists a unique common fixed point of the pair mappings (S, T) which is the solution of the equations (76) and (77).

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare no conflict of interest.

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