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Research Article

On New Modifications Governed by Quantum Hahn's Integral Operator Pertaining to Fractional Calculus

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In the article, we present several generalizations for the generalized Čebyšev type inequality in the frame of quantum fractional Hahn's integral operator by using the quantum shift operator ${}_{\sigma}\Psi_{\mathfrak{q}}(\varsigma) = \mathfrak{q}\varsigma + (1-\mathfrak{q})\sigma(\varsigma \in [l_1, l_2], \sigma = l_1 + \omega/(1-\mathfrak{q}), 0 < \mathfrak{q} < 1, \omega \geq 0)$. As applications, we provide some associated variants to illustrate the efficiency of quantum Hahn's integral operator and compare our obtained results and proposed technique with the previously known results and existing technique. Our ideas and approaches may lead to new directions in fractional quantum calculus theory.

1. Introduction

In 1882, Čebyšev discovered a fascinating and significantly valuable integral inequality as follows:

$$\frac{1}{l_1 - l_2}\int_{l_1}^{l_2} \mathcal{Q}(x)\mathcal{U}(x)dx \leq \left(\frac{1}{l_1 - l_2}\int_{l_1}^{l_2} \mathcal{Q}(x)dx\right)\left(\frac{1}{l_1 - l_2}\int_{l_1}^{l_2} \mathcal{U}(x)dx\right), \tag{1}$$

if $\mathcal{Q}$ and $\mathcal{U}$ are two integrable and synchronous functions on $[l_1, l_2]$, where the functions $\mathcal{Q}$ and $\mathcal{U}$ are said to be synchronous on $[l_1, l_2]$ if

$$(\mathcal{Q}(x) - \mathcal{Q}(y))(\mathcal{U}(x) - \mathcal{U}(y)) \geq 0 \tag{2}$$

for all $x, y \in [l_1, l_2]$.

It is well known that the Čebyšev inequality (1) has wild applications in the fields of pure and applied mathematics [1–10]. Recently, the generalizations and variants for the Čebyšev inequality (1) have attracted the attention of many researchers [11–20].

Quantum difference operators are receiving an increase of interest due to their applications [21, 22]. Roughly speaking, quantum calculus can substitute the classical derivative by a difference operator, which allows to deal nondifferentiation functions.

Let $\mathfrak{q} \in (0,1)$, $\omega \geq 0$, $I \subseteq \mathbb{R}$ be an interval such that $\omega_0 = \omega/(1-\mathfrak{q}) \in I$, and $\mathscr{H}_1 : I \longrightarrow \mathbb{R}$ be a real-valued function. Then, the Hahn difference operator $\mathscr{D}_{\mathfrak{q},\omega}$ [23] is defined by

$$\mathscr{D}_{\mathfrak{q},\omega}\mathscr{H}_1(\varsigma) = \begin{cases} \dfrac{\mathscr{H}_1(\mathfrak{q}\varsigma + \omega) - \mathscr{H}_1(\varsigma)}{\varsigma(\mathfrak{q}-1) + \omega}, & \varsigma \neq \omega_0, \\ \mathscr{H}_1'(\omega_0), & \varsigma = \omega_0, \end{cases} \tag{3}$$

if $\mathscr{H}_1$ is differentiable at ω_0.

The Hahn difference operator (3) unifies (in the limit) the Jackson q-difference derivative $\mathscr{D}_{\mathbf{q}}$ [24] for $\mathbf{q} \in (0,1)$ and the forward difference $\mathscr{D}_\omega$ for $\mathbf{q} \longrightarrow 1$, which are defined by

$$\mathscr{D}_{\mathbf{q}}\mathscr{H}_1(\varsigma) = \begin{cases} \dfrac{\mathscr{H}_1(\varsigma) - \mathscr{H}_1(\mathbf{q}\varsigma)}{\varsigma(1-\mathbf{q})}, & \varsigma \neq 0, \\ \mathscr{H}_1'(0), & \varsigma = 0, \end{cases} \tag{4}$$

if $\mathscr{H}_1'(0)$ exists for $\omega = 0$, and

$$\mathscr{D}_\omega \mathscr{H}_1(\varsigma) = \frac{\mathscr{H}_1(\varsigma+\omega) - \mathscr{H}_1(\varsigma)}{\omega} \tag{5}$$

for $\omega > 0$.

The Hahn difference operator has been applied successfully in the construction of families of orthogonal polynomials as well as in approximation problems [25–28].

In [29], the authors introduced some concepts of fractional quantum calculus in terms of a $\mathbf{q}$-shifting operator ${}_\sigma\Psi_{\mathbf{q}}(\varsigma) = \mathbf{q}\varsigma + (1-\mathbf{q})\sigma$.

Let $I = [l_1, l_2] \subseteq \mathbb{R}$ be an interval. Then, the point σ of Hahn calculus on the interval $[l_1, l_2]$ generated by the quantum numbers $0 < \mathbf{q} < 1$ and $\omega \geq 0$ is given by

$$\sigma = l_1 + \frac{\omega}{1-\mathbf{q}}. \tag{6}$$

We state that $\sigma \in [l_1, l_2]$ for all consequences of our investigation; the quantum Hahn shifting operator is defined by

$${}_\sigma\Psi_{\mathbf{q}}(\varsigma) = \mathbf{q}\varsigma + (1-\mathbf{q})\sigma, \quad \varsigma \in [l_1, l_2], \tag{7}$$

and the iterated κ-times quantum shifting is given by

$${}_\sigma\Psi_{\mathbf{q}}^{\kappa}(\varsigma) = {}_\sigma\Psi_{\mathbf{q}}^{\kappa-1}\left({}_\sigma\Psi_{\mathbf{q}}(\varsigma)\right) = \mathbf{q}^\kappa\varsigma + (1-\mathbf{q}^\kappa)\sigma, \tag{8}$$

with ${}_\sigma\Psi_{\mathbf{q}}^0(\varsigma) = \varsigma$ for $\varsigma \in [l_1, l_2]$.

Let us recall the basic knowledge of quantum Hahn calculus on an interval $[l_1, l_2]$ (see [30]).

Definition 1. Let $\mathscr{H}_1$ be a function defined on $[l_1, l_2]$. Then, the quantum Hahn difference operator is defined by

$${}_{l_1}\mathscr{D}_{\mathbf{q},\omega}\mathscr{H}_1(\varsigma) = \begin{cases} \dfrac{\mathscr{H}_1(\varsigma) - \mathscr{H}_1\left({}_\sigma\Psi_{\mathbf{q}}(\varsigma)\right)}{\varsigma - {}_\sigma\Psi_{\mathbf{q}}(\varsigma)}, & \varsigma \neq \sigma, \\ \mathscr{H}_1'(\sigma), & \varsigma = \sigma, \end{cases} \tag{9}$$

if $\mathscr{H}_1$ is differentiable at σ.

Definition 2. Let $\mathscr{H}_1 : [l_1, l_2] \longrightarrow \mathbb{R}$ be a given function and $x, y \in [l_1, l_2]$. Then, the $\mathbf{q}, \omega$-quantum Hahn integral of $\mathscr{H}_1$ from x to y is defined by

$$\int_x^y \mathscr{H}_1(s)_{l_1} d_{\mathbf{q},\omega}s = \int_\sigma^y \mathscr{H}_1(s)_{l_1} d_{\mathbf{q},\omega}s - \int_\sigma^x \mathscr{H}_1(s)_{l_1} d_{\mathbf{q},\omega}s, \tag{10}$$

where

$$\int_\sigma^\varsigma \mathscr{H}_1(s)_{l_1} d_{\mathbf{q},\omega}s = \left[\varsigma - {}_\sigma\Psi_{\mathbf{q}}(\varsigma)\right] \sum_{i=0}^{\infty} \mathbf{q}^i \mathscr{H}_1\left({}_\sigma\Psi_{\mathbf{q}}^i(\varsigma)\right) \tag{11}$$

for $\varsigma \in [l_1, l_2]$ provided that the series converge at $\varsigma = x$ and $\varsigma = y$. The function $\mathscr{H}_1$ is said to be $\mathbf{q}, \omega$-integrable on $[l_1, l_2]$ if (11) exists for all $\varsigma \in [l_1, l_2]$.

Before approaching the main definitions of fractional quantum Hahn calculus on $[l_1, l_2]$, we present the σ-power function which is stated as

$$\begin{aligned} (n-m)_\sigma^{(0)} &= 1, \\ (n-m)_\sigma^{(\kappa)} &= \prod_{i=0}^{\kappa-1}\left(n - {}_\sigma\Psi_{\mathbf{q}}^i(m)\right), \\ &\kappa \in \mathbb{N} \cup \{\infty\}. \end{aligned} \tag{12}$$

Precisely, if $\alpha \in \mathbb{R}$, then

$$(n-m)_\sigma^{(\alpha)} = \prod_{i=0}^{\infty} \frac{\left(n - {}_\sigma\Psi_{\mathbf{q}}^i(m)\right)}{\left(n - {}_\sigma\Psi_{\mathbf{q}}^{\alpha+i}(m)\right)}, \tag{13}$$

with ${}_\sigma\Psi_{\mathbf{q}}^\varsigma(m) = \mathbf{q}^\varsigma m + (1-\mathbf{q}^\varsigma)\sigma$ for $\varsigma \in \mathbb{R}$.

The $\mathbf{q}$-gamma function is defined as

$$\Gamma_{\mathbf{q}}(\eta) = \frac{(1-\mathbf{q})_0^{\eta-1}}{(1-\mathbf{q})^{\eta-1}} \tag{14}$$

for $\eta \in \mathbb{R} \setminus \{0, -1, -2, \cdots\}$.

Obviously, $\Gamma_{\mathbf{q}}(\eta+1) = [\alpha]_{\mathbf{q}}\Gamma_{\mathbf{q}}(\eta)$, where $[c]_{\mathbf{q}} = (1-\mathbf{q}^c)/(1-\mathbf{q})(c \in \mathbb{R})$ and $\mathbf{q}$ is the quantum number.

Now, we introduce the concepts of fractional quantum Hahn derivative and integral of Riemann-Liouville type [31].

Definition 3 (see [31]). Suppose that $\alpha \geq 0$ and $\mathscr{H}_1 : [l_1, l_2] \longrightarrow \mathbb{R}$ is a real-valued function. Then, the fractional quantum Hahn derivative $\left({}_{l_1}\mathscr{D}_{\mathbf{q},\omega}^\alpha \mathscr{H}_1\right)(\varsigma)$ of the Riemann-Liouville type of order α is defined by

$$\begin{aligned} &\left({}_{l_1}\mathscr{D}_{\mathbf{q},\omega}^\alpha \mathscr{H}_1\right)(\varsigma) \\ &= \frac{1}{\Gamma_{\mathbf{q}}(n-\alpha)_{l_1}} \mathscr{D}_{\mathbf{q},\omega}^n \int_{l_1}^{\varsigma} \left(\varsigma - {}_\sigma\Psi_{\mathbf{q}}(s)\right)_\sigma^{n-\alpha-1} \mathscr{H}_1(s)_{l_1} d_{\mathbf{q},\omega}, \end{aligned} \tag{15}$$

where n is the smallest integer greater than or equal to α.

Definition 4 (see [31]). Let $\alpha \geq 0$ and $\mathscr{H}_1 : [l_1, l_2] \longrightarrow \mathbb{R}$ be a real-valued function. Then, the fractional quantum Hahn integral $\left({}_{l_1}\mathcal{I}^{\alpha}_{\mathfrak{q},\omega}\mathcal{H}_1\right)(\varsigma)$ of the Riemann-Liouville type of order α is defined by

$$\left({}_{l_1}\mathcal{I}^{\alpha}_{\mathfrak{q},\omega}\mathcal{H}_1\right)(\varsigma) = \frac{1}{\Gamma_{\mathfrak{q}}(\alpha)}\int_{l_1}^{\varsigma}\left(\varsigma -_{\sigma}\Psi_{\mathfrak{q}}(s)\right)_{\sigma}^{\alpha-1}\mathcal{H}_1(s)_{l_1}d_{\mathfrak{q},\omega}, \quad \varsigma \in [l_1, l_2], \tag{16}$$

with $\left({}_{l_1}\mathcal{I}^{l_1}_{\mathfrak{q},\omega}\mathcal{H}_1\right)(\varsigma) = \mathcal{H}_1(\varsigma)$ if the right-hand side exists.

Theorem 5 (see [31]). *Let $\alpha, \beta \in \mathbb{R}^+$, $\vartheta \in (-1,\infty)$, and $\sigma \in [l_1, l_2]$. Then, one has*

$$\begin{aligned} \left({}_{l_1}\mathcal{I}^{\alpha}_{\mathfrak{q},\omega}(x-l_1)^{\vartheta}_{\sigma}\right)(\varsigma) &= \frac{\Gamma_{\mathfrak{q}}(\vartheta+1)}{\Gamma_{\mathfrak{q}}(\alpha+\vartheta+1)}(\varsigma - l_1)^{\alpha+\vartheta}_{\sigma}, \\ \left({}_{l_1}\mathcal{D}^{\alpha}_{\mathfrak{q},\omega}(x-l_1)^{\vartheta}_{\sigma}\right)(\varsigma) &= \frac{\Gamma_{\mathfrak{q}}(\vartheta+1)}{\Gamma_{\mathfrak{q}}(\alpha-\vartheta+1)}(\varsigma - l_1)^{\alpha-\vartheta}_{\sigma}. \end{aligned} \tag{17}$$

Fractional calculus is invariably important in almost all fields of mathematics and applied sciences. Also, the fractional differential equations can provide adequate models for many physical problems in areas such as heat equation, wave equation, Poisson equation and Laplace equation, fluid mechanics, biological populations, viscoelasticity, advection-diffusion, and signal processing [32, 33].

Inequality plays an irreplaceable role in the development of mathematics. Very recently, many new inequalities such as Hermite-Hadamard type inequality [34–38], Petrović type inequality [39], Pólya-Szegö type inequality [40], Ostrowski type inequality [41], reverse Minkowski inequality [42], Jensen type inequality [43, 44], Bessel function inequality [45], trigonometric and hyperbolic function inequalities [46], fractional integral inequality [47–51], complete and generalized elliptic integral inequalities [52–57], generalized convex function inequality [58–60], and mean value inequality [61–63] have been discovered by many researchers. In particular, the applications of integral inequalities have gained considerable importance among researchers for fixed-point theorems; the existence and uniqueness of solutions for differential equations [64–68] and numerous numerical and analytical methods have been recommended for the advancement of integral inequalities [69–75].

Asawasamrit et al. [76] expounded the concept of $\mathfrak{q}$-derivative over the interval $[l_1, l_2] \subset \mathbb{R}$ and derived several inequalities on quantum analogues, for example, $\mathfrak{q}$-Cauchy-Schwarz inequality, $\mathfrak{q}$-Grüss-Čebyšev integral inequality, $\mathfrak{q}$-Grüss inequality, and other integral inequalities, by use of the convexity theory.

The main purpose of the article is to provide the novel versions of the generalized Čebyšev inequalities and present the associated variants via quantum Hahn's fractional integral operator.

To end this section, we give the definition of the one-sided fractional quantum Hahn integral in the Riemann-Liouville sense.

Definition 6. Let $\alpha \geq 0$ and $\mathcal{H}_1 : [l_1, l_2] \longrightarrow \mathbb{R}$ be a real-valued function. Then, the one-sided fractional quantum Hahn integral $\left({}_{0^+}\mathcal{I}^{\alpha}_{\mathfrak{q},\omega}\mathcal{H}_1\right)(\varsigma)$ of Riemann-Liouville type of order α is defined by

$$\left({}_{0^+}\mathcal{I}^{\alpha}_{\mathfrak{q},\omega}\mathcal{H}_1\right)(\varsigma) = \frac{1}{\Gamma_{\mathfrak{q}}(\alpha)}\int_{0}^{\varsigma}\left(\varsigma -_{\sigma}\Psi_{\mathfrak{q}}(s)\right)_{\sigma}^{\alpha-1}\mathcal{H}_1(s)_{l_1}d_{\mathfrak{q},\omega}, \quad s > 0. \tag{18}$$

2. Certain Extended Weighted Čebyšev Fractional Quantum Hahn Integral Operator

In this section, we provide several new generalizations for the weighted extensions of Čebyšev functionals via a quantum Hahn integral operator.

Theorem 7. *Let $s, s_1, u, u_1, r, r_1 > 1$ with $s^{-1} + s_1{}^{-1} = r^{-1} + r_1{}^{-1} = u^{-1} + u_1{}^{-1} = 1$, $\mathfrak{q} \in (0,1)$, $\omega \geq 0$, $\mathcal{H}_1$ be a positive $\mathfrak{q},\omega$-integrable function defined on $[0,\infty)$, and $\mathcal{Q}$ and $\mathcal{U}$ be two $\mathfrak{q},\omega$-differentiable functions defined on $[0,\infty)$ such that $\mathcal{Q}' \in L_s([0,\infty))$ and $\mathcal{U}' \in L_r([0,\infty))$. Then, the inequalities*

$$\begin{aligned} 2\Big|&\left(\left({}_{0^+}\mathcal{I}^{\alpha}_{\mathfrak{q},\omega}\mathcal{H}_1\right)(\varsigma)\left({}_{0^+}\mathcal{I}^{\alpha}_{\mathfrak{q},\omega}\mathcal{Q}\mathcal{U}\right)(\varsigma) - \left({}_{0^+}\mathcal{I}^{\alpha}_{\mathfrak{q},\omega}\mathcal{H}_1\mathcal{Q}\right)(\varsigma)\left({}_{0^+}\mathcal{I}^{\alpha}_{\mathfrak{q},\omega}\mathcal{H}_1\mathcal{U}\right)(\varsigma)\right)\Big| \\ &\leq \left(\frac{\|\mathcal{Q}'\|_s^u}{\Gamma_{\mathfrak{q}}^u(\alpha)}\int_0^{\varsigma}\int_0^{\varsigma}\left(\varsigma -_{\sigma}\Psi_{\mathfrak{q}}(x)\right)_{\sigma}^{\alpha-1}\left(\varsigma -_{\sigma}\Psi_{\mathfrak{q}}(y)\right)_{\sigma}^{(\alpha-1)}\Phi'(x) \cdot \Phi'(y)\mathcal{H}_1(x)\mathcal{H}_1(y)|x-y|^{1/s_1+1/r_1}d_{\mathfrak{q},\omega}x d_{\mathfrak{q},\omega}y\right)^{1/u} \\ &\quad \times \left(\frac{\|\mathcal{U}'\|_r^{u_1}}{\Gamma_{\mathfrak{q}}^{u_1}(\alpha)}\int_0^{\varsigma}\int_0^{\varsigma}\left(\varsigma -_{\sigma}\Psi_{\mathfrak{q}}(x)\right)_{\sigma}^{\alpha-1}\left(\varsigma -_{\sigma}\Psi_{\mathfrak{q}}(y)\right)_{\sigma}^{\alpha-1} \cdot \mathcal{H}_1(x)\mathcal{H}_1(y)|x-y|^{1/s_1+1/r_1}d_{\mathfrak{q},\omega}x d_{\mathfrak{q},\omega}y\right)^{1/u_1} \\ &\leq \frac{\|\mathcal{Q}'\|_s^u\|\mathcal{U}'\|_r^{u_1}}{\left(\Gamma_{\mathfrak{q}}(\alpha)\right)^2}\left(\int_0^{\varsigma}\int_0^{\varsigma}\left(\varsigma -_{\sigma}\Psi_{\mathfrak{q}}(x)\right)_{\sigma}^{\alpha-1}\left(\varsigma -_{\sigma}\Psi_{\mathfrak{q}}(y)\right)_{\sigma}^{\alpha-1} \cdot \mathcal{H}_1(x)\mathcal{H}_1(y)|x-y|^{1/s_1+1/r_1}d_{\mathfrak{q},\omega}x d_{\mathfrak{q},\omega}y\right) \end{aligned} \tag{19}$$

hold for all $\varsigma > 0$.

Proof. Let $x, y \in (0, \varsigma)$ and

$$\mathcal{G}(x,y) = \left(\mathcal{Q}(x) - \mathcal{Q}(y)\right)\left(\mathcal{U}(x) - \mathcal{U}(y)\right). \tag{20}$$

Then, $\mathcal{G}(x,y)$ can be written as

$$\mathcal{G}(x,y) = \mathcal{Q}(x)\mathcal{U}(x) - \mathcal{Q}(x)\mathcal{U}(y) - \mathcal{Q}(y)\mathcal{U}(x) - \mathcal{U}(y)\mathcal{Q}(y). \tag{21}$$

Multiplying both sides of (21) by $((\varsigma{-_\sigma}\Psi_{\mathfrak{q}}(x))_\sigma^{\alpha-1}/\Gamma_{\mathfrak{q}}(\alpha))\mathscr{H}_1(x)$ and then performing the $\mathfrak{q},\omega$-integration with respect to x over $(0,\varsigma)$, we have

$$\begin{aligned}
\int_0^\varsigma \frac{(\varsigma{-_\sigma}\Psi_{\mathfrak{q}}(x))_\sigma^{\alpha-1}}{\Gamma_{\mathfrak{q}}(\alpha)}\mathscr{H}_1(x)\mathscr{G}(x,y)d_{\mathfrak{q},\omega}x
&= \int_0^\varsigma \frac{(\varsigma{-_\sigma}\Psi_{\mathfrak{q}}(x))_\sigma^{\alpha-1}}{\Gamma_{\mathfrak{q}}(\alpha)}\mathscr{H}_1(x)\mathscr{Q}(x)\mathscr{U}(x)d_{\mathfrak{q},\omega}x \\
&\quad - \int_0^\varsigma \frac{(\varsigma{-_\sigma}\Psi_{\mathfrak{q}}(x))_\sigma^{\alpha-1}}{\Gamma_{\mathfrak{q}}(\alpha)}\mathscr{H}_1(x)\mathscr{Q}(x)\mathscr{U}(y)d_{\mathfrak{q},\omega}x \\
&\quad - \int_0^\varsigma \frac{(\varsigma{-_\sigma}\Psi_{\mathfrak{q}}(x))_\sigma^{\alpha-1}}{\Gamma_{\mathfrak{q}}(\alpha)}\mathscr{H}_1(x)\mathscr{Q}(y)\mathscr{U}(x)d_{\mathfrak{q},\omega}x \\
&\quad - \mathscr{U}(y)\mathscr{Q}(y)\int_0^\varsigma \frac{(\varsigma{-_\sigma}\Psi_{\mathfrak{q}}(x))_\sigma^{\alpha-1}}{\Gamma_{\mathfrak{q}}(\alpha)}\mathscr{H}_1(x)d_{\mathfrak{q},\omega}x.
\end{aligned} \tag{22}$$

Inequality (22) can be rewritten as

$$\begin{aligned}
\int_0^\varsigma \frac{(\varsigma{-_\sigma}\Psi_{\mathfrak{q}}(x))_\sigma^{\alpha-1}}{\Gamma_{\mathfrak{q}}(\alpha)}\mathscr{H}_1(x)\mathscr{G}(x,y)d_{\mathfrak{q},\omega}x
&= \left({}_{0^+}\mathscr{I}_{\mathfrak{q},\omega}^\alpha\mathscr{H}_1\mathscr{Q}\mathscr{U}\right)(\varsigma) - \mathscr{U}(y)\left({}_{0^+}\mathscr{I}_{\mathfrak{q},\omega}^\alpha\mathscr{H}_1\mathscr{Q}\right)(\varsigma) \\
&\quad - \mathscr{Q}(y)\left({}_{0^+}\mathscr{I}_{\mathfrak{q},\omega}^\alpha\mathscr{H}_1\mathscr{U}\right)(\varsigma) + \mathscr{Q}(y)\mathscr{U}(y)\left({}_{0^+}\mathscr{I}_{\mathfrak{q},\omega}^\alpha\mathscr{H}_1\right)(\varsigma).
\end{aligned} \tag{23}$$

Multiplying both sides of (23) by $((\varsigma{-_\sigma}\Psi_{\mathfrak{q}}(y))_\sigma^{\alpha-1}/\Gamma_{\mathfrak{q}}(\alpha))\mathscr{H}_1(y)$ and then performing the $\mathfrak{q},\omega$-integration with respect to y over $(0,\varsigma)$, we get

$$\begin{aligned}
&\frac{1}{(\Gamma_{\mathfrak{q}}(\alpha))^2}\int_0^\varsigma\int_0^\varsigma (\varsigma{-_\sigma}\Psi_{\mathfrak{q}}(x))_\sigma^{\alpha-1}(\varsigma{-_\sigma}\Psi_{\mathfrak{q}}(y))_\sigma^{\alpha-1} \\
&\quad \cdot \mathscr{H}_1(x)\mathscr{H}_1(y)\mathscr{G}(x,y)d_{\mathfrak{q},\omega}x d_{\mathfrak{q},\omega}y \\
&\quad = 2\left(\left({}_{0^+}\mathscr{I}_{\mathfrak{q},\omega}^\alpha\mathscr{H}_1\right)(\varsigma)\left({}_{0^+}\mathscr{I}_{\mathfrak{q},\omega}^\alpha\mathscr{Q}\mathscr{U}\right)(\varsigma) - \left({}_{0^+}\mathscr{I}_{\mathfrak{q},\omega}^\alpha\mathscr{H}_1\mathscr{Q}\right)(\varsigma)\left({}_{0^+}\mathscr{I}_{\mathfrak{q},\omega}^\alpha\mathscr{H}_1\mathscr{U}\right)(\varsigma)\right).
\end{aligned} \tag{24}$$

Similarly, we have

$$\mathscr{G}(x,y) = \int_x^y\int_x^y \mathscr{Q}'(\theta)\mathscr{U}'(\vartheta)d\theta d\vartheta. \tag{25}$$

Taking into account the Hölder inequality, we have

$$|\mathscr{Q}(x)-\mathscr{Q}(y)| \le |x-y|^{1/s_1}\left|\int_x^y|\mathscr{Q}'(\theta)|^s d\theta\right|^{1/s}, \tag{26}$$

$$|\mathscr{U}(x)-\mathscr{U}(y)| \le |x-y|^{1/r_1}\left|\int_x^y|\mathscr{U}'(\vartheta)|^r d\vartheta\right|^{1/r}. \tag{27}$$

It follows from (26) and (27) that

$$\begin{aligned}
|\mathscr{G}(x,y)| &\le |(\mathscr{Q}(x)-\mathscr{Q}(y))(\mathscr{U}(x)-\mathscr{U}(y))| \\
&\le |x-y|^{1/s_1+1/r_1}\left|\int_x^y|\mathscr{Q}'(\theta)|^s d\theta\right|^{1/s}\left|\int_x^y|\mathscr{U}'(\vartheta)|^r d\vartheta\right|^{1/r}.
\end{aligned} \tag{28}$$

From (24) and (28), we obtain

$$\begin{aligned}
&2\left|\left({}_{0^+}\mathscr{I}_{\mathfrak{q},\omega}^\alpha\mathscr{H}_1\right)(\varsigma)\left({}_{0^+}\mathscr{I}_{\mathfrak{q},\omega}^\alpha\mathscr{Q}\mathscr{U}\right)(\varsigma) - \left({}_{0^+}\mathscr{I}_{\mathfrak{q},\omega}^\alpha\mathscr{H}_1\mathscr{Q}\right)(\varsigma)\left({}_{0^+}\mathscr{I}_{\mathfrak{q},\omega}^\alpha\mathscr{H}_1\mathscr{U}\right)(\varsigma)\right| \\
&= \frac{1}{(\Gamma_{\mathfrak{q}}(\alpha))^2}\int_0^\varsigma\int_0^\varsigma (\varsigma{-_\sigma}\Psi_{\mathfrak{q}}(x))_\sigma^{\alpha-1}(\varsigma{-_\sigma}\Psi_{\mathfrak{q}}(y))_\sigma^{\alpha-1} \cdot \mathscr{H}_1(x)\mathscr{H}_1(y)|\mathscr{G}(x,y)|d_{\mathfrak{q},\omega}x d_{\mathfrak{q},\omega}y \\
&\le \frac{1}{(\Gamma_{\mathfrak{q}}(\alpha))^2}\int_0^\varsigma\int_0^\varsigma (\varsigma{-_\sigma}\Psi_{\mathfrak{q}}(x))_\sigma^{\alpha-1}(\varsigma{-_\sigma}\Psi_{\mathfrak{q}}(y))_\sigma^{\alpha-1}\mathscr{H}_1(x)\mathscr{H}_1(y) \\
&\quad \times |x-y|^{1/s_1+1/r_1}\left|\int_x^y|\mathscr{Q}'(\theta)|^s d\theta\right|^{1/s}\left|\int_x^y|\mathscr{U}'(\vartheta)|^r d\vartheta\right|^{1/r}d_{\mathfrak{q},\omega}x d_{\mathfrak{q},\omega}y.
\end{aligned} \tag{29}$$

Making use of the Hölder inequality for bivariate integral, we have

$$\begin{aligned}
&2\left|\left({}_{0^+}\mathscr{I}_{\mathfrak{q},\omega}^\alpha\mathscr{H}_1\right)(\varsigma)\left({}_{0^+}\mathscr{I}_{\mathfrak{q},\omega}^\alpha\mathscr{Q}\mathscr{U}\right)(\varsigma) - \left({}_{0^+}\mathscr{I}_{\mathfrak{q},\omega}^\alpha\mathscr{H}_1\mathscr{Q}\right)(\varsigma)\left({}_{0^+}\mathscr{I}_{\mathfrak{q},\omega}^\alpha\mathscr{H}_1\mathscr{U}\right)(\varsigma)\right| \\
&\le \frac{1}{(\Gamma_{\mathfrak{q}}(\alpha))^2}\left(\int_0^\varsigma\int_0^\varsigma (\varsigma{-_\sigma}\Psi_{\mathfrak{q}}(x))_\sigma^{\alpha-1} \cdot (\varsigma{-_\sigma}\Psi_{\mathfrak{q}}(y))_\sigma^{\alpha-1}\mathscr{H}_1(x)\mathscr{H}_1(y) \times |x-y|^{1/s_1+1/r_1}\left|\int_x^y|\mathscr{Q}'(\theta)|^s d\theta\right|^{u/s}d_{\mathfrak{q},\omega}x d_{\mathfrak{q},\omega}y\right)^{1/u} \\
&\quad \times \left(\int_0^\varsigma\int_0^\varsigma (\varsigma{-_\sigma}\Psi_{\mathfrak{q}}(x))_\sigma^{\alpha-1}(\varsigma{-_\sigma}\Psi_{\mathfrak{q}}(y))_\sigma^{\alpha-1}\mathscr{H}_1(x)\mathscr{H}_1(y) \times |x-y|^{1/s_1+1/r_1}\left|\int_x^y|\mathscr{U}'(\vartheta)|^r d\vartheta\right|^{u_1/r}d_{\mathfrak{q},\omega}x d_{\mathfrak{q},\omega}y\right)^{1/u_1}.
\end{aligned} \tag{30}$$

It follows from (30) and the inequalities

$$\left|\int_x^y|\mathscr{Q}'(\theta)|^s d\theta\right|^{1/s} \le \|\mathscr{Q}'\|_s, \left|\int_x^y|\mathscr{U}'(\vartheta)|^r d\vartheta\right|^{1/r} \le \mathrm{a}\|\mathscr{U}'\|_r \tag{31}$$

that

$$
\begin{aligned}
&2|\left(\left({}_{0^+}\mathcal{I}^{\alpha}_{\mathfrak{q},\omega}\mathcal{H}_1\right)(\varsigma)\left({}_{0^+}\mathcal{I}^{\alpha}_{\mathfrak{q},\omega}\mathcal{Q}\mathcal{U}\right)(\varsigma)\right.\\
&\quad\left.-\left({}_{0^+}\mathcal{I}^{\alpha}_{\mathfrak{q},\omega}\mathcal{H}_1\mathcal{Q}\right)(\varsigma)\left({}_{0^+}\mathcal{I}^{\alpha}_{\mathfrak{q},\omega}\mathcal{H}_1\mathcal{U}\right)(\varsigma)\right)|\\
&\le\left(\frac{\|\mathcal{Q}'\|_s^u}{\Gamma^u_{\mathfrak{q}}(\alpha)}\int_0^{\varsigma}\int_0^{\varsigma}\left(\varsigma-_{\sigma}\Psi_{\mathfrak{q}}(x)\right)^{\alpha-1}_{\sigma}\left(\varsigma-_{\sigma}\Psi_{\mathfrak{q}}(y)\right)^{\alpha-1}_{\sigma}\Phi'(x)\Phi'(y)\right.\\
&\quad\left.\cdot\mathcal{H}_1(x)\mathcal{H}_1(y)|x-y|^{1/s_1+1/r_1}d_{\mathfrak{q},\omega}x d_{\mathfrak{q},\omega}y\right)^{1/u}\\
&\quad\times\left(\frac{\|\mathcal{U}'\|_r^{u_1}}{\Gamma^{u_1}_{\mathfrak{q}}(\alpha)}\int_0^{\varsigma}\int_0^{\varsigma}\left(\varsigma-_{\sigma}\Psi_{\mathfrak{q}}(x)\right)^{\alpha-1}_{\sigma}\left(\varsigma-_{\sigma}\Psi_{\mathfrak{q}}(y)\right)^{\alpha-1}_{\sigma}\right.\\
&\quad\left.\cdot\mathcal{H}_1(x)\mathcal{H}_1(y)|x-y|^{1/s_1+1/r_1}d_{\mathfrak{q},\omega}x d_{\mathfrak{q},\omega}y\right)^{1/u_1}.
\end{aligned}
\tag{32}
$$

Therefore, we get the desired inequality

$$
\begin{aligned}
&2|\left(\left({}_{0^+}\mathcal{I}^{\alpha}_{\mathfrak{q},\omega}\mathcal{H}_1\right)(\varsigma)\left({}_{0^+}\mathcal{I}^{\alpha}_{\mathfrak{q},\omega}\mathcal{Q}\mathcal{U}\right)(\varsigma)\right.\\
&\quad\left.-\left({}_{0^+}\mathcal{I}^{\alpha}_{\mathfrak{q},\omega}\mathcal{H}_1\mathcal{Q}\right)(\varsigma)\left({}_{0^+}\mathcal{I}^{\alpha}_{\mathfrak{q},\omega}\mathcal{H}_1\mathcal{U}\right)(\varsigma)\right)|\\
&\le\frac{\|\mathcal{Q}'\|_s^u\|\mathcal{U}'\|_r^{u_1}}{\left(\Gamma_{\mathfrak{q}_1}(\alpha)\right)^2}\left(\int_0^{\varsigma}\int_0^{\varsigma}\left(\varsigma-_{\sigma}\Psi_{\mathfrak{q}}(x)\right)^{(\alpha-1)}_{\sigma}\left(\varsigma-_{\sigma}\Psi_{\mathfrak{q}}(y)\right)^{\alpha-1}_{\sigma}\right.\\
&\quad\left.\cdot\mathcal{H}_1(x)\mathcal{H}_1(y)|x-y|^{1/s_1+1/r_1}d_{\mathfrak{q},\omega}x d_{\mathfrak{q},\omega}y\right).
\end{aligned}
\tag{33}
$$

Let $\omega=1$. Then, Theorem 7 leads to Corollary 8 which provide a new result for $\mathfrak{q}$-fractional integral operator.hold for all $\varsigma>0$.

Corollary 8. *Let $s, s_1, u, u_1, r, r_1>1$ with $s^{-1}+s_1^{-1}=r^{-1}+r_1^{-1}=u^{-1}+u_1^{-1}=1$, $\mathfrak{q}\in(0,1)$, $\mathcal{H}_1$ be a positive $\mathfrak{q}, 1$-integrable function defined on $[0,\infty)$, and $\mathcal{Q}$ and $\mathcal{U}$ be two $\mathfrak{q}, 1$-differentiable functions defined on $[0,\infty)$ such that $\mathcal{Q}'\in L_s([0,\infty))$ and $\mathcal{U}'\in L_r([0,\infty))$. Then, the inequalities*

$$
\begin{aligned}
&2\left|\left(\left({}_{0^+}\mathcal{I}^{\alpha}_{\mathfrak{q},1}\mathcal{H}_1\right)(\varsigma)\left({}_{0^+}\mathcal{I}^{\alpha}_{\mathfrak{q},1}\mathcal{Q}\mathcal{U}\right)(\varsigma)\right.\right.\\
&\quad\left.\left.-\left({}_{0^+}\mathcal{I}^{\alpha}_{\mathfrak{q},1}\mathcal{H}_1\mathcal{Q}\right)(\varsigma)\left({}_{0^+}\mathcal{I}^{\alpha}_{\mathfrak{q},1}\mathcal{H}_1\mathcal{U}\right)(\varsigma)\right)\right|\\
&\le\left(\frac{\|\mathcal{Q}'\|_s^u}{\Gamma^u_{\mathfrak{q}}(\alpha)}\int_0^{\varsigma}\int_0^{\varsigma}\left(\varsigma-_{\sigma}\Psi_{\mathfrak{q}}(x)\right)^{(\alpha-1)}_{\sigma}\left(\varsigma-_{\sigma}\Psi_{\mathfrak{q}}(y)\right)^{\alpha-1}_{\sigma}\right.\\
&\quad\left.\cdot\Phi'(x)\Phi'(y)\mathcal{H}_1(x)\mathcal{H}_1(y)|x-y|^{1/s_1+1/r_1}d_q x d_{\mathfrak{q}}y\right)^{1/u}\\
&\quad\times\left(\frac{\|\mathcal{U}'\|_r^{u_1}}{\Gamma^{u_1}_{\mathfrak{q}}(\alpha)}\int_0^{\varsigma}\int_0^{\varsigma}\left(\varsigma-_{\sigma}\Psi_{\mathfrak{q}}(x)\right)^{\alpha-1}_{\sigma}\left(\varsigma-_{\sigma}\Psi_{\mathfrak{q}}(y)\right)^{\alpha-1}_{\sigma}\right.\\
&\quad\left.\cdot\mathcal{H}_1(x)\mathcal{H}_1(y)|x-y|^{1/s_1+1/r_1}d_{\mathfrak{q}}x d_{\mathfrak{q}}y\right)^{1/u_1}\\
&\le\frac{\|\mathcal{Q}'\|_s^u\|\mathcal{U}'\|_r^{u_1}}{\left(\Gamma_{\mathfrak{q}}(\alpha)\right)^2}\left(\int_0^{\varsigma}\int_0^{\varsigma}\left(\varsigma-_{\sigma}\Psi_{\mathfrak{q}}(x)\right)^{\alpha-1}_{\sigma}\left(\varsigma-_{\sigma}\Psi_{\mathfrak{q}}(y)\right)^{\alpha-1}_{\sigma}\right.\\
&\quad\left.\cdot\mathcal{H}_1(x)\mathcal{H}_1(y)|x-y|^{1/s_1+1/r_1}d_{\mathfrak{q}}x d_{\mathfrak{q}}y\right)
\end{aligned}
\tag{34}
$$

hold for all $\varsigma>0$.

Remark 9. If $\mathfrak{q}=\omega=1$, then Theorem 7 reduces to the result for the Riemann-Liouville fractional integral operator given in [77]. Some results given in the literature [13, 78] can also be obtained from Theorem 7 immediately.

Theorem 10. *Let $s, s_1, u, u_1, r, r_1>1$ with $s^{-1}+s_1^{-1}=r^{-1}+r_1^{-1}=u^{-1}+u_1^{-1}=1$, $\mathfrak{q}_i\in(0,1)$, $\omega_i\ge 0(i=1,2)$, $\mathcal{H}_1$ and $\mathcal{H}_2$ be the positive $\mathfrak{q}_i, \omega_i$-integrable functions defined on $[0,\infty)$, and $\mathcal{Q}$ and $\mathcal{U}$ be the $\mathfrak{q}_i, \omega_i$-differentiable functions defined on $[0,\infty)$ such that $\mathcal{Q}'\in L_s([0,\infty))$ and $\mathcal{U}'\in L_r([0,\infty))$. Then, the inequality*

$$
\begin{aligned}
&|\left({}_{0^+}\mathcal{I}^{\beta}_{\mathfrak{q}_2,\omega_2}\mathcal{H}_2\right)(\varsigma)\left({}_{0^+}\mathcal{I}^{\alpha}_{\mathfrak{q}_1,\omega_1}\mathcal{H}_1\mathcal{Q}\mathcal{U}\right)(\varsigma)\\
&\quad-\left({}_{0^+}\mathcal{I}^{\beta}_{\mathfrak{q}_2,\omega_2}\mathcal{H}_1\mathcal{U}\right)(\varsigma)\left({}_{0^+}\mathcal{I}^{\alpha}_{\mathfrak{q}_1,\omega_1}\mathcal{H}_1\mathcal{Q}\right)(\varsigma)\\
&\quad-\left({}_{0^+}\mathcal{I}^{\beta}_{\mathfrak{q}_2,\omega_2}\mathcal{H}_2\mathcal{Q}\right)(\varsigma)\left({}_{0^+}\mathcal{I}^{\alpha}_{\mathfrak{q}_1,\omega_1}\mathcal{H}_1\mathcal{U}\right)(\varsigma)\\
&\quad+\left({}_{0^+}\mathcal{I}^{\beta}_{\mathfrak{q}_2,\omega_2}\mathcal{H}_2\mathcal{Q}\mathcal{U}\right)(\varsigma)\left({}_{0^+}\mathcal{I}^{\alpha}_{\mathfrak{q}_1,\omega_1}\mathcal{H}_1\right)(\varsigma)|\\
&\le\frac{\|\mathcal{Q}\|_s\|\mathcal{U}\|_r}{\Gamma_{\mathfrak{q}_1}(\alpha)\Gamma_{\mathfrak{q}_2}(\beta)}\int_0^{\varsigma}\int_0^{\varsigma}\left(\varsigma-{}_{\sigma_1}\Psi_{\mathfrak{q}_1}(x)\right)^{\alpha-1}_{\sigma_1}\left(\varsigma-_{\sigma_2}\Psi_{\mathfrak{q}_2}(y)\right)^{\beta-1}_{\sigma_2}\\
&\quad\times|x-y|^{1/s_1+1/r_1}\mathcal{H}_1(x)\mathcal{H}_2(y)d_{\mathfrak{q}_1,\omega_1}x d_{\mathfrak{q}_2,\omega_2}y
\end{aligned}
\tag{35}
$$

holds for all $\varsigma>0$.

Proof. Multiplying both sides of (23) by $\left((\varsigma-_{\sigma_2}\Psi_{\mathfrak{q}_2}(y))^{\beta-1}_{\sigma_2}/\Gamma_{\mathfrak{q}_2(\beta)}\right)\mathcal{H}_2(y)$ and performing $\mathfrak{q}_2, \omega_2$-integration with respect to y over $(0,\varsigma)$, we have

$$
\begin{aligned}
&\frac{1}{\Gamma_{\mathfrak{q}_1}(\alpha)\Gamma_{\mathfrak{q}_2}(\beta)}\int_0^{\varsigma}\int_0^{\varsigma}\left(\varsigma-_{\sigma_1}\Psi_{\mathfrak{q}_1}(x)\right)^{\alpha-1}_{\sigma_1}\left(\varsigma-_{\sigma_2}\Psi_{\mathfrak{q}_2}(y)\right)^{\beta-1}_{\sigma_2}\\
&\quad\cdot\mathcal{H}_1(x)\mathcal{H}_2(y)\mathcal{G}(x,y)d_{\mathfrak{q}_1,\omega_1}x d_{\mathfrak{q}_2,\omega_{29}}y\\
&=\left({}_{0^+}\mathcal{I}^{\beta}_{\mathfrak{q}_2,\omega_2}\mathcal{H}_2\right)(\varsigma)\left({}_{0^+}\mathcal{I}^{\alpha}_{\mathfrak{q}_1,\omega_1}\mathcal{H}_1\mathcal{Q}\mathcal{U}\right)(\varsigma)\\
&\quad-\left({}_{0^+}\mathcal{I}^{\beta}_{\mathfrak{q}_2,\omega_2}\mathcal{H}_1\mathcal{U}\right)(\varsigma)\left({}_{0^+}\mathcal{I}^{\alpha}_{\mathfrak{q}_1,\omega_1}\mathcal{H}_1\mathcal{Q}\right)(\varsigma)\\
&\quad-\left({}_{0^+}\mathcal{I}^{\beta}_{\mathfrak{q}_2,\omega_2}\mathcal{H}_2\mathcal{Q}\right)(\varsigma)\left({}_{0^+}\mathcal{I}^{\alpha}_{\mathfrak{q}_1,\omega_1}\mathcal{H}_1\mathcal{U}\right)(\varsigma)\\
&\quad+\left({}_{0^+}\mathcal{I}^{\beta}_{\mathfrak{q}_2,\omega_2}\mathcal{H}_2\mathcal{Q}\mathcal{U}\right)(\varsigma)\left({}_{0^+}\mathcal{I}^{\alpha}_{\mathfrak{q}_1,\omega_1}\mathcal{H}_1\right)(\varsigma).
\end{aligned}
\tag{36}
$$

Taking modulus on both sides of (36), one has

$$
\begin{aligned}
&|\left({}_{0^+}\mathcal{I}^{\beta}_{\mathfrak{q}_2,\omega_2}\mathcal{H}_2\right)(\varsigma)\left({}_{0^+}\mathcal{I}^{\alpha}_{\mathfrak{q}_1,\omega_1}\mathcal{H}_1\mathcal{Q}\mathcal{U}\right)(\varsigma)\\
&\quad-\left({}_{0^+}\mathcal{I}^{\beta}_{\mathfrak{q}_2,\omega_2}\mathcal{H}_1\mathcal{U}\right)(\varsigma)\left({}_{0^+}\mathcal{I}^{\alpha}_{\mathfrak{q}_1,\omega_1}\mathcal{H}_1\mathcal{Q}\right)(\varsigma)\\
&\quad-\left({}_{0^+}\mathcal{I}^{\beta}_{\mathfrak{q}_2,\omega_2}\mathcal{H}_2\mathcal{Q}\right)(\varsigma)\left({}_{0^+}\mathcal{I}^{\alpha}_{\mathfrak{q}_1,\omega_1}\mathcal{H}_1\mathcal{U}\right)(\varsigma)\\
&\quad+\left({}_{0^+}\mathcal{I}^{\beta}_{\mathfrak{q}_2,\omega_2}\mathcal{H}_2\mathcal{Q}\mathcal{U}\right)(\varsigma)\left({}_{0^+}\mathcal{I}^{\alpha}_{\mathfrak{q}_1,\omega_1}\mathcal{H}_1\right)(\varsigma)|\\
&=\frac{1}{\Gamma_{\mathfrak{q}_1}(\alpha)\Gamma_{\mathfrak{q}_2}(\beta)}\int_0^{\varsigma}\int_0^{\varsigma}\left(\varsigma-_{\sigma_1}\Psi_{\mathfrak{q}_1}(x)\right)^{\alpha-1}_{\sigma_1}\left(\varsigma-_{\sigma_2}\Psi_{\mathfrak{q}_2}(y)\right)^{\beta-1}_{\sigma_2}\\
&\quad\cdot\mathcal{H}_1(x)\mathcal{H}_2(y)|\mathcal{G}(x,y)|d_{\mathfrak{q}_1,\omega_1}x d_{\mathfrak{q}_2,\omega_2}y\\
&\le\frac{1}{\Gamma_{\mathfrak{q}_1}(\alpha)\Gamma_{\mathfrak{q}_2}(\beta)}\int_0^{\varsigma}\int_0^{\varsigma}\left(\varsigma-_{\sigma_1}\Psi_{\mathfrak{q}_1}(x)\right)^{\alpha-1}_{\sigma_1}\left(\varsigma-{}_{\sigma_2}\Psi_{\mathfrak{q}_2}(y)\right)^{\beta-1}_{\sigma_2}\\
&\quad\times|x-y|^{1/s_1+1/r_1}\left|\int_x^y|\mathcal{Q}'(\theta)|^s d\theta\right|^{1/s}\left|\int_x^y|\mathcal{U}'(\vartheta)|^r d\vartheta\right|^{1/r}\\
&\quad\cdot\mathcal{H}_1(x)\mathcal{H}_2(y)d_{\mathfrak{q}_1,\omega_1}x d_{\mathfrak{q}_2,\omega_2}y\\
&=\frac{\|\mathcal{Q}\|_s\|\mathcal{U}\|_r}{\Gamma_{\mathfrak{q}_1}(\alpha)\Gamma_{\mathfrak{q}_2}(\beta)}\int_0^{\varsigma}\int_0^{\varsigma}\left(\varsigma-_{\sigma_1}\Psi_{\mathfrak{q}_1}(x)\right)^{\alpha-1}_{\sigma_1}\left(\varsigma-{}_{\sigma_2}\Psi_{\mathfrak{q}_2}(y)\right)^{\beta-1}_{\sigma_2}\\
&\quad\times|x-y|^{1/s_1+1/r_1}\mathcal{H}_1(x)\mathcal{H}_2(y)d_{\mathfrak{q}_1,\omega_1}x d_{\mathfrak{q}_2,\omega_2}y.
\end{aligned}
\tag{37}
$$

Let $\omega_i = 1$ for $i = 1, 2$. Then, Theorem 10 leads to Corollary 11 which provide a new result for q_i-fractional integral operator.

Corollary 11. *Let $s, s_1, u, u_1, r, r_1 > 1$ with $s^{-1} + s_1^{-1} = r^{-1} + r_1^{-1} = u^{-1} + u_1^{-1} = 1$, $\mathsf{q}_i \in (0, 1)(i = 1, 2)$, $\mathscr{H}_1$ and $\mathscr{H}_2$ be the positive q_i-integrable functions defined on $[0, \infty)$, and $\mathscr{Q}$ and $\mathscr{U}$ be the q_i-differentiable functions defined on $[0, \infty)$ such that $\mathscr{Q}' \in L_s([0, \infty))$ and $\mathscr{U}' \in L_r([0, \infty))$. Then, the inequality*

$$\begin{aligned}
&\left|\left({}_{0^+}\mathscr{I}^{\beta}_{\mathsf{q}_2}\mathscr{H}_2\right)(\varsigma)\left({}_{0^+}\mathscr{I}^{\alpha}_{\mathsf{q}_1}\mathscr{H}_1\mathscr{Q}\mathscr{U}\right)(\varsigma)\right.\\
&\quad-\left({}_{0^+}\mathscr{I}^{\beta}_{\mathsf{q}_2}\mathscr{H}_1\mathscr{U}\right)(\varsigma)\left({}_{0^+}\mathscr{I}^{\alpha}_{\mathsf{q}_1}\mathscr{H}_1\mathscr{Q}\right)(\varsigma)\\
&\quad-\left({}_{0^+}\mathscr{I}^{\beta}_{\mathsf{q}_2}\mathscr{H}_2\mathscr{Q}\right)(\varsigma)\left({}_{0^+}\mathscr{I}^{\alpha}_{\mathsf{q}_1}\mathscr{H}_1\mathscr{U}\right)(\varsigma)\\
&\quad\left.+\left({}_{0^+}\mathscr{I}^{\beta}_{\mathsf{q}_2}\mathscr{H}_2\mathscr{Q}\mathscr{U}\right)(\varsigma)\left({}_{0^+}\mathscr{I}^{\alpha}_{\mathsf{q}_1}\mathscr{H}_1\right)(\varsigma)\right|\\
&\le\frac{\|\mathscr{Q}\|_s\|\mathscr{U}\|_r}{\Gamma_{\mathsf{q}_1}(\alpha)\Gamma_{\mathsf{q}_2}(\beta)}\int_0^{\varsigma}\int_0^{\varsigma}\left(\varsigma-{}_{\sigma_1}\Psi_{\mathsf{q}_1}(x)\right)_{\sigma_1}^{\alpha-1}\left(\varsigma-{}_{\sigma_2}\Psi_{\mathsf{q}_2}(y)\right)_{\sigma_2}^{\beta-1}\\
&\quad\times|x-y|^{1/s_1+1/r_1}\mathscr{H}_1(x)\mathscr{H}_2(y)d_{\mathsf{q}_1}x d_{\mathsf{q}_2}y
\end{aligned}\tag{38}$$

holds for all $\varsigma > 0$.

Remark 12. Let $\mathsf{q}_i = \omega_i = 1$. Then, Theorem 10 becomes Theorem 14 of [77].

3. Some New Generalizations by Fractional Quantum Hahn Integral Operator

Theorem 13. *Let $r, s > 1$ with $1/r + 1/s = 1$, $\mathsf{q}_i \in (0, 1)$ and $\omega_i \ge 0$ for $i = 1, 2$, and $\mathscr{Q}$ and $\mathscr{U}$ be the the positive q_i, ω_i-integral functions. Then the following inequalities*

(A1) $(1/r)({}_{0^+}\mathscr{I}^{\alpha}_{\mathsf{q}_1,\omega_1}\mathscr{Q}^r)(\varsigma)({}_{0^+}\mathscr{I}^{\beta}_{\mathsf{q}_2,\omega_2}\mathscr{U}^r)(\varsigma) + (1/s)({}_{0^+}\mathscr{I}^{\alpha}_{\mathsf{q}_1,\omega_1}\mathscr{U}^s)(\varsigma)({}_{0^+}\mathscr{I}^{\beta}_{\mathsf{q}_2,\omega_2}\mathscr{Q}^s)(\varsigma) \ge ({}_{0^+}\mathscr{I}^{\alpha}_{\mathsf{q}_1,\omega_1}\mathscr{Q}\mathscr{U})(\varsigma)({}_{0^+}\mathscr{I}^{\beta}_{\mathsf{q}_2,\omega_2}\mathscr{Q}\mathscr{U})(\varsigma)$

(A2) $(1/r)({}_{0^+}\mathscr{I}^{\alpha}_{\mathsf{q}_1,\omega_1}\mathscr{Q}^r)(\varsigma)({}_{0^+}\mathscr{I}^{\beta}_{\mathsf{q}_2,\omega_2}\mathscr{U}^s)(\varsigma) + (1/s)({}_{0^+}\mathscr{I}^{\alpha}_{\mathsf{q}_1,\omega_1}\mathscr{U}^s)(\varsigma)({}_{0^+}\mathscr{I}^{\beta}_{\mathsf{q}_2,\omega_2}\mathscr{Q}^r)(\varsigma) \ge ({}_{0^+}\mathscr{I}^{\alpha}_{\mathsf{q}_1,\omega_1}\mathscr{Q}\mathscr{U})(\varsigma)({}_{0^+}\mathscr{I}^{\beta}_{\mathsf{q}_2,\omega_2}\mathscr{Q}^{r-1}\mathscr{U}^{s-1})(\varsigma)$

(A3) $(1/r)({}_{0^+}\mathscr{I}^{\alpha}_{\mathsf{q}_1,\omega_1}\mathscr{Q}^r)(\varsigma)({}_{0^+}\mathscr{I}^{\beta}_{\mathsf{q}_2,\omega_2}\mathscr{U}^2)(\varsigma) + (1/s)({}_{0^+}\mathscr{I}^{\alpha}_{\mathsf{q}_1,\omega_1}\mathscr{Q}^2)(\varsigma)({}_{0^+}\mathscr{I}^{\beta}_{\mathsf{q}_2,\omega_2}\mathscr{U}^s)(\varsigma) \ge ({}_{0^+}\mathscr{I}^{\alpha}_{\mathsf{q}_1,\omega_1}\mathscr{Q}\mathscr{U})(\varsigma)({}_{0^+}\mathscr{I}^{\beta}_{\mathsf{q}_2,\omega_2}\mathscr{Q}^{2/s}\mathscr{U}^{2/r})(\varsigma)$

(A4) $(1/r)({}_{0^+}\mathscr{I}^{\alpha}_{\mathsf{q}_1,\omega_1}\mathscr{Q}^2)(\varsigma)({}_{0^+}\mathscr{I}^{\beta}_{\mathsf{q}_2,\omega_2}\mathscr{Q}^r)(\varsigma) + (1/s)({}_{0^+}\mathscr{I}^{\alpha}_{\mathsf{q}_1,\omega_1}\mathscr{U}^2)(\varsigma)({}_{0^+}\mathscr{I}^{\beta}_{\mathsf{q}_2,\omega_2}\mathscr{U}^s)(\varsigma) \ge ({}_{0^+}\mathscr{I}^{\alpha}_{\mathsf{q}_1,\omega_1}\mathscr{Q}^{2/r}\mathscr{U}^{2/s})(\varsigma)({}_{0^+}\mathscr{I}^{\beta}_{\mathsf{q}_2,\omega_2}\mathscr{Q}\mathscr{U})(\varsigma)$

hold for all $\varsigma > 0$.

Proof. Considering the well-known Young inequality

$$\frac{1}{r}\rho_1^r + \frac{1}{s}\rho_2^s \ge \rho_1\rho_2 \tag{39}$$

for all $\rho_1, \rho_2 \ge 0$ and $r, s > 1$ with $r^{-1} + s^{-1} = 1$.

Substituting $\rho_1 = \mathscr{Q}(x)\mathscr{U}(y)$ and $\rho_2 = \mathscr{Q}(y)\mathscr{U}(x)(x, y > 0)$ into (39) gives

$$\frac{1}{r}\mathscr{Q}^r(x)\mathscr{U}^r(y) + \frac{1}{s}\mathscr{Q}^s(y)\mathscr{U}^s(x) \ge \mathscr{Q}(x)\mathscr{U}(y)\mathscr{Q}(y)\mathscr{U}(x). \tag{40}$$

Multiplying both sides of (40) by

$$\frac{\left(\varsigma-{}_{\sigma_1}\Psi_{\mathsf{q}_1}(x)\right)_{\sigma_1}^{\alpha-1}\left(\varsigma-{}_{\sigma_2}\Psi_{\mathsf{q}_2}(y)\right)_{\sigma_2}^{\beta-1}}{\Gamma_{\mathsf{q}_1}(\alpha)\Gamma_{\mathsf{q}_2}(\beta)} \tag{41}$$

and then performing the q_i, ω_i-integration with respect to x and y over $(0, \varsigma)$ lead to the conclusion that

$$\begin{aligned}
&\frac{1}{r}\int_0^{\varsigma}\frac{\left(\varsigma-{}_{\sigma_1}\Psi_{\mathsf{q}_1}(x)\right)_{\sigma_1}^{\alpha-1}\left(\varsigma-{}_{\sigma_2}\Psi_{\mathsf{q}_2}(y)\right)_{\sigma_2}^{\beta-1}}{\Gamma_{\mathsf{q}_1}(\alpha)\Gamma_{\mathsf{q}_2}(\beta)}\\
&\quad\cdot\mathscr{Q}^r(x)\mathscr{U}^r(y)d_{\mathsf{q}_1,\omega_1}x d_{\mathsf{q}_2,\omega_2}y\\
&\quad+\frac{1}{s}\int_0^{\varsigma}\frac{\left(\varsigma-{}_{\sigma_1}\Psi_{\mathsf{q}_1}(x)\right)_{\sigma_1}^{\alpha-1}\left(\varsigma-{}_{\sigma_2}\Psi_{\mathsf{q}_2}(y)\right)_{\sigma_2}^{\beta-1}}{\Gamma_{\mathsf{q}_1}(\alpha)\Gamma_{\mathsf{q}_2}(\beta)}\\
&\quad\cdot\mathscr{Q}^s(y)\mathscr{U}^s(x)d_{\mathsf{q}_1,\omega_1}x d_{\mathsf{q}_2,\omega_2}y\\
&\ge\int_0^{\varsigma}\frac{\left(\varsigma-{}_{\sigma_1}\Psi_{\mathsf{q}_1}(x)\right)_{\sigma_1}^{\alpha-1}\left(\varsigma-{}_{\sigma_2}\Psi_{\mathsf{q}_2}(y)\right)_{\sigma_2}^{\beta-1}}{\Gamma_{\mathsf{q}_1}(\alpha)\Gamma_{\mathsf{q}_2}(\beta)}\\
&\quad\cdot\mathscr{Q}(x)\mathscr{U}(y)\mathscr{Q}(y)\mathscr{U}(x)d_{\mathsf{q}_1,\omega_1}x d_{\mathsf{q}_2,\omega_2}y.
\end{aligned}\tag{42}$$

Consequently, we have

$$\begin{aligned}
&\frac{1}{r}\left({}_{0^+}\mathscr{I}^{\alpha}_{\mathsf{q}_1,\omega_1}\mathscr{Q}^r\right)(\varsigma)\left({}_{0^+}\mathscr{I}^{\beta}_{\mathsf{q}_2,\omega_2}\mathscr{U}^r\right)(\varsigma)\\
&\quad+\frac{1}{s}\left({}_{0^+}\mathscr{I}^{\alpha}_{\mathsf{q}_1,\omega_1}\mathscr{U}^s\right)(\varsigma)\left({}_{0^+}\mathscr{I}^{\beta}_{\mathsf{q}_2,\omega_2}\mathscr{Q}^s\right)(\varsigma)\\
&\ge\left({}_{0^+}\mathscr{I}^{\alpha}_{\mathsf{q}_1,\omega_1}\mathscr{Q}\mathscr{U}\right)(\varsigma)\left({}_{0^+}\mathscr{I}^{\beta}_{\mathsf{q}_2,\omega_2}\mathscr{Q}\mathscr{U}\right)(\varsigma),
\end{aligned}\tag{43}$$

which implies (A_1). The remaining inequalities can be derived by adopting the similar argument and accompanying the selection of parameters in Young inequality as follows:

$(A_2)\rho_1 = \mathscr{Q}(x)/\mathscr{Q}(y)$, $\rho_2 = \mathscr{U}(x)/\mathscr{U}(y)$,

$(A_3)\rho_1 = \mathscr{Q}(x)\mathscr{U}^{2/r}(y)$, $\rho_2 = \mathscr{Q}^{2/s}(x)\mathscr{U}(y)$,

$(A_4)\rho_1 = \mathscr{Q}^{2/r}(x)\mathscr{Q}(y)$, $\rho_2 = \mathscr{U}^{2/s}(x)\mathscr{U}(y)$.

Theorem 14. *Let $r, s > 0$ with $r + s = 1$, $\mathsf{q}_i \in (0, 1)$ and $\omega_i \ge 0$ for $i = 1, 2$, and $\mathscr{Q}$ and $\mathscr{U}$ be the positive q_i, ω_i-integral functions defined on $[0, \infty)$. Then, the inequalities.*

(A_5) $r({}_{0^+}\mathscr{I}^{\alpha}_{\mathsf{q}_1,\omega_1}\mathscr{Q})(\varsigma)({}_{0^+}\mathscr{I}^{\beta}_{\mathsf{q}_2,\omega_2}\mathscr{U})(\varsigma) + s({}_{0^+}\mathscr{I}^{\alpha}_{\mathsf{q}_1,\omega_1}\mathscr{U})(\varsigma)({}_{0^+}\mathscr{I}^{\beta}_{\mathsf{q}_2,\omega_2}\mathscr{Q})(\varsigma) \ge ({}_{0^+}\mathscr{I}^{\alpha}_{\mathsf{q}_1,\omega_1}\mathscr{Q}^r\mathscr{U}^s)(\varsigma)({}_{0^+}\mathscr{I}^{\beta}_{\mathsf{q}_2,\omega_2}\mathscr{Q}^s\mathscr{U}^r)(\varsigma)$

(A_6) $r({}_{0^+}\mathscr{I}^{\alpha}_{\mathsf{q}_1,\omega_1}\mathscr{Q})(\varsigma)({}_{0^+}\mathscr{I}^{\beta}_{\mathsf{q}_2,\omega_2}\mathscr{U})(\varsigma) + s({}_{0^+}\mathscr{I}^{\alpha}_{\mathsf{q}_1,\omega_1}\mathscr{U})(\varsigma)({}_{0^+}\mathscr{I}^{\beta}_{\mathsf{q}_2,\omega_2}\mathscr{Q})(\varsigma) \ge ({}_{0^+}\mathscr{I}^{\alpha}_{\mathsf{q}_1,\omega_1}\mathscr{Q}^r\mathscr{U}^s)(\varsigma)({}_{0^+}\mathscr{I}^{\beta}_{\mathsf{q}_2,\omega_2}\mathscr{Q}^{r-1}\mathscr{U}^{s-1})(\varsigma)$

(A_7) $r({}_{0^+}\mathscr{I}^{\alpha}_{\mathsf{q}_1,\omega_1}\mathscr{Q})(\varsigma)({}_{0^+}\mathscr{I}^{\beta}_{\mathsf{q}_2,\omega_2}\mathscr{U}^{2/r})(\varsigma) + s({}_{0^+}\mathscr{I}^{\alpha}_{\mathsf{q}_1,\omega_1}\mathscr{U})(\varsigma)({}_{0^+}\mathscr{I}^{\beta}_{\mathsf{q}_2,\omega_2}\mathscr{Q}^{2/s})(\varsigma) \ge ({}_{0^+}\mathscr{I}^{\alpha}_{\mathsf{q}_1,\omega_1}\mathscr{Q}^r\mathscr{U}^s)(\varsigma)({}_{0^+}\mathscr{I}^{\beta}_{\mathsf{q}_2,\omega_2}\mathscr{Q}^2\mathscr{U}^2)(\varsigma)$

(A_8) $r({}_{0^+}\mathscr{I}^{\alpha}_{\mathsf{q}_1,\omega_1}\mathscr{Q}^{2/r})(\varsigma)({}_{0^+}\mathscr{I}^{\beta}_{\mathsf{q}_2,\omega_2}\mathscr{Q})(\varsigma) + s({}_{0^+}\mathscr{I}^{\alpha}_{\mathsf{q}_1,\omega_1}\mathscr{U}^{2/s})(\varsigma)({}_{0^+}\mathscr{I}^{\beta}_{\mathsf{q}_2,\omega_2}\mathscr{U})(\varsigma) \ge ({}_{0^+}\mathscr{I}^{\alpha}_{\mathsf{q}_1,\omega_1}\mathscr{Q}^2\mathscr{U}^2)(\varsigma)({}_{0^+}\mathscr{I}^{\beta}_{\mathsf{q}_2,\omega_2}\mathscr{Q}^s\mathscr{U}^r)(\varsigma)$

hold for all $\varsigma > 0$.

Proof. Considering the well-known weighted $AM - GM$ inequality

$$r\rho_1 + s\rho_2 \ge \rho_1^r\rho_2^s \quad (\rho_1, \rho_2 \ge 0, r, s > 0, r + s = 1). \tag{44}$$

Substituting $\rho_1 = \mathcal{Q}(x)\mathcal{U}(y)$ and $\rho_2 = \mathcal{Q}(y)\mathcal{U}(x)(x, y > 0)$ yields

$$r\mathcal{Q}(x)\mathcal{U}(y) + s\mathcal{Q}(y)\mathcal{U}(x) \geq \mathcal{Q}^r(x)\mathcal{U}^r(y)\mathcal{Q}^s(y)\mathcal{U}^s(x). \quad (45)$$

Conducting product on both sides of (45) by

$$\frac{\left(\varsigma - {}_{\sigma_1}\Psi_{\mathfrak{q}_1}(x)\right)_{\sigma_1}^{\alpha-1}\left(\varsigma - {}_{\sigma_2}\Psi_{\mathfrak{q}_2}(y)\right)_{\sigma_2}^{\beta-1}}{\Gamma_{\mathfrak{q}_1}(\alpha)\Gamma_{\mathfrak{q}_2}(\beta)}, \quad (46)$$

and then performing the $\mathfrak{q}_i, \omega_i$-integration with respect to x and y over $(0, \varsigma)$, we obtain

$$\begin{aligned}
&r\int_0^{\varsigma} \frac{\left(\varsigma - {}_{\sigma_1}\Psi_{\mathfrak{q}_1}(x)\right)_{\sigma_1}^{\alpha-1}\left(\varsigma - {}_{\sigma_2}\Psi_{\mathfrak{q}_2}(y)\right)_{\sigma_2}^{\beta-1}}{\Gamma_{\mathfrak{q}_1}(\alpha)\Gamma_{\mathfrak{q}_2}(\beta)} \\
&\quad \cdot \mathcal{Q}(x)\mathcal{U}(y)d_{\mathfrak{q}_1,\omega_1}x d_{\mathfrak{q}_2,\omega_2}y \\
&\quad + s\int_0^{\varsigma} \frac{\left(\varsigma - {}_{\sigma_1}\Psi_{\mathfrak{q}_1}(x)\right)_{\sigma_1}^{\alpha-1}\left(\varsigma - {}_{\sigma_2}\Psi_{\mathfrak{q}_2}(y)\right)_{\sigma_2}^{\beta-1}}{\Gamma_{\mathfrak{q}_1}(\alpha)\Gamma_{\mathfrak{q}_2}(\beta)} \\
&\quad \cdot \mathcal{Q}(y)\mathcal{U}(x)d_{\mathfrak{q}_1,\omega_1}x d_{\mathfrak{q}_2,\omega_2}y \\
&\geq \int_0^{\varsigma} \frac{\left(\varsigma - {}_{\sigma_1}\Psi_{\mathfrak{q}_1}(x)\right)_{\sigma_1}^{\alpha-1}\left(\varsigma - {}_{\sigma_2}\Psi_{\mathfrak{q}_2}(y)\right)_{\sigma_2}^{\beta-1}}{\Gamma_{\mathfrak{q}_1}(\alpha)\Gamma_{\mathfrak{q}_2}(\beta)} \\
&\quad \cdot \mathcal{Q}^r(x)\mathcal{U}^r(y)\mathcal{Q}^s(y)\mathcal{U}^s(x)d_{\mathfrak{q}_1,\omega_1}x d_{\mathfrak{q}_2,\omega_2}y.
\end{aligned} \quad (47)$$

Consequently, we have

$$\begin{aligned}
&r\left({}_{0^+}\mathcal{I}_{\mathfrak{q}_1,\omega_1}^{\alpha}\mathcal{Q}\right)(\varsigma)\left({}_{0^+}\mathcal{I}_{\mathfrak{q}_2,\omega_2}^{\beta}\mathcal{U}\right)(\varsigma) \\
&\quad + s\left({}_{0^+}\mathcal{I}_{\mathfrak{q}_1,\omega_1}^{\alpha}\mathcal{U}\right)(\varsigma)\left({}_{0^+}\mathcal{I}_{\mathfrak{q}_2,\omega_2}^{\beta}\mathcal{Q}\right)(\varsigma) \\
&\geq \left({}_{0^+}\mathcal{I}_{\mathfrak{q}_1,\omega_1}^{\alpha}\mathcal{Q}^r\mathcal{U}^s\right)(\varsigma)\left({}_{0^+}\mathcal{I}_{\mathfrak{q}_2,\omega_2}^{\beta}\mathcal{Q}^s\mathcal{U}^r\right)(\varsigma),
\end{aligned} \quad (48)$$

which implies (A_5). The rest of variants can be derived by adopting the similar strategy and accompanying the selection of parameters in $AM - GM$ inequality as follows:

$(A_6)\rho_1 = \mathcal{Q}(x)/\mathcal{Q}(y), \rho_2 = \mathcal{U}(x)/\mathcal{U}(y)$,

$(A_7)\rho_1 = \mathcal{Q}(x)\mathcal{U}^{2/r}(y), \rho_2 = \mathcal{Q}^{2/s}(x)\mathcal{U}(y)$,

$(A_8)\rho_1 = \mathcal{Q}^{2/r}(x)\mathcal{Q}(y)$,

$\rho_2 = \mathcal{U}^{2/s}(x)\mathcal{U}(y)$.

Theorem 15. *Let $\mathfrak{q} \in (0, 1)$, $\omega \geq 0$, $\mathcal{Q}$ and $\mathcal{U}$ be the positive $\mathfrak{q}, \omega$-integrable functions defined on $[0, \infty)$, and*

$$e = \min_{0\leq x\leq\varsigma} \frac{\mathcal{Q}(\varsigma)}{\mathcal{U}(\varsigma)}, \quad (49)$$

$$\mathcal{E} = \max_{0\leq x\leq\varsigma} \frac{\mathcal{Q}(\varsigma)}{\mathcal{U}(\varsigma)}. \quad (50)$$

Then, the inequalities

(A_9) $0 \leq ({}_{0^+}\mathcal{I}_{\mathfrak{q},\omega}^{\alpha}\mathcal{Q}^2)(\varsigma)({}_{0^+}\mathcal{I}_{\mathfrak{q},\omega}^{\alpha}\mathcal{U}^2)(\varsigma) \leq ((e + \mathcal{E})^2/4e\mathcal{E})({}_{0^+}\mathcal{I}_{\mathfrak{q},\omega}^{\alpha}(\mathcal{Q}\mathcal{U}))^2(\varsigma)$

(A_{10}) $0 \leq \sqrt{({}_{0^+}\mathcal{I}_{\mathfrak{q},\omega}^{\alpha}\mathcal{Q}^2)(\varsigma)({}_{0^+}\mathcal{I}_{\mathfrak{q},\omega}^{\alpha}\mathcal{U}^2)(\varsigma)} - ({}_{0^+}\mathcal{I}_{\mathfrak{q},\omega}^{\alpha}\mathcal{Q}\mathcal{U})(\varsigma) \leq ((\sqrt{\mathcal{E}} - \sqrt{e})^2/2\sqrt{e\mathcal{E}})({}_{0^+}\mathcal{I}_{\mathfrak{q},\omega}^{\alpha}\mathcal{Q}\mathcal{U})(\varsigma)$

(A_{11}) $0 \leq ({}_{0^+}\mathcal{I}_{\mathfrak{q},\omega}^{\alpha}\mathcal{Q}^2)(\varsigma)({}_{0^+}\mathcal{I}_{\mathfrak{q},\omega}^{\alpha}\mathcal{U}^2)(\varsigma) - ({}_{0^+}\mathcal{I}_{\mathfrak{q},\omega}^{\alpha}\mathcal{Q}\mathcal{U})^2(\varsigma) \leq ((\mathcal{E} - e)^2/4e\mathcal{E})({}_{0^+}\mathcal{I}_{\mathfrak{q},\omega}^{\alpha}(\mathcal{Q}\mathcal{U}))^2(\varsigma)$

hold for all $\varsigma > 0$.

Proof. From (49) and (50), we clearly see that

$$\left(\frac{\mathcal{Q}(x)}{\mathcal{U}(x)} - e\right)\left(\mathcal{E} - \frac{\mathcal{Q}(x)}{\mathcal{U}(x)}\right)\mathcal{U}^2(x) \geq 0(0 \leq x < \varsigma). \quad (51)$$

Conducting product on both sides of (51) by $(\varsigma - {}_{\sigma}\Psi_{\mathfrak{q}}(x))_{\sigma}^{(\alpha-1)}/\Gamma_{\mathfrak{q}}(\alpha)$ and then performing the $\mathfrak{q}, \omega$-integration with respect to x over $(0, \varsigma)$ yield

$$\left({}_{0^+}\mathcal{I}_{\mathfrak{q},\omega}^{\alpha}\mathcal{Q}^2\right)(\varsigma) + e\mathcal{E}\left({}_{0^+}\mathcal{I}_{\mathfrak{q},\omega}^{\alpha}\mathcal{U}^2\right)(\varsigma) \leq (e + \mathcal{E})\left({}_{0^+}\mathcal{I}_{\mathfrak{q},\omega}^{\alpha}\mathcal{Q}\mathcal{U}\right)(\varsigma). \quad (52)$$

It follows from $e\mathcal{E} > 0$ and

$$\sqrt{\left({}_{0^+}\mathcal{I}_{\mathfrak{q},\omega}^{\alpha}\mathcal{Q}^2\right)(\varsigma)} - \sqrt{e\mathcal{E}\left({}_{0^+}\mathcal{I}_{\mathfrak{q},\omega}^{\alpha}\mathcal{U}^2\right)(\varsigma)} \geq 0 \quad (53)$$

that

$$\begin{aligned}
&2\sqrt{\left({}_{0^+}\mathcal{I}_{\mathfrak{q},\omega}^{\alpha}\mathcal{Q}^2\right)(\varsigma)}\sqrt{e\mathcal{E}\left({}_{0^+}\mathcal{I}_{\mathfrak{q},\omega}^{\alpha}\mathcal{U}^2\right)(\varsigma)} \\
&\quad \leq \left({}_{0^+}\mathcal{I}_{\mathfrak{q},\omega}^{\alpha}\mathcal{Q}^2\right)(\varsigma) + (e\mathcal{E})\left({}_{0^+}\mathcal{I}_{\mathfrak{q},\omega}^{\alpha}\mathcal{U}^2\right)(\varsigma).
\end{aligned} \quad (54)$$

From (52) and (54), we conclude that

$$\left({}_{0^+}\mathcal{I}_{\mathfrak{q},\omega}^{\alpha}\mathcal{Q}^2\right)(\varsigma)\left({}_{0^+}\mathcal{I}_{\mathfrak{q},\omega}^{\alpha}\mathcal{U}^2\right)(\varsigma) \leq \frac{(e + \mathcal{E})^2}{4e\mathcal{E}}\left({}_{0^+}\mathcal{I}_{\mathfrak{q},\omega}^{\alpha}(\mathcal{Q}\mathcal{U})\right)^2(\varsigma), \quad (55)$$

which implies (A_9). By making few changes in (A_9), we can get (A_{10}) and (A_{11}).

Theorem 16. *Let $\mathfrak{q} \in (0, 1)$, $\omega \geq 0$, and $\mathcal{Q}$ and $\mathcal{U}$ be the positive $\mathfrak{q}, \omega$-integrable functions defined on $[0, \infty)$ such that*

$$0 < \delta_1 \leq \mathcal{Q}(x) \leq \delta_2 < \infty, \quad (56)$$

$$0 < \Delta_1 \leq \mathcal{U}(x) \leq \Delta_2 < \infty. \quad (57)$$

Then, the inequalities

$(A_{12})0 \leq ({}_{0^+}\mathcal{I}_{\mathfrak{q},\omega}^{\alpha}\mathcal{Q}^2)(\varsigma)({}_{0^+}\mathcal{I}_{\mathfrak{q},\omega}^{\alpha}\mathcal{U}^2)(\varsigma) \leq ((\delta_1\Delta_1 + \delta_2\Delta_2)^2/4\delta_1\Delta_1\delta_2\Delta_2)({}_{0^+}\mathcal{I}_{\mathfrak{q},\omega}^{\alpha}(\mathcal{Q}\mathcal{U}))^2(\varsigma)$

$(A_{13})0 \leq \sqrt{({}_{0^+}\mathcal{I}_{\mathfrak{q},\omega}^{\alpha}\mathcal{Q}^2)(\varsigma)({}_{0^+}\mathcal{I}_{\mathfrak{q},\omega}^{\alpha}\mathcal{U}^2)(\varsigma)} - ({}_{0^+}\mathcal{I}_{\mathfrak{q},\omega}^{\alpha}\mathcal{Q}\mathcal{U})(\varsigma) \leq ((\sqrt{\delta_2\Delta_2} - \sqrt{\delta_1\Delta_1})^2/2\sqrt{\delta_1\Delta_1\delta_2\Delta_2})({}_{0^+}\mathcal{I}_{\mathfrak{q},\omega}^{\alpha}\mathcal{Q}\mathcal{U})(\varsigma)$

$(A_{14})0 \leq ({}_{0^+}\mathcal{I}_{\mathfrak{q},\omega}^{\alpha}\mathcal{Q}^2)(\varsigma)({}_{0^+}\mathcal{I}_{\mathfrak{q},\omega}^{\alpha}\mathcal{U}^2)(\varsigma) - ({}_{0^+}\mathcal{I}_{\mathfrak{q},\omega}^{\alpha}\mathcal{Q}\mathcal{U})^2(\varsigma) \leq ((\delta_2\Delta_2 - \delta_1\Delta_1)^2/4\delta_1\Delta_1\delta_2\Delta_2)({}_{0^+}\mathcal{I}_{\mathfrak{q},\omega}^{\alpha}(\mathcal{Q}\mathcal{U}))^2(\varsigma)(A_{14})$

hold for all $\varsigma > 0$.

Proof. It follows from (56) and (57) that

$$\frac{\delta_1}{\Delta_2} \leq \frac{\mathcal{Q}(x)}{\mathcal{U}(x)} \leq \frac{\delta_2}{\Delta_1}. \tag{58}$$

The proof can be derived by following Theorem 15.

Theorem 17. *Let $r, s > 0$ with $r + s = 1$, $\mathfrak{q} \in (0, 1)$, $\omega \geq 0$, $\mathcal{P}, \mathcal{Q}$ and $\mathcal{U}$ be the positive $\mathfrak{q}, \omega$-integrable functions defined on $[0, \infty)$ such that (56) and (57) hold. Then, the inequalities*

$(A_{15})({}_{0^+}\mathcal{I}^{\alpha}_{\mathfrak{q},\omega}\mathcal{P}\mathcal{Q})^s(\varsigma)({}_{0^+}\mathcal{I}^{\alpha}_{\mathfrak{q},\omega}(\mathcal{P}/\mathcal{Q}))^r(\varsigma) \leq (r\delta_1 + s\delta_2/(\delta_1\delta_2)^r)({}_{0^+}\mathcal{I}^{\alpha}_{\mathfrak{q},\omega}\mathcal{P})(\varsigma)$

$(A_{16})({}_{0^+}\mathcal{I}^{\alpha}_{\mathfrak{q},\omega}\mathcal{P}\mathcal{Q})^s(\varsigma)({}_{0^+}\mathcal{I}^{\alpha}_{\mathfrak{q},\omega}\mathcal{P}\mathcal{U})^r(\varsigma) \leq (r\delta_1\Delta_1 + s\delta_2\Delta_2/(\delta_1\delta_2)^r(\Delta_1\Delta_2)^s)({}_{0^+}\mathcal{I}^{\alpha}_{\mathfrak{q},\omega}\mathcal{P}\mathcal{Q}\mathcal{U})^s(\varsigma)(A_{16})$

hold for all $\varsigma > 0$.

Proof. It follows from $(s\mathcal{Q}(x) - r\delta_1)(\mathcal{Q}(x) - \delta_2) \leq 0$ that

$$s\mathcal{Q}^2(x) - (r\delta_1 + s\delta_2)\mathcal{Q}(x) + r\delta_1\delta_2 \leq 0. \tag{59}$$

Multiplying both sides of (59) by $\mathcal{P}(x)/\mathcal{Q}(x)$, we have

$$s\mathcal{P}(x)\mathcal{Q}(x) + r\delta_1\delta_2\frac{\mathcal{P}(x)}{\mathcal{Q}(x)} \leq (r\delta_1 + s\delta_2)\mathcal{P}(x). \tag{60}$$

Taking into account the well-known $AM - GM$ inequality for (60), we get

$$\begin{aligned}
&\left(\int_0^{\varsigma}\frac{(\varsigma - {}_{\sigma}\Psi_{\mathfrak{q}}(x))_{\sigma}^{\alpha-1}}{\Gamma_{\mathfrak{q}}(\alpha)}\mathcal{P}(x)\mathcal{Q}(x)d_{\mathfrak{q},\omega}x\right)^s \\
&\quad\cdot\left(\int_0^{\varsigma}\frac{(\varsigma - {}_{\sigma}\Psi_{\mathfrak{q}}(x))_{\sigma}^{\alpha-1}}{\Gamma_{\mathfrak{q}}(\alpha)}\frac{\mathcal{P}(x)}{\mathcal{Q}(x)}d_{\mathfrak{q},\omega}x\right)^r \\
&= \frac{1}{(\delta_1\delta_2)^r}\left(\int_0^{\varsigma}\frac{(\varsigma - {}_{\sigma}\Psi_{\mathfrak{q}}(x))_{\sigma}^{\alpha-1}}{\Gamma_{\mathfrak{q}}(\alpha)}\mathcal{P}(x)\mathcal{Q}(x)d_{\mathfrak{q},\omega}x\right)^s \\
&\quad\cdot\left(\delta_1\delta_2\int_0^{\varsigma}\frac{(\varsigma - {}_{\sigma}\Psi_{\mathfrak{q}}(x))_{\sigma}^{\alpha-1}}{\Gamma_{\mathfrak{q}}(\alpha)}\frac{\mathcal{P}(x)}{\mathcal{Q}(x)}d_{\mathfrak{q},\omega}x\right)^r \\
&\leq \frac{1}{(\delta_1\delta_2)^r}\left(\frac{s}{\Gamma_{\mathfrak{q}}(\alpha)}\int_0^{\varsigma}(\varsigma - {}_{\sigma}\Psi_{\mathfrak{q}}(x))_{\sigma}^{\alpha-1}\mathcal{P}(x)\mathcal{Q}(x)d_{\mathfrak{q},\omega}x\right. \\
&\quad\left.+\frac{r\delta_1\delta_2}{\Gamma_{\mathfrak{q}}(\alpha)}\int_0^{\varsigma}(\varsigma - {}_{\sigma}\Psi_{\mathfrak{q}}(x))_{\sigma}^{\alpha-1}\frac{\mathcal{P}(x)}{\mathcal{Q}(x)}d_{\mathfrak{q},\omega}x\right) \\
&\leq \frac{r\delta_1 + s\delta_2}{(\delta_1\delta_2)^r}\left(\frac{1}{\Gamma_{\mathfrak{q}}(\alpha)}\int_0^{\varsigma}(\varsigma - {}_{\sigma}\Psi_{\mathfrak{q}}(x))_{\sigma}^{\alpha-1}\mathcal{P}(x)d_{\mathfrak{q},\omega}x\right),
\end{aligned} \tag{61}$$

which implies (A_{15}).

Replacing, respectively, $\mathcal{P}$ and $\mathcal{Q}$ by $\mathcal{P}\mathcal{Q}\mathcal{U}$ and $\mathcal{Q}/\mathcal{U}$ in (61) and (56), we attain the required inequality (A_{16}):

$$\begin{aligned}
&\left(\int_0^{\varsigma}\frac{(\varsigma - {}_{\sigma}\Psi_{\mathfrak{q}}(x))_{\sigma}^{\alpha-1}}{\Gamma_{\mathfrak{q}}(\alpha)}\mathcal{P}(x)\mathcal{Q}(x)d_{\mathfrak{q},\omega}x\right)^s \\
&\quad\cdot\left(\int_0^{\varsigma}\frac{(\varsigma - {}_{\sigma}\Psi_{\mathfrak{q}}(x))_{\sigma}^{\alpha-1}}{\Gamma_{\mathfrak{q}}(\alpha)}\mathcal{P}(x)\mathcal{U}(x)d_{\mathfrak{q},\omega}x\right)^r \\
&\leq \frac{r\delta_1\Delta_1 + s\delta_2\Delta_2}{(\delta_1\delta_2)^r(\Delta_1\Delta_2)^s}\left(\int_0^{\varsigma}\frac{(\varsigma - {}_{\sigma}\Psi_{\mathfrak{q}}(x))_{\sigma}^{\alpha-1}}{\Gamma_{\mathfrak{q}}(\alpha)}\mathcal{P}(x)\mathcal{Q}(x)\mathcal{U}(x)d_{\mathfrak{q},\omega}x\right)^s.
\end{aligned} \tag{62}$$

Theorem 18. *Let $\mathfrak{q} \in (0, 1)$, $\omega \geq 0$, and $\mathcal{P}, \mathcal{Q}$ and $\mathcal{U}$ be the $\mathfrak{q}, \omega$-integrable functions defined on $[0, \infty)$ such that $\mathcal{P}(x) \geq 0$ and (56) holds. Then, for all $\varsigma > 0$, we have*

$(A_{17})\delta_1\delta_2({}_{0^+}\mathcal{I}^{\alpha}_{\mathfrak{q},\omega}\mathcal{P}\mathcal{U}^2)(\varsigma) + \Delta_1\Delta_2({}_{0^+}\mathcal{I}^{\alpha}_{\mathfrak{q},\omega}\mathcal{P}\mathcal{Q}^2)(\varsigma) \leq (\delta_1\Delta_1 + \delta_2\Delta_2)({}_{0^+}\mathcal{I}^{\alpha}_{\mathfrak{q},\omega}\mathcal{P}\mathcal{Q}\mathcal{U})(\varsigma) \leq |\delta_1\Delta_1 + \delta_2\Delta_2|[({}_{0^+}\mathcal{I}^{\alpha}_{\mathfrak{q},\omega}\mathcal{P}\mathcal{Q}^2)(\varsigma) + ({}_{0^+}\mathcal{I}^{\alpha}_{\mathfrak{q},\omega}\mathcal{P}\mathcal{U}^2)(\varsigma)](A_{17})$

$(A_{18})\sqrt{\delta_1\delta_2/\Delta_1\Delta_2}({}_{0^+}\mathcal{I}^{\alpha}_{\mathfrak{q},\omega}\mathcal{P}\mathcal{U}^2)(\varsigma) + \sqrt{\Delta_1\Delta_2/\delta_1\delta_2}({}_{0^+}\mathcal{I}^{\alpha}_{\mathfrak{q},\omega}\mathcal{P}\mathcal{Q}^2)(\varsigma) \leq (\sqrt{\delta_1\Delta_1/\delta_2\Delta_2} + \sqrt{\delta_2\Delta_2/\delta_1\Delta_1})({}_{0^+}\mathcal{I}^{\alpha}_{\mathfrak{q},\omega}\mathcal{P}\mathcal{Q}\mathcal{U})(\varsigma)(A_{18})$

$(A_{19})({}_{0^+}\mathcal{I}^{\alpha}_{\mathfrak{q},\omega}\mathcal{P}\mathcal{U}^2)(\varsigma)({}_{0^+}\mathcal{I}^{\alpha}_{\mathfrak{q},\omega}\mathcal{P}\mathcal{Q}^2)(\varsigma) \leq (\delta_1\Delta_1 + \delta_2\Delta_2/2\delta_1\Delta_1\delta_2\Delta_2)^2({}_{0^+}\mathcal{I}^{\alpha}_{\mathfrak{q},\omega}\mathcal{P}\mathcal{Q}\mathcal{U})(\varsigma)(A_{19})$

Proof. We clearly see that

$$\mathcal{P}(x)(\delta_2\mathcal{U}(x) - \Delta_1\mathcal{Q}(x))(\Delta_2\mathcal{Q}(x) - \delta_1\mathcal{U}(x)) \geq 0 (x \geq 0). \tag{63}$$

Inequality (63) can be written as

$$\begin{aligned}
&\delta_1\delta_2\mathcal{P}(x)\mathcal{U}^2(x) + \Delta_1\Delta_2\mathcal{P}(x)\mathcal{Q}^2(x) \\
&\quad\leq (\delta_1\Delta_1 + \delta_2\Delta_2)\mathcal{P}(x)\mathcal{Q}(x)\mathcal{U}(x).
\end{aligned} \tag{64}$$

Conducting product on both sides of (64) by $(\varsigma - {}_{\sigma}\Psi_{\mathfrak{q}}(x))_{\sigma}^{(\alpha-1)}/\Gamma_{\mathfrak{q}}(\alpha)$ and then performing the $\mathfrak{q}, \omega$-integration with respect to x over $(0, \varsigma)$ yield

$$\begin{aligned}
&\delta_1\delta_2\left({}_{0^+}\mathcal{I}^{\alpha}_{\mathfrak{q},\omega}\mathcal{P}\mathcal{U}^2\right)(\varsigma) + \Delta_1\Delta_2\left({}_{0^+}\mathcal{I}^{\alpha}_{\mathfrak{q},\omega}\mathcal{P}\mathcal{Q}^2\right)(\varsigma) \\
&\quad\leq (\delta_1\Delta_1 + \delta_2\Delta_2)\left({}_{0^+}\mathcal{I}^{\alpha}_{\mathfrak{q},\omega}\mathcal{P}\mathcal{Q}\mathcal{U}\right)(\varsigma).
\end{aligned} \tag{65}$$

Also, by Cauchy inequality, we get

$$\begin{aligned}
&\delta_1\delta_2\left({}_{0^+}\mathcal{I}^{\alpha}_{\mathfrak{q},\omega}\mathcal{P}\mathcal{U}^2\right)(\varsigma) + \Delta_1\Delta_2\left({}_{0^+}\mathcal{I}^{\alpha}_{\mathfrak{q},\omega}\mathcal{P}\mathcal{Q}^2\right)(\varsigma) \\
&\quad\leq (\delta_1\Delta_1 + \delta_2\Delta_2)\left({}_{0^+}\mathcal{I}^{\alpha}_{\mathfrak{q},\omega}\mathcal{P}\mathcal{Q}\mathcal{U}\right)(\varsigma) \\
&\quad\leq |\delta_1\Delta_1 + \delta_2\Delta_2|\left[\left({}_{0^+}\mathcal{I}^{\alpha}_{\mathfrak{q},\omega}\mathcal{P}\mathcal{Q}^2\right)(\varsigma) + \left({}_{0^+}\mathcal{I}^{\alpha}_{\mathfrak{q},\omega}\mathcal{P}\mathcal{U}^2\right)(\varsigma)\right].
\end{aligned} \tag{66}$$

Multiplying both sides of the inequality (66) by $1/\sqrt{\delta_1\delta_2\Delta_1\Delta_2}$, we get (A_{17}).

Alternately, it follows from $\delta_1\delta_2\Delta_1\Delta_2 > 0$ and

$$\left(\sqrt{\delta_1\delta_2\left({}_{0^+}\mathcal{I}^{\alpha}_{\mathbf{q},\omega}\mathcal{P}\mathcal{U}^2\right)(\varsigma)}-\sqrt{\Delta_1\Delta_2\left({}_{0^+}\mathcal{I}^{\alpha}_{\mathbf{q},\omega}\mathcal{P}\mathcal{Q}^2\right)(\varsigma)}\right)^2\geq 0 \tag{67}$$

that

$$\begin{aligned}&2\sqrt{\delta_1\delta_2\left({}_{0^+}\mathcal{I}^{\alpha}_{\mathbf{q},\omega}\mathcal{P}\mathcal{U}^2\right)(\varsigma)}\sqrt{\Delta_1\Delta_2\left({}_{0^+}\mathcal{I}^{\alpha}_{\mathbf{q},\omega}\mathcal{P}\mathcal{Q}^2\right)(\varsigma)}\\&\quad\leq\delta_1\delta_2\left({}_{0^+}\mathcal{I}^{\alpha}_{\mathbf{q},\omega}\mathcal{P}\mathcal{U}^2\right)(\varsigma)+\Delta_1\Delta_2\left({}_{0^+}\mathcal{I}^{\alpha}_{\mathbf{q},\omega}\mathcal{P}\mathcal{Q}^2\right)(\varsigma).\end{aligned} \tag{68}$$

From (A_{17}) and (68), we get

$$\begin{aligned}&4\delta_1\delta_2\Delta_1\Delta_2\left({}_{0^+}\mathcal{I}^{\alpha}_{\mathbf{q},\omega}\mathcal{P}\mathcal{U}^2\right)(\varsigma)\left({}_{0^+}\mathcal{I}^{\alpha}_{\mathbf{q},\omega}\mathcal{P}\mathcal{Q}^2\right)(\varsigma)\\&\quad\leq(\delta_1\Delta_2+\delta_2\Delta_1)_{0^+}\mathcal{I}^{\alpha}_{\mathbf{q},\omega}\mathcal{P}\mathcal{U}\mathcal{Q}^2(\varsigma),\end{aligned} \tag{69}$$

which conclude (A_{19}). Similarly, we can prove the inequality (A_{18}).

Theorem 19. *Let $\mathbf{q}\in(0,1)$, $\omega\geq 0$, and $\mathcal{P},\mathcal{Q}$ and $\mathcal{U}$ be the $\mathbf{q},\omega$-integrable functions defined on $[0,\infty)$ such that $\mathcal{P}(x)\geq 0$ and*

$$(\delta_2\mathcal{U}(x)-\Delta_1\mathcal{Q}(y))(\Delta_2\mathcal{Q}(y)-\delta_1\mathcal{U}(x))\geq 0 \tag{70}$$

for all $x,y>0$. Then, for all $\varsigma>0$, we have

$$\begin{aligned}&\delta_1\delta_1\left({}_{0^+}\mathcal{I}^{\alpha}_{\mathbf{q},\omega}\mathcal{P}\right)(\varsigma)\left({}_{0^+}\mathcal{I}^{\alpha}_{\mathbf{q},\omega}\mathcal{P}\mathcal{U}^2\right)(\varsigma)\\&\quad+\Delta_1\Delta_2\left({}_{0^+}\mathcal{I}^{\alpha}_{\mathbf{q},\omega}\mathcal{P}\right)(\varsigma)\left({}_{0^+}\mathcal{I}^{\alpha}_{\mathbf{q},\omega}\mathcal{P}\mathcal{Q}^2\right)(\varsigma)\\&\quad\leq(\delta_1\Delta_1+\delta_2\Delta_2)\left({}_{0^+}\mathcal{I}^{\alpha}_{\mathbf{q},\omega}\mathcal{P}\mathcal{Q}\right)\left(\varsigma\left({}_{0^+}\mathcal{I}^{\alpha}_{\mathbf{q},\omega}\mathcal{P}\mathcal{U}\right)(\varsigma)\right),\end{aligned} \tag{71}$$

$$\begin{aligned}&\delta_1\delta_2\left({}_{0^+}\mathcal{I}^{\alpha}_{\mathbf{q},\omega}\mathcal{P}\mathcal{U}\right)^2(\varsigma)+\Delta_1\Delta_2\left({}_{0^+}\mathcal{I}^{\alpha}_{\mathbf{q},\omega}\mathcal{P}\mathcal{Q}\right)^2(\varsigma)\\&\quad\leq(\delta_1\Delta_1+\delta_2\Delta_2)\left({}_{0^+}\mathcal{I}^{\alpha}_{\mathbf{q},\omega}\mathcal{P}\right)(\varsigma)\left({}_{0^+}\mathcal{I}^{\alpha}_{\mathbf{q},\omega}\mathcal{P}\mathcal{Q}\mathcal{U}\right)(\varsigma)\end{aligned} \tag{72}$$

$$\begin{aligned}&\delta_1\delta_2\left({}_{0^+}\mathcal{I}^{\alpha}_{\mathbf{q},\omega}\mathcal{P}\mathcal{U}\right)^2(\varsigma)+\Delta_1\Delta_2\left({}_{0^+}\mathcal{I}^{\alpha}_{\mathbf{q},\omega}\mathcal{P}\mathcal{Q}\right)^2(\varsigma)\\&\quad\leq(\delta_1\Delta_1+\delta_2\Delta_2)\left({}_{0^+}\mathcal{I}^{\alpha}_{\mathbf{q},\omega}\mathcal{P}\mathcal{Q}\right)(\varsigma)\left({}_{0^+}\mathcal{I}^{\alpha}_{\mathbf{q},\omega}\mathcal{P}\mathcal{Q}\mathcal{U}\right)(\varsigma).\end{aligned} \tag{73}$$

Proof. Under the given assumption, we have

$$\mathcal{P}(x)\mathcal{P}(y)(\delta_2\mathcal{U}(x)-\Delta_1\mathcal{Q}(y))(\Delta_2\mathcal{Q}(y)-\delta_1\mathcal{U}(x))\geq 0, \tag{74}$$

which implies that

$$\begin{aligned}&\delta_1\delta_1\mathcal{P}(x)\mathcal{P}(y)\mathcal{U}^2(x)+\Delta_1\Delta_2\mathcal{P}(x)\mathcal{P}(y)\mathcal{Q}^2(y)\\&\quad\leq\delta_1\Delta_1\mathcal{P}(x)\mathcal{P}(y)\mathcal{Q}(y)\mathcal{U}(x)+\delta_2\Delta_2\mathcal{P}(x)\mathcal{P}(y)\mathcal{Q}(y)\mathcal{U}(x).\end{aligned} \tag{75}$$

Conducting product on both sides of (75) by

$$\frac{\left(\varsigma-{}_{\sigma}\Psi_{\mathbf{q}}(x)\right)_{\sigma}^{\alpha-1}\left(\varsigma-{}_{\sigma}\Psi_{\mathbf{q}}(y)\right)_{\sigma}^{\alpha-1}}{\Gamma_{\mathbf{q}}^2(\alpha)}, \tag{76}$$

and then performing the $\mathbf{q},\omega$-integration with respect to x and y over $(0,\varsigma)$, we get (71).

From (65), (71), and the Cauchy inequality

$$\begin{aligned}&\left({}_{0^+}\mathcal{I}^{\alpha}_{\mathbf{q},\omega}\mathcal{P}\mathcal{Q}\right)^2(\varsigma)\leq\left({}_{0^+}\mathcal{I}^{\alpha}_{\mathbf{q},\omega}\mathcal{P}\right)(\varsigma)\left({}_{0^+}\mathcal{I}^{\alpha}_{\mathbf{q},\omega}\mathcal{P}\mathcal{Q}^2\right)(\varsigma),\\&\left({}_{0^+}\mathcal{I}^{\alpha}_{\mathbf{q},\omega}\mathcal{P}\mathcal{U}\right)^2(\varsigma)\leq\left({}_{0^+}\mathcal{I}^{\alpha}_{\mathbf{q},\omega}\mathcal{P}\right)(\varsigma)\left({}_{0^+}\mathcal{I}^{\alpha}_{\mathbf{q},\omega}\mathcal{P}\mathcal{U}^2\right)(\varsigma),\end{aligned} \tag{77}$$

we get

$$\begin{aligned}&\delta_1\delta_2\left({}_{0^+}\mathcal{I}^{\alpha}_{\mathbf{q},\omega}\mathcal{P}\mathcal{U}\right)^2(\varsigma)+\Delta_1\Delta_2\left({}_{0^+}\mathcal{I}^{\alpha}_{\mathbf{q},\omega}\mathcal{P}\mathcal{Q}\right)^2(\varsigma)\\&\quad\leq\delta_1\delta_2\left({}_{0^+}\mathcal{I}^{\alpha}_{\mathbf{q},\omega}\mathcal{P}\right)(\varsigma)\left({}_{0^+}\mathcal{I}^{\alpha}_{\mathbf{q},\omega}\mathcal{P}\mathcal{Q}^2\right)(\varsigma)\\&\qquad+\Delta_1\Delta_2\left({}_{0^+}\mathcal{I}^{\alpha}_{\mathbf{q},\omega}\mathcal{P}\right)(\varsigma)\left({}_{0^+}\mathcal{I}^{\alpha}_{\mathbf{q},\omega}\mathcal{P}\mathcal{U}^2\right)(\varsigma)\\&\quad\leq(\delta_1\Delta_1+\delta_2\Delta_2)\left({}_{0^+}\mathcal{I}^{\alpha}_{\mathbf{q},\omega}\mathcal{P}\right)(\varsigma)\left({}_{0^+}\mathcal{I}^{\alpha}_{\mathbf{q},\omega}\mathcal{P}\mathcal{Q}\mathcal{U}\right)(\varsigma),\\&\delta_1\delta_2\left({}_{0^+}\mathcal{I}^{\alpha}_{\mathbf{q},\omega}\mathcal{P}\mathcal{U}\right)^2(\varsigma)+\Delta_1\Delta_2\left({}_{0^+}\mathcal{I}^{\alpha}_{\mathbf{q},\omega}\mathcal{P}\mathcal{Q}\right)^2(\varsigma)\\&\quad\leq(\delta_1\Delta_1+\delta_2\Delta_2)\left({}_{0^+}\mathcal{I}^{\alpha}_{\mathbf{q},\omega}\mathcal{P}\mathcal{Q}\right)(\varsigma)\left({}_{0^+}\mathcal{I}^{\alpha}_{\mathbf{q},\omega}\mathcal{P}\mathcal{U}\right)(\varsigma),\end{aligned} \tag{78}$$

which completes the proof of (72) and (73).

4. Conclusion

We have discovered several generalizations for the generalized Čebyšev type inequality via quantum fractional Hahn's integral operator by using the quantum shift operator ${}_{\sigma}\Psi_{\mathbf{q}}(\varsigma)=\mathbf{q}\varsigma+(1-\mathbf{q})\sigma(\varsigma\in[l_1,l_2],\sigma=l_1+(\omega/(1-\mathbf{q})),0<\mathbf{q}<1,\omega\geq 0)$, provided some associated variants to show the efficiency of quantum Hahn's integral operator, and compared our obtained results and proposed technique with the previously known results and existing technique. The outcome shows that the proposed plans are extremely important and computationally appealing to deal with several sorts of differential equations. As a future research course of this paper, the new techniques obtained in the present paper can be prolonged to attain analytical solutions of quantum mechanics introduced in different works distributed currently connected with high-dimensional fractional equations.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors' Contributions

All authors contributed equally to writing of this paper. All authors read and approved the final manuscript.

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