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Research Article

Solutions for a Singular Hadamard-Type Fractional Differential Equation by the Spectral Construct Analysis

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In this paper, we consider the existence of positive solutions for a Hadamard-type fractional differential equation with singular nonlinearity. By using the spectral construct analysis for the corresponding linear operator and calculating the fixed point index of the nonlinear operator, the criteria of the existence of positive solutions for equation considered are established. The interesting point is that the nonlinear term possesses singularity at the time and space variables.

1. Introduction

In this paper, we focus on the existence of positive solutions for the following singular Hadamard-type fractional differential equation:

$$\begin{cases} \mathscr{D}_{\mathbf{t}}{}^{\alpha}\mathscr{D}_{\mathbf{t}}{}^{\beta}z(t)=f\left(t,z(t),-\mathscr{D}_{\mathbf{t}}{}^{\beta}z(t)\right), & 1<t<e, \\ z(1)=\sigma z(1)=\sigma z(e)=0, \mathscr{D}_{\mathbf{t}}{}^{\beta}z(1)=\sigma\mathscr{D}_{\mathbf{t}}{}^{\beta}z(1)=\sigma\mathscr{D}_{\mathbf{t}}{}^{\beta}z(e)=0, \end{cases} \tag{1}$$

where $2<\alpha,\beta\le 3$; σ is a differential operator denoted by $t(d/dt)$, that is, $\sigma z(t)=t(d/dt)z(t)$; $\mathscr{D}_{\mathbf{t}}{}^{\alpha}, \mathscr{D}_{\mathbf{t}}{}^{\beta}$ are the Hadamard fractional derivative of order α, β; and $f\in(1,e)\times(0,+\infty)\times(0,+\infty), [0,+\infty)$ is a continuous function with singularity at $t=0,1$ and $u=0, v=0$.

Singularity refers to a point or a domain where the given mathematical object is not defined or not "well-behaved." Near a singular point or zone, a minor change of the variable will lead to major changes of the property of the target object. Many physical phenomena in natural sciences and engineering often exhibit some singular behaviour. For example, Fisk [1] found that in certain materials the quantum fluctuations at absolute zero may push a system into a different phase or state, as result, the process loses its continuity, and then, the singular behaviour happens near the quantum critical points. In fluid mechanics, when a fluid is subjected to a severe impact to form a fracture, singular points or singular domains also follow the fracture. Normally, at singular points and domains, the extreme behaviour such as blow-up phenomena [2, 3], impulsive influence [4–9], and chaotic system [10–13], often leads to some difficulties for people in understanding and predicting the corresponding natural problems. Hence, the study of singularity for complex systems governed by differential equations [14–27] is important and interesting in deepening the understanding of the internal laws of dynamic system.

On the other hand, since the fractional differential operator is nonlocal, some often use it to describe viscoelastic behaviour and memory phenomena in various natural science fields such as the silicone gel with the property of weak frequency dependency [28, 29] and advection dispersion in anomalous diffusion [30–34]. In most cases, some are interested in the qualitative properties of solutions for the corresponding fractional equations; for the detail, see [35–65]. In particular, in order to obtain the qualitative properties of solutions, many nonlinear analysis methods, such as fixed point theorems [66–71], iterative techniques [72–80],variational methods [81–98], and upper and lower solution methods [29, 44], have been developed and employed to study the qualitative properties and numerical results of solutions for various types of differential equations. For example, by using the fixed point index theory, Wang [69] established the existence and multiplicity of positive solutions for the following nonlocal singular fractional differential equation:

$$\begin{cases} D_{0_+}^{\alpha}u(t)+f(t,u(t)=0, 0<t<1, n-1<\alpha\le n, \\ u(0)=u'(0)==u^{(n-2)}(0)=0, D_{0_+}^{\beta}u(1)=\int_0^{\eta}a(t)D_{0_+}^{\gamma}u(t)dV(t), \end{cases} \tag{2}$$

where $D_{0_+}^{\alpha}$ denotes the standard Riemann-Liouville fractional derivative, $n\ge 3, 0<\beta<1, 0\le\gamma<\alpha-1, \eta\in(0,1], f(t,x)$ may be singular at $t=0,1$, and $x=0, a(t)\in L^1[0,1]\cap C(0,1)$ is nonnegative. In recent years, to enrich the theory of fractional calculus, the fractional derivative and integral were extended to many different forms such as Hadamard, Erdelyi-Kober, Hilfer derivatives, and integrals. In particular, it is more difficult to obtain the qualitative properties of solutions for Hadamard-type fractional differential equations since Hadamard derivatives possess a singular logarithmic kernel [99–106].

In this paper, we focus on the existence of positive solutions for the Hadamard-type fractional differential equation (1) with singularity in space variables. Our work has some new contributions. Firstly, the equation contains a Hadamard-type fractional derivative which has a singular logarithmic kernel. Secondly, the nonlinearity can have strong singularity in time and space variables. Thirdly, a new limit condition of integral type is introduced to overcome the difficulty of singularity. The rest of this paper is organized as follows. In Section 2, we firstly introduce the concept of Hadamard fractional integral and differential operators and then give the logarithmic kernel and Green function of the boundary value problem and their properties. Our main results are summarized in Section 3.

2. Preliminaries and Lemmas

Before the main results, we firstly recall the definition of the Hadamard-type fractional integrals and derivatives; for detail, see [107].

Let $\alpha\in\mathbb{C}, \mathbf{R}e(\alpha)>0, n=[\mathbf{R}e(\alpha)]$ and (a,b) be a finite or infinite interval of $\mathbb{R}^+$. The α-order left Hadamard fractional integral is defined by

$$(I_a^{\alpha}x)(t)=\frac{1}{\Gamma(\alpha)}\int_a^t\left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{x(s)}{s}ds, t\in(a,b), \tag{3}$$

and the α left Hadamard fractional derivative is defined by

$$(\mathscr{D}_{\mathbf{t}}{}^{\alpha})x(t)=\frac{1}{\Gamma(n-\alpha)}\left(t\frac{d}{dt}\right)^n\int_a^t\left(\ln\frac{t}{s}\right)^{n-\alpha-1}\frac{x(s)}{s}ds, t\in(a,b). \tag{4}$$

In what follows, we consider the following linear auxiliary problem:

$$\begin{cases} -\mathscr{D}_{\mathbf{t}}{}^{\beta}z(t)=x(t), 1<t<e, \\ z(1)=\sigma z(1)=\sigma z(e)=0. \end{cases} \tag{5}$$

It follows from [99] that problem (5) has a unique solution

$$z(t)=\int_1^e H(t,s)x(s)\frac{ds}{s}, \tag{6}$$

where

$$H(t,s)=\frac{1}{\Gamma(\beta)}\begin{cases} (\ln t)^{\beta-1}(1-\ln s)^{\beta-2}-(\ln t-\ln s)^{\beta-1}, & 1\le s\le t\le e, \\ (\ln t)^{\beta-1}(1-\ln s)^{\beta-2}, & 1\le t\le s\le e, \end{cases} \tag{7}$$

is Green's function of equation (5). Now, let $x(t)=-\mathscr{D}_{\mathbf{t}}{}^{\beta}z(t)$, then the Hadamard-type fractional differential equation (1) reduces to the following convenient form:

$$\begin{cases} -\mathscr{D}_{\mathbf{t}}{}^{\alpha}x(t)=f\left(t,\int_1^e H(t,s)x(s)\frac{ds}{s},x(t)\right), t\in(1,e), \\ x(1)=\sigma z(1)=\sigma x(e)=0. \end{cases} \tag{8}$$

It follows from (6) that equation (8) is equivalent to the following integral equation:

$$x(t)=\int_1^e G(t,s)\left[f\left(s,\int_1^e H(s,\tau)x(\tau)\frac{d\tau}{\tau},x(s)\right)\right]\frac{ds}{s}, \tag{9}$$

where

$$G(t,s)=\frac{1}{\Gamma(\alpha)}\begin{cases} (\ln t)^{\alpha-1}(1-\ln s)^{\alpha-2}-(\ln t-\ln s)^{\alpha-1}, & 1\le s\le t\le e, \\ (\ln t)^{\alpha-1}(1-\ln s)^{\alpha-2}, & 1\le t\le s\le e. \end{cases} \tag{10}$$

As a result, in order to find the positive solutions of equation (1), it is sufficient to search the fixed point of the following operator:

$$Tx(t)=\int_1^e G(t,s)\left[f\left(s,\int_1^e H(s,\tau)x(\tau)\frac{d\tau}{\tau},x(s)\right)\right]\frac{ds}{s}. \tag{11}$$

Lemma 1 (see [99]). *Let $\psi_i(t)=\ln t(1-\ln t)^{i-2}, i=\alpha,\beta, t\in[1,e]$. Then, Green's functions H, G has the following properties:*

$$H, G \in C([1,e] \times [1,e], \mathbb{R}^+). \tag{12}$$

(i) For all $t, s \in [1,e]$, the following inequalities hold:

$$\begin{aligned}\frac{1}{\Gamma(\beta)}(\ln t)^{\beta-1}\psi_\beta(s) \le H(t,s) \le \frac{1}{\Gamma(\beta)}\psi_\beta(s),\\ \frac{1}{\Gamma(\alpha)}(\ln t)^{\alpha-1}\psi_\alpha(s) \le G(t,s) \le \frac{1}{\Gamma(\alpha)}\psi_\alpha(s)\end{aligned} \tag{13}$$

Let Q be a cone of Banach space E, for any $0 < r < R < \infty$, define

$$\begin{aligned}\bar{Q}_r = \{x \in Q : \|x\| < r\}, \partial Q_r = \{x \in Q : \|x\| = r\}, \bar{Q}_R \setminus Q_r\\ = \{x \in Q : r \le \|x\| \le R\}.\end{aligned} \tag{14}$$

Now, we state the following lemmas which will be used in the rest of the paper.

Lemma 2 (see [108]). *Assume $T : \bar{Q}_r \to Q$ is a completely continuous operator.*

(i) If there exists $x_0 \in Q \setminus \{\theta\}$ such that

$$x - Tx \ne \mu x_0, x \in \partial Q_r, \mu \ge 0, \tag{15}$$

then, the fixed point index $i(T, Q_r, Q) = 0$

(ii) If

$$Tx \ne \mu x, x \in \partial Q_r, \mu \ge 1, \tag{16}$$

then, the fixed point index $i(T, Q_r, Q) = 1$

Lemma 3 (Krein-Rutmann, see [108]). *Let $L : E \to E$ be a continuous linear operator, P be a total cone, and $L(P) \subset P$. If there exist $\psi \in E \setminus (-P)$ and a positive constant c such that $cL(\psi) \ge \psi$, then the spectral radius $\rho(L) \ne 0$ and has a positive eigenfunction corresponding to its first eigenvalue $\lambda = \rho(L)^{-1}$.*

Lemma 4 (Gelfand's formula, see [108]). *For a bounded linear operator L and the operator norm $\|\cdot\|$, the spectral radius of L^n satisfies*

$$\rho(L) = \lim_{n\to+\infty} \|L^n\|^{1/n}. \tag{17}$$

In this paper, we use the following assumption:

(B1) $f \in C((0,1) \times (0,+\infty) \times (0,+\infty), [0,+\infty))$ and for any $0 < r < R < +\infty$,

$$\lim_{n\to+\infty} \sup_{u,v\in \bar{Q}_R Q_r} \int_{\Omega_n} \psi_\alpha(s) f(s, u(s), v(s)) ds = 0, \tag{18}$$

where $\Omega_n = [1, 1 + (1/n)] \cup [e - (1/n), e]$.

Now, let $E = C[1,e]$, then E is a Banach space equipped with the norm $\|x\| = \max_{t\in[1,e]} | x(t) |$. Let $P = \{x \in E : x(t) \ge 0, t \in [1,e]\}$ and

$$Q = \{x \in P : x(t) \ge (\ln t)^{\alpha-1} \|x\|, t \in [1,e]\}, \tag{19}$$

then, Q is a cone in the Banach space E and $\bar{Q}_R Q_r \subset Q \subset P$. Let us define a nonlinear operator $T : \bar{Q}_R \setminus Q_r \to P$ and a linear operator $L : E \to E$:

$$\begin{aligned}(Tx)(t) &= \int_1^e G(t,s)\left[f\left(s, \int_1^e H(s,\tau)x(\tau)\frac{d\tau}{\tau}, x(s)\right)\right]\frac{ds}{s}, t \in [1,e],\\ (Lx)(t) &= \int_1^e G(t,s)x(s)\frac{ds}{s}, t \in [1,e].\end{aligned} \tag{20}$$

Thus, in order to solve equation (1), we only need to find the fixed point of operator equation $x = Tx$. To do this, we firstly establish some lemmas.

Lemma 5. *$L : Q \to Q$ is a completely continuous operator with the spectral radius $\rho(L) \ne 0$. Moreover, L has a positive eigenfunction ω^* corresponding to the first eigenvalue $\mu_1 = (\rho(L))^{-1}$.*

Proof. Firstly, it follows from Lemma 2 that, for any $x \in Q$, one has

$$\begin{aligned}\|Lx\| = \max_{t\in[1,e]} \int_1^e G(t,s)x(s)\frac{ds}{s} \le \frac{1}{\Gamma(\alpha)}\int_1^e \psi_\alpha(s)x(s)\frac{ds}{s},\\ Lx(t) \ge \frac{1}{\Gamma(\alpha)}(\ln t)^{\alpha-1}\int_1^e \psi_\alpha(s)x(s)\frac{ds}{s},\end{aligned} \tag{21}$$

which imply that $L : Q \to Q$. Since $H(t,s), t \in [1,e] \times [1,e]$ is uniform continuity, then the operator $L : Q \to Q$ is completely continuous.

On the other hand, by (10), we know that there exists a $\tau_0 \in (1,e)$ such that $G(\tau_0, \tau_0) > 0$. Thus, from the continuity of G, there exists a closed interval $[c,d] \subset (1,e)$ such that $\tau_0 \in (c,d)$ and $G(t,s) > 0$ for all $t, s \in [c,d]$. Now, we take $x \in Q$ such that $x(\tau_0) > 0$ and $x(t) = 0$ for all $t\in[c,d]$. Then, for all $t \in [c,d]$, one has

$$(Lx)(t) = \int_1^e G(t,s)x(s)\frac{ds}{s} \ge \int_c^d G(t,s)x(s)\frac{ds}{s} > 0. \tag{22}$$

Thus, there exists $\mu > 0$ such that $\mu(Lx)(t) \ge x(t)$ for $t \in [0,1]$. From Krein-Rutmann's theorem, we know that the spectral radius $\rho(L) \neq 0$ and T have a positive eigenfunction ω^* that satisfies $\mu_1 L\omega^* = \omega^*$, where $\mu_1 = (\rho(L))^{-1}$ is its first eigenvalue. The proof is completed.

Lemma 6. *Suppose that (B1) holds, then the operator $T : \bar{Q}_R \setminus Q_r \to Q$ is completely continuous.*

Proof. Firstly, for any $x \in \bar{Q}_R \setminus Q_r$, $t \in [1, e]$, it follows from Lemma 2 that

$$\begin{aligned}(Tx)(t) &= \int_1^e G(t,s)\left[f\left(s, \int_1^e H(s,\tau)x(\tau)\frac{d\tau}{\tau}, x(s)\right)\right]\frac{ds}{s}\\ &\leq \frac{1}{\Gamma(\alpha)}\int_1^e \psi_\alpha(s)\left[f\left(s, \int_1^e H(s,\tau)x(\tau)\frac{d\tau}{\tau}, x(s)\right)\right]\frac{ds}{s},\end{aligned} \tag{23}$$

that is

$$\|Tx\| \leq \frac{1}{\Gamma(\alpha)}\int_1^e \psi_\alpha(s)\left[f\left(s, \int_1^e H(s,\tau)x(\tau)\frac{d\tau}{\tau}, x(s)\right)\right]\frac{ds}{s}. \tag{24}$$

Similarly, one also has

$$\begin{aligned}(Tx)(t) &= \int_1^e G(t,s)\left[f\left(s, \int_1^e H(s,\tau)x(\tau)\frac{d\tau}{\tau}, x(s)\right)\right]\frac{ds}{s}\\ &\geq \frac{1}{\Gamma(\alpha)}(\ln t)^{\alpha-1}\int_1^e \psi_\alpha(s)\left[f\left(s, \int_1^e H(s,\tau)x(\tau)\frac{d\tau}{\tau}, x(s)\right)\right]\frac{ds}{s}\\ &\geq (\ln t)^{\alpha-1}\|Tx\|,\end{aligned} \tag{25}$$

which implies $T(Q) \subset Q$ and then $T(\bar{Q}_R \setminus Q_r) \subset Q$.

On the other hand, from (B1), we know that there exists a natural number l such that

$$\sup_{x\in\bar{Q}_R\setminus Q_r}\int_{\Omega_l}\psi_\alpha(s)\left[f\left(s, \int_1^e H(s,\tau)x(\tau)\frac{d\tau}{\tau}, x(s)\right)\right]\frac{ds}{s} < \frac{1}{2}. \tag{26}$$

Thus, for any $x \in \bar{Q}_R Q_r$ and $1 + (1/l) \leq t \leq e - 1/l$, we have

$$\left(\ln\left(1 + \frac{1}{l}\right)\right)^{\alpha-1} r \leq x(t) \leq \|x\| = R, t \in [1, e], \tag{27}$$

$$\begin{aligned}&\frac{(\ln(1+(1/l)))^{\alpha-1}(\ln(1+(1/l)))^{\beta-1}r}{(\beta-1)\Gamma(\beta+1)} \leq \frac{(\ln t)^{\alpha-1}(\ln t)^{\beta-1}}{\Gamma(\beta)}\\ &\cdot \int_1^e \psi_\beta(\tau)x(\tau)\frac{d\tau}{\tau}\|x\| \leq \int_1^e H(s,\tau)x(\tau)\frac{d\tau}{\tau}\\ &\leq \frac{1}{\Gamma(\beta)}\int_1^e \psi_\beta(\tau)x(\tau)\frac{d\tau}{\tau} \leq \frac{R}{(\beta-1)\Gamma(\beta+1)}, t \in [1, e].\end{aligned} \tag{28}$$

Take

$$M_1 = \max_{(t,x,y)\in\Omega_l^*}\{f(t,x,y)\}, \tag{29}$$

where

$$\begin{aligned}\Omega_l^* = &\left[1 + \frac{1}{l}, e - \frac{1}{l}\right] \times \left[\frac{(\ln(1+(1/l)))^{\alpha-1}(\ln(1+(1/l)))^{\beta-1}R}{(\beta-1)\Gamma(\beta+1)},\right.\\ &\left.\frac{r}{(\beta-1)\Gamma(\beta+1)}\right] \times \left[\left(\ln\left(1+\frac{1}{l}\right)\right)^{\alpha-1} r, R\right].\end{aligned} \tag{30}$$

So, it follows from (26), (27) and (28) that

$$\begin{aligned}&\sup_{x\in\bar{Q}_R\setminus Q_r}\int_1^e \psi_\alpha(s)\left[f\left(s, \int_1^e H(s,\tau)x(\tau)\frac{d\tau}{\tau}, x(s)\right)\right]\frac{ds}{s}\\ &\leq \sup_{x\in\bar{Q}_R\setminus Q_r}\int_{\Omega_l}\psi_\alpha(s)\left[f\left(s, \int_1^e H(s,\tau)x(\tau)\frac{d\tau}{\tau}, x(s)\right)\right]\frac{ds}{s}\\ &\quad + \sup_{x\in\bar{Q}_R\setminus Q_r}\int_{1+1/l}^{e-1/l}\psi_\alpha(s)\left[f\left(s, \int_1^e H(s,\tau)x(\tau)\frac{d\tau}{\tau}, x(s)\right)\right]\frac{ds}{s}\\ &\leq \frac{1}{2} + M_1\int_1^e \psi_\alpha(s)\frac{ds}{s} < +\infty,\end{aligned} \tag{31}$$

which implies T is uniformly bounded for any bounded set.

Secondly, we shall prove that $T : \bar{Q}_R \setminus Q_r \to Q$ is continuous. To do this, let $x_n, x_0 \in \bar{Q}_R Q_r$ and $\|x_n - x_0\| \to 0$ $(n \to \infty)$. For any $\varepsilon > 0$, it follows from (B1) that there exists a natural number $m > 0$ such that

$$\sup_{x\in\bar{Q}_R\setminus Q_r}\int_{\Omega_m}\psi_\alpha(s)\left[f\left(s, \int_1^e H(s,\tau)x(\tau)\frac{d\tau}{\tau}, x(s)\right)\right]\frac{ds}{s} < \frac{\varepsilon}{4}. \tag{32}$$

On the other hand, it follows form the fact that $f(t, u, v)$ is uniformly continuous on$\Omega_m^* = [1 + (1/m), e - 1/m] \times [((\ln(1 + (1/m)))^{\alpha-1}(\ln(1 + (1/m)))^{\beta-1}R/(\beta-1)\Gamma(\beta+1)), (r/(\beta-1)\Gamma(\beta+1))] \times [(\ln(1 + (1/m)))^{\alpha-1}r, R]$, (21) that

$$\begin{aligned}\lim_{n\to+\infty}\bigg|&f\left(s, \int_1^e H(s,\tau)x_n(\tau)\frac{d\tau}{\tau}, x_n(s)\right)\\ &- f\left(s, \int_1^e H(s,\tau)x_0(\tau)\frac{d\tau}{\tau}, x_0(s)\right)\bigg| = 0,\end{aligned} \tag{33}$$

holds uniformly on $s \in [1 + (1/m), e - (1/m)]$. Thus, the

Lebesgue control convergence theorem ensures

$$\int_{1+(1/m)}^{e-(1/m)} \psi_\alpha(s) \left| f\left(s, \int_1^e H(s,\tau)x_n(\tau)\frac{d\tau}{\tau}, x_n(s)\right) - f\left(s, \int_1^e H(s,\tau)x_0(\tau)\frac{d\tau}{\tau}, x_0(s)\right)\right| \frac{ds}{s} \to 0, \text{ as } n \to \infty. \tag{34}$$

In other words, for the above $\varepsilon > 0$, there exists a positive integer N such that $n > N$, we have

$$\int_{1+1/m}^{e-1/m} \psi_\alpha(s) \left| f\left(s, \int_1^e H(s,\tau)x_n(\tau)\frac{d\tau}{\tau}, x_n(s)\right) - f\left(s, \int_1^e H(s,\tau)x_0(\tau)\frac{d\tau}{\tau}, x_0(s)\right)\right| \frac{ds}{s} < \frac{\varepsilon}{2}. \tag{35}$$

Thus, by (32) and (35), for any $n > N$, we have

$$\begin{aligned}\|Tx_n - Tx_0\| &\le 2 \sup_{x\in \bar{Q}_R\setminus Q_r} \int_{\Omega_m} \psi_\alpha(s)\left[f\left(s, \int_1^e H(s,\tau)x_0(\tau)\frac{d\tau}{\tau}, x_0(s)\right)\right]\frac{ds}{s} \\ &+ \int_{1+(1/m)}^{e-(1/m)} \psi_\alpha(s) \left| f\left(s, \int_1^e H(s,\tau)x_n(\tau)\frac{d\tau}{\tau}, x_n(s)\right) \right. \\ &\left. - f\left(s, \int_1^e H(s,\tau)x_0(\tau)\frac{d\tau}{\tau}, x_0(s)\right)\right| \frac{ds}{s} < 2\times\frac{\varepsilon}{4} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned} \tag{36}$$

Therefore, $T : \bar{Q}_R Q_r \to Q$ is continuous.

In the end, we shall prove that T is equicontinuous. Firstly, it follows from (B1) that for any $\varepsilon > 0$, there exists a positive integer k such that

$$\sup_{x\in \bar{Q}_R\setminus Q_r} \int_{\Omega_k} \psi_\alpha(s)\left[f\left(s, \int_1^e H(s,\tau)x(\tau)\frac{d\tau}{\tau}, x(s)\right)\right]\frac{ds}{s} < \frac{\varepsilon}{4}. \tag{37}$$

Take

$$M_2 = \max_{(t,x,y)\in\Omega_k^*} \{f(t,x,y)\}, \tag{38}$$

where

$$\Omega_k^* = \left[1 + \frac{1}{k}, e - \frac{1}{k}\right] \times \left[\frac{(\ln(1+1/k))^{\alpha-1}(\ln(1+(1/k)))^{\beta-1} r}{(\beta-1)\Gamma(\beta+1)}, \frac{R}{(\beta-1)\Gamma(\beta+1)}\right] \times \left[\left(\ln\left(1+\frac{1}{k}\right)\right)^{\alpha-1} r, R\right]. \tag{39}$$

Notice that $G(t,s)$ is uniformly continuous on $[1,e]\times[1,e]$, so for the above $\varepsilon > 0$ and fixed $s \in [1+(1/k), e-(1/k)]$, there exists $\delta > 0$, when $|t_1 - t_2| < \delta$, $t_1, t_2 \in [1,e]$, we have

$$|G(t_1,s) - G(t_2,s)| \le [2M_2(e-1)]^{-1}\varepsilon. \tag{40}$$

It follows from the above argument that, for $|t_1 - t_2| < \delta$, $t_1, t_2 \in [1,e]$, one has

$$\begin{aligned}|Tx(t) - Tx(t')| &\le 2 \sup_{x\in \bar{Q}_R\setminus Q_r} \int_{\Omega_k} \psi_\alpha(s)\left[f\left(s, \int_1^e H(s,\tau)x(\tau)\frac{d\tau}{\tau}, x(s)\right)\right]\frac{ds}{s} \\ &+ \sup_{x\in \bar{Q}_R\setminus Q_r} \int_{1+(1/k)}^{e-(1/k)} |G(t,s) - G(t',s)| \left[f\left(s, \int_1^e H(s,\tau)x(\tau)\frac{d\tau}{\tau}, x(s)\right)\right]\frac{ds}{s} \\ &< 2\times\frac{\varepsilon}{4} + \frac{\varepsilon}{2} = \varepsilon,\end{aligned} \tag{41}$$

which implies that T is an equicontinuous operator. According to the Arzela-Ascoli Theorem, $T : \bar{Q}_R \setminus Q_r \to Q$ is completely continuous. The proof is completed.

3. Main Results

We state the main results of this paper as follows.

Theorem 7. *Let μ_1 be the first eigenvalue of L, assume (B1) holds and*

$$\limsup_{\substack{u+v\to+\infty \\ v\to+\infty}} \frac{f(t,u,v)}{v} < \mu_1 < \liminf_{\substack{u\to 0^+ \\ v\to 0^+}} \frac{f(t,u,v)}{u+v}, \tag{42}$$

uniformly holds on $t \in [1,e]$. Then, equation (1) has at least one positive solution.

Proof. Firstly, by Lemma 6, we know that $T : \bar{Q}_R \setminus Q_r \to Q$ is completely continuous. Thus, it follows from the extension theorem of a completely continuous operator (see Theorem 8.3 on page 56 of [108]) that, for any $R > 0$, there exists a completely continuous operator $T^* : \bar{Q}_R \to Q$. Thus, without loss of the generality, we shall still write it as T.

Next, it follows from (42) that there exists $r > 0$ such that

$$f(t,u,v) \ge \mu_1(u+v),\ |v| \le \frac{r}{(\beta-1)\Gamma(\beta+1)},\ |x| \le r,\ t\in[1,e]. \tag{43}$$

Since, for any $x \in \partial Q_r$, we have

$$\left|\int_1^e H(s,\tau)x(\tau)\frac{d\tau}{\tau}\right| \le \frac{r}{(\beta-1)\Gamma(\beta+1)}, |x(s)| \le r, \tag{44}$$

thus, for $x \in \partial Q_r$, from (43) and (44), one gets

$$\begin{aligned}(Tx)(t) &= \int_1^e G(t,s)\left[f\left(s, \int_1^e H(s,\tau)x(\tau)\frac{d\tau}{\tau}, x(s)\right)\right]\frac{ds}{s} \\ &\ge \mu_1 \int_1^e G(t,s)\left|\int_1^e H(s,\tau)x(\tau)\frac{d\tau}{\tau} + x(s)\right|\frac{ds}{s} \\ &\ge \mu_1(Lx)(t), t\in[1,e].\end{aligned} \tag{45}$$

Let ω^* be the positive eigenfunction corresponding to μ_1, i.e., $\omega^* = \mu_1 L\omega^*$. In what follows, we shall use the contradiction method to show

$$x - Tx \neq \mu\omega^*, x \in \partial Q_r, \mu \geq 0. \tag{46}$$

Firstly, we can suppose that T has no fixed points on $x \in \partial Q_r$ (otherwise, the proof is finished). In this case, let us suppose there exist $x_0 \in \partial Q_r$ and $\mu_0 \geq 0$ such that $x_0 - Tx_0 = \mu_0\omega^*$. This implies that $\mu_0 > 0$ and then

$$x_0 = Tx_0 + \mu_0\omega^* \geq \mu_0\omega^*. \tag{47}$$

Let $\bar{\mu} = \sup\{\mu \mid x_0 \geq \mu\omega^*\}$, then $\bar{\mu} \geq \mu_0$, $x_0 \geq \bar{\mu}\omega^*$, $\mu_1 Lx_0 \geq \mu_1\bar{\mu}L\omega^* = \bar{\mu}\omega^*$. Thus, it follows from (45) that

$$x_0 = Tx_0 + \mu_0\omega^* \geq \mu_1 Lx_0 + \mu_0\omega^* \geq \bar{\mu}\omega^* + \mu_0\omega^* = (\bar{\mu} + \mu_0)\omega^*, \tag{48}$$

which contradicts with the definition of $\bar{\mu}$. So (46) holds and from Lemma 2, we have

$$i(T, Q_r, Q) = 0. \tag{49}$$

On the other hand, it follows from (42) that there exists $R_1 > r$ and $0 < \kappa < 1$ such that

$$f(t, u, v) \leq \kappa\mu_1|v|, \text{for } |u + v| \geq R_1,\ |v| \geq R_1. \tag{50}$$

Let $\tilde{L}x = \kappa\mu_1 Lx$. Obviously, $\tilde{L} : E \to E$ is also a bounded linear operator and $\tilde{L}(Q) \subset Q$. Since μ_1 is the first eigenvalue of operator L and $0 < \kappa < 1$, we have

$$r^{-1}(\tilde{L}) = (\kappa\mu_1 r(L))^{-1} = \kappa^{-1} > 1. \tag{51}$$

From Gelfand's formula, we have

$$\kappa = \lim_{n\to+\infty} \left\|\tilde{L}^n\right\|^{1/n}. \tag{52}$$

Now, choose $\varepsilon_0 = 1/2(1 - \kappa)$, it follows from (52) that there exists an enough large integer $N > 0$ such that when $n \geq N$, one has

$$\|\tilde{L}^n\| \leq [\kappa + \varepsilon_0]^n. \tag{53}$$

Let $\tilde{L}^0 = I$ be the identity operator and define

$$\|x\|^* = \sum_{i=1}^{N} [\kappa + \varepsilon_0]^{N-i}\|\tilde{L}^{i-1}x\|, x \in E, \tag{54}$$

then $\|\cdot\|^*$ is still the norm of E. Let

$$M = \sup_{x\in\partial Q_{R_1}} \int_1^e \psi_\alpha(s)\left[f\left(s, \int_1^e H(s,\tau)x(\tau)\frac{d\tau}{\tau}, x(s)\right)\right]\frac{ds}{s}, \tag{55}$$

then from (31), we have $M < +\infty$. Take

$$R_2 > \max\left\{R_1, \frac{2}{\varepsilon_0}M^*\right\}, \tag{56}$$

where $M^* = \|M\|^*$. Notice that $\|x\|^* > [\kappa + \varepsilon_0]^{N-1}\|x\|$, so we can choose a large enough $R > R_2$ such that $\|x\| \geq R$ implies $\|x\|^* > R_2$.

Next, we shall show

$$Tx \neq \mu x, x \in \partial Q_R, \mu \geq 1. \tag{57}$$

Otherwise, there exist $x_1 \in \partial Q_R$ and $\mu^* \geq 1$ such that $Tx_1 = \mu^* x_1$. Let $\tilde{x}(t) = \min\{x_1(t), R_1\}$ and

$$D(x_1) = \{t \in [1, e]: x_1(t) > R_1\}. \tag{58}$$

Since $x_1 \in C[1, e]$, $x_1(t) \leq \|x_1\| = R$, there exists $1 < t_0 \leq e$ such that $x_1(t_0) = R$. So for any $t \in [1, e]$, we get $\tilde{x}(t) = \min\{x_1(t), R_1\} \leq \min\{R, R_1\} = R_1$ and $\tilde{x}(t_0) = \min\{x_1(t_0), R_1\} = \min\{R, R_1\} = R_1$, which implies $\|\tilde{x}(t)\| = R_1$, i.e., $\tilde{x} \in \partial Q_{R_1}$. Thus by Lemma 2, we have

$$\begin{aligned}
\mu^* x_1 = (Tx_1)(t) &= \int_1^e G(t,s)\left[f\left(s, \int_1^e H(s,\tau)x_1(\tau)\frac{d\tau}{\tau}, x_1(s)\right)\right]\frac{ds}{s} \\
&\leq \int_{D(x_1)} G(t,s)\left[f\left(s, \int_1^e H(s,\tau)x_1(\tau)\frac{d\tau}{\tau}, x_1(s)\right)\right]\frac{ds}{s} \\
&\quad + \int_{[1,e]\setminus D(x_1)} \psi_\alpha(s)\left[f\left(s, \int_1^e H(s,\tau)\tilde{x}(\tau)\frac{d\tau}{\tau}, \tilde{x}(s)\right)\right]\frac{ds}{s} \\
&\leq \kappa\mu_1\int_1^e G(t,s)x_1(s)\frac{ds}{s} + \int_1^e \psi_\alpha(s)\left[f\left(s, \int_1^e H(s,\tau)\tilde{x}(\tau)\frac{d\tau}{\tau}, \tilde{x}(s)\right)\right]\frac{ds}{s} \\
&\leq (\tilde{L}x_1)(t) + M, t \in [1, e].
\end{aligned} \tag{59}$$

So it follows from $\tilde{L}(Q) \subset Q$ and (59), we have

$$0 \leq \left(\tilde{L}^j(Tx_1)\right)(t) \leq \left(\tilde{L}^j(\tilde{L}x_1 + M)\right)(t), j = 0, 1, 2, \cdots, N - 1. \tag{60}$$

Since Q is a normal cone with normality constant 1 and (60), one has

$$\left\|\tilde{L}^j(Tx_1)\right\| \leq \left\|\tilde{L}^j(\tilde{L}x_1 + M)\right\|, j = 0, 1, 2, \cdots, N - 1, \tag{61}$$

which leads to

$$\begin{aligned}
\|Tx_1\|^* &= \sum_{i=1}^{N} [\kappa + \varepsilon_0]^{N-i}\left\|\tilde{L}^{i-1}(Tx_1)\right\| \\
&\leq \sum_{i=1}^{N} [\kappa + \varepsilon_0]^{N-i}\left\|\tilde{L}^{i-1}(\tilde{L}x_1 + M)\right\| = \left\|\tilde{L}x_1 + M\right\|^*.
\end{aligned} \tag{62}$$

According to the selection of R_2, we have $M^* < (\varepsilon_0/2)R_2$. Thus, it follows from the fact $\|x\| \geq R$ implies $\|x\|^* > R_2$ and

(53), (54), and (62) that

$$\begin{aligned}\mu^* \|x_1\|^* &= \|Tx_1\|^* \leq \|\tilde{L}x_1\|^* + M^* = \sum_{i=1}^{N} [\kappa + \varepsilon_0]^{N-i} \|\tilde{L}^i x_1\| + M^* \\ &= [\kappa + \varepsilon_0] \sum_{i=1}^{N-1} [\kappa + \varepsilon_0]^{N-i-1} \|\tilde{L}^i x_1\| + \|\tilde{L}^N x_1\| + M^* \\ &\leq [\kappa + \varepsilon_0] \sum_{i=1}^{N-1} [\kappa + \varepsilon_0]^{N-i-1} \|\tilde{L}^i x_1\| + [\kappa + \varepsilon_0]^N \|x_1\| + M^* \\ &= [\kappa + \varepsilon_0] \sum_{i=1}^{N} [\kappa + \varepsilon_0]^{N-i} \|\tilde{L}^{i-1} x_1\| + M^* \leq [\kappa + \varepsilon_0] \|x_1\|^* \\ &\quad + \frac{\varepsilon_0}{2} R_2 \leq [\kappa + \varepsilon_0] \|x_1\|^* + \frac{\varepsilon_0}{2} \|x_1\|^* = \left[\frac{1}{4}\kappa + \frac{3}{4}\right] \|x_1\|^*. \end{aligned} \tag{63}$$

Notice that $\mu^* \geq 1$, we have $(1/4)\kappa + 3/4 \geq 1$, and then $\kappa \geq 1$, which is a contradiction with $0 < \kappa < 1$. So (57) is indeed valid and it follows from Lemma 2 that

$$i(T, Q_R, Q) = 1. \tag{64}$$

Thus, (49) and (64) lead to

$$i\left(T, Q_R \setminus \bar{Q}_r, Q\right) = i(T, Q_R, Q) - i(T, Q_r, Q) = 1, \tag{65}$$

which implies T has at least one fixed point on $Q_R \setminus \bar{Q}_r$. Consequently, equation (1) has at least one positive solution.

Theorem 8. *Let μ_1 be the first eigenvalue of L, suppose (B1) holds, and*

$$\limsup_{\substack{u\to 0^+\\ v\to 0^+}} \frac{f(t,u,v)}{v} < \mu_1, \quad \liminf_{u+v\to+\infty} \frac{f(t,u,v)}{u+v} > \tilde{\mu}_1 \tag{66}$$

uniformly on $t \in [1, e]$, where $\tilde{\mu}_1$ is any eigenvalues of L. Then, equation (1) has at least one positive solution.

To prove Theorem 8, we need some preliminaries and lemma. For any enough small $0 < \varepsilon < 1$, define

$$(L_\varepsilon x)(t) = \int_{1+\varepsilon}^{e-\varepsilon} G(t,s) x(s) \frac{ds}{s}, \ t \in [1, e]. \tag{67}$$

By Lemma 5, we know $L_\varepsilon : Q \to Q$ is still a completely continuous linear operator with the spectral radius $\rho(L_\varepsilon) \neq 0$, and L_ε has a positive eigenfunction ω_ε corresponding to its first eigenvalue $\mu_\varepsilon = (\rho(L_\varepsilon))^{-1}$.

Lemma 9. *There exists an eigenvalue $\tilde{\mu}_1$ of L such that*

$$\lim_{\varepsilon\to 0^+} \mu_\varepsilon = \tilde{\mu}_1. \tag{68}$$

Proof. Let $\varepsilon_m \to 0\,(m \to \infty)$ satisfying $\varepsilon_1 \geq \varepsilon_2 \geq \cdots \geq \varepsilon_m \geq \cdots$. Thus, for $\omega \in Q$ and $k > m$, one has

$$\left(L_{\varepsilon_m}\omega\right)(t) \leq \left(L_{\varepsilon_k}\omega\right)(t) \leq (L\omega)(t), \quad t \in [1, e], \tag{69}$$

and then

$$\left(L^n_{\varepsilon_m}\omega\right)(t) \leq \left(L^n_{\varepsilon_k}\omega\right)(t) \leq (L^n\omega)(t), \quad t \in [1, e], n = 2, 3, \cdots, \tag{70}$$

where $L^n_{\varepsilon_m} = L(L^{n-1}_{\varepsilon_m})$, $n = 2, 3, \cdots$. It follows from the fact that Q is a normal cone with normality constant 1 that

$$\|L^n_{\varepsilon_m}\| \leq \|L^n_{\varepsilon_k}\| \leq \|L^n\|, n = 1, 2, \cdots. \tag{71}$$

By Gelfand's formula, we have $\mu_{\varepsilon_m} \geq \mu_{\varepsilon_k} \geq \mu_1$, where μ_1 is the first eigenvalue of L, which implies $\{\mu_{\varepsilon_m}\}$ is monotonous with lower boundedness μ_1. Let

$$\lim_{m\to+\infty} \mu_{\varepsilon_m} = \tilde{\mu}_1, \tag{72}$$

we assert $\tilde{\mu}_1$ is an eigenvalue of L.

In fact, let ω_{ε_m} be positive eigenfunctions of L_{ε_m} corresponding to μ_{ε_m} with $\|\omega_{\varepsilon_m}\| = 1$, $m = 1, 2, \cdots$. So we have

$$\omega_{\varepsilon_m}(t) = \mu_{\varepsilon_m} L_{\varepsilon_m} \omega_{\varepsilon_m}(t) = \mu_{\varepsilon_m} \int_{1+\varepsilon_m}^{e-\varepsilon_m} G(t,s) \omega_{\varepsilon_m}(s) \frac{ds}{s}, \quad t \in [1, e]. \tag{73}$$

Since

$$\begin{aligned}\left\|L_{\varepsilon_m}\omega_{\varepsilon_m}\right\| &= \max_{1\leq t\leq e} \int_{1+\varepsilon_m}^{e-\varepsilon_m} G(t,s)\omega_{\varepsilon_m}(s)\frac{ds}{s} \leq \frac{1}{\Gamma(\alpha)} \int_1^e \psi_\alpha(s) \frac{ds}{s} \\ &< +\infty, (m = 1, 2, \cdots),\end{aligned} \tag{74}$$

$\{L_{\varepsilon_m}\omega_{\varepsilon_m}\} \subset Q$ is uniformly bounded.

Moreover, for any m and $t_1, t_2 \in [1, e]$, one has

$$|L_{\varepsilon_m}\omega_{\varepsilon_m}(t_1) - L_{\varepsilon_m}\omega_{\varepsilon_m}(t_2)| \leq \int_{\varepsilon_m}^{1-\varepsilon_m} |G(t_1,s) - G(t_2,s)| \omega_{\varepsilon_m}(s) \frac{ds}{s}. \tag{75}$$

Notice that $G(t,s)$ is uniformly continuous on $[1,e] \times [1,e]$; then, (75) implies $\{L_{\varepsilon_m}\omega_{\varepsilon_m}\} \subset Q$ is equicontinuous. Thus, from the Arzela-Ascoli theorem and $\lim_{m\to+\infty}\mu_{\varepsilon_m} = \tilde{\mu}_1$, we know $\omega_{\varepsilon_m} \to \omega_0$ as $m \to +\infty$, which yields $\|\omega_0\| = 1$. Take the limit for two sides of (73), we obtain

$$\omega_0(t) = \tilde{\mu}_1 \int_1^e G(t,s)\omega_0(s)\frac{ds}{s}, \quad t \in [1, e], \tag{76}$$

that is, $\omega_0 = \tilde{\mu}_1 L\omega_0$, and $\tilde{\mu}_1$ is an eigenvalue of L.

Proof of Theorem 8. Firstly, it follows from (66) that for any $t \in [1, e]$, there exists $r > 0$ such that

$$f(t, u, v) \le \mu_1 v, |u| \le \frac{r}{(\beta-1)\Gamma(\beta+1)}, |v| \le r. \quad (77)$$

Thus, for any $x \in \partial Q_r$, since

$$\left| \int_1^e H(s, \tau) x(\tau) \frac{d\tau}{\tau} \right| \le \frac{r}{(\beta-1)\Gamma(\beta+1)}, |x(s)| \le \|x\| = r, \quad (78)$$

we have

$$\begin{aligned} (Tx)(t) &= \int_1^e G(t, s) \left[f\left(s, \int_1^e H(s, \tau) x(\tau) \frac{d\tau}{\tau}, x(s) \right) \right] \frac{ds}{s} \\ &\le \mu_1 \int_1^e G(t, s) x(s) \frac{ds}{s} = \mu_1 (Lx)(t), t \in [1, \mathrm{e}]. \end{aligned} \quad (79)$$

In the following, we prove that

$$Tx \ne \mu x, \text{ for any } x \in \partial Q_r, \mu \ge 1. \quad (80)$$

Firstly, we may suppose that T has no fixed point on ∂Q_r (otherwise, the proof is completed). If (80) is not true, i.e., there exist $x_0 \in \partial Q_r$ and $\mu_0 \ge 1$ satisfying $Tx_0 = \mu_0 x_0$. Thus $\mu_0 > 1$, it follows from (79) that

$$\mu_0 x_0 = Tx_0 \le \mu_1 L x_0. \quad (81)$$

From induction, we have

$$\mu_0^n x_0 \le \mu_1^n L^n x_0, n = 1, 2, \cdots, \quad (82)$$

which implies that

$$\|L^n\| \ge \frac{\|L^n x_0\|}{\|x_0\|} \ge \frac{\mu_0^n \|x_0\|}{\mu_1^n \|x_0\|} = \frac{\mu_0^n}{\mu_1^n}. \quad (83)$$

Thus, by Gelfand's formula, we have

$$r(L) = \lim_{n\to\infty} \sqrt[n]{\|L^n\|} \ge \frac{\mu_0}{\mu_1} > \frac{1}{\mu_1}, \quad (84)$$

which contradicts with $r(L) = \mu_1^{-1}$. Therefore, (80) is valid and from Lemma 2, we get

$$i(T, Q_r, Q) = 1. \quad (85)$$

On the other hand, for any fixed small enough $0 < \varepsilon < 1$, take

$$\rho_\varepsilon = \frac{\Gamma(\alpha-1)(\ln(1+\varepsilon))^{\beta-1}}{(\beta-1)\Gamma(\alpha+\beta-2)} + (\ln(1+\varepsilon))^{\alpha-1}. \quad (86)$$

By (66) and $\lim_{\varepsilon\to 0^+} \mu_\varepsilon = \tilde{\mu}_1$, we can find a sufficiently small $\varepsilon > 0$ and $R > r$ such that

$$f(t, u, v) \ge \mu_\varepsilon (u+v), u+v \ge \rho_\varepsilon R, t \in [1, e], \quad (87)$$

where μ_ε is the first eigenvalue of L_ε.

Let ω_ε be the positive eigenfunction of L_ε corresponding to μ_ε, which implies $\omega_\varepsilon = \mu_\varepsilon L_\varepsilon \omega_\varepsilon$. For any $x \in \partial Q_R$, $s \in [1+\varepsilon, e-\varepsilon]$, we have

$$\begin{aligned} &\int_1^e H(s, \tau) x(\tau) \frac{d\tau}{\tau} + x(s) \ge \frac{1}{\Gamma(\beta)} (\ln s)^{\beta-1} \int_1^e \psi_\beta(\tau) (\ln \tau)^{\alpha-1} \frac{d\tau}{\tau} \|x\| \\ &\quad + (\ln s)^{\alpha-1} \|x\| \ge \left(\frac{\Gamma(\alpha-1)(\ln(1+\varepsilon))^{\beta-1}}{(\beta-1)\Gamma(\alpha+\beta-2)} + (\ln(1+\varepsilon))^{\alpha-1} \right) R = \rho_\varepsilon R. \end{aligned} \quad (88)$$

Thus, it follows from (87) and (88) that

$$\begin{aligned} (Tx)(t) &= \int_1^e G(t, s) \left[f\left(s, \int_1^e H(s, \tau) x(\tau) \frac{d\tau}{\tau}, x(s) \right) \right] \frac{ds}{s} \\ &\ge \int_{1+\varepsilon}^{e-\varepsilon} G(t, s) \left[f\left(s, \int_1^e H(s, \tau) x(\tau) \frac{d\tau}{\tau}, x(s) \right) \right] \frac{ds}{s} \\ &\ge \mu_\varepsilon \int_{1+\varepsilon}^{e-\varepsilon} G(t, s) \left(\int_1^e H(s, \tau) x(\tau) \frac{d\tau}{\tau} + x(s) \right) \frac{ds}{s} \\ &\ge \mu_\varepsilon \int_{1+\varepsilon}^{e-\varepsilon} G(t, s) x(s) \frac{ds}{s} = \mu_\varepsilon (L_\varepsilon x)(t), t \in [1, e]. \end{aligned} \quad (89)$$

Similar to the proof of Theorem 7, we obtain

$$x - Tx \ne \mu \omega_\varepsilon, x \in \partial Q_R, \mu \ge 0. \quad (90)$$

According to Lemma 2, we have

$$i(T, Q_R, Q) = 0. \quad (91)$$

Combining (80) and (91), one has

$$i\left(T, Q_R \setminus \bar{Q}_r, Q\right) = i(T, Q_R, Q) - i(T, Q_r, Q) = -1. \quad (92)$$

Hence, T has at least one fixed point on $Q_R \setminus \bar{Q}_r$, and then equation (1) has at least a positive solution. The proof is completed.

4. Conclusion

Singular behaviour is a class of important natural phenomena in many physical science, mathematics, engineering, and bioscience. So, the study for singularity is an interesting and challenging problem. In this paper, we consider the existence of positive solutions for a Hadamard-type fractional differential equation with singular nonlinearity by introducing a new limit-type growth condition. The main advantage of the assumption is that it provides an effective method for handling the singularity at space variables. This assumption is valid and reasonable and easier to get the solution of the target equation.

Data Availability

Not applicable.

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

Authors' Contributions

The study was carried out by the collaboration of all authors. All authors read and approved the final manuscript.

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