Research Article

Hyponormality on a Weighted Bergman Space

Houcine Sadraoui , Borhen Halouani , Mubariz T. Garayev , and Adel AlShehri

Department of Mathematics, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia

Correspondence should be addressed to Borhen Halouani; halouani@ksu.edu.sa

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A bounded Hilbert space operator T is hyponormal if $T^*T - TT^*$ is a positive operator. We consider the hyponormality of Toeplitz operators on a weighted Bergman space. We find a necessary condition for hyponormality in the case of a symbol of the form $f + \bar{g}$ where f and g are bounded analytic functions on the unit disk. We then find sufficient conditions when f is a monomial.

1. Introduction

Let D_R denote the disk of radius R in the complex plane, $d\mu_\alpha(z) = (1/\pi R^2)(1 - |z/R|^2)^\alpha dA(z)$ where $dA(z)$ is the Lebesgue measure on D_R and $\alpha > -1$. $L^2(D_R, d\mu_\alpha(z))$ denotes the Hilbert space of complex-valued functions on D_R that are square integrable with respect to μ . We write $\|f\|^2 = \int_{D_R} |f(z)|^2 d\mu(z)$. When f is analytic on D_R , we have

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad \|f\|^2 = \sum_{n=0}^{\infty} R^{2n} \frac{n! \Gamma(\alpha + 1)}{\Gamma(n + \alpha + 2)} |a_n|^2. \quad (1)$$

Denote by A_α^2 the space of analytic functions on D_R such that $\|f\|^2 < \infty$. It is known that A_α^2 is a Hilbert space [1], and an orthonormal basis is given by $e_n(z) = (\sqrt{\Gamma(n + \alpha + 2)}/R^n \sqrt{n! \Gamma(\alpha + 1)}) z^n$. The Toeplitz operator with symbol f on A_α^2 is defined by $T_f(k) = P(fk)$ where f is bounded and measurable on D_R , k is in A_α^2 , and P is the orthogonal projection of $L^2(D_R, d\mu_\alpha(z))$ onto A_α^2 . Hankel operators are defined by $H_f(k) = (I - P)(fk)$, f and k as before. Recall that a bounded operator T on a Hilbert space is hyponormal if $T^*T - TT^*$ is a positive operator. Hyponormality on the Hardy space is studied by Cowen in [2, 3]. Unweighted Bergman spaces on

the unit disk ($\alpha = 0, R = 1$) and Toeplitz operators on these spaces were considered in [4, 5] and [6]. Hyponormality was first considered by Sadraoui in [7]. An improvement of the necessary condition is due to Ahern and Cuckovic [8]. A new necessary condition in a special case is found by Cuckovic and Curto in [9]. Many results on hyponormality on weighted Bergman spaces treat special cases. We cite for example [1, 10]. In this work for simplicity's sake, we consider the case $\alpha = 1$ and $R = 1$. Under a smoothness assumption, we give a fairly general necessary condition for the hyponormality of Toeplitz operators with a symbol of the form $f + \bar{g}$ where f and g are bounded and analytic. We also give sufficient conditions when f is a monomial and g a polynomial without assuming $R = 1$.

We begin by recalling some general properties of Toeplitz operators.

2. Some General Properties

We assume f, g are in $L^\infty(D_R)$. Then, we have:

- (1) $T_{f+g} = T_f + T_g$
- (2) $T_f^* = T_{\bar{f}}$
- (3) $T_{\bar{f}} T_g = T_{\bar{f}g}$ if f or g analytic on D_R

The use of these properties leads to describing hyponormality in more than one form. These are easy to prove by using Douglas lemma [11].

Proposition 1. *Let f, g be bounded and analytic on D_R . Then, the following are equivalent: [label=]*

- (i) $T_{f+\bar{g}}$ is hyponormal
- (ii) $H_{\bar{g}}^* H_{\bar{g}} \leq H_{\bar{f}}^* H_{\bar{f}}$
- (iii) $\|(I - P)(\bar{g}k)\| \leq \|(I - P)(\bar{f}k)\|$ for any k in A_α^2
- (iv) $\|\bar{g}k\|^2 - \|P(\bar{g}k)\|^2 \leq \|\bar{f}k\|^2 - \|P(\bar{f}k)\|^2$ for any k in A_α^2
- (v) $H_{\bar{g}} = CH_{\bar{f}}$ where C is of norm less than or equal to one

We also need the following lemmas.

Lemma 2. *For s and t integers, we have*

$$P(\bar{z}^t z^s) = \begin{cases} R^{2t} \frac{s! \Gamma(s - t + \alpha + 2)}{\Gamma(s + \alpha + 2)(s - t)!} z^{s-t}, & \text{if } s \geq t \\ 0, & \text{if } s < t \end{cases}. \quad (2)$$

Proof. We have

$$\begin{aligned} \langle P(\bar{z}^t z^s), z^m \rangle &= \langle z^s, z^{m+t} \rangle = \frac{1}{\pi R^2} \int_0^{2\pi} \int_0^R r^{s+m+t+1} e^{i(s-t-m)\theta} \\ &\quad \cdot \left(1 - \left(\frac{r}{R}\right)^2\right)^\alpha dr d\theta. \end{aligned} \quad (3)$$

We see that $\langle P(\bar{z}^t z^s), z^m \rangle = 0$ if $s < t$. For $s \geq t$, $\langle P(\bar{z}^t z^s), z^m \rangle \neq 0$ if and only if $m = s - t$. In this case, we get

$$\begin{aligned} \langle P(\bar{z}^t z^s), z^{s-t} \rangle &= \frac{2}{R^2} \int_0^R r^{2s+1} \left(1 - \left(\frac{r}{R}\right)^2\right)^\alpha dr \\ &= 2R^{2s} \int_0^1 u^{2s+1} (1 - u^2)^\alpha du \\ &= R^{2s} \int_0^1 x^s (1 - x)^\alpha dx = R^{2s} \beta(s + 1, \alpha + 1) \\ &= R^{2s} \frac{\Gamma(s + 1) \Gamma(\alpha + 1)}{\Gamma(s + \alpha + 2)}. \end{aligned} \quad (4)$$

On the other hand, if $P(\bar{z}^t z^s) = \lambda z^{s-t}$, then

$$\langle P(\bar{z}^t z^s), z^{s-t} \rangle = \lambda \|z^{s-t}\|^2, \quad (5)$$

so $\lambda = \langle P(\bar{z}^t z^s), z^{s-t} \rangle / \|z^{s-t}\|^2$. A computation shows

$$\|z^{s-t}\|^2 = R^{2(s-t)} \frac{(s-t)! \Gamma(\alpha + 1)}{\Gamma(s - t + \alpha + 2)}, \quad (6)$$

so $\lambda = R^{2t} (s! \Gamma(s - t + \alpha + 2) / ((s - t)! \Gamma(s + \alpha + 2)))$, and the result follows.

Lemma 3. *If $f = \sum_{n=0}^\infty a_n z^n$ is bounded analytic on the disk D_R , then the (i, j) th entry of the matrix of $H_{\bar{f}}^* H_{\bar{f}}$ with respect to the orthonormal basis $\{(\sqrt{\Gamma(n + \alpha + 2)} / R^n \sqrt{n! \Gamma(\alpha + 1)}) z^n\}_{n \geq 0}$ is given by:*

$$\begin{aligned} &\frac{\sqrt{\Gamma(j + \alpha + 2)}}{R^j \sqrt{j!}} \frac{R^i \sqrt{\Gamma(i + \alpha + 2)}}{\sqrt{i!}} \sum_{m \geq j-i, m \geq 0} a_{m+i-j} \bar{a}_m R^{2m} \\ &\quad \cdot \frac{(m+i)!}{\Gamma(m+i+\alpha+2)} - \frac{\sqrt{j!} \sqrt{i!} R^{j-i}}{\sqrt{\Gamma(j+\alpha+2)} \sqrt{\Gamma(i+\alpha+2)}} \\ &\quad \cdot \sum_{0 \leq m, i-j \leq m \leq i} R^{2m} a_m \bar{a}_{m+ji} \frac{\Gamma(i-m+\alpha+2)}{(i-m)!}. \end{aligned} \quad (7)$$

Proof. We have:

$$\begin{aligned} &\bar{f} f \frac{\sqrt{\Gamma(j + \alpha + 2)}}{R^j \sqrt{j! \Gamma(\alpha + 1)}} z^j \\ &= \frac{\sqrt{\Gamma(j + \alpha + 2)}}{R^j \sqrt{j! \Gamma(\alpha + 1)}} \sum_{n,m} a_n \bar{a}_m z^{n+j} z^m \\ &= \frac{\sqrt{\Gamma(j + \alpha + 2)}}{R^j \sqrt{j! \Gamma(\alpha + 1)}} \sum_{p \geq j, m \geq 0} a_{p-j} \bar{a}_m z^p z^m. \end{aligned} \quad (8)$$

Thus, by Lemma 2,

$$\begin{aligned} &T_{\bar{f}} T_f \left(\frac{\sqrt{\Gamma(j + \alpha + 2)}}{R^j \sqrt{j! \Gamma(\alpha + 1)}} z^j \right) \\ &= \frac{\sqrt{\Gamma(j + \alpha + 2)}}{R^j \sqrt{j! \Gamma(\alpha + 1)}} \sum_{p \geq j, p \geq m, m \geq 0} a_{p-j} \bar{a}_m R^{2m} \\ &\quad \cdot \frac{p! \Gamma(p - m + \alpha + 2)}{\Gamma(p + \alpha + 2)(p - m)!} z^{p-m} \\ &= \frac{\sqrt{\Gamma(j + \alpha + 2)}}{R^j \sqrt{j! \Gamma(\alpha + 1)}} \sum_{p \geq j, m \geq 0, p \geq m} a_{p-j} \bar{a}_m R^{2m} \\ &\quad \cdot \frac{p! \Gamma(p - m + \alpha + 2)}{\Gamma(p + \alpha + 2)(p - m)!} z^{p-m}. \end{aligned} \quad (9)$$

Thus, noting that $\langle z^{p-m}, z^i \rangle \neq 0$ if $p = m + i$, we get

$$\begin{aligned} & \left\langle T_{\bar{f}} T_f \left(\frac{\sqrt{\Gamma(j+\alpha+2)}}{R^j \sqrt{j! \Gamma(\alpha+1)}} z^j \right), \frac{\sqrt{\Gamma(i+\alpha+2)}}{R^i \sqrt{i! \Gamma(\alpha+1)}} z^i \right\rangle \\ &= \frac{\sqrt{\Gamma(j+\alpha+2)}}{R^j \sqrt{j! \Gamma(\alpha+1)}} \frac{\sqrt{\Gamma(i+\alpha+2)}}{R^i \sqrt{i! \Gamma(\alpha+1)}} \sum_{m \geq j-i, m \geq 0} a_{m+i-j} \bar{a}_m R^{2m} \\ & \quad \cdot \frac{(m+i)! \Gamma(i+\alpha+2)}{\Gamma(m+i+\alpha+2) i!} R^{2i} \frac{i! \Gamma(\alpha+1)}{\Gamma(i+\alpha+2)} \\ &= \frac{\sqrt{\Gamma(j+\alpha+2)}}{R^j \sqrt{j!}} \frac{\sqrt{\Gamma(i+\alpha+2)}}{\sqrt{i!}} \sum_{m \geq j-i, m \geq 0} a_{m+i-j} \bar{a}_m R^{2m} \\ & \quad \cdot \frac{(m+i)!}{\Gamma(m+i+\alpha+2)}. \end{aligned} \quad (10)$$

We also have by Lemma 2

$$\begin{aligned} P \left(\bar{f} \frac{\sqrt{\Gamma(j+\alpha+2)}}{R^j \sqrt{j! \Gamma(\alpha+1)}} z^j \right) &= \frac{\sqrt{\Gamma(j+\alpha+2)}}{R^j \sqrt{j! \Gamma(\alpha+1)}} P \left(\sum_0^\infty \bar{a}_n z^n z^j \right) \\ &= \frac{\sqrt{\Gamma(j+\alpha+2)}}{R^j \sqrt{j! \Gamma(\alpha+1)}} \sum_0^j \bar{a}_n R^{2n} \frac{j! \Gamma(j-n+\alpha+2)}{\Gamma(j+\alpha+2) \cdot (j-n)!} z^{j-n}. \end{aligned} \quad (11)$$

Then,

$$\begin{aligned} & \left\langle T_{\bar{f}} \left(\frac{\sqrt{\Gamma(j+\alpha+2)}}{R^j \sqrt{j! \Gamma(\alpha+1)}} z^j \right), T_{\bar{f}} \left(\frac{\sqrt{\Gamma(i+\alpha+2)}}{R^i \sqrt{i! \Gamma(\alpha+1)}} z^i \right) \right\rangle \\ &= \frac{\sqrt{\Gamma(j+\alpha+2)}}{R^j \sqrt{j! \Gamma(\alpha+1)}} \frac{\sqrt{\Gamma(i+\alpha+2)}}{\sqrt{i! \Gamma(\alpha+1)}} \sum_{m=1}^\infty a_m \bar{a}_n \langle P(\bar{z}^n z^j), P(\bar{z}^m z^i) \rangle. \end{aligned} \quad (12)$$

The above inner product is nonzero if $i - m = j - n \geq 0$. We get by Lemma 2

$$\begin{aligned} & \left\langle T_{\bar{f}} T_{\bar{f}} \left(\frac{\sqrt{\Gamma(j+\alpha+2)}}{R^j \sqrt{j! \Gamma(\alpha+1)}} z^j \right), \frac{\sqrt{\Gamma(i+\alpha+2)}}{R^i \sqrt{i! \Gamma(\alpha+1)}} z^i \right\rangle \\ &= \frac{\sqrt{j!} \sqrt{i!} R^{j-i}}{\sqrt{\Gamma(j+\alpha+2)} \sqrt{\Gamma(i+\alpha+2)}} \\ & \quad \cdot \sum_{0 \leq m, i-j \leq m \leq i} R^{2m} a_m \bar{a}_{m+j-i} \frac{\Gamma(i-m+\alpha+2)}{(i-m)!}. \end{aligned} \quad (13)$$

When $R = 1$, we set $D_1 = D$. We obtain in this case the following corollary.

Corollary 4. If $f = \sum_{n=0}^{+\infty} a_n z^n$ is bounded and analytic on the unit disk D , and if $\alpha = 1$. Then, the matrix of $H_{\bar{f}}^* H_{\bar{f}}$ with respect to the orthonormal basis

$\{\sqrt{(n+2)(n+1)} z^n\}_{n \geq 0}$ is given by

$$\begin{aligned} \Delta_{i,j} &= \sqrt{(j+2)(j+1)} \sqrt{(i+2)(i+1)} \\ & \quad \cdot \sum_{m \geq j-i, m \geq 0} a_{m+i-j} \bar{a}_m \frac{1}{(m+i+2)(m+i+1)} \\ & \quad - \frac{1}{\sqrt{(j+2)(j+1)} \sqrt{(i+2)(i+1)}} \\ & \quad \cdot \sum_{0 \leq m, i-j \leq m \leq i} a_m \bar{a}_{m+j-i} (i-m+2)(i-m+1). \end{aligned} \quad (14)$$

Recall that the Hardy space H^2 is defined by

$$H^2 = \left\{ f = \sum_{n=0}^{+\infty} a_n z^n \text{ analytic on the unit disk such that } \sum_{n=0}^{+\infty} |a_n|^2 < +\infty \right\}. \quad (15)$$

In the following theorem, the Toeplitz operator $T_{|f'|^2}$ on H^2 is not necessarily bounded. However, its matrix $(\Gamma_{i,j})_{i,j}$ in the usual orthonormal basis of H^2 is defined.

The following theorem is an extension of Lemma 2.4.2 [7] to the weighted Bergman space.

Theorem 5. Assume f bounded analytic on the unit disk D and $f' \in H^2$. Let $(\Gamma_{i,j})$ be the matrix of the Toeplitz operator on H^2 , $T_{|f'|^2}$. Then, $n^2 \Delta_{i+n,j+n} \rightarrow 2\Gamma_{i,j}$ as $n \rightarrow \infty$.

Proof. The symmetric matrix of $H_{\bar{f}}^* H_{\bar{f}}$ with respect to the orthonormal basis $\{\sqrt{(n+2)(n+1)} z^n\}_{n \geq 0}$ is given by (set $j = i + p$ in Corollary 4 with $p \geq 0$.)

$$\begin{aligned} \Delta_{i,i+p} &= \sqrt{(i+p+2)(i+p+1)} \sqrt{(i+2)(i+1)} \\ & \quad \cdot \sum_{m \geq p} a_{m-p} \bar{a}_m \frac{1}{(m+i+2)(m+i+1)} \\ & \quad - \frac{1}{\sqrt{(i+p+2)(i+p+1)} \sqrt{(i+2)(i+1)}} \\ & \quad \cdot \sum_{0 \leq m \leq i} a_m \bar{a}_{m+p} (i-m+2)(i-m+1). \end{aligned} \quad (16)$$

To simplify, set $C_{n,i,p} = (i+n+1)(i+n+2)(i+n+p+1)(i+n+p+2)$. Then,

$$\begin{aligned} \Delta_{i+n,i+n+p} &= \sqrt{C_{n,i,p}} \sum_{m \geq p} a_{m-p} \bar{a}_m \frac{1}{(m+i+n+2)(m+i+n+1)} \\ & \quad - \frac{1}{\sqrt{C_{n,i,p}}} \sum_{0 \leq m \leq i+n} a_m \bar{a}_{m+p} (i+n-m+2) \\ & \quad \cdot (i+n-m+1) \end{aligned} \quad (17)$$

for any integer n . Set $m = p + l$ in the first sum, and $m = l$ in the second. With some obvious notations, we get

$$\begin{aligned}\Delta_{i+n,i+n+p} &= \sqrt{C_{n,i,p}} \sum_{l \geq 0} a_l \overline{a_{p+l}} \frac{1}{(p+l+i+n+2)(p+l+i+n+1)} - \frac{1}{\sqrt{C_{n,i,p}}} \sum_{0 \leq l \leq i+n} a_l \overline{a_{p+l}} (i+n-l+2)(i+n-l+1) \\ &= \sum_{l \leq i+n} a_l \overline{a_{p+l}} \frac{C_{n,i,p} - (i+n-l+2)(i+n-l+1)(p+l+i+n+2)(p+l+i+n+1)}{\sqrt{C_{n,i,p}}(p+l+i+n+2)(p+l+i+n+1)} \\ &\quad + \sqrt{C_{n,i,p}} \sum_{l \geq i+n+1} a_l \overline{a_{p+l}} \frac{1}{(p+l+i+n+2)(p+l+i+n+1)} = \sum_{l \leq i+n} A_{n,l,p,i} a_l \overline{a_{p+l}} + \sum_{l \geq i+n+1} B_{n,l,p,i} a_l \overline{a_{p+l}}.\end{aligned}\quad (18)$$

A computation shows that

$$A_{n,l,p,i} = \frac{l(l+p)((i+n+1)^2 + (i+n+2)^2) + lp(l+p)(2n+2i+3) + l^2(l+p)^2}{\sqrt{C_{n,i,p}}(p+l+i+n+2)(p+l+i+n+1)}.\quad (19)$$

It is not difficult to see that

$$n^2 A_{n,l,p,i} a_l \overline{a_{p+l}} \xrightarrow{n \rightarrow \infty} 2l(l+p) a_l \overline{a_{p+l}},\quad (20)$$

and that the following inequality holds:

$$\begin{aligned}n^2 A_{n,l,p,i} |a_l \overline{a_{p+l}}| &\leq 5l(l+p) |a_l| |a_{p+l}| \\ &\leq 5/2 (l^2 |a_l|^2 + (l+p)^2 |a_{p+l}|^2)\end{aligned}\quad (21)$$

for $l \leq i+n$. We have that $\sum l^2 |a_l|^2 + (l+p)^2 |a_{p+l}|^2 < \infty$ since $f' \in H^2$. Writing the sum $\sum_{l \leq i+n} n^2 A_{n,l,p,i} a_l \overline{a_{p+l}}$ as an integral, where the measure is the counting measure, and using the dominated convergence theorem, we get that

$$\sum_{l \leq i+n} n^2 A_{n,l,p,i} a_l \overline{a_{p+l}} \xrightarrow{n \rightarrow \infty} 2 \sum l(l+p) a_l \overline{a_{p+l}}.\quad (22)$$

Clearly, $\sqrt{C_{n,i,p}} \leq (i+n+p+1)(i+n+p+2)$. So we have for $l \geq i+n+1$,

$$\begin{aligned}n^2 |B_{n,l,p,i} a_l \overline{a_{p+l}}| &= \frac{n^2 \sqrt{C_{n,i,p}}}{(p+l+i+n+2)(p+l+i+n+1)} |a_l \overline{a_{p+l}}| \\ &\leq l(l+p) |a_l| |a_{p+l}|\end{aligned}\quad (23)$$

for $l \geq i+n+1$. Thus,

$$\begin{aligned}\sum_{l \geq i+n+1} n^2 |B_{n,l,p,i} a_l \overline{a_{p+l}}| &\leq \sum_{l \geq i+n+1} l(l+p) |a_l| |a_{p+l}| \\ &\leq \frac{1}{2} \sum_{l \geq i+n+1} l^2 |a_l|^2 + \frac{1}{2} \sum_{l \geq i+n+1} (l+p)^2 |a_{p+l}|^2.\end{aligned}\quad (24)$$

This shows that

$$\sum_{l \geq i+n+1} n^2 B_{n,l,p,i} a_l \overline{a_{p+l}} \xrightarrow{n \rightarrow \infty} 0.\quad (25)$$

We deduce that $n^2 \Delta_{i+n,i+n+p} \xrightarrow{n \rightarrow \infty} 2 \sum l(l+p) a_l \overline{a_{p+l}} = 2\Gamma_{i,i+p}$.

3. The Results

Using the previous theorem leads us to our main result which extends Theorem 5.4.3 [7].

Theorem 6. Let f and g be bounded and analytic on the unit disk D , and such that $f' \in H^2$. If $T_{f+\bar{g}}$ is hyponormal on A_1^2 then $g' \in H^2$, and $|g'| \leq |f'|$ a.e. on the unit circle.

Proof. Set $g = \sum_n b_n z^n$. All diagonal terms of the matrix of $H_f^* H_{\bar{g}} - H_{\bar{g}}^* H_g$ are positive; thus, using the same notation as in the previous proof, we have

$$\begin{aligned} & \sum_{l \leq i+n} n^2 A_{n,l,0,i} |a_l|^2 + \sum_{l \geq i+n+1} n^2 B_{n,l,0,i} |a_l|^2 \\ & \geq \sum_{l \leq i+n} n^2 A_{n,l,0,i} |b_l|^2 + \sum_{l \geq i+n+1} n^2 B_{n,l,0,i} |b_l|^2. \end{aligned} \quad (26)$$

We have seen that the left hand side tends to $2 \sum l^2 |a_l|^2$, a finite sum since $f' \in H^2$. Writing $\sum_{l \leq i+n} n^2 A_{n,l,0,i} |b_l|^2$ as an integral with respect to the counting measure, noticing that $\chi_{\{0, \dots, i+n\}}(l) n^2 A_{n,l,0,i} |b_l|^2 \xrightarrow{n \rightarrow \infty} 2l^2 |b_l|^2$, and using Fatou's lemma, we deduce that $2 \sum l^2 |b_l|^2 \leq 2 \sum l^2 |a_l|^2$ and $g' \in H^2$. If $(\Omega_{i,j})$ denotes the matrix of $H_{\bar{f}}^* H_{\bar{f}} - H_{\bar{g}}^* H_{\bar{g}}$, then $n^2 \Omega_{i+n,j+n} \rightarrow \Lambda_{i,j}$, where $(\Lambda_{i,j})$ is the matrix of the Toeplitz operator on H^2 , $T_{|f'|^2 - |g'|^2}$. Since the matrix of $H_{\bar{f}}^* H_{\bar{f}} - H_{\bar{g}}^* H_{\bar{g}}$ is positive by hypothesis, it follows that the matrix of $T_{|f'|^2 - |g'|^2}$ is also positive. A Toeplitz matrix is positive if its symbol is positive [12]. It follows that $|f'| \geq |g'|$ a.e on the unit circle.

To obtain sufficient conditions when f is a monomial, we need the following lemma.

Lemma 7. *Let q be a positive integer, the matrix of $H_{z^q}^* H_{z^q}$ is diagonal and given by:*

$$d_i = \begin{cases} R^{2q} \frac{(q+i)! \Gamma(i+\alpha+2)}{i! \Gamma(q+i+\alpha+2)}, & \text{if } i < q \\ R^{2q} \left(\frac{(q+i)! \Gamma(i+\alpha+2)}{i! \Gamma(q+i+\alpha+2)} - \frac{i! \Gamma(i-q+\alpha+2)}{(i-q)! \Gamma(i+\alpha+2)} \right), & \text{if } i \geq q \end{cases}. \quad (27)$$

In what follows, we will take $\alpha = 1$. It follows from Proposition 1 that, if $p > q$, the hyponormality of $T_{z^q + \lambda \bar{z}^p}$ is equivalent to the following inequalities:

$$|\lambda|^2 \frac{(i+2)(i+1)}{(p+i+2)(p+i+1)} \leq R^{2(q-p)} \frac{(i+2)(i+1)}{(q+i+2)(q+i+1)} \quad i < q; \quad (28)$$

$$\begin{aligned} |\lambda|^2 \frac{(i+2)(i+1)}{(p+i+2)(p+i+1)} & \leq R^{2(q-p)} \left(\frac{(i+2)(i+1)}{(q+i+2)(q+i+1)} \right. \\ & \quad \left. - \frac{(i-q+2)(i-q+1)}{(i+2)(i+1)} \right) \quad q \leq i < p; \end{aligned} \quad (29)$$

$$\begin{aligned} |\lambda|^2 & \left(\frac{(i+2)(i+1)}{(p+i+2)(p+i+1)} - \frac{(i-p+2)(i-p+1)}{(i+2)(i+1)} \right) \\ & \leq R^{2(q-p)} \left(\frac{(i+2)(i+1)}{(q+i+2)(q+i+1)} \right. \\ & \quad \left. - \frac{(i-q+2)(i-q+1)}{(i+2)(i+1)} \right) \quad p \leq i. \end{aligned} \quad (30)$$

Lemma 8. *The inequality (28) holds if and only if $|\lambda| \leq R^{q-p} \sqrt{(p+q+1)(p+q)/2q(2q+1)}$.*

Proof. Set $g(i) = (p+i+2)(p+i+1)/(q+i+2)(q+i+1)$. Then, (28) is equivalent to $|\lambda| \leq R^{q-p} \min \{ \sqrt{g(i)} i < q \}$. Using logarithmic differentiation, it is easy to see that g decreases with i , the minimum is thus assumed at $i = q-1$; hence, the result follows.

Lemma 9. *The inequality (29) holds if and only if*

$$|\lambda| \leq \sqrt{2} \frac{q}{p+1} R^{q-p} \sqrt{\frac{(2p+1)(2p^2+2p+1-q^2)}{p(p+q)(p+q+1)}}. \quad (31)$$

Proof. A computation shows that inequality (29) is equivalent to

$$|\lambda|^2 \leq q^2 R^{2(q-p)} \frac{(p+i+2)(p+i+1)(2i^2+6i+5-q^2)}{(q+i+1)(q+i+2)(i+1)^2(i+2)^2} \quad q \leq i < p. \quad (32)$$

Using logarithmic differentiation, we can see that the right hand side of the last inequality is a decreasing function of i . Thus, the minimum is assumed at $i = p-1$, and this proves the lemma.

A similar argument shows the following.

Lemma 10. *The inequality (30) holds if and only if $|\lambda| \leq (q/p) R^{q-p}$.*

Proposition 11. *If $q < p$, then $T_{z^q + \lambda \bar{z}^p}$ is hyponormal if and only if $|\lambda| \leq (q/p) R^{q-p}$.*

Proof. Using elementary computations and the previous lemma, it is easy to see that if $|\lambda| \leq (q/p) R^{q-p}$, then inequalities (28) and (29) are satisfied. This proves the result.

Remark 12. The result remains true if $p = q$.

Denote by U_1 the unit ball of $A_1^{2\perp}$.

Definition 13. For $f \in A_1^2$, we denote by G_f the set

$$G_f = \left\{ g \in A_1^2 : \sup_{l \in U_1} |\langle \bar{g}, \bar{k}l \rangle| \leq \sup_{l \in U_1} |\langle \bar{f}, \bar{k}l \rangle| \text{ for any } k \in H^\infty \right\}. \quad (33)$$

We see, from the density of H^∞ in A_1^2 and Proposition 1, that when g and f are in H^∞ , $g \in G_f$ is equivalent to $T_{f+\bar{g}}$ is hyponormal. The following proposition lists some properties of G_f [7]. We provide the proof for the sake of completeness.

Proposition 14. *For $f \in A_1^2$ the following holds: [label=()]*

- (a) G_f is convex and balanced
- (b) If $g \in G_f$, then $g + c$ is in G_f for any complex number c
- (c) $f \in G_f$.

(d) G_f is closed in the weak topology of $L^2(D_R, d\mu_l(z))$

Proof. Only (d) needs to be verified. Assume (g_μ) is a net in G_f and $g_\mu \rightarrow g$. We have for l_0 in U_1 , $|\langle \bar{g}, \bar{k}l_0 \rangle| = \lim |\langle \bar{g}_\mu, \bar{k}l_0 \rangle|$ and $|\langle \bar{g}_\mu, \bar{k}l_0 \rangle| \leq \sup \{|\langle \bar{g}_\mu, \bar{k}l \rangle|, l \in U_1\} \leq \sup \{|\langle \bar{f}, \bar{k}l \rangle|, l \in U_1\}$. The result follows by taking the limit and taking the supremum on U_1 .

Using the two previous propositions, we obtain the following.

Theorem 15. Let (λ_n) be a sequence of complex numbers satisfying $\sum_q^\infty |\lambda_n| \leq 1$ where q is a positive integer. Then, $T_{z^q + \sum_q^\infty \lambda_n (q/n) R^{q-n} z^n}$ is hyponormal.

We now consider the hyponormality of $T_{z^q + \lambda z^p}$ when $p < q$. In this case, hyponormality is equivalent to the following three inequalities:

$$\begin{aligned} |\lambda|^2 &\leq R^{2(q-p)} \frac{(p+i+2)(p+i+1)}{(q+i+2)(q+i+1)} \quad i < p, \\ |\lambda|^2 &\leq \frac{1}{p^2} R^{2(q-p)} \frac{(p+i+2)(p+i+1)(i+2)^2(i+1)^2}{(q+i+2)(q+i+1)(2i^2+6i+5-p^2)} \quad p \leq i < q, \\ |\lambda|^2 &\leq \frac{q^2}{p^2} R^{2(q-p)} \frac{(p+i+2)(p+i+1)(2i^2+6i+5-q^2)}{(q+i+2)(q+i+1)(2i^2+6i+5-p^2)} \quad q \leq i. \end{aligned} \quad (34)$$

A computation, using logarithmic differentiation, allows us to see that the right hand side of each of these inequalities increases with i . This leads to the following inequalities:

$$\begin{aligned} |\lambda| &\leq R^{q-p} \sqrt{\frac{(p+2)(p+1)}{(q+2)(q+1)}}; \\ |\lambda| &\leq R^{q-p} \frac{(p+1)(p+2)}{p} \sqrt{\frac{2(2p+1)}{(p+q+1)(p+q+2)(p+5)}}; \\ |\lambda| &\leq R^{q-p} \frac{q}{p} \sqrt{\frac{(q+5)(p+q+1)(p+q+2)}{2(2q^2+6q+5-p^2)(2q+1)}}. \end{aligned} \quad (35)$$

Set

$$\begin{aligned} A_{p,q} &= \sqrt{\frac{(p+2)(p+1)}{(q+2)(q+1)}}, \\ B_{p,q} &= \frac{(p+1)(p+2)}{p} \sqrt{\frac{2(2p+1)}{(p+q+1)(p+q+2)(p+5)}}, \\ C_{p,q} &= \frac{q}{p} \sqrt{\frac{(q+5)(p+q+1)(p+q+2)}{2(2q^2+6q+5-p^2)(2q+1)}}. \end{aligned} \quad (36)$$

Write

$$B_{p,q} = \sqrt{\frac{(2p+1)(2p+2)(p+1)(p^2+4p+4)}{(p+q+1)(p+q+2)p^2(p+5)}}. \quad (37)$$

Since $g(i) = (p+i+2)(p+i+1)/(q+i+2)(q+i+1)$ increases with i , we get

$$\frac{(p+1)(p+2)}{(q+1)(q+2)} \leq \frac{(2p+1)(2p+2)}{(p+q+1)(p+q+2)}. \quad (38)$$

Also, $(p+1)(p^2+4p+4) \geq p^2(p+5)$. Thus, $A_{p,q} < B_{p,q}$ for $p < q$. We have

$$C_{p,q} = \sqrt{\frac{q^2(p+q+2)(p+q+1)(q+1)(q+5)}{2p^2(q+1)(2q+1)(2q^2+6q+5-p^2)}}, \quad (39)$$

and we can readily verify that for $p < q$, we have

$$q^2(q+1)(q+5) \geq p^2(2q^2+6q+5-p^2). \quad (40)$$

As before, $g(i) = (p+i+2)(p+i+1)/(q+i+2)(q+i+1)$ increases with i if $p < q$. We get $g(0) \leq g(q)$ which leads to the inequality

$$\frac{(p+2)(p+1)}{(q+1)(q+2)} \leq \frac{(p+q+2)((p+q+1))}{2(q+1)(2q+1)}. \quad (41)$$

Combining the last inequality with inequality (40), we deduce the following:

Proposition 16. If $p < q$ then $T_{z^q + \lambda z^p}$ is hyponormal if and only if $|\lambda| \leq R^{q-p} \sqrt{(p+2)(p+1)/(q+2)(q+1)}$.

Remark 17. The result holds also if $p = q$.

From the properties of G_{z^q} , noted in Proposition 14, we deduce the following corollaries.

Corollary 18. Let $\lambda_1, \lambda_2, \dots, \lambda_q$ be complex numbers satisfying $\sum_{i=1}^q |\lambda_i| \leq 1$. Then, $T_{z^q + \sum_{i=1}^q \lambda_i R^{q-n} \sqrt{(n+2)(n+1)/(q+2)(q+1)} z^n}$ is hyponormal.

Corollary 19. Let (λ_n) be a sequence of complex numbers satisfying $\sum_{i=1}^\infty |\lambda_i| \leq 1$. Then, $T_{z^q + \sum_{i=1}^q \lambda_i R^{q-n} \sqrt{(n+2)(n+1)/(q+2)(q+1)} z^n + \sum_{q+1}^\infty \lambda_n (q/n) R^{q-n} z^n}$ is hyponormal.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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