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Research Article

Nontrivial Solutions of the Kirchhoff-Type Fractional *p*-Laplacian Dirichlet Problem

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In this article, we consider the new results for the Kirchhoff-type p-Laplacian Dirichlet problem containing the Riemann-Liouville fractional derivative operators. By using the mountain pass theorem and the genus properties in the critical point theory, we get some new results on the existence and multiplicity of nontrivial weak solutions for such Dirichlet problem.

1. Introduction

In this paper, we are concerned with the existence and multiplicity of nontrivial weak solutions for the Kirchhoff-type fractional Dirichlet problem with p-Laplacian of the form

$$\begin{cases} \left(a + b\int_0^T |{}_0D_t^\alpha u(t)|^p dt\right)^{p-1} {}_tD_T^\alpha \phi_p({}_0D_t^\alpha u(t)) = f(t, u(t)), & t \in (0, T), \\ u(0) = u(T) = 0, \end{cases} \tag{1}$$

where $a, b > 0$, and $p > 1$ are constants, ${}_0D_t{}^\alpha$ and ${}_tD_T{}^\alpha$ are the left and right Riemann-Liouville fractional derivatives of order $\alpha \in (1/p, 1]$, respectively, and $\phi_p : \mathbb{R} \longrightarrow \mathbb{R}$ is the p-Laplacian [1] defined by

$$\phi_p(s) = |s|^{p-2}s, \quad s \neq 0, \phi_p(0) = 0, \tag{2}$$

and $f \in C([0, T] \times \mathbb{R}, \mathbb{R})$. It should be pointed out that the weak solutions of the boundary value problem (BVP for short) (1) mean the critical points of the associated energy functional.

The fractional derivative ${}_0D_t^\alpha$ is nonlocal and reduces to the local first-order differential operator when $\alpha = 1$. Moreover, the p-Laplacian ϕ_p is nonlinear and reduces to the linear identity operator when $p = 2$. If $b = 0$, BVP (1) reduces to the following fractional p-Laplacian BVP [2]:

$$\begin{cases} {}_tD_T^\alpha \phi_p({}_0D_t^\alpha u(t)) = f(t, u(t)), & t \in (0, T), \\ u(0) = u(T) = 0. \end{cases} \tag{3}$$

In contrast to BVP (3), if $b \neq 0$, another nonlocal term,

$$\int_0^T |{}_0D_t^\alpha u(t)|^p dt, \tag{4}$$

makes BVP (1) rough when one deals with it by the variational methods.

Recently, many important results on the fractional differential equations [3–18] and the Kirchhoff equations [19–24] have been obtained. Motivated by the above works, we study the solvability of BVP (1). More precisely, we prove that BVP (1) possesses at least one nontrivial weak solution when $f(t, x)$ is $(p^2 - 1)$-superlinear or $(p^2 - 1)$-sublinear in x at infinity and possesses infinitely many nontrivial weak solutions when $f(t, x)$ is $(p^2 - 1)$-sublinear in x at infinity. The main ingredients used here are the mountain pass theorem and the genus properties in the critical point theory.

Note that, since the Kirchhoff-type p-Laplacian is a nonlinear operator, it is usually difficult to verify the Palais-Smale condition ((PS)-condition for short). Now, we make the following assumptions on the nonlinearity f.

(H_{11}). There exist two constants $\mu > p^2, R > 0$ such that

$$0 < \mu F(t,x) \le x f(t,x), \quad \forall t \in [0,T],\ x \in \mathbb{R} \text{ with } |x| \ge R, \tag{5}$$

where $F(t,x) = \int_0^x f(t,s)ds$.

(H_{12}). $f(t,x) = o(|x|^{p-1})$ as $|x| \longrightarrow 0$ uniformly for $\forall t \in [0,T]$.

(H_{21}). There exists a constant $1 < r_1 < p^2$ and a function $d \in L^1([0,T],\mathbb{R}^+)$ such that

$$|f(t,x)| \le r_1 d(t)|x|^{r_1-1}, \quad \forall (t,x) \in [0,T]\times\mathbb{R}. \tag{6}$$

(H_{22}). There exists an open interval $\mathbb{I} \subset [0,T]$ and three constants $\eta, \delta > 0$, $1 < r_2 < p^2$ such that

$$F(t,x) \ge \eta |x|^{r_2}, \quad \forall (t,x) \in \mathbb{I}\times[-\delta,\delta]. \tag{7}$$

(H_{23}). $f(t,x) = -f(t,-x)$, $\forall (t,x) \in [0,T]\times\mathbb{R}$.

We are now to state our main results.

Theorem 1. *Let (H_{11}) and (H_{12}) be satisfied. Then, BVP (1) possesses at least one nontrivial weak solution.*

Theorem 2. *Let (H_{21}) and (H_{22}) be satisfied. Then, BVP (1) possesses at least one nontrivial weak solution.*

Theorem 3. *Let (H_{21})–(H_{23}) be satisfied. Then, BVP (1) possesses infinitely many nontrivial weak solutions.*

2. Preliminaries

2.1. Fractional Sobolev Space. In this subsection, we present some basic definitions and notations of the fractional calculus [25, 26]. Moreover, we introduce a fractional Sobolev space and some properties of this space [14].

Definition 4 (see [25]). For $\beta > 0$, the left and right Riemann-Liouville fractional integrals of order β of a function $u : [a,b] \longrightarrow \mathbb{R}$ are given by

$$\begin{aligned} {}_aD_t^{-\beta}u(t) &= \frac{1}{\Gamma(\beta)}\int_a^t (t-s)^{\beta-1}u(s)ds, \\ {}_tD_b^{-\beta}u(t) &= \frac{1}{\Gamma(\beta)}\int_t^b (s-t)^{\beta-1}u(s)ds, \end{aligned} \tag{8}$$

respectively, provided that the right-hand-side integrals are pointwise defined on $[a,b]$, where $\Gamma(\cdot)$ is the gamma function.

Definition 5 (see [25]). For $n-1 \le \beta < n$ $(n \in \mathbb{N})$, the left and right Riemann-Liouville fractional derivatives of order β of a function $u : [a,b] \longrightarrow \mathbb{R}$ are given by

$$\begin{aligned} {}_aD_t^{\beta}u(t) &= \frac{d^n}{dt^n}\,{}_aD_t^{\beta-n}u(t), \\ {}_tD_b^{\beta}u(t) &= (-1)^n\frac{d^n}{dt^n}\,{}_tD_b^{\beta-n}u(t). \end{aligned} \tag{9}$$

Remark 6. When $\beta = 1$, one can obtain from Definitions 4 and 5 that

$$\begin{aligned} {}_aD_t^{\,1}u(t) &= u'(t), \\ {}_tD_b^{\,1}u(t) &= -u'(t), \end{aligned} \tag{10}$$

where u^0 is the usual first-order derivative of u.

Definition 7 (see [14]). For $0 < \alpha \le 1$ and $1 < p < \infty$, the fractional derivative space $E_0^{\alpha,p}$ is defined by the closure of $C_0^\infty((0,T),\mathbb{R})$ with respect to the following norm:

$$\|u\|_{E^{\alpha,p}} = \left(\|u\|_{L^p}^p + \|{}_0D_t^\alpha u\|_{L^p}^p\right)^{1/p}, \tag{11}$$

where $\|u\|_{L^p} = \left(\int_0^T |u(t)|^p dt\right)^{1/p}$ is the norm of $L^p((0,T),\mathbb{R})$.

Remark 8. It is obvious that, for $u \in E_0^{\alpha,p}$, one has

$$u, {}_0D_t^{\,\alpha}u \in L^p((0,T),\mathbb{R}), \quad u(0) = u(T) = 0. \tag{12}$$

Lemma 9 (see [14]). *Let $0 < \alpha \le 1$ and $1 < p < \infty$. The fractional derivative space $E_0^{\alpha,p}$ is a reflexive and separable Banach space.*

Lemma 10 (see [14]). *Let $0 < \alpha \le 1$ and $1 < p < \infty$. For $u \in E_0^{\alpha,p}$, one has*

$$\|u\|_{L^p} \le C_p \|{}_0D_t^\alpha u\|_{L^p}, \tag{13}$$

where

$$C_p = \frac{T\alpha}{\Gamma(\alpha+1)} > 0, \tag{14}$$

is a constant. Moreover, if $\alpha > 1/p$, then

$$\|u\|_\infty \le C_\infty \|{}_0D_t^\alpha u\|_{L^p}, \tag{15}$$

where $\|u\|_\infty = \max_{t\in 0,T}|u(t)|$ is the norm of $C([0,T],\mathbb{R})$,

$$C_\infty = \frac{T^{\alpha-(1/p)}}{\Gamma(\alpha)(\alpha q - q + 1)^{1/q}} > 0, \quad q = \frac{p}{p-1} > 1. \tag{16}$$

Remark 11. By (13), we can consider the space $E_0^{\alpha,p}$ with norm

$$\|u\|_{E^{\alpha,p}} = \|{}_0D_t^\alpha u\|_{L^p}, \tag{17}$$

in what follows.

Lemma 12 (see [14]). *Let $1/p < \alpha \leq 1$ and $1 < p < \infty$. The imbedding of $E_0^{\alpha,p}$ in $C([0,T],\mathbb{R})$ is compact.*

2.2. Critical Point Theory. Now, we present some necessary definitions and theorems of the critical point theory [27, 28].

Let X be a real Banach space, and $I \in C^1(X,\mathbb{R})$ which means that I is a continuously Fréchet differentiable functional. Moreover, let $B_\rho(0)$ be an open ball in X and $\partial B_\rho(0)$ denote its boundary.

Definition 13 (see [27]). Let $I \in C^1(X,\mathbb{R})$. If any sequence $\{u_k\} \subset X$ for which $\{I(u_k)\}$ is bounded and $I'(u_k) \longrightarrow 0$ as $k \longrightarrow \infty$ possesses a convergent subsequence in X, then we say that I satisfies the (PS)-condition.

Lemma 14 (see [28]). *Let X be a real Banach space, and $I \in C^1(X,\mathbb{R})$ satisfying the (PS)-condition. Suppose that $I(0) = 0$ and (C_1), there are constants $\rho, \sigma > 0$ such that $I|_{\partial B\rho(0)} \geq \sigma$; (C_2) there is an $e \in X \setminus \overline{B_\rho(0)}$ such that $I(e) \leq 0$.*

Then, I possesses a critical value $c \geq \sigma$. Moreover, c can be characterized as

$$c = \inf_{\gamma\in\Gamma} \max_{s\in[0,1]} I(\gamma(s)), \tag{18}$$

where

$$\Gamma = \{\gamma \in C([0,1],X) | \gamma(0) = 0,\ \gamma(1) = e\}. \tag{19}$$

Lemma 15 (see [27]). *Let X be a real Banach space, and $I \in C^1(X,\mathbb{R})$ satisfies the (PS)-condition. If I is bounded from below, then $c = \inf_X I$ is a critical value of I.*

In order to find the infinitely many critical points of I, we introduce the following genus properties. Let

$$\begin{aligned}
&\Sigma = \{A \subset X - \{0\} | A \text{ is closed in } X \text{ and symmetric with respect to } 0\},\\
&K_c = \left\{u \in X | I(u) = c, I'(u) = 0\right\},\\
&I^c = \{u \in X | I(u) \leq c\}.
\end{aligned} \tag{20}$$

Definition 16 (see [28]). For $A \in \Sigma$, we say that the genus of A is n denoted by $\gamma(A) = n$ if there is an odd map $G \in C(A, \mathbb{R}^n \setminus \{0\})$ and n is the smallest integer with this property.

Lemma 17 (see [28]). *Let I be an even C^1 functional on X and satisfy the (PS)-condition. For any $n \in \mathbb{N}$, set*

$$\begin{aligned}
\Sigma_n &= \{A \in \Sigma | \gamma(A) \geq n\},\\
c_n &= \inf_{A\in\sum_n} \sup_{u\in A} I(u).
\end{aligned} \tag{21}$$

(i) *If $\Sigma_n \neq 0$ and $c_n \in \mathbb{R}$, then c_n is a critical value of I*

(ii) *If there exists $l \in \mathbb{N}$ such that $c_n = c_{n+1} = \cdots = c_{n+l} = c \in \mathbb{R}$, and $c \neq I(0)$, then $\gamma(K_c) \geq l + 1$*

Remark 18. From Remark 7.3 in [28], we know that if $K_c \in \Sigma$ and $\gamma(K_c) > 1$, then K_c contains infinitely many distinct points; that is, I has infinitely many distinct critical points in X.

3. Proof of Theorem 1

In this section, we discuss the existence of nontrivial weak solutions of BVP (1) when the nonlinearity $f(t,x)$ is $(p^2 - 1)$-superlinear in x at infinity.

Define the functional $I : E_0^{\alpha,p} \longrightarrow \mathbb{R}$ by

$$\begin{aligned}
I(u) &= \frac{1}{bp^2}\left(a + b\int_0^T |{}_0D_t^\alpha u(t)|^p dt\right)^p\\
&\quad - \int_0^T F(t,u(t))dt - \frac{a^p}{bp^2}\\
&= \frac{1}{bp^2}\left(a + b\|u\|_{E^{\alpha,p}}^p - \int_0^T F(t,u(t))dt - \frac{a^p}{bp^2}\right).
\end{aligned} \tag{22}$$

It is easy to verify from (15), (17), and $f \in C([0,T] \times \mathbb{R},\mathbb{R})$ that the functional I is well defined on $E_0^{\alpha,p}$ and is a continuously Fréchet differentiable functional; that is, $I \in C^1(E_0^{\alpha,p},\mathbb{R})$. Furthermore, we have

$$\begin{aligned}
\left\langle I'(u), v\right\rangle &= \left(a + b\|u\|_{E^{\alpha,p}}^p\right)^{p-1}\int_0^T \phi_p({}_0D_t^\alpha u(t))\,{}_0D_t^\alpha v(t)dt\\
&\quad - \int_0^T f(t,u(t))v(t)dt, \quad \forall u, v \in E_0^{\alpha,p},
\end{aligned} \tag{23}$$

which yields

$$\left\langle I'(u), u\right\rangle = \left(a + b\|u\|_{E^{\alpha,p}}^p\right)^{p-1}\|u\|_{E^{\alpha,p}}^p - \int_0^T f(t,u(t))u(t)dt. \tag{24}$$

In the following, for simplicity, let

$$\begin{aligned}
M_{pk} &= a + b\|uk\|_{E^{\alpha,p}}^p,\\
M_p &= a + b\|u\|_{E^{\alpha,p}}^p.
\end{aligned} \tag{25}$$

Lemma 19. *Assume that (H_{11}) holds. Then, I satisfies the (PS)-condition in $E_0^{\alpha,p}$.*

Proof. Let $\{u_k\} \subset E_0^{\alpha,p}$ be a sequence such that

$$\begin{aligned}
&|I(u_k)| \leq K,\\
&I'(u_k) \longrightarrow 0 \text{ as } k \longrightarrow \infty,
\end{aligned} \tag{26}$$

where $K > 0$ is a constant. We first prove that $\{u_k\}$ is bounded in $E_0^{\alpha,p}$. From the continuity of $\mu F(t,x) - xf(t,x)$ and (H_{11}), we obtain that there exists a constant $c > 0$ such that

$$F(t,x) \le \frac{1}{\mu} xf(t,x) + c, \quad \forall (t,x) \in [0,T] \times \mathbb{R}. \tag{27}$$

Thus, by (22) and (24), we have

$$\begin{aligned} K &\ge I(u_k) \ge \frac{1}{bp^2} M_{pk}^p - \frac{1}{\mu} \int_0^T f(t,u_k(t)) u_k(t) dt - cT - \frac{a^p}{bp^2} \\ &= \frac{1}{bp^2} M_{pk}^p - \frac{1}{\mu} M_{pk}^{p-1} \|u_k\|_{E^{\alpha,p}}^p + \frac{1}{\mu} \left\langle I'(u_k), u_k \right\rangle \\ &\quad - cT - \frac{a^p}{bp^2} \ge M_{pk}^{p-1} \left(\left(\frac{1}{p^2} - \frac{1}{\mu} \right) \|u_k\|_{E^{\alpha,p}}^p + \frac{a}{bp^2} \right) \\ &\quad - \frac{1}{\mu} \|I'(u_k)\|_{(E_0^{\alpha,p})^*} \|u_k\|_{E^{\alpha,p}} - cT - \frac{a^p}{bp^2}, \end{aligned} \tag{28}$$

which together with $I'(u_k) \longrightarrow 0$ as $k \longrightarrow \infty$ yields

$$\begin{aligned} K &\ge M_{pk}^{p-1} \left(\left(\frac{1}{p^2} - \frac{1}{\mu} \right) \|u_k\|_{E^{\alpha,p}}^p + \frac{a}{bp^2} \right) \\ &\quad - \|u_k\|_{E^{\alpha,p}} - cT - \frac{a^p}{bp^2} \text{ as } k \longrightarrow \infty. \end{aligned} \tag{29}$$

Then, it follows from $\mu > p^2$ that $\{u_k\}$ is bounded in $E_0^{\alpha,p}$. Since $E_0^{\alpha,p}$ is a reflexive Banach space (see Lemma 9), going if necessary to a subsequence, we can assume $u_k \longrightarrow u$ in $E_0^{\alpha,p}$. Hence, from $I'(u_k) \longrightarrow 0$ as $k \longrightarrow \infty$ and the definition of weak convergence, we have

$$\begin{aligned} &\left\langle I'(u_k) - I'(u), u_k - u \right\rangle \\ &\quad = \left\langle I'(u_k), u_k - u \right\rangle - \left\langle I'(u), u_k - u \right\rangle \\ &\quad \le \|I'(u_k)\|_{(E_0^{\alpha,p})^*} \|u_k - u\|_{E^{\alpha,p}} \\ &\quad - \left\langle I'(u), u_k - u \right\rangle \longrightarrow 0 \text{ as } k \longrightarrow \infty. \end{aligned} \tag{30}$$

In addition, we obtain from (15), (17), and Lemma 12 that $\{u_k\}$ is bounded in $C([0,T],\mathbb{R})$ and $\|u_k - u\|_\infty \longrightarrow 0$ as $k \longrightarrow \infty$. Thus, there exists a constant $c_1 > 0$ such that

$$|f(t,u_k(t)) - f(t,u(t))| \le c_1, \quad \forall t \in [0,T], \tag{31}$$

which yields

$$\begin{aligned} &\left| \int_0^T (f(t,u_k(t)) - f(t,u(t)))(u_k(t) - u(t)) dt \right| \\ &\quad \le c_1 T \|u_k - u\|_\infty \longrightarrow 0 \text{ as } k \longrightarrow \infty. \end{aligned} \tag{32}$$

Moreover, by the boundedness of $\{u_k\}$ in $E_0^{\alpha,p}$, one has

$$\begin{aligned} &\left(M_{pk}^{p-1} - M_p^{p-1} \right) \int_0^T \phi_p({}_0D_t^\alpha u(t))({}_0D_t^\alpha u_k(t) - {}_0D_t^\alpha u(t)) dt \\ &\quad = \left(M_{pk}^{p-1} - M_p^{p-1} \right) \left\langle I_1'(u), u_k - u \right\rangle \longrightarrow 0 \text{ as } k \longrightarrow \infty, \end{aligned} \tag{33}$$

where I_1' is the Fréchet derivative of $I_1 : E_0^{\alpha,p} \to \mathbb{R}$ defined by

$$I_1(u) = \frac{1}{p} \int_0^T |{}_0D_t^\alpha u(t)|^p dt. \tag{34}$$

From (23), we have

$$\begin{aligned} &\left\langle I'(u_k) - I'(u), u_k - u \right\rangle \\ &\quad + \int_0^T (f(t,u_k(t)) - f(t,u(t)))(u_k(t) - u(t)) \\ &\quad \cdot dt = M_{pk}^{p-1} \int_0^T \phi_p({}_0D_t^\alpha u_k(t))({}_0D_t^\alpha u_k(t) - {}_0D_t^\alpha u(t)) \\ &\quad \cdot dt - M_p^{p-1} \int_0^T \phi_p({}_0D_t^\alpha u(t))({}_0D_t^\alpha u_k(t) - {}_0D_t^\alpha u(t)) \\ &\quad \cdot dt = M_{pk}^{p-1} \int_0^T \left(\phi_p({}_0D_t^\alpha u_k(t)) - \phi_p({}_0D_t^\alpha u(t)) \right) \\ &\quad \cdot ({}_0D_t^\alpha u_k(t) - {}_0D_t^\alpha u(t)) dt + \left(M_{pk}^{p-1} - M_p^{p-1} \right) \int_0^T \phi_p({}_0D_t^\alpha u(t)) \\ &\quad \cdot ({}_0D_t^\alpha u_k(t) - {}_0D_t^\alpha u(t)) dt, \end{aligned} \tag{35}$$

which together with (30)–(33) yields

$$\int_0^T \left(\phi_p({}_0D_t^\alpha u_k(t)) - \phi_p({}_0D_t^\alpha u(t)) \right) ({}_0D_t^\alpha u_k(t) - {}_0D_t^\alpha u(t)) dt \longrightarrow 0, \tag{36}$$

as $k \longrightarrow \infty$.

Following (2.10) in [29], there exist two constants c_2, $c_3 > 0$ such that

$$\begin{aligned} &\int_0^T \left(\phi_p({}_0D_t^\alpha u_k(t)) - \phi_p({}_0D_t^\alpha u(t)) \right) ({}_0D_t^\alpha u_k(t) - {}_0D_t^\alpha u(t)) \\ &\quad \cdot dt \ge \begin{cases} c_2 \displaystyle\int_0^T |{}_0D_t^\alpha u_k(t) - {}_0D_t^\alpha u(t)|^p dt, & p \ge 2, \\ c_3 \displaystyle\int_0^T \frac{|{}_0D_t^\alpha u_k(t) - {}_0D_t^\alpha u(t)|^2}{(|{}_0D_t^\alpha u_k(t)| + |{}_0D_t^\alpha u(t)|)^{2-p}} dt, & 1 < p < 2. \end{cases} \end{aligned} \tag{37}$$

When $1 < p < 2$, based on the Hölder inequality, we get

$$\begin{aligned}
&\int_0^T |{}_0D_t^\alpha u_k(t) - {}_0D_t^\alpha u(t)|^p \\
&\quad \cdot dt \le \left(\int_0^T \frac{|{}_0D_t^\alpha u_k(t) - {}_0D_t^\alpha u(t)|^2}{(|{}_0D_t^\alpha u_k(t)| + |{}_0D_t^\alpha u(t)|)^{2-p}} dt\right)^{p/2} \\
&\quad \cdot \left(\int_0^T (|{}_0D_t^\alpha u_k(t)| + |{}_0D_t^\alpha u(t)|)^p dt\right)^{(2-p)/2} \\
&\le c_4 \left(\|u_k\|_{E^{\alpha,p}}^p + \|u\|_{E^{\alpha,p}}^p\right)^{(2-p)/2} \\
&\quad \cdot \left(\int_0^T \frac{|{}_0D_t^\alpha u_k(t) - {}_0D_t^\alpha u(t)|^2}{(|{}_0D_t^\alpha u_k(t)| + |{}_0D_t^\alpha u(t)|)^{2-p}} dt\right)^{p/2},
\end{aligned} \tag{38}$$

where $c_4 = 2^{(p-1)(2-p)/2} > 0$ is a constant, which together with (37) implies

$$\begin{aligned}
&\int_0^T \left(\phi_p({}_0D_t^\alpha u_k(t)) - \phi_p({}_0D_t^\alpha u(t))\right) \\
&\quad \cdot ({}_{20}D_t^\alpha u_k(t) - {}_0D_t^\alpha u(t)) \\
&\quad \cdot dt \ge c_3 c_4^{-2/p} \left(\|u_k\|_{E^{\alpha,p}}^p + \|u\|_{E^{\alpha,p}}^p\right)^{(p-2)/p} \|u_k - u\|_{E^{\alpha,p}}^2, \\
&\qquad 1 < p < 2.
\end{aligned} \tag{39}$$

When $p \ge 2$, by (37), we have

$$\begin{aligned}
&\int_0^T \left(\phi_p({}_0D_t^\alpha u_k(t)) - \phi_p({}_0D_t^\alpha u(t))\right)({}_0D_t^\alpha u_k(t) - {}_0D_t^\alpha u(t)) \\
&\quad \cdot dt \ge c_2 \|u_k - u\|_{E^{\alpha,p}}^p, \quad p \ge 2.
\end{aligned} \tag{40}$$

Then, it follows from (36), (39), and (40) that

$$\|u_k - u\|_{E^{\alpha,p}} \longrightarrow 0 \text{ as } k \longrightarrow \infty. \tag{41}$$

Hence, I satisfies the (PS)-condition.

Proof of Theorem 1. From (H_{12}), there exist two constants $0 < \varepsilon < 1, \delta > 0$ such that

$$F(t,x) \le \frac{(1-\varepsilon)a^{p-1}}{pC_p^p}|x|^p, \quad \forall t \in [0,T],\ x \in \mathbb{R} \text{ with } |x| \le \delta, \tag{42}$$

where $C_p > 0$ is a constant defined in (13). Let $\rho = \delta/C_\infty$ and $\sigma = \varepsilon\alpha^{p-1}\rho^p/p$, where $C_\infty > 0$ is a constant defined in (15). Then, by (15) and (17), we have

$$\|u\|_\infty \le C_\infty \|u\|_{E^{\alpha,p}} = \delta, \quad \forall u \in E_0^{\alpha,p} \text{ with } \|u\|_{E^{\alpha,p}} = \rho, \tag{43}$$

which together with (13), (17), (22), and (42) implies

$$\begin{aligned}
I(u) &= \frac{1}{bp^2}\left(a + b\|u\|_{E^{\alpha,p}}^p\right)^p - \int_0^T F(t,u(t)) \\
&\quad \cdot dt - \frac{a^p}{bp^2} \ge \frac{a^{p-1}}{p}\|u\|_{E^{\alpha,p}}^p - \frac{(1-\varepsilon)a^{p-1}}{pC_p^p}\int_0^T |u(t)|^p \\
&\quad \cdot dt \ge \frac{a^{p-1}}{p}\|u\|_{E^{\alpha,p}}^p - \frac{(1-\varepsilon)a^{p-1}}{p}\|u\|_{E^{\alpha,p}}^p = \frac{\varepsilon a^{p-1}}{p}\|u\|_{E^{\alpha,p}}^p = \sigma, \\
&\qquad \forall u \in E_0^{\alpha,p} \text{ with } \|u\|_{E^{\alpha,p}} = \rho,
\end{aligned} \tag{44}$$

which means that the condition (C_1) in Lemma 14 is satisfied.

From (H_{11}), a simple argument using the very definition of the derivative shows that there exist two constants $c_1, c_2 > 0$ such that

$$F(t,x) \ge c_1|x|^\mu - c_2, \quad \forall (t,x) \in [0,T] \times \mathbb{R}. \tag{45}$$

Then, for any $u \in E_0^{\alpha,p} \setminus \{0\}$, $\xi \in \mathbb{R}^+$, we can obtain from (22) and $\mu > p^2$ that

$$\begin{aligned}
I(\xi u) &= \frac{1}{bp^2}\left(a + b\|\xi u\|_{E^{\alpha,p}}^p\right)^p - \int_0^T F(t,\xi u(t)) \\
&\quad \cdot dt - \frac{a^p}{bp^2} \le \frac{1}{bp^2}\left(a + b\|\xi u\|_{E^{\alpha,p}}^p\right)^p \\
&\quad - c_1\int_0^T |\xi u(t)|^\mu dt + c_2 T - \frac{a^p}{bp^2} \\
&= \frac{1}{bp^2}\left(a + b\xi^p\|u\|_{E^{\alpha,p}}^p\right)^p - c_1\xi^\mu\|u\|_{L^\mu}^\mu \\
&\quad + c_2 T - \frac{a^p}{bp^2} \longrightarrow -\infty \text{as } \xi \longrightarrow \infty.
\end{aligned} \tag{46}$$

Thus, taking ξ_0 large enough and letting $e = \xi_0 u$, we have $I(e) \le 0$. Hence, the condition (C_2) in Lemma 14 is also satisfied.

Finally, by $I(0) = 0$, Lemmas 14 and 19, we get a critical point u^* of I satisfying $I(u^*) \ge \sigma > 0$, and so u^* is a nontrivial solution of BVP (1).

4. Proofs of Theorems 2 and 3

In this section, we discuss the existence and multiplicity of nontrivial weak solutions of BVP (1) when the nonlinearity $f(t,x)$ is (p^2-1)-sublinear in x at infinity.

Lemma 20. *Suppose that (H_{21}) is satisfied. Then, I is bounded from below in $E_0^{\alpha,p}$.*

Proof. From (H_{21}), one has

$$|F(t,u)| \le d(t)|u|^{r_1}, \quad \forall (t,u) \in [0,T] \times \mathbb{R}, \tag{47}$$

which together with (15)–(22) yields

$$I(u) \geq \frac{1}{bp^2}\left(a+b\|u\|_{E^{\alpha,p}}^{p}\right)^{p} - \int_0^T d(t)|u(t)|^{r_1} \cdot dt - \frac{a^p}{bp^2} \geq \frac{1}{bp^2}\left(a+b\|u\|_{E^{\alpha,p}}^{p}\right)^{p} - \|d\|_{L^1}\|u\|_{\infty}^{r_1} - \frac{a^p}{bp^2} \geq \frac{1}{bp^2}\left(a+b\|u\|_{E^{\alpha,p}}^{p}\right)^{p} - C_{\infty}^{r_1}\|d\|_{L^1}\|u\|_{E^{\alpha,p}}^{r_1} - \frac{a^p}{bp^2}. \tag{48}$$

Since $1 < r_1 < p^2$, (48) yields $I(u) \longrightarrow \infty$ as $|u|_{E^{\alpha,p}} \longrightarrow \infty$. Hence, I is bounded from below.

Lemma 21. *Assume that (H_{21}) holds. Then, I satisfies the (PS)-condition in $E_0^{\alpha,p}$.*

Proof. Let be a sequence such that

$$\begin{aligned} |I(u_k)| &\leq K, \\ I'(u_k) &\longrightarrow 0 \text{ as } k \longrightarrow \infty, \end{aligned} \tag{49}$$

where $K > 0$ is a constant. Then, (48) implies that $\{u_k\}$ is bounded in $E_0^{\alpha,p}$. The remainder of proof is similar to the proof of Lemma 19, so we omit the details.

Proof of Theorem 2. From Lemmas 15, 20, and 21, we obtain $c = \inf_{E_0^{\alpha,p}} I(u)$ which is a critical value of I; that is, there exists a critical point $u^* \in E_0^{\alpha,p}$ such that $I(u^*) = c$.

Now, we show $u^* \neq 0$. Let $u_0 \in (W_0^{1,2}(\mathbb{I}, \mathbb{R}) \cap E_0^{\alpha,p}) \setminus \{0\}$ and $\|u_0\|_{\infty} = 1$, from (22) and (H_{22}), we get

$$I(su_0) = \frac{1}{bp^2}\left(a+b\|su_0\|_{E^{\alpha,p}}^{p}\right)^{p} - \int_0^T F(t, su_0(t)) \cdot dt - \frac{a^p}{bp^2} = \frac{1}{bp^2}\left(a+bs^p\|u_0\|_{E^{\alpha,p}}^{p}\right)^{p} - \int_{\mathbb{I}} F(, su_0(t)) \cdot dt - \frac{a^p}{bp^2} \leq \frac{1}{bp^2}\left(a+bs^p\|u_0\|_{E^{\alpha,p}}^{p}\right)^{P} - \eta s^{r_2}\int_{\mathbb{I}} |u_0(t)|^{r_2} \cdot dt - \frac{a^p}{bp^2}, \quad 0 < s \leq \delta. \tag{50}$$

Since $1 < r_2 < p^2$, (50) implies $I(su_0) < 0$ for $s > 0$ small enough. Then, $I(u^*) = c < 0$; hence, u^* is a nontrivial critical point of I, and so u^* is a nontrivial solution of BVP (1).

Proof of Theorem 3. From Lemmas 20 and 21, we obtain that $I \in C^1(E_0^{\alpha,p}, \mathbb{R})$ is bounded from below and satisfies the (PS)-condition. In addition, (22) and (H_{23}) show that I is even and $I(0) = 0$.

Fixing $n \in \mathbb{N}$, we take n disjoint open intervals $\mathbb{I}_i$ such that $\cup_{i=1}^{n} \mathbb{I}_i \subset \mathbb{I}$.

Let $u_i \in (W_0^{1,2}(\mathbb{I}_i, \mathbb{R}) \cap E_0^{\alpha,p}) \setminus \{0\}$ and $\|u_i\|_{E^{\alpha,p}} = 1$, and

$$\begin{aligned} E_n &= \operatorname{span}\{u_1, u_2, \cdots, u_n\}, \\ S_n &= \left\{u \in E_n \,\middle|\, \|u\|_{E^{\alpha,p}} = 1\right\}. \end{aligned} \tag{51}$$

For $u \in E_n$, there exists $\lambda_i \in \mathbb{R}$, such that

$$u(t) = \sum_{i=1}^{n} \lambda_i u_i(t), \quad \forall t \in [0, T]. \tag{52}$$

Thus, we get

$$\|u\|_{E^{\alpha,p}}^{p} = \int_0^T |{}_0D_t^{\alpha} u(t)|^p dt = \sum_{i=1}^{n} |\lambda_i|^p \int_{\mathbb{I}_i} |{}_0D_t^{\alpha} u_i(t)|^p \cdot dt = \sum_{i=1}^{n} |\lambda_i|^p \int_0^T |{}_0D_t^{\alpha} u_i(t)|^p dt = \sum_{i=1}^{n} |\lambda_i|^p \|u_i\|_{E^{\alpha,p}}^{p} = \sum_{i=1}^{n} |\lambda_i|^p, \quad \forall u \in E_n. \tag{53}$$

From (15)–(22), (52), and (H_{22}), for $u \in S_n$, one has

$$I(su) = \frac{1}{bp^2}\left(a+b\|su\|_{E^{\alpha,p}}^{p}\right)^{p} - \int_0^T F(t, su(t)) \cdot dt - \frac{a^p}{bp^2} = \frac{1}{bp^2}\left(a+bs^p\right)^{p} - \sum_{i=1}^{n} \int_{\mathbb{I}_i} F(t, s\lambda_i u_i(t)) \cdot dt - \frac{a^p}{bp_2} \leq \frac{1}{bp^2}\left(a+bs^p\right)^{p} - \eta s^{r_2} \sum_{i=1}^{n} |\lambda_i|^{r_2} \int_{\mathbb{I}_i} |u_i(t)|^{r_2} \cdot dt - \frac{a^p}{bp^2}, \quad 0 < s \leq \frac{\delta}{C_{\infty}\lambda *}, \tag{54}$$

where $\lambda^* = \max_{i \in \{1,2,\cdots,n\}} |\lambda_i| > 0$ is a constant. Since $1 < r_2 < p^2$, it follows from (54) that there exist constants $\epsilon, \sigma > 0$ such that

$$I(\sigma u) < -\varepsilon, \quad \forall u \in S_n. \tag{55}$$

Let

$$\begin{aligned} S_n^{\sigma} &= \{\sigma u | u \in S_n\}, \\ \Lambda &= \left\{(\lambda_1, \lambda_2, \cdots, \lambda_n) \in \mathbb{R}^n \,\middle|\, \sum_{i=1}^{n} |\lambda_i|^p < \sigma^p\right\}. \end{aligned} \tag{56}$$

Then, we obtain from (55) that

$$I(u) < -\varepsilon, \quad \forall u \in S_n^{\sigma}, \tag{57}$$

which, together with the fact that I is even and $I(0) = 0$, yields

$$S_n^{\sigma} \subset I^{-\varepsilon} \in \sum. \tag{58}$$

From (53), it is seen that the mapping $(\lambda_1, \lambda_2, \cdots, \lambda_n) \longrightarrow \sum_{i=1}^{n} \lambda_i u_i(t)$ from $\partial \Lambda$ to S_n^σ is odd and homeomorphic. Hence, by some properties of the genus (Propositions 7.5 and 7.7 in [28]), we deduce that

$$\gamma(I^{-\epsilon}) \geq \gamma(S_n^\sigma) = n. \tag{59}$$

Thus, $I^{-\varepsilon} \in \sum_n$ and so $\Sigma_n \neq 0$. Let

$$c_n = \inf_{A \in \sum_n} \sup_{u \in A} I(u). \tag{60}$$

It follows from I is bounded from below that $-\infty < c_n \leq -\varepsilon < 0$. That is, for any $n \in \mathbb{N}$, c_n is a real negative number. Hence, by Lemma 17, I admits infinitely many nontrivial critical points, and so BVP (1) possesses infinitely many nontrivial negative energy solutions.

Obviously the following assumption implies (H_{21})–(H_{23}).

(H_{24}). $f(t, x) = r g(t)|x|^{r-2} x$, where $1 < r < p^2$ is a constant, $g \in C([0, T], \mathbb{R})$, and there exists an open interval $I \subset [0, T]$ such that $g(t) > 0, \forall t \in I$.

As a direct result, we have the following result.

Corollary 22. *Let (H_{24}) be satisfied. Then, BVP (1) possesses infinitely many nontrivial weak solutions.*

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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