Research Article

Multiple Solution Results for Perturbed Fractional Differential Equations with Impulses

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The multiplicity of classical solutions for impulsive fractional differential equations has been studied by many scholars. Using Morse theory, Brezis and Nirenberg's Linking Theorem, and Clark theorem, we aim to solve this kind of problems. By this way, we obtain the existence of at least three classical solutions and k distinct pairs of classical solutions. Finally, an example is presented to illustrate the feasibility of the main results in this paper.

1. Introduction

Consider the multiple solutions of fractional order impulsive systems as follows:

$$\begin{cases} -\frac{1}{2} \frac{d}{dt} ({}_0D_t^{-\beta} + {}_tD_T^{-\beta}) u'(t) = a(t)u(t) + \nabla F(t, u(t)), & t \neq t_j, \text{ a.e. } t \in [0, T], \\ \Delta({}_tD_t^\alpha u)(t_j) = I_j(u(t_j)), & t_j \in [0, T], j = 1, 2, \dots, l, \\ u(0) = u(T) = 0, \end{cases} \quad (1)$$

where $\beta \in [0, 1)$, $\alpha = 1 - \beta/2 \in (1/2, 1]$; ${}_0D_t^{-\beta}$, ${}_tD_T^{-\beta}$ are the left and right Riemann-Liouville fractional integrals of order β , ${}_0D_t^\alpha$, ${}_tD_T^\alpha$ are used to denote the left and right Caputo fractional derivatives of order α , $0 = t_0 < t_1 < \dots < t_l < t_{l+1} = T$, $a \in C[0, T]$, $F: [0, T] \times \mathbb{R}^N \rightarrow \mathbb{R}$ is a given function, $\nabla F(t, x)$ is the gradient of F at x , there are constants a_0, a_1 with $0 < a_0 \leq a(t) \leq a_1$,

$$({}_tD_t^\alpha u)(t) = \frac{1}{2} \{ {}_0D_t^{\alpha-1} ({}_0D_t^\alpha u) - {}_tD_T^{\alpha-1} ({}_tD_T^\alpha u) \}(t),$$

$$\begin{aligned} \Delta({}_tD_t^\alpha u)(t_j) &= \frac{1}{2} \{ {}_0D_t^{\alpha-1} ({}_0D_t^\alpha u) - {}_tD_T^{\alpha-1} ({}_tD_T^\alpha u) \}(t_j^+) \\ &\quad - \frac{1}{2} \{ {}_0D_t^{\alpha-1} ({}_0D_t^\alpha u) - {}_tD_T^{\alpha-1} ({}_tD_T^\alpha u) \}(t_j^-), \end{aligned}$$

$$\begin{aligned} &\{ {}_0D_t^{\alpha-1} ({}_0D_t^\alpha u) - {}_tD_T^{\alpha-1} ({}_tD_T^\alpha u) \}(t_j^+) \\ &= \lim_{t \rightarrow t_j^+} \{ {}_0D_t^{\alpha-1} ({}_0D_t^\alpha u) - {}_tD_T^{\alpha-1} ({}_tD_T^\alpha u) \}(t), \end{aligned}$$

$$\begin{aligned} &\{ {}_0D_t^{\alpha-1} ({}_0D_t^\alpha u) - {}_tD_T^{\alpha-1} ({}_tD_T^\alpha u) \}(t_j^-) \\ &= \lim_{t \rightarrow t_j^-} \{ {}_0D_t^{\alpha-1} ({}_0D_t^\alpha u) - {}_tD_T^{\alpha-1} ({}_tD_T^\alpha u) \}(t), \end{aligned} \quad (2)$$

for $j = 1, 2, \dots, l$.

The problem (1) arises from the phenomena of advection dispersion and was first scrutinized by Erwin and Roop in [1]. From then on, more and more scholars began to pay attention to the problem in [1] and the related problems.

Fractional calculus is different from integral calculus in nature. It has nonlocal characteristics and is very suitable for describing materials and processes with memory effect and genetic properties. Therefore, fractional differential equations are widely used in many domains, for instance,

biomedicine, economic mathematics, and technology science [2, 3]. In recent years, the variational methods and critical point theory have been widely used to study fractional differential equations [4–8].

In [8], the authors discussed the following fractional order differential systems:

$$\begin{cases} -\frac{1}{2} \frac{d}{dt} \left({}_0 D_t^{-\beta} + {}_t D_T^{-\beta} \right) u'(t) = \nabla F(t, u(t)), & \text{a.e. } t \in [0, T], \\ u(0) = u(T) = 0. \end{cases} \quad (3)$$

They used the critical point theory and other tools to verify the existence of solutions. From then on, a number of scholars began to use such methods for research, as shown in [9–11].

In [12], the authors discussed the following problems:

$$\begin{cases} {}_t D_T^\alpha (a(t) {}_0 D_t^\alpha u(x)) = \lambda u(t) \nabla F(t, u(t)), & \text{a.e. } t \in [0, T], \\ u(0) = u(T) = 0. \end{cases} \quad (4)$$

They proved that there are at least k pairs of weak solutions and two weak solutions by using the Clark Theorem and other methods.

An impulsive phenomenon is a common phenomenon in nature and engineering applications. The models reflected in mathematics are impulsive differential equations. The most prominent feature of impulsive differential equation is that it can fully consider the impact of instantaneous mutation on the state. Therefore, in recent decades, impulsive differential equation theory has been widely used in biological mathematics, theoretical mechanics, biomedicine, and economic mathematics (see [13–18]).

For the past few years, very few scholars used the variational method and critical point theory to discuss impulsive fractional differential equations and their boundary value problems. Moreover, few papers discuss the fractional order system by using Morse theory (see [19–24]).

In [23], the authors discussed the following problems:

$$\begin{cases} {}_t D_T^\alpha ({}_0^c D_t^\alpha u(t)) + k(t)u(t) = f(t, u(t)), & 0 < t < T, t \neq t_j, \\ \Delta({}_t D_T^{\alpha-1} ({}_0^c D_t^\alpha u)) (t_j) = I_j(u(t_j)), & j = 1, 2, \dots, m, \\ u(0) = u(T) = 0. \end{cases} \quad (5)$$

The multiple solutions of this problem are verified with Morse theory and the Clark theorem by the authors.

In [25], the sufficient conditions for the existence of infinite solutions to the system (1) are obtained by using the variational method.

Based on the above literatures, in the present paper, we will discuss the existence of multiple classical solutions for (1) by using Morse theory, Clark theorem, and Brezis and Nirenberg's Linking Theorem.

First of all, we give some assumptions.

(H1) $I_j \in C([0, T], R)$, $I_j(0) = 0$, there exist some constants $e_j, \gamma_j \in [0, 1)$, $a_j, b_j > 0$, such that $\lim_{|u| \rightarrow 0} (|I_j(u)|/|u|^{e_j}) = b_j$, $|I_j(u)| \leq a_j |u|^{\gamma_j}$, $j = 1, \dots, l$, and $\int_0^u I_j(s) ds \geq 0$, for $\forall u \in R$

(H2) $F \in C([0, T], R)$ and $F(t, 0) = 0$, $\lim_{|u| \rightarrow \infty} \sup (F(t, u)/|u|^2) < 1/2(I^2(\alpha + 1)|\cos(\pi\alpha)/T^{2\alpha}| - a_1)$ uniformly on $t \in [0, T]$

(H3) $\lim_{|u| \rightarrow 0} \sup (|F(t, u)|/|u|^2) < I^2(\alpha + 1)/2T^{2\alpha}$ uniformly for $t \in [0, T]$. There exist four constants $C > 0, r, r_0, \gamma \in (1,$

$\max_{j \in \{1, 2, \dots, l\}} \{\gamma_j + 1\})$ such that $F(t, u) \geq C|u|^\gamma$, $r \leq |u| \leq r_0$, a.e. $t \in [0, T]$

(H4) $F(t, -u) = F(t, u)$ and $I_j(-u) = -I_j(u)$ ($j = 1, 2, \dots, l$), for $u \in R$

The key outcomes are as follows.

Theorem 1. *Let (H1)–(H3) hold. Then, the problem (1) has at least three classical solutions.*

Theorem 2. *Let (H1)–(H4) hold. Then, the problem (1) has at least k distinct pairs of classical solutions.*

Note that the methods in this article are distinct from [25] and our results are richer. The problems in this paper we studied are different from the problems in [23]. Compared with [23], classical solutions are investigated in this paper.

The structure of this article is as below. In Section 2, we provide some preliminary knowledge, which are helpful to the proof of the key outcomes. We prove the key outcomes in Section 3. Finally, an example is given to illustrate the main results.

2. Preliminaries

Similar to [25], we first convert system (1) into a new format as follows:

$$\begin{cases} \frac{d}{dt} \left\{ \frac{1}{2} {}_0 D_t^{\alpha-1} ({}_0^c D_t^\alpha u(t)) - \frac{1}{2} {}_t D_T^{\alpha-1} ({}_t^c D_T^\alpha u(t)) \right\} + a(t)u(t) + \nabla F(t, u(t)) = 0, & t \neq t_j, \text{ a.e. } t \in [0, T], \\ \Delta(D_t^\alpha u)(t_j) = I_j(u(t_j)), & t_j \in (0, T), j = 1, 2, \dots, l, \\ u(0) = u(T) = 0. \end{cases} \quad (6)$$

Remark 3. Because of the equivalence of system (1) and system (6), we know that the solutions of system (6) are the solutions of system (1).

We first build the function spaces as below, the goal of which is to establish the variational framework of system (6).

Let us recall that for any fixed $1 \leq p \leq \infty$ and $t \in [0, T]$, $\|u\|_\infty = \max_{t \in [0, T]} |u(t)|$, $\|u\|_{L^p} = (\int_0^T |u(s)|^p ds)^{1/p}$.

Definition 4 (see [22]). Let $0 < \alpha \leq 1$, we define the fractional derivative space E_0^α by the closure of $C_0^\infty([0, T], R)$ with $u(0) = u(T)$ under the norm

$$\|u\|_\alpha = \left(\int_0^T |{}_0^c D_t^\alpha u(t)|^2 dt + \int_0^T |u(t)|^2 dt \right)^{1/2}, \quad \forall u \in E_0^\alpha. \quad (7)$$

Lemma 5 (see [25]). Let $0 < \alpha \leq 1$, and E_0^α is a Banach space with reflexive and separable.

Definition 6 (see [25]). We define that the function $u \in E_0^\alpha$ is a weak solution of the system (6) if the following holds:

$$\begin{aligned} & \int_0^T \left\{ -\frac{1}{2} [{}_0^c D_t^\alpha u(t) {}_t^c D_T^\alpha v(t) + {}_t^c D_T^\alpha u(t) {}_0^c D_t^\alpha v(t)] - a(t)u(t)v(t) \right\} dt \\ & + \sum_{j=1}^l I_j(u(t_j))v(t_j) - \int_0^T \nabla F(t, u(t))v(t) dt = 0 \end{aligned} \quad (8)$$

for $\forall v \in E_0^\alpha$.

We define $\Phi : E_0^\alpha \longrightarrow R$ as

$$\begin{aligned} \Phi(u) = & \int_0^T -\frac{1}{2} {}_0^c D_t^\alpha u(t) {}_t^c D_T^\alpha u(t) dt - \frac{1}{2} \int_0^T a(t)u^2(t) dt \\ & - \int_0^T F(t, u(t)) dt + \sum_{j=1}^l \int_0^{u(t_j)} I_j(s) ds. \end{aligned} \quad (9)$$

From (H1), (H2), we know the functional Φ is continuously differentiable. So $\forall u, v \in E_0^\alpha$, we have

$$\begin{aligned} \langle \Phi'(u), v \rangle = & \int_0^T -\frac{1}{2} [{}_0^c D_t^\alpha u(t) {}_t^c D_T^\alpha v(t) + {}_t^c D_T^\alpha u(t) {}_0^c D_t^\alpha v(t)] dt \\ & - \int_0^T a(t)u(t)v(t) dt + \sum_{j=1}^l I_j(u(t_j))v(t_j) \\ & - \int_0^T \nabla F(t, u(t))v(t) dt. \end{aligned} \quad (10)$$

Remark 7. Obviously, from (10), we know that the critical points of functional Φ are the weak solutions of system (6).

Definition 8 (see [25]). We define

$$u \in \left\{ u \in AC([0, T]): \int_{t_j}^{t_{j+1}} \left(\int_0^T |{}_0^c D_t^\alpha u(t)|^2 + \int_0^T |u(t)|^2 \right)^{1/2} dt < \infty, j=0, \dots, l \right\}, \quad (11)$$

as a classic solution of system (6) if it satisfies the following conditions:

$$(i) \quad 1/2 \{ {}_0 D_t^{\alpha-1} ({}_0^c D_t^\alpha u) - {}_t D_T^{\alpha-1} ({}_t^c D_T^\alpha u) \} (t_j^+) = \lim_{t \rightarrow t_j^+} 1/2 \{ {}_0 D_t^{\alpha-1} ({}_0^c D_t^\alpha u) - {}_t D_T^{\alpha-1} ({}_t^c D_T^\alpha u) \} (t),$$

$$\begin{aligned} & \frac{1}{2} \{ {}_0 D_t^{\alpha-1} ({}_0^c D_t^\alpha u) - {}_t D_T^{\alpha-1} ({}_t^c D_T^\alpha u) \} (t_j^-) \\ & = \lim_{t \rightarrow t_j^-} \frac{1}{2} \{ {}_0 D_t^{\alpha-1} ({}_0^c D_t^\alpha u) - {}_t D_T^{\alpha-1} ({}_t^c D_T^\alpha u) \} (t), \end{aligned}$$

$$\Delta(D_t^\alpha u)(t_j) = I_j(u(t_j)),$$

$$u(0) = u(T) = 0 \quad (12)$$

$$(ii) \quad u \text{ content system (1) a.e. on } t \in [0, T] \setminus \{t_1, t_2, \dots, t_l\}$$

Lemma 9 (see [25]). The function $u \in E_0^\alpha$ is a classical solution of system (6) when u is a weak solution of system (6).

Remark 10. Combine Remarks 3 and 7 and Lemma 9, we know that the critical point of functional Φ is the classical solution of the system (1). Therefore, we will directly discuss the critical point of Φ as below.

Lemma 11 (see [8]). Let $1/2 < \alpha \leq 1$ and $1 < p < \infty$, for all $u \in E_0^\alpha$, one has

$$\|u\|_{L^p} \leq \frac{T^\alpha}{\Gamma(\alpha+1)} \|{}_0^c D_t^\alpha u\|_{L^p}. \quad (13)$$

Moreover, if $\alpha > 1/p$, $1/p + 1/q = 1$, then

$$\|u\|_\infty \leq \frac{T^{\alpha-1/p}}{\Gamma(\alpha)((\alpha-1)q+1)^{1/q}} \|{}_0^c D_t^\alpha u\|_{L^p}. \quad (14)$$

In particular, if $p = 2$, then

$$\begin{aligned} \|u\|_{L^2}^2 & \leq \frac{T^{2\alpha}}{\Gamma^2(\alpha+1)} \|u\|_\alpha^2, \\ \|u\|_\infty^2 & \leq \frac{T^{2\alpha-1}}{\Gamma^2(\alpha)(2\alpha-1)} \|u\|_\alpha^2. \end{aligned} \quad (15)$$

It is easy to prove that the norm $\|u\|_\alpha = (\int_0^T |{}_0^c D_t^\alpha u(t)|^2 dt + \int_0^T |u(t)|^2 dt)^{1/2}$ is equivalent to $\|u\|_\alpha =$

$(\int_0^T |{}_0^c D_t^\alpha u(t)|^2 dt)^{1/2}$, $\forall u \in E_0^\alpha$. Next, we will use $\|u\|_\alpha = (\int_0^T |{}_0^c D_t^\alpha u(t)|^2 dt)^{1/2}$ as the norm in E_0^α .

Lemma 12 (see [8]). Let $1/2 < \alpha \leq 1$, $\forall u \in E_0^\alpha$, have

$$|\cos(\pi\alpha)| \|u\|_\alpha^2 \leq - \int_0^T {}_0^c D_t^\alpha u(t) {}_i^c D_T^\alpha u(t) dt \leq \frac{1}{|\cos(\pi\alpha)|} \|u\|_\alpha^2. \quad (16)$$

Lemma 13 (see [8]). Let $1/2 < \alpha \leq 1$. Assume the sequence $\{u_n\}$ converges weakly to u in E_0^α . Then, $u_n \rightarrow u$ strongly in $C([0, T], R)$, i.e., $\|u_n - u\|_\infty \rightarrow 0$, as $n \rightarrow \infty$.

Definition 14 (see [23]). We say that Φ satisfies the (PS) condition in E_0^α , if any $\{u_n\}_{n \in \mathbb{N}} \subset E_0^\alpha$, for which $\{\Phi(u_n)\}_{n \in \mathbb{N}}$ is bounded and $\Phi'(u_n) \rightarrow 0$ as $n \rightarrow \infty$ owns a strongly convergent subsequence in E_0^α .

Lemma 15 (see [26]). Let E have a direct sum decomposition $E = V \oplus W$, and $k = \dim V < \infty$. Let 0 be a critical point of Φ with $\Phi(0) = 0$, Φ is bounded below and satisfying (PS) condition. Suppose that, for some $\rho > 0$,

$$\begin{aligned} \Phi(u) &\leq 0, \quad \forall u \in V, \|u\| \leq \rho, \\ \Phi(u) &> 0, \quad \forall u \in W, \|u\| \leq \rho. \end{aligned} \quad (17)$$

Also, assume that $\inf_E \Phi < 0$. Then, Φ has at least two nonzero critical points and $C_k(\Phi, 0) \neq 0$.

Lemma 16 (see [27]). Let E be a real Banach space, $\Phi \in C^1(E, R)$; assume that Φ is even, bounded from below, and satisfying (PS) condition. Assume $\Phi(0) = 0$, there exists a set $E' \subset E$ such that E' is homeomorphic to S^{k-1} by an odd map, and $\sup_{E'} \Phi < 0$. Then, Φ has at least k distinct pairs of critical points.

3. Proofs of Main Results

Lemma 17. Suppose (H1), (H2) hold, if $\{u_n\}$ is a (PS) sequence, then $\{u_n\}$ is bounded.

Proof. If $\{u_n\}$ is a (PS) sequence, that is,

$$\Phi(u_n) \text{ is bounded, } \Phi'(u_n) \rightarrow 0, \text{ as } n \rightarrow \infty. \quad (18)$$

From (H2), for some $\xi > 0$ small enough, there is a constant $C_\xi > 0$, for any $u \in R$, $t \in [0, T]$ such that

$$|F(t, u)| \leq \frac{1}{2} \left(\frac{\Gamma^2(\alpha+1)}{T^{2\alpha}} |\cos(\pi\alpha)| - a_1 - \xi \right) |u|^2 + C_\xi. \quad (19)$$

According to (19), for $u \in E_0^\alpha$, $u \neq 0$, one has

$$\begin{aligned} \Phi(u) &= \int_0^T -\frac{1}{2} {}_0^c D_t^\alpha u(t) {}_i^c D_T^\alpha u(t) dt - \frac{1}{2} \int_0^T a(t) u^2(t) dt \\ &\quad - \int_0^T F(t, u(t)) dt + \sum_{j=1}^l \int_0^{u(t_j)} I_j(s) ds \geq \frac{1}{2} |\cos(\pi\alpha)| \|u\|_\alpha^2 \\ &\quad - \frac{1}{2} \left(\frac{\Gamma^2(\alpha+1)}{T^{2\alpha}} |\cos(\pi\alpha)| - a_1 - \xi \right) \int_0^T |u(t)|^2 dt \\ &\quad - \frac{a_1}{2} \int_0^T u^2(t) dt - C_\xi T = \frac{1}{2} |\cos(\pi\alpha)| \|u\|_\alpha^2 \\ &\quad - \frac{1}{2} \left(\frac{\Gamma^2(\alpha+1)}{T^{2\alpha}} |\cos(\pi\alpha)| - \xi \right) \|u\|_\alpha^2 - C_\xi T \\ &\geq \frac{1}{2} |\cos(\pi\alpha)| \|u\|_\alpha^2 - \frac{1}{2} \left(|\cos(\pi\alpha)| - \frac{T^{2\alpha}}{\Gamma^2(\alpha+1)} \xi \right) \|u\|_\alpha^2 \\ &\quad - C_\xi T \geq \frac{\xi T^{2\alpha}}{2\Gamma^2(\alpha+1)} \|u\|_\alpha^2 - C_\xi T. \end{aligned} \quad (20)$$

Because $\Phi(u)$ is bounded, by (20), we can get $\{u_n\}$ is bounded in E_0^α and Φ is bounded from below. The proof is completed.

Lemma 18. Assume (H1), (H2) hold, then Φ satisfies the (PS) condition.

Proof. If $\{u_n\}$ is a (PS) sequence, from Lemma 17, we get $\{u_n\}$ is a bounded sequence in E_0^α . By Lemma 5, we get $\{u_n\}$ has a weakly convergent subsequence. Without loss of generality, we also assume that u_n converges weakly to u_0 in E_0^α , then from (9) and (18), we know

$$\begin{aligned} \langle \Phi'(u_n) - \Phi'(u_0), u_n - u_0 \rangle &\leq \|\Phi'(u_n)\| \cdot \|u_n - u_0\| \\ &\rightarrow 0, \quad n \rightarrow \infty. \end{aligned} \quad (21)$$

By Lemma 13, we can obtain that $u_n \rightarrow u_0$ in $C([0, T], R)$, as $n \rightarrow \infty$, i.e.,

$$\|u_n - u_0\|_\infty \rightarrow 0, \text{ as } n \rightarrow \infty. \quad (22)$$

From (10), we have

$$\begin{aligned} &\langle \Phi'(u_n) - \Phi'(u_0), u_n - u_0 \rangle \\ &= \int_0^T -\frac{1}{2} [{}_0^c D_t^\alpha (u_n - u_0) {}_i^c D_T^\alpha (u_n - u_0) + {}_i^c D_T^\alpha (u_n - u_0) {}_0^c D_t^\alpha (u_n - u_0)] \\ &\quad - \int_0^T (a(t)(u_n - u_0))(u_n - u_0) dt \\ &\quad - \int_0^T [\nabla F(t, u_n) - \nabla F(t, u_0)](u_n - u_0) dt \\ &\quad + \sum_{j=1}^l [I_j(u_n(t_j)) - I_j(u_0(t_j))](u_n(t_j) - u_0(t_j)) \\ &\geq \frac{1}{2} |\cos(\pi\alpha)| \|u_n - u_0\|_\alpha^2 - \left| \int_0^T [\nabla F(t, u_n) - \nabla F(t, u_0)] \right| \|u_n - u_0\|_\infty \\ &\quad + \left| \sum_{j=1}^l [I_j(u_n(t_j)) - I_j(u_0(t_j))] \right| \|u_n - u_0\|_\infty. \end{aligned} \quad (23)$$

By (21), (22), and (23), we can infer that $\|u_n - u_0\|_\alpha^2 \rightarrow 0$, as $n \rightarrow \infty$, i.e., u_n strongly converges to u_0 . Therefore, Φ satisfies the (PS) condition.

By Lemma 5, we can obtain that there is an orthogonal basis $\{e_i\}$ of E_0^α such that $E_0^\alpha = \overline{\text{span}\{e_i : i = 1, 2, \dots\}}$. We define $X_i := \text{span}\{e_i\}$, $V_k = \bigoplus_{i=1}^k X_i$, $Y_k = \bigoplus_{i=k}^\infty X_i$ ($k = 1, 2, \dots$). Then, $E_0^\alpha = V_k \oplus Y_k$.

Proof of Theorem 1. From (H1), (H2), one knows $F(t, 0) = 0$ and $I_j(0) = 0$, $j = 1, \dots, l$. We find out Φ has a critical point at 0. Therefore, we can get the linking $E_0^\alpha = V_k \oplus Y_k$ of Φ at 0.

According to the equivalence of norm of normed space in finite dimension, there exist positive constants M_1, M_2, M'_1, M'_2 such that

$$\begin{aligned} M_1 \|u\|_\alpha &\leq \|u\|_\infty \leq M_2 \|u\|_\alpha, \\ M'_1 \|u\|_\alpha &\leq \|u\|_{L^p} \leq M'_2 \|u\|_\alpha, \end{aligned} \quad (24)$$

$$u \in V_k.$$

First, let $u \in V_k$. Because V_k is finite dimensional, there exists $0 < \rho_1 < 1$ small for $r_0 > 0$, such that

$$|u(t)| \leq \|u\|_\infty \leq M_2 \|u\|_\alpha < M_2 \rho_1 < r_0, \quad u \in V_k, \|u\|_\alpha < \rho_1. \quad (25)$$

For any $r \in (0, r_0)$, we set $\Omega_1 = \{t \in [0, T] : |u| \leq r\}$, $\Omega_2 = \{t \in [0, T] : r \leq |u| \leq r_0\}$, $\Omega_3 = \{t \in [0, T] : r_0 \leq |u|\}$, where $[0, T] = \bigcup_{i=1}^3 \Omega_i$ and Ω_i ($i = 1, 2, 3$) are pairwise disjoint.

Set $F^*(t, u) = F(t, u) - C|u|^\gamma$, for $\|u\|_\alpha < \rho_1$, $u \in V_k$, combine (H1), (H3) and Lemma 12, we have

$$\begin{aligned} \Phi(u) &= \int_0^T -\frac{1}{2} {}^c D_t^\alpha u(t) {}^c D_t^\alpha u(t) dt - \frac{1}{2} \int_0^T a(t) u^2(t) dt \\ &\quad - \int_0^T F(t, u(t)) dt + \sum_{j=1}^l \int_0^{u(t_j)} I_j(s) ds \leq \frac{1}{2|\cos(\pi\alpha)|} \|u\|_\alpha^2 \\ &\quad + \sum_{j=1}^l \frac{a_j}{\gamma_j + 1} |u(t_j)|^{\gamma_j+1} - \int_0^T C|u|^\gamma dt - \int_{\Omega_1} F^*(t, u(t)) dt \\ &\quad - \int_{\Omega_2} F^*(t, u(t)) dt - \int_{\Omega_3} F^*(t, u(t)) dt \leq \frac{1}{2|\cos(\pi\alpha)|} \|u\|_\alpha^2 \\ &\quad + \sum_{j=1}^l \frac{a_j A_0^{\gamma_j+1}}{\gamma_j + 1} \|u\|_\alpha^{\gamma_j+1} - C\|u\|_{L^p}^\gamma - \int_{\Omega_1} F^*(t, u(t)) dt \\ &\quad - \int_{\Omega_2} F^*(t, u(t)) dt - \int_{\Omega_3} F^*(t, u(t)) dt, \end{aligned} \quad (26)$$

where $A_0 = T^{\alpha-1/2}/\Gamma(\alpha)\sqrt{2\alpha-1}$.

According to (25) and the definition of Ω_3 is empty set, we have $\int_{\Omega_3} F^*(t, u(t)) dt = 0$, for any $u \in V_k$. By (H3), one has $\int_{\Omega_2} F^*(t, u(t)) dt \geq C|u|^\gamma - C|u|^\gamma = 0$. On Ω_1 , $|u| < r$. From (H3), we can get $\int_{\Omega_1} F^*(t, u(t)) dt \rightarrow 0$, as $r \rightarrow 0$.

Then, $\forall u \in V_k, r \in (0, r_0)$, $\|u\|_\alpha \leq \rho \leq 1$, $1 < \gamma < \max\{\gamma_j + 1\} < 2$, according to (26), we can get

$$\begin{aligned} \Phi(u) &\leq \frac{1}{2|\cos(\pi\alpha)|} \|u\|_\alpha^2 + \sum_{j=1}^l \frac{a_j A_0^{\gamma_j+1}}{\gamma_j + 1} \|u\|_\alpha^{\gamma_j+1} - M_1^\gamma C \|u\|_\alpha^\gamma \\ &\leq \|u\|_\alpha^\gamma \left(\frac{1}{2|\cos(\pi\alpha)|} \|u\|_\alpha^{2-\gamma} + \sum_{j=1}^l \frac{a_j A_0^{\gamma_j+1}}{\gamma_j + 1} \|u\|_\alpha^{\gamma_j+1-\gamma} - M_1^\gamma C \right) \leq 0. \end{aligned} \quad (27)$$

Hence,

$$\Phi(u) \leq 0, \quad \forall u \in V_k, \|u\|_\alpha \leq \rho_1. \quad (28)$$

Next, set $u \in Y_k$. Because $E_0^\alpha \rightarrow C_0^\infty([0, T], \mathbb{R})$ is continuous compact embedding. Hence, for $u \in Y_k, \varepsilon > 0$, there exists $0 < \rho_2 < 1$ small such that $|u| \leq \|u\|_\infty \leq T^{2\alpha-1} \|u\|_\alpha / \Gamma^2(\alpha) (2\alpha-1) < T^{2\alpha-1} \rho_2 / \Gamma^2(\alpha) (2\alpha-1) < \varepsilon$, for $\|u\|_\alpha < \rho_2$.

From (H3), $\forall |u| < \varepsilon, u \in Y_k, \|u\|_\alpha \leq \rho_2, t \in [0, T]$, there is $\xi \in (0, |\cos(\pi\alpha)|)$, one has

$$|F(t, u)| \leq (|\cos(\pi\alpha)| - \xi) \frac{\Gamma^2(\alpha+1)}{2T^{2\alpha}} |u|^2. \quad (29)$$

From (H1), $\forall |u| < \varepsilon, u \in Y_k, \|u\|_\alpha \leq \rho_2$, one has

$$\frac{|I_j(u)|}{|u|^{e_j}} > \frac{1}{2} b_j. \quad (30)$$

Let $b = \min_{j=1}^l b_j, e = \max_{j=1}^l e_j, \forall |u| < \varepsilon, u \in Y_k, \|u\|_\alpha \leq \rho_2 < 1$, by Lemmas 11 and 12 and (29) and (30), we obtain

$$\begin{aligned} \Phi(u) &= \int_0^T -\frac{1}{2} {}^c D_t^\alpha u(t) {}^c D_t^\alpha u(t) dt - \frac{1}{2} \int_0^T a(t) u^2(t) dt \\ &\quad - \int_0^T F(t, u(t)) dt + \sum_{j=1}^l \int_0^{u(t_j)} I_j(s) ds \geq \frac{1}{2} |\cos(\pi\alpha)| \|u\|_\alpha^2 \\ &\quad - (|\cos(\pi\alpha)| - \xi) \frac{\Gamma^2(\alpha+1)}{2T^{2\alpha}} \|u\|_{L^2}^2 - \frac{1}{2} a_1 T |u|^2 + \frac{1}{2} \frac{bl}{e+1} |u|^{e+1} \\ &\geq \frac{1}{2} |\cos(\pi\alpha)| \|u\|_\alpha^2 - \frac{1}{2} (|\cos(\pi\alpha)| - \xi) \|u\|_\alpha^2 - \frac{1}{2} a_1 T |u|^2 \\ &\quad + \frac{1}{2} \frac{bl}{e+1} |u|^{e+1} \geq \frac{1}{2} \left(\xi \|u\|_\alpha^2 + |u|^{e+1} \left(\frac{bl}{e+1} - a_1 T |u|^{1-e} \right) \right) > 0. \end{aligned} \quad (31)$$

Hence,

$$\Phi(u) > 0, \forall u \in Y_k, \|u\|_\alpha \leq \rho_2. \quad (32)$$

Let $\rho = \min\{\rho_1, \rho_2\}$, from (28) and (32), we obtain

$$\begin{aligned} \Phi(u) &\leq 0, \quad \forall u \in V_k, \|u\|_\alpha \leq \rho, \\ \Phi(u) &> 0, \quad \forall u \in Y_k, \|u\|_\alpha \leq \rho. \end{aligned} \quad (33)$$

It follows from Lemmas 17 and 18 that Φ is bounded from below and satisfies the (PS) condition. Then, from Lemma 15,

we can get Φ has at least two nonzero critical points, and $C_k(\Phi, 0) \cong 0$, so $u = 0$ is a homological nontrivial point of Φ . Hence, the system (1) has at least three classical solutions.

Proof of Theorem 2. According to (H4), we can deduce that Φ is even. By Lemmas 17 and 18, we know Φ is bounded from below and satisfies the (PS) condition. For given $\rho > 0$, set $E' = S_\rho = \{u \in V_k : \|u\| = \rho\}$. By (27), if ρ is small enough, one has $\sup_{E'} \Phi(u) < 0$. Clearly, $\dim V_k = k$. Then, we can conclude that Φ has at least k distinct pairs of critical points from Lemma 16. Hence, the system (1) has at least k distinct pairs of classical solutions. We complete the proof.

Example 19.

$$\begin{cases} -\frac{1}{2} \frac{d}{dt} ({}_0 D_t^{-0.6} + {}_t D_T^{0.6}) u'(t) = \frac{2}{5} u(t) + \nabla F(t, u(t)), & t \neq t_j, \text{ a.e. } t \in [0, 1], \\ \Delta({}_t D_t^{0.7} u)(t_j) = I_j(u(t_j)), & t_j \in [0, 1], j = 1, 2, \dots, l, \\ u(0) = u(T) = 0. \end{cases} \quad (34)$$

According to (34), we can see that $\beta = 0.6$, $\alpha = 0.7$, $a(t) = 2/5$, $T = 1$.

Let $I_j(u) = (4/3)u^{1/3}$, then the condition (H1) holds with $a_j = 3/2$, $\gamma_j = 1/3$, $b_j = 4/3$, $e_j = 1/3$.

Let $F(t, u(t)) = 1/15(1 + \sin^2 t)|u|^2(1/\ln(|u|^{2/3} + 1.5))$, $a_1 = 9/20$. By simple calculations, we can get $\Gamma^2(\alpha + 1)/T^{2\alpha} \approx 0.84474$, $|\cos(0.7\pi)| \approx 0.58778$,

$$\lim_{|u| \rightarrow \infty} \sup \frac{F(t, u)}{|u|^2} \rightarrow 0 < \frac{1}{2} \left(\frac{\Gamma^2(\alpha + 1)}{T^{2\alpha}} |\cos(\pi\alpha)| - a_1 \right) \approx 0.02326; \quad (35)$$

then, the condition (H2) holds.

By (H3), we know

$$\lim_{|u| \rightarrow 0} \sup \frac{|F(t, u)|}{|u|^2} \leq 0.32884 < \frac{\Gamma^2(\alpha + 1)}{2T^{2\alpha}} \approx 0.42237. \quad (36)$$

Let $r = e - 1.7$, $r_0 = e - 1.5$, $C = 2/25$, $\gamma = 6/5$; then, for $e - 1.7 \leq |u| \leq e - 1.5$, we have

$$F(t, u) > 0.14471|u|^2 \geq \frac{2}{25}|u|^{6/5}; \quad (37)$$

then, the condition (H3) is satisfied.

It easy to see that the condition (H4) holds.

According to Theorem 1, the system (1) exists at least three classical solutions. According to Theorem 2, the system (1) possesses at least k distinct pairs of classical solutions.

4. Conclusions

In this work, we study perturbed fractional differential equation with impulses. We give sufficient conditions of the

existence of at least three classical solutions and at least k distinct pairs of classical solutions for problems (1), where k is the dimension of V_k .

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare no conflict of interest.

Authors' Contributions

The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

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