Hindawi
Journal of Function Spaces
Volume 2020, Article ID 8512636, 7 pages
https://doi.org/10.1155/2020/8512636

*Research Article*

# Monotonicities in Orlicz Spaces Equipped with Mazur-Orlicz $F$-Norm

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Received 25 December 2019; Revised 29 April 2020; Accepted 4 May 2020; Published 31 May 2020

Academic Editor: Gen Q. Xu



Some basic properties in Orlicz spaces and Orlicz sequence spaces that are generated by monotone function equipped with the Mazur-Orlicz $F$-norm are studied in this paper. We give some relationships between the modulus and the Mazur-Orlicz $F$-norm. We obtain an interesting result that the norm of an element in line segments is formed by two elements on the unit sphere less than or equal to 1 if and only if that the monotone function is a convex function. The criterion that Orlicz spaces and Orlicz sequence spaces that are generated by monotone function equipped with the Mazur-Orlicz $F$-norm are strictly monotone or lower locally uniform monotone is presented.

## 1. Introduction

Geometry of Banach space has important applications in the control theory, fixed-point theory, ergodic theory, probability theory, and vector analytic function theory. Recall that the monotonicity properties are restrictions of appropriate rotundity properties to the set of couples of comparable elements in the positive cone of a Banach lattice (see [1–5]). Consequently, then in many cases, good convex properties can be replaced successfully by respective monotonicity properties (see [6–8]). It is well known that monotonicity properties (strict and uniform monotonicity) play an analogous role in the best dominated approximation problems in Banach lattices as the respective rotundity properties (strict and uniform rotundity) in the best approximation problems in Banach spaces (see [9, 10]). Moreover, monotonicity properties are applicable in the ergodic theory, since they provide a tool for estimating a norm (see [11–13]). In this paper, based on the paper in 2019 by Cui et al., after removing the continuous condition of generating function (see [6]), we found some conclusions completely different from those in the Banach spaces we studied in the past, which provided us with a new understanding and enlightenment for studying geometric properties of seminormed linear spaces in the future.

## 2. General Auxiliary and New Results

*Definition 1.* *Let $(G, \Sigma, \mu)$ be a $\sigma$ -finite measure space and $L^0(\mu)$ be the space of all (equivalence) classes of $\Sigma$ -measurable real-valued funcions defined onG. A function $\Phi : (-\infty,\infty) \longrightarrow [0,\infty]$ is called an Orlicz function if $\Phi(u) > 0$ for all $u \neq 0$ $\lim_{u\to 0+0} \Phi(u) = 0$ , and even $\Phi$ is the nondecreasing function and $\lim_{u\to\infty} \Phi(u) = \infty$ .Any Orlicz function $\Phi$ determines a functional $I_\Phi : L^0(\mu) \longrightarrow [0,\infty]$ defined by the formula $I_\Phi(f) = \int_G \Phi(|f(t)|)d\mu(t)$ and called the modular. The order ideal*

$$L^\Phi(\mu) = \left\{f \in L^0(\mu) \colon I_\Phi(rf)<\infty \text{ for some } r > 0\right\} \quad (1)$$

in $L^0(\mu)$ is called an Orlicz space.
Let us define a lattice ideal $E^\Phi(\mu)$ in $L^\Phi(\mu)$as

$$E^\Phi(\mu) = \left\{f \in L^0(\mu) \colon I_\Phi(rf)<\infty \text{ for all } r > 0\right\}. \quad (2)$$

The space $L^\Phi(\mu)$ is linear with respect to the following lattice $F$-norm, called the Mazur-Orlicz $F$-norm (see [6]):

$$\|f\|_F = \inf\{\lambda > 0 : I_\Phi(f/\lambda) \le \lambda\}. \quad (3)$$

Note that if $\Phi$ is a convex function, then $L^\Phi(\mu)$, endowed with the equivalent lattice norm (called the Luxemburg norm) $\|f\|_\Phi = \inf\{\lambda > 0 : I_\Phi(f/\lambda) \le 1\}$, becomes a Banach lattice. If $\Phi$ is right continuous and $X$ is a Banach space, increasing function on $[0, +\infty]$ such that $\Phi(0) = 0$. Define

$$L_\Phi(\mu, X) = \left\{ f \in L_0(\mu, X) : \int_G \Phi\left(\frac{\|f(x)\|}{\rho}\right) d\mu(x) < \infty \text{ for some } \rho > 0 \right\}. \quad (4)$$

The properties of $\Phi$ yield that if for $f \in L_\Phi(\mu, X)$, we put

$$\|f\|_\Phi = \inf\left\{ \rho > 0 : \int_G \Phi\left(\frac{\|f(x)\|}{\rho}\right) d\mu(x) < \rho \right\}, \quad (5)$$

then $\|\cdot\|_\Phi$ satisfies the axioms of a $\Delta$-norm in $L_\Phi(\mu, X)$ (see [14]). Moreover, $\|\cdot\|_\Phi$ is equivalent to an $F$-norm under which $L_\Phi(\mu, X)$ is complete, whence $(L_\Phi(\mu, X), \|\cdot\|_\Phi)$ is an $F$-space.

In the case of the counting measure space, the space $L^\Phi(\mu) = l^\Phi$ is always nontrivial and

$$l^\Phi = \left\{ x \in l^0 : \forall_{\lambda > 0} \exists_{n_\lambda \in \mathbb{N}} \sum_{n = n_\lambda}^{\infty} \Phi(\lambda x(n)) < \infty \right\} =: h^\Phi. \quad (6)$$

*Remark 2. If $\Phi$ is not right continuous at zero, then $\lim_{u \to 0+0} \Phi(u) = a > 0$. If we take $x_n(t) = (1/n)\chi_G(t)$, then $x_n(t)$ is convergent to 0 uniformly, but $I_\Phi(x_n) = \int_G \Phi((1/n)\chi_G(t))dt = \Phi(1/n)m(G) \ge am(G)$ for all $n \in N$. This means that the case $\lim_{u \to 0+0} \Phi(u) > 0$. We do not even have the fact that the uniform convergence of the sequence $\{x_n\}$ implies its modular convergence.*

**Lemma 3.** *$L^\Phi(\mu)$ is a linear space.*

*Proof.* Obviously, if $\Phi$ is a monotone increasing function, then for all $u, v \in R$ and $\alpha \in (0, 1)$ we have

$$\Phi(\alpha u + (1 - \alpha)v) \le \Phi(u) + \Phi(v). \quad (7)$$

Take any $f, g \in L^\Phi(\mu)$, by the definition of the $L^\Phi(\mu)$; there exists $\lambda_1, \lambda_2 > 0$ such that

$$\int_G \Phi(\lambda_1 f(x))dx < \infty, \int_G \Phi(\lambda_2 f(x))dx < \infty. \quad (8)$$

We choose $\lambda_0 = \min\{\lambda_1, \lambda_2\}$, that is,

$$\int_G \Phi(\lambda_0 f(x))dx < \infty, \int_G \Phi(\lambda_0 f(x))dx < \infty. \quad (9)$$

Assume now that $a \ne 0$ or $b \ne 0$, and we conclude that

$$\int_G \Phi\left(\frac{\lambda_0}{|a| + |b|}(af(x) + bg(x))\right) \cdot dx \le \int_G \Phi\left(\frac{\lambda_0}{|a| + |b|}(|a||f(x)| + |b||g(x)|)\right) \cdot dx \le \int_G \Phi(\lambda_0 f(x))dx + \int_G \Phi(\lambda_0 g(x))dx < \infty. \quad (10)$$

Hence, $L^\Phi(\mu)$ is a linear space.

*Definition 4* (see [6]). *An $F$-normed Orlicz space $(L^\Phi(\mu), \|\cdot\|_F)$is said to be strictly monotone (we write $L^\Phi(\mu) \in (SM)$ ) if for any $x, y \in L^\Phi(\mu)$ such that $0 \le y \le x$ , we have $\|y\|_F < \|x\|_F$whenever $y \ne x$ (or equivalently $\|x - y\|_E < \|x\|_E$ whennever $y \ne 0$ ).*

*Definition 5* (see [6]). An *F-normed Orlicz space*$(L^\Phi(\mu), \|\cdot\|_F)$ *is said to be lower locally uniformly monotone (we write$L^\Phi(\mu) \in (LLUM)$) if for any$x \in L^\Phi(\mu)$ and $(x_n)_{n=1}^\infty$ in$L^\Phi(\mu)$ such that $0 \le x_n \le x$for all$n \in \mathbb{N}$ and $\|x_n\|_F \longrightarrow \|x\|_F$as$n \longrightarrow \infty$, the condition$\|x - x_n\|_F \longrightarrow 0$as$n \longrightarrow \infty$ holds.*

*Definition 6. We say that $\Phi$ satisfies $\Delta_2$ -condition ( $\Phi \in \Delta_2$ for short) whenever there are constants $K > 0$ and $u_0 > 0$* such that (see [15])

$$\Phi(2u) \le K\Phi(u)(u \ge u_0). \quad (11)$$

*Definition 7. We say that $\Phi$ satisfies $\Delta_l$ -condition ( $\Phi \in \Delta_l$ for short) whenever there are constants $K_1 > 0$ and $u_1 > 0$ such that*

$$\Phi(lu) \le K_1\Phi(u)(u \ge u_1). \quad (12)$$

**Lemma 8.** *$\Phi \in \Delta_2$ if and only if $\Phi \in \Delta_l$ for all $l > 1$.*

*Proof.* Sufficiency: given that $l > 1$, choose an integer $n_0$ such that $l^{n_0} > 2$. Then, by (12), we have

$$\Phi(2u) \le \Phi(l^{n_0}u) < K_1^{n_0}\Phi(u)(u \ge u_1). \quad (13)$$

Necessity: now choose an integer $n_1$ such that $2^{n_1} > l$. Then, by (11) we get

$$\Phi(lu) \le \Phi(2^{n_1}u) \le K^{n_1}\Phi(u)(u \ge u_0) \quad (14)$$

which ends the proof.

*Definition 9. We say that $\Phi$ satisfies $\delta_2$ -condition ( $\Phi \in \delta_2$ for short) whenever there are constants $k_1 > 0$ and $u_2 > 0$ such that*

$$\Phi(2u) \le k_1\Phi(u)(|u| \le u_2). \quad (15)$$

**Theorem 10.** *If $\Phi$ is an Orlicz function such that $\Phi \in \Delta_2$, then* *for any $\varepsilon > 0$ , there exists $K > 0$ such that*

$$I_\Phi(2f) \le K I_\Phi(f) + \varepsilon, \tag{16}$$

*for all $f \in L^\Phi(\mu)$.*

*Proof.* By $\Phi \in \Delta_2$, there exists $u_1 > 0$ and $K_1 > 0$ such that $\Phi(2u) \le K_1\Phi(u)$ $(\forall u \ge u_1)$. Let us assume that $0 < u_0 < u_1$ and for any $u \in [u_0, u_1]$ and $\Phi(u_0) > 0$, we get

$$\frac{\Phi(2u)}{\Phi(u)} \le \frac{\Phi(2u_1)}{\Phi(u_0)}. \tag{17}$$

We choose $K = \max\{K_1, \Phi(2u_1)/\Phi(u_0)\}$. We can hence easily get that $\Phi(2u) \le K\Phi(u)$ for any $u \ge u_0$. By $\lim_{u\to 0+0} \Phi(u) = 0$, for any $\varepsilon > 0$, there exists $\delta > 0$ such that $\Phi(2u_0)m(G) < \varepsilon$, whenever $u_0 < \delta$. Defining $G_n = \{x \in G : | f(x)| \le u_0\}$, we obtain

$$\begin{aligned} I_\Phi(2f) &= \int_{G_n} \Phi(2f(x))dx + \int_{G\backslash G_n} \Phi(2f(x)) \\ &\cdot dx \le \Phi(2u_0)m(G) + K\int_G \Phi(f(x)) \\ &\cdot dx \le K I_\Phi(f) + \varepsilon. \end{aligned} \tag{18}$$

**Theorem 11.** *For any $f \in L^\Phi(\mu)$,$I_\Phi(f/\|f\|_F) = \|f\|_F$if and only if$\Phi$is right continuous and$\Phi \in \Delta_2$.*

*Proof.* Sufficiency: for any $f \in L^\Phi(\mu) \setminus \{0\}$ the function $F(\lambda) = I_\Phi(\lambda f/\|f\|_F)$ is right continuous on the interval $(0, \infty)$. By the definition of $\|f\|_F$, we have $F(1) \le \|f\|_F$ and for any $\lambda > 1$, we have $F(\lambda) > \|f\|_F$. Using $\lim_{\lambda\to 1+0} \Phi(\lambda f(x)) \longrightarrow \Phi(f(x))$ and the Lebesgue dominated convergence theorem, we obtain

$$\|f\|_F \ge I_\Phi\left(\frac{f}{\|f\|_F}\right) = \lim_{\lambda\to 1^+} \int_G \Phi\left(\frac{\lambda f(t)}{\|f\|_F}\right)dt, \tag{19}$$

which finishes the proof of the implication $I_\Phi(f/\|f\|_F) = \|f\|_F$.

Necessity: assume that $\Phi$ does not satisfy the $\Delta_2$-condition. This yields that there exists a strictly increasing sequence $\{u_n\}_{n=1}^\infty$ of position numbers such that $\Phi(u_1)m(G) \ge 1$ and

$$\Phi\left(\left(1 + \frac{1}{n}\right)u_n\right) > 2^n\Phi(u_n)(n = 1, 2, 3, \cdots). \tag{20}$$

Next, choose a sequence $\{G_n\}_{n=1}^\infty$ of pairwise disjoint sets in $G$ such that $m(G_n) = 1/2^n\Phi(u_n)(n = 1, 2, 3, \cdots)$. Defining $x = \sum_{n=1}^\infty u_n\chi_{G_n}$, we have

$$I_\Phi(x) = \sum_{n=1}^\infty \int_{G_n} \Phi(x(t))dt = \sum_{n=1}^\infty \Phi(u_n)m(G_n) = \sum_{n=1}^\infty \frac{1}{2^n} = 1. \tag{21}$$

On the other hand, we have that for any $\lambda \in (0, 1)$, there exists $n_1 \in N$ such that $1 + (1/n) < \lambda$ for any $n \ge n_1$. Hence,

$$\begin{aligned} I_\Phi(\lambda x) &= \int_G \Phi(\lambda x(t))dt = \sum_{n=1}^\infty \Phi(\lambda u_n)m(G_n) \\ &\ge \sum_{n=1}^\infty \Phi\left(\left(1 + \frac{1}{n}\right)u_n\right)m(G_n) \\ &\ge \sum_{n=1}^\infty 2^n\Phi(u_n)m(G_n) = \sum_{n=1}^\infty 1 = \infty. \end{aligned} \tag{22}$$

Consequently, this shows that $\Phi \in \Delta_2$ is a necessary condition.

Next, we will prove the necessary of the right-continuous function of $\Phi$. If $\Phi$ is not right continuous on $R$, then there exists a point $u_0 \in (0,+\infty)$ such that $\Phi(u_0) < \lim_{u\to u_0+0} \Phi(u) = b$; that is, for any $\lambda < 1$, we get $\Phi(u_0/\lambda) \ge b$. Take any $G_0 \subset G$ with $\Phi(u_0)m(G_0) < 1$. Let us take $c > 0$ large enough. Since $\lim_{u\to\infty} \Phi(u) = +\infty$, we have

$$\Phi(u_0)m(G_0) + \Phi(c)m(GG_0) > 1. \tag{23}$$

Since $G$ is nonatomic, we can find $G_2 \subset G \setminus G_0$ such that

$$\Phi(u_0)m(G_0) + \Phi(c)m(G_1) = 1 - \frac{b - \Phi(u_0)}{2}m(G_0). \tag{24}$$

Let us define

$$x = u_0\chi_{G_0} + c\chi_{G_1} \tag{25}$$

Then, $I_\Phi(x) = \Phi(u_0)m(G_0) + \Phi(c)m(G_1) < 1$, whence we obtain that $x \in L^\Phi(\mu)$ and $\|x\|_F \le 1$. On the other hand, for any $\lambda > 1$, we have

$$\begin{aligned} I_\Phi\left(\frac{x}{\lambda}\right) &= \Phi\left(\frac{u_0}{\lambda}\right)m(G_0) + \Phi\left(\frac{c}{\lambda}\right)m(G_1) > bm(G_0) \\ &+ \Phi(c)m(G_1) = bm(G_0) + \left(1 - \frac{b - \Phi(u_0)}{2}\right)m(G_0) \\ &- \Phi(u_0)m(G_0) = 1 + \frac{b - \Phi(u_0)}{2}m(G_0) > 1, \end{aligned} \tag{26}$$

which gives that $\|x\|_F \ge 1$ yields $\|x\|_F = 1$, a contradiction, which finishes the proof of necessity.

For convenience, from now on, we write $S(L^\Phi(\mu)) = \{f \in L^\Phi(\mu) : \|f\|_F = 1\}$.which contradicts the assumption $\|y\|_F = \|x\|_F$. Therefore, $\|y\|_F < \|x\|_F$.

**Theorem 12.** *Assume that $x, y \in S(L^\Phi(\mu))$ and $\alpha \in (0, 1)$. Then, $\|\alpha x + (1 - \alpha)y\|_F \le 1$ if and only if $\Phi$ is a convex function on $R$.*

*Proof.* Sufficiency: for any $x, y \in S(L^{\Phi}(\mu))$, we have $I_{\Phi}(x) \le 1$ and $I_{\Phi}(y) \le 1$. Assume that $z = \alpha x + (1-\alpha)y$. Then, $I_{\Phi}(z) \le \alpha I_{\Phi}(x) + (1-\alpha)I_{\Phi}(y) \le 1$. Hence, $\|\alpha x + (1-\alpha)y\|_F \le 1$. This finishes the proof.

Necessity: if $\Phi$ is not a convex function, then there exists $u > v$, such that $\Phi((u+v)/2) > (1/2)(\Phi(u) + \Phi(v))$. Let us choose $G_1, G_2 \subset G$ such that $m(G_1) = m(G_2)$ and $\Phi(u)m(G_1) + \Phi(v)m(G_2) \le 1$. Take any $c > 0$ such that

$$\Phi(u)m(G_1) + \Phi(v)m(G_2) + \Phi(c)m(G_3) = 1. \tag{27}$$

Define

$$x = u\chi_{G_1} + v\chi_{G_2} + c\chi_{G_3}, y = v\chi_{G_1} + u\chi_{G_2} + c\chi_{G_3}, z = \frac{x+y}{2}. \tag{28}$$

We obtain

$$\begin{aligned} I_{\Phi}(z) &= \Phi\left(\frac{u+v}{2}\right)m(G_1) + \Phi\left(\frac{u+v}{2}\right)m(G_2) + \Phi(c)m(G_3) \\ &> \frac{1}{2}(\Phi(u) + \Phi(v))m(G_1) + \frac{1}{2}(\Phi(v) + \Phi(u))m(G_2) \\ &\quad + \Phi(c)m(G_3) = \frac{1}{2}(I_{\Phi}(x) + I_{\Phi}(y)) = 1. \end{aligned} \tag{29}$$

Consequently, $\|x\|_F > 1$ is a contradiction.

*Example 13. Put*

$$\Phi(u) = \begin{cases} u, & |u| \le 1, \\ 1, & 1 \le u < 2, \\ 2, & 2 \le u \le 3, \\ 3u, & u > 3. \end{cases} \tag{30}$$

Then, $\Phi$ satisfies the $\Delta_2$-condition and $\Phi$ is right continuous. Define

$$x(t) = \begin{cases} 1, & x \in [0,1], \\ 0, & x \in (1,2], \end{cases} \qquad y(t) = \begin{cases} 3, & x \in \left[0, \frac{1}{2}\right], \\ 0, & x \in \left(\frac{1}{2}, 2\right]. \end{cases} \tag{31}$$

Then, $y \ne x$ and $I_{\Phi}(x) = \Phi(1)m([0,1]) = 1$ and $I_{\Phi}(y) = \Phi(3)m([0,1/2]) = 1$. Moreover, $\|y\|_F = \|x\|_F = 1$. Now let us define

$$z(t) = \frac{1}{3}x + \frac{2}{3}y = \begin{cases} \frac{7}{3}, & x \in \left[0, \frac{1}{2}\right], \\ \frac{1}{3}, & x \in \left(\frac{1}{2}, 1\right], \\ 0, & x \in (1,2]. \end{cases} \tag{32}$$

Then, $I_{\Phi}(z) = \Phi(7/3)m([0,1/2]) + \Phi(1/3)m((1/2,1]) = 2 \cdot (1/2) + (1/3)\cdot(1/2) = 1 + (1/6) > 1$. This means that $\|z\|_F > 1$.

*Remark 13. This example shows that the norm of an element in a line segment $[x,y]$ consisting of two elements x and y in a unit sphere may be greater than 1. This is essentially different from Banach space X. For Banach space X ,for any $x, y \in S(X)$ and for any $\lambda \in [0,1]$ ,we get that $[x,y] \subset B(X) = \{x \in X : \|x\|_F \le 1\}$.Using* Theorem 12, *if$\Phi$is a convex function, we conclude that if$\|x\|_F \le 1$, $\|y\|_F \le 1$, then$\|\lambda x + (1-\lambda)y\|_F \le 1$.*

**Theorem 14.** *The Orlicz space $L^{\Phi}(\mu)$ equipped with the Mazur-Orlicz F -norm $\|\cdot\|_F$ is strictly monotone if and only if $\Phi$ is strictly increasing on $R^+$ and $\Phi$ satisfies the $\Delta_2$ -condition and $\Phi$ is right continuous.*

*Proof.* Sufficiency: let $0 \le y \le x$, $y \ne x$, then there exists $e_0 \subset G$ and $m(e_0) > 0$ such that $0 \le y(t) < x(t) (t \in e_0)$. Assuming for the contrary that $\|y\|_F = \|x\|_F$, we have by Theorem 11,

$$\begin{aligned} \|y\|_F &= \int_{G\setminus e_0} \Phi\left(\frac{y(t)}{\|y\|_F}\right)dt + \int_{e_0} \Phi\left(\frac{y(t)}{\|y\|_F}\right) \\ &\quad \cdot dt < \int_{G\setminus e_0} \Phi\left(\frac{x(t)}{\|y\|_F}\right)dt + \int_{e_0} \Phi\left(\frac{x(t)}{\|y\|_F}\right) \\ &\quad \cdot dt = \int_{G\setminus e_0} \Phi\left(\frac{x(t)}{\|x\|_F}\right)dt + \int_{e_0} \Phi\left(\frac{x(t)}{\|x\|_F}\right)dt = \|x\|_F, \end{aligned} \tag{33}$$

Necessity: if $\Phi$ is not right continuous on $R$, then there exists a point $u_0 \in (0, +\infty)$ such that $\Phi(u_0) < \lim_{u \to u_0 + 0} \Phi(u) = b$; that is,for any $\lambda < 1$, we get $\Phi(u_0/\lambda) \ge b$. Take any $G_0 \subset G$ with $\Phi(u_0)m(G_0) < 1$. Let us take $c > 0$ large enough. Since $\lim_{u\to\infty} \Phi(u) = +\infty$, we obtain

$$\Phi(u_0)m(G_0) + \Phi(c)m(GG_0) > 1. \tag{34}$$

Since $G$ is nonatomic, we can find $G_1 \subset G \setminus (G_0 \cup G_1)$ such that

$$\Phi(u_0)m(G_0) + \Phi(c)m(G_1) = 1 - \frac{b - \Phi(u_0)}{2}m(G_0). \tag{35}$$

Moreover, take $G_2 \subset G \setminus (G_0 \cup G_1)$, where $d$ is a fixed positive number such that

$$\Phi(u_0)m(G_0) + \Phi(c)m(G_1) + \Phi(d)m(G_2) = 1. \quad (36)$$

Let us define

$$x = u_0\chi_{G_0} + c\chi_{G_1}, y = u_0\chi_{G_0} + c\chi_{G_1} + d\chi_{G_2}. \quad (37)$$

Then, $I_\Phi(x) = \Phi(u_0)m(G_0) + \Phi(c)m(G_1) < 1$, whence we obtain that $x \in L^\Phi(\mu)$ and $\|x\|_F \le 1$. On the other hand, we have that for any $\lambda > 1$,

$$\begin{aligned} I_\Phi\left(\frac{x}{\lambda}\right) &= \Phi\left(\frac{u_0}{\lambda}\right)m(G_0) + \Phi\left(\frac{c}{\lambda}\right)m(G_1) > bm(G_0) \\ &+ \Phi(c)m(G_1) = bm(G_0) + \left(1 - \frac{b-\Phi(u_0)}{2}\right)m(G_0) \\ &- \Phi(u_0)m(G_0) = 1 + \frac{b-\Phi(u_0)}{2}m(G_0) > 1, \end{aligned} \quad (38)$$

which gives that $\|x\|_F \ge 1$ yields $\|x\|_F = 1$. By (36), we obtain $\|x\|_F = \|y\|_F = 1$. Since $0 \le x \le y$ and $x \ne y$, this means that the space $(L^\Phi(\mu), \|\cdot\|_F)$ is not strictly monotone.

We will prove the necessity of the $\Delta_2$-condition. Assume first that $\Phi$ does not satisfy the $\Delta_2$-condition. This yields that there exists a strictly increasing sequence $(u_n)_{n=1}^\infty$ of positive numbers such that $\Phi(u_1)m(G) \ge 1$ and $\Phi((1+(1/n))u_n) > 2^n\Phi(u_n)$. Let us choose first $G_1 \subset G$ such that $\Phi(u_1)m(G_1) = 1/2$. We have $\Phi(u_2)m(GG_1) \ge \Phi(u_1)m(G_1) \ge 1/2$. Since $G$ is nonatomic, we can find $G_2 \subset G \setminus G_1$ such that $\Phi(u_2)m(G_2) = 1/4$. Continuing this procedure by induction, we can find a sequence $\{G_n\}_{n=1}^\infty$ of measurable sets such that $G_n \subset G \setminus \cup_{k=1}^{n-1} G_n$ and $\Phi(u_n)m(G_n) = 2^{-n}$, for any $n \in N$, $n \ge 2$. Let us define

$$y = \sum_{n=2}^\infty u_n\chi_{G_n}, x = \sum_{n=1}^\infty u_n\chi_{G_n}. \quad (39)$$

Then, $0 \le y \le x$, $y \ne x$ and $I_\Phi(x) = \sum_{n=1}^\infty \Phi(u_n)m(G_n) = \sum_{n=1}^\infty 2^{-n} = 1$, whence we obtain that $x, y \in L^\Phi(\mu)$ and $\|x\|_F \le 1$. On the other hand, we have that for any $\lambda \in (0,1)$, there exists $m \in N$, $m \ge 2$, such that $(1/\lambda) > 1 + (1/n)$ for any $n \ge m$, whence

$$\begin{aligned} I_\Phi\left(\frac{y}{\lambda}\right) &\ge \sum_{n=m}^\infty \Phi\left(\frac{u_n}{\lambda}\right)m(G_n) \ge \sum_{n=m}^\infty \Phi\left(\left(1+\frac{1}{n}\right)u_n\right)m(G_n) \\ &> \sum_{n=m}^\infty 2^n\Phi(u_n)m(G_n) = \sum_{n=m}^\infty 1 = \infty, \end{aligned} \quad (40)$$

which gives that $\|y\|_F \ge \lambda$ and by the arbitrariness of $\lambda \in (0,1)$, we get $\|y\|_F \ge 1$. The inequalities $0 \le y \le x$, $\|x\|_F \le 1$, and $\|y\|_F \ge 1$ yield $\|y\|_F = \|x\|_F$. This equality shows that space $(L^\Phi(\mu), \|\cdot\|_F)$ is not strictly monotone.

Now, we will prove the necessity of strict monotonicity of $\Phi$ on $R^+$. Assume for the contrary that $\Phi$ is constant on some interval $[a,b] \subset R^+$, where $0 < a < b < \infty$, so we have $\Phi(a) > 0$. Take $A_1 \in \Sigma$ such that $0 < \Phi(a)m(A_1) < 1$, and find a positive number $c$ such that $0 < \Phi(a)m(A_1) + \Phi(c)m(G \setminus A_1) > 1$; using the measure space $\{G, \Sigma, \mu\}$ which is nonatomic, there exists $A_2 \subset GA_1$ such that

$$\Phi(a)m(A_1) + \Phi(c)m(A_2) = 1. \quad (41)$$

Define

$$x(t) = a\chi_{A_1} + c\chi_{A_2}, y(t) = b\chi_{A_1} + c\chi_{A_2}. \quad (42)$$

We obviously have $0 \le x \le y$, $y \ne x$, $I_\Phi(y) = I_\Phi(x) = 1$ which yields $\|x\|_F = \|y\|_F = 1$; this means that $\|\cdot\|_F$ is not strictly monotone.

**Theorem 15.** *$L^\Phi(\mu)$ is lower locally uniformly monotone if and only if $\Phi$ is strictly monotone and a right continuous function and $\Phi$ satisfies the$\Delta_2$ -condition.*

*Proof.* In order to prove the sufficiency of the theorem, assume that $0 \le x_n \le x \in L^\Phi(\mu) \setminus \{0\}$ and $\|x_n\|_F \longrightarrow \|x_n\|_F$ as $n \longrightarrow \infty$. We need to prove that $\|x_n - x\|_F \longrightarrow 0$ as $n \longrightarrow \infty$.

First, we will prove that $x_n \longrightarrow x$ in measure. Assuming that $x_n \nrightarrow x$ in measure, without loss of generality, then there exists $\varepsilon_0 > 0$, $\delta_0 > 0$ such that $m(\{t \in G : x(t) - x_n(t) \ge \varepsilon_0\}) \ge \delta_0$. Define

$$G_n = \{t \in G : x(t) - x_n(t) \ge \varepsilon_0\} \quad (43)$$

and $z_n(t) = x(t)\chi_{G\setminus G_n} + (x(t) - \varepsilon_0)\chi_{G_n}$ for each $n \in N$. It is clear that $0 \le x_n(t) \le z_n(t) \le x(t)$ hold. Using condition $\Phi \in \Delta_2$, we know that $\int_G \Phi(x(t))dt < \infty$ and then, there exists $D > 0$ such that

$$m(\{t \in G : |x(t)| > D\}) < \frac{\delta_0}{3}. \quad (44)$$

Therefore, by conditions (43) and (44), we get that

$$m(\{t \in G : x(t) \le D, z_n(t) \le D, x(t) - z_n(t) \ge \varepsilon_0\}) \ge \frac{\delta_0}{3}. \quad (45)$$

Next, we will prove that there exists $\delta_1 > 0$ such that

$$\Phi(u + \varepsilon_0) \ge \Phi(u) + \delta_1 0 \le u \le D. \quad (46)$$

It is obvious that $\Phi(u + \varepsilon_0) > \Phi(u)$, since $\Phi$ is strictly increasing. If (46) does not hold, then $\exists u_n \in [0, D]$, such that

$$\Phi(u_n + \varepsilon_0) \le \Phi(u_n) + \frac{1}{n}. \quad (47)$$

By the compactness theorem, assume that $u_n \longrightarrow u_0$. If $u_n \searrow u_0$, since $\Phi$ is right continuous, we get $\Phi(u_0+\varepsilon_0) \le \Phi(u_0)$, a contradiction. If $u_n \nearrow u_0$, since $\lim_{u_n \to u_0} \Phi(u_n) \le \Phi(u_0)$, we have

$$\lim_{n\to\infty} \Phi(u_n) = b \le \Phi(u_0). \quad (48)$$

Moreover, $u_n+\varepsilon_0 \nearrow u_0+\varepsilon_0$; we have known that there exists $n_1 \in N, n > n_1$ such that $\Phi(u_n+\varepsilon_0) > \Phi(u_0)$, which gives $\lim_{n\to\infty} \Phi(u_n+\varepsilon_0) > \Phi(u_0)$. This shows that

$$\Phi(u_0) < \lim_{n\to\infty} \Phi(u_n+\varepsilon_0) \le \lim_{n\to\infty} \Phi(u_n) = b \le \Phi(u_0), \quad (49)$$

a contradiction.

Put $G_n(D) = \{t \in G : x_n(t)/\|x_n\|_F \ge D\}$ for each $n \in N$. From condition (44) there exists a $D > 0$ such that $m(G_n(D)) \le (\delta_n/2)$. Put $G_{n_1} = G_n \setminus G_n(D)$ for each $n \in N$. Then,

$$m(G_{n_1}) \ge m(G_n) - m(G_n(D)) \ge \frac{\delta_0}{2}. \quad (50)$$

Therefore,

$$\frac{x(t)}{\|x_n\|_F} - \frac{x_n(t)}{\|x_n\|_F} \ge \frac{\varepsilon_0}{\|x_n\|_F} \ge \frac{\varepsilon_0}{\|x\|_F} \left(t \in G_{n_1}\right). \quad (51)$$

Without loss of generality, we may assume that $(\|x\|_F/2) \le \|x_n\|_F$ for each $n \in N$. Put $\varepsilon_1 = 2\varepsilon_0/\|x\|_F$. Applying condition (46), there exists a $\delta_1 \ge 0$ such that $\Phi(u+\varepsilon_1) \ge \Phi(u) + \delta_1$ when $u \in [0, D]$.

Using $0 \le x_n(t) \le x(t)$ and $\|x_n\|_F \nearrow \|x\|_F$ as $n \longrightarrow \infty$, we have $\Phi(x(t)/\|x_n\|_F) \longrightarrow \Phi(x(t)/\|x_n\|_F)$ thanks to $\Phi$ that is right continuous. Then, there exists a constant $\alpha > 0$ such that $\Phi(x(t)/\|x_n\|_F) \le \Phi(\alpha x(t))$ for all $n \in N$ and $\mu - a.e.t \in \Omega$. Since $\Phi \in \Delta_2$, we have $\int_G \Phi(\alpha x(t))dt < \infty$. By the Lebesgue dominated convergence theorem, we obtain

$$\int_G \Phi\left(\frac{x(t)}{\|x_n\|_F}\right) dt \longrightarrow \int_G \Phi\left(\frac{x(t)}{\|x\|_F}\right) dt = \|x\|_F. \quad (52)$$

Consequently,

$$\begin{aligned}
\|x\|_F &\longleftarrow \int_G \Phi\left(\frac{x(t)}{\|x_n\|_F}\right) dt \ge \int_{G_{n_1}} \Phi\left(\frac{x_n(t)+\varepsilon_0}{\|x_n\|_F}\right) \\
&\cdot dt + \int_{G\setminus G_{n_1}} \Phi\left(\frac{x(t)}{\|x_n\|_F}\right) dt \ge \int_{G_{n_1}} \Phi\left(\frac{x_n(t)}{\|x_n\|_F} + \frac{\varepsilon_0}{\|x\|_F}\right) \\
&\cdot dt + \int_{G\setminus G_{n_1}} \Phi\left(\frac{x(t)}{\|x_n\|_F}\right) dt \ge \int_{G_{n_1}} \Phi\left(\frac{x_n(t)}{\|x_n\|_F}\right) \\
&\cdot dt + \int_{G_{n_1}} \delta_1 dt + \int_{G\setminus G_{n_1}} \Phi\left(\frac{x(t)}{\|x_n\|_F}\right) \\
&\cdot dt = \int_{G_{n_1}} \Phi\left(\frac{x_n(t)}{\|x_n\|_F}\right) dt + \frac{\delta_0\delta_1}{2} + \int_{G\setminus G_{n_1}} \Phi\left(\frac{x(t)}{\|x_n\|_F}\right) \\
&\cdot dt \ge \int_{G_{n_1}} \Phi\left(\frac{x_n(t)}{\|x_n\|_F}\right) dt + \frac{\delta_0\delta_1}{2} + \int_{G\setminus G_{n_1}} \Phi\left(\frac{x_n(t)}{\|x_n\|_F}\right) \\
&\cdot dt = \int_G \Phi\left(\frac{x_n(t)}{\|x_n\|_F}\right) dt + \frac{\delta_0\delta_1}{2} = \|x_n\|_F + \frac{\delta_0\delta_1}{2},
\end{aligned} \quad (53)$$

which contradicts with the assumption $\lim_{n\to\infty} \|x_n\|_F = \|x\|_F$. Therefore, $x_n$ is convergent to $x$ in measure.

Hence, $x_n/\|x_n\|_F \stackrel{\mu}{\longrightarrow} x/\|x\|_F$. By the fact that there is absolute continuity of integral, for any $\varepsilon > 0$, there exists $\delta_2 > 0$ such that

$$\int_{E_0} \Phi\left(\frac{x(t)}{\|x\|_F}\right) dt < \varepsilon, \quad (54)$$

for every measureable set $E \subset G$ with $m(E) < \delta_2$.

Consequently, this implies $\lim_{n\to\infty} \int_{E_0} \Phi(x_n(t)/\|x_n\|_F)dt = \int_{E_0} \Phi(x(t)/\|x\|_F)dt$. By the Egorov theorem, for the above $\delta_2 > 0$, there exists $E_0 \subset G$ such that $m(E_0) < \delta_2$ and

$$\frac{x_n(t)}{\|x_n\|_F} \longrightarrow \frac{x(t)}{\|x\|_F} \text{uniformly in t} \in G \setminus E_0. \quad (55)$$

Since $\lim_{u\to 0+0} \Phi(u) = 0$, we get

$$\lim_{n\to\infty} \int_{G\setminus E_0} \Phi\left(\frac{x(t)}{\|x\|_F} - \frac{x_n(t)}{\|x_n\|_F}\right) dt = 0. \quad (56)$$

Consequently,

$$\begin{aligned}
\int_G \Phi\left(\frac{x(t)}{\|x\|_F} - \frac{x_n(t)}{\|x_n\|_F}\right) dt &= \int_{G\setminus E_0} \Phi\left(\frac{x(t)}{\|x\|_F} - \frac{x_n(t)}{\|x_n\|_F}\right) \\
&\cdot dt + \int_{E_0} \Phi\left(\frac{x(t)}{\|x\|_F} - \frac{x_n(t)}{\|x_n\|_F}\right) dt \le \int_{G\setminus E_0} \Phi\left(\frac{x(t)}{\|x\|_F} - \frac{x_n(t)}{\|x_n\|_F}\right) \\
&\cdot dt + \int_{E_0} \Phi\left(\frac{x(t)}{\|x\|_F}\right) dt \le \varepsilon + \varepsilon = 2\varepsilon.
\end{aligned} \quad (57)$$

By the arbitrariness of $\varepsilon$, we obtain that $\|x - x_n\|_E \longrightarrow 0$ as $n \longrightarrow \infty$ holds.

In virtue of Theorem 14, we have a necessity that is obvious.

**Theorem 16.** *For any $x \in l_\Phi \setminus \{0\}$, $I_\Phi(x/\|x\|_F) = \|x\|_F$ if and only if*

*(i) $\Phi \in \delta_2$*

*(ii) $\Phi$ is right continuous*

*Proof.* Necessity:

(i) If $\Phi \notin \delta_2$, then there exists an element $x \in l_\Phi$ such that $I_\Phi(x) < 1$ and $I_\Phi(x/\lambda) = \infty$ for any $0 < \lambda < 1$. Hence, $\|x\|_F = 1$ and $I_\Phi(x/\|x\|_F) < \|x\|_F$

(ii) If there exists a point $u_0 \in (0,+\infty)$ such that $\Phi(u_0) < \lim_{u \to u_0+0} \Phi(u) = b$, that is, for any $0 < \lambda < 1$, we get $\Phi(u_0/\lambda) \geq b$

Put $x = (bu_0, 0, \cdots)$. Then, $I_\Phi(x/b) = \Phi(u_0) < b$ and for any $0 < \lambda < 1$, we have $I_\Phi(x/\lambda b) = \Phi(bu_0/\lambda b) \geq b$. Hence, $\|x\|_F = b$ and $I_\Phi(x/\|x\|_F) < \|x\|_F$.

The sufficiency of the proof is similar as Theorem 11.

**Theorem 17.** *The following statements are equivalent:*

*(1) The Orlicz sequence space $l_\Phi$ equipped with the Mazur-Orlicz F-norm is lower locally uniformly monotone*

*(2) The Orlicz sequence space $l_\Phi$ equipped with the Mazur-Orlicz F-norm is strictly monotone*

*(3) (a) $\Phi$ is strictly increasing on $R^+$*

*(b) $\Phi \in \delta_2$*
*(c) $\Phi$ is right continuous*

*Proof.* The implication (1)⟹(2) is clear.

(2)⟹(3) The proof is similar as the proof of Theorem 15. We only give the proof that $\Phi$ is right continuous, if there exists a point $u_0 \in (0,+\infty)$ such that $\Phi(u_0) < \lim_{u \to u_0+0} \Phi(u) = b$. That is, for any $0 < \lambda < 1$, we get $\Phi(u_0/\lambda) \geq b$. Put $x = (bu_0, 0, \cdots)$. Then, $I_\Phi(x/\|x\|_F) < \|x\|_F$, Since $\lim_{u \to 0+0} \Phi(u) = 0$, there exists a $c > 0$ such that $\Phi(bu_0/b) + \Phi(c/b) \leq \|x\|_F = b$. Put $y = (bu_0, c, 0, \cdots)$. Then, $b = \|x\|_F \leq \|y\|_F \leq b$, $0 \leq x \leq y$ and $x \neq y$.

(3)⟹(1) The idea is the same as the proof of Theorem 15, and the proof is simpler than Theorem 15. We only need to replace convergence by the measure of convergence by coordinate.

## Data Availability

No data were used to support this study.

## Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

## Acknowledgments

This work is supported by the National Natural Science Foundation of China under Grant no. 11871181.

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