Hindawi
Journal of Function Spaces
Volume 2020, Article ID 8598323, 8 pages
https://doi.org/10.1155/2020/8598323

Research Article

Variational Method to p-Laplacian Fractional Dirichlet Problem with Instantaneous and Noninstantaneous Impulses

Yiru Chen, Haibo Gu, and Lina Ma

School of Mathematics Sciences, Xinjiang Normal University, Urumqi, Xinjiang 830017, China

Correspondence should be addressed to Haibo Gu; hbgu_math@163.com

Received 10 April 2020; Accepted 1 June 2020; Published 21 July 2020

Academic Editor: Xinguang Zhang

In this paper, a research has been done about the existence of solutions to the Dirichlet boundary value problem for p-Laplacian fractional differential equations which include instantaneous and noninstantaneous impulses. Based on the critical point principle and variational method, we provide the equivalence between the classical and weak solutions of the problem, and the existence results of classical solution for our equations are established. Finally, an example is given to illustrate the major result.

1. Introduction

Fractional calculus involves arbitrary order derivatives and integration, so it plays a very important role in various fields such as physical engineering, medical image processing, mathematics, chemical engineering, and electricity. For this reason, many scholars did research on the theory of fractional differential equations continuously and have made enormous achievements; readers who are interested in these kinds of researches can refer to relevant literature (see [1–6] and the references therein). Of course, in the science field, an impulsive phenomenon has also been spread widely when dealing with practical problems, and it has become a very effective tool for describing changes in sudden discontinuous jumps. In addition, from the perspective of the duration of the change, impulses can be divided into instantaneous impulses and noninstantaneous impulses. The difference between them is that the duration of the instantaneous impulse continuous change is relatively short compared to the duration of the entire process, and the noninstantaneous impulse change is to keep moving from any fixed point and at a certain time interval. In [7], Agarwal et al. provide a more detailed introduction to these two impulsive differential equations. We also noticed that some experts combine the theory of fractional calculus, of noncompactness, with Sadovskii's fixed-point theorem and even more excellent methods have obtained more and better properties for fractional order equations with impulsive conditions, which greatly promotes its development (see, for example, [8–15], and the references therein).

For a long time, many scholars have conducted in-depth research on instantaneous impulsive differential equations. By using fixed-point theorems, critical point theorems, and variational methods, they obtained the existence of solutions (see [16–18]). But in many cases, instantaneous impulses cannot describe the development of certain dynamics. In 2013, Hernández and O'Regan first proposed the concept of noninstantaneous impulsive differential equations (see [19]). Since then, the existence of solutions to noninstantaneous impulsive problems has been gradually expanded by using some methods such as fixed-point theory and analytical semigroup theory, but the variational structure of general noninstantaneous impulsive differential equations has not received widespread attention. Among them, the existence of solutions for second-order differential equations with instantaneous and noninstantaneous impulses was investigated. Tian and Zhang obtained the existence results through the principle of variation (see [20]). In addition, it is worth noting that by using the critical point theory and variational method, Zhao et al. proved the existence and multiplicity of nontrivial solutions to the problem of nonlinear nontransient impulsive differential equations (see [21]). At the same time, some scholars have obtained the existence results of solutions for noninstantaneous impulsive differential equations by variational methods (see [22–24]).

In [25], Zhang and Liu generalized linear fractional differential equations to nonlinearities on the interval $t \in (s_i, t_i + 1]$ and removed the restriction ${}_tD_T^{\alpha-1}({}_0^CD_t^\alpha u(0)) = \text{constant}$. By using a variational method, they considered the following Dirichlet problems with instantaneous and noninstantaneous impulse differential equations:

$$\begin{cases} {}_tD_T^\alpha \left({}_0^CD_t^\alpha\right) u(t) = f_i(t, u(t)), & t \in (s_i, t_i + 1], i = 0, 1, 2, \cdots, n, \\ \Delta_t D_T^{\alpha-1}\left({}_0^CD_t^\alpha u\right)(t_i) = I_i(u(t_i)), & i = 1, 2, \cdots, n, \\ {}_tD_T^{\alpha-1}\left({}_0^CD_t^\alpha u\right)(t) = {}_tD_T^{\alpha-1}\left({}_0^CD_t^\alpha u\right)(t_i^+), & t \in (t_i, s_i], i = 1, 2, \cdots, n, \\ {}_tD_T^{\alpha-1}\left({}_0^CD_t^\alpha u\right)(s_i^-) = {}_tD_T^{\alpha-1}\left({}_0^CD_t^\alpha u\right)(s_i^+), & i = 1, 2, \cdots, n, \\ u(0) = u(T) = 0. \end{cases} \tag{1}$$

In [26], Zhou et al. discussed the existence of solutions for fractional differential equations of p-Laplacian with instantaneous and noninstantaneous impulses. The innovation was that when $p = 2$, problem (2) can be regarded as problem (1), and when $\alpha = 1$, it can be simplified to a more general integer order case. Finally, they got the classical solution and prove that the weak solution was equivalent to the classical solution:

$$\begin{cases} {}_tD_T^\alpha \phi_P\left({}_0^CD_t^\alpha u(t)\right) + g(t)|u|^{P-2}u(t) = f_i(t, u(t)), & t \in (s_i, t_{i+1}], i = 0, 1, 2, \cdots, n, \\ \Delta_t D_T^{\alpha-1}\phi_P\left({}_0^CD_t^\alpha u\right)(t_i) = I_i(u(t_i)), & i = 1, 2, \cdots, n, \\ {}_tD_T^{\alpha-1}\phi_P\left({}_0^CD_t^\alpha u\right)(t) = {}_tD_T^{\alpha-1}\phi_P\left({}_0^CD_t^\alpha u\right)(t_i^+), & t \in (t_i, s_i], i = 1, 2, \cdots, n, \\ {}_tD_T^{\alpha-1}\phi_P\left({}_0^CD_t^\alpha u\right)(s_i^-) = {}_tD_T^{\alpha-1}\phi_P\left({}_0^CD_t^\alpha u\right)(s_i^+), & i = 1, 2, \cdots, n, \\ u(0) = u(T) = 0. \end{cases} \tag{2}$$

Motivated by the above-mentioned work, the paper focuses on the existence of solutions for the following p-Laplacian fractional differential equations with instantaneous and noninstantaneous impulses, and if $D_xF_i(t, u(t) - u(t_i + 1)) = f_i(t, u(t))$, the following problem reduces to (2):

$$\begin{cases} {}_tD_T^\alpha \phi_P\left({}_0^CD_t^\alpha u(t)\right) + \lambda(t)|u|^{p-2}u(t) = D_xF_i(t, u(t) - u(t_i + 1)), \\ t \in (s_i, t_{i+1}], \quad i = 0, 1, 2, \cdots, N, \\ \Delta_t D_T^{\alpha-1}\phi_P\left({}_0^CD_t^\alpha u\right)(t_i) = I_i(u(t_i)), & i = 1, 2, \cdots, N, \\ {}_tD_T^{\alpha-1}\phi_P\left({}_0^CD_t^\alpha u\right)(t) = {}_tD_T^{\alpha-1}\phi_P\left({}_0^CD_t^\alpha u\right)(t_i^+), & t \in (t_i, s_i], i = 1, 2, \cdots, N, \\ {}_tD_T^{\alpha-1}\phi_P\left({}_0^CD_t^\alpha u\right)(s_i^-) = {}_tD_T^{\alpha-1}\phi_P\left({}_0^CD_t^\alpha u\right)(s_i^+), & i = 1, 2, \cdots, N, \\ u(0) = u(T) = 0, \end{cases} \tag{3}$$

where $\alpha \in (1/p, 1], p \geq 2, {}_tD_T^\alpha$, and ${}_0^CD_T^\alpha$ denote the right Riemann-Liouville fractional derivatives and left Caputo fractional derivatives. $0 = s_0 < t_0 < s_1 < t_1 < \cdots < t_N = T, I_i \in C(R, R)$, and there exists $i \in \{1, 2, \cdots, N\}$ such that $I(u(t_i)) \neq 0. f_i \in C((s_i, t_{i+1}] \times R, R), \phi_p : R \longrightarrow R$ is the p-Laplacian function defined as $\phi_p(s) = |s|^{p-2}s(s \neq 0), \phi_p(0) = 0. \lambda \in L^\infty [0, T]$, $\text{essinf}_{t\in(0,T)}\lambda(t) > -(\Gamma(\alpha + 1)/T^\alpha)^p$. The nonlinear functions $D_xF_i(t, x)$ are the derivatives of $F_i(t, x)$, for every

$i = 0, 1, 2, \cdots, N$. To the best of our knowledge, the instantaneous impulses start abruptly at point t_i and the noninstantaneous impulses continue during the intervals $(t_i, s_i]$, and

$$\begin{aligned}
&\Delta_t D_T^{\alpha-1}\phi_p\left({}_0^C D_t^\alpha u\right)(t_i) = {}_t D_T^{\alpha-1}\phi_p\left({}_0^C D_t^\alpha u\right)(t_i^+) - {}_t D_T^{\alpha-1}\phi_p\left({}_0^C D_t^\alpha u\right)(t_i^-),\\
&{}_t D_T^{\alpha-1}\phi_p\left({}_0^C D_t^\alpha u\right)(t_i^\pm) = \lim_{t\to t_i^\pm}\left({}_t D_T^{\alpha-1}\phi_p\left({}_0^C D_t^\alpha u\right)(t)\right),\\
&{}_t D_T^{\alpha-1}\phi_p\left({}_0^C D_t^\alpha u\right)(s_i^\pm) = \lim_{t\to s_i^\pm}\left({}_t D_T^{\alpha-1}\phi_p\left({}_0^C D_t^\alpha u\right)(t)\right).
\end{aligned} \tag{4}$$

The rest of the paper is organized as follows. In Section 2, some basic knowledge and lemmas used in the latter are presented. In Section 3, we first give the equivalent form of solution of problem (3), and secondly, we establish the equivalence of the classical solution and the weak solution by using the critical point principle for the variational structure of problem (3) and finally prove the existence of solution. An illustrative example is given to show the practical usefulness of the analytical results in Section 4.

2. Preliminaries

In this section, we introduce some important definitions, lemmas, and theorems that are important to use later. For the definitions of the left and right fractional integrals and derivatives, we can refer to References [7, 23].

Definition 1 (see [26]). Let $\alpha \in (1/p, 1], p \in [2,\infty)$, the fractional space

$$E_0^{\alpha,p} = \left\{u : [0, T] \longrightarrow R\big|_U, {}_0^C D_t^\alpha u \in L^p([0, T], R), u(0) = u(T)\right\} \tag{5}$$

is defined by the closure of $C_0^\infty([0, T], R)$ with the norm

$$\|u\|_{\alpha,p} = \left(\int_0^T \lambda(t)|u(t)|^p dt + \int_0^T |{}_0^c D_t^\alpha u(t)|^p dt\right)^{1/p}. \tag{6}$$

Definition 2 (see [26]). Let $0 < \alpha \le 1$, the fractional derivative space $E_0^{\alpha,p}$ is a reflexive as well as separable Banach space.

Remark 3 (see [25]). Let $p = 2$, we define $E_0^\alpha = E_0^{\alpha,2}$. It is obvious that the fractional derivative space E_0^α is the space of functions $u \in L^2([0, T], R)$ having an α-order Caputo fractional derivative ${}_0^c D_t^\alpha u \in L^2([0, T], R)$ and $u(0) = u(T)$.

Proposition 4 (see [26]). *Let $0 < \alpha \le 1$ and $u \in AC([a, b])$, then for $t \in [a, b]$,*

$$\begin{aligned}
{}_a^c D_t^\alpha u(t) &= {}_a D_t^\alpha u(t) - \frac{u(a)}{\Gamma(1-\alpha)}(t-a)^{-\alpha},\\
{}_t^c D_b^\alpha u(t) &= {}_t D_b^\alpha u(t) - \frac{u(b)}{\Gamma(1-\alpha)}(b-t)^{-\alpha}.
\end{aligned} \tag{7}$$

Proposition 5 (see [26]). *Let $\alpha \in (0, 1]$ and $p \in (1,+\infty)$, for any $u \in E_0^{\alpha,p}$, we have*

$$\|u\|_{L_p} \le \frac{T^\alpha}{\Gamma(\alpha+1)}\left(\int_0^T \left|{}_0^C D_t^\alpha u(t)\right|^p dt\right)^{1/p}. \tag{8}$$

Moreover, if $\alpha > 1/p$ and $1/p + 1/q = 1$, then

$$\|u\|_\infty \le \Omega\left(\int_0^T \left|{}_0^C D_t^\alpha u(t)\right|^p dt\right)^{1/q}, \tag{9}$$

where $\Omega = T^{\alpha-(1/p)}/\Gamma(\alpha)[(\alpha-1)q+1]^{1/q}$.

Proposition 6 (see [26]). *Let $\alpha \in (1/p, 1)$, the space $E_0^{\alpha,p}$ is compactly embedded in $C([0, T], R)$.*

Proposition 7 (see [26]). *Let $\alpha \in (0, 1]$, and $u, v \in L^p(a, b)$:*

$$\int_a^b ({}_t D_b^\alpha u(t))v(t)dt = \int_a^b ({}_a D_t^\alpha v(t))u(t)dt. \tag{10}$$

Definition 8. A function

$$u \in \left\{u \in AC([0, T]) : \int_{s_i}^{t_{i+1}} \left(|u(t)|^p + \left|{}_0^C D_t^\alpha u(t)\right|^p\right)dt < +\infty, i = 0, 1, 2, \cdots, N\right\} \tag{11}$$

is a classical solution of problem (3) if u satisfies the conditions of problem (3) and the boundary condition $u(0) = u(T) = 0$ holds.

Lemma 9. *Let X be a reflexive Banach space and let $\varphi : X \longrightarrow (-\infty, +\infty]$ be weakly lower semicontinuous on X. If φ has a bounded minimizing sequence, then φ has a minimum on X.*

Lemma 10. *If $\varphi \longrightarrow (-\infty, +\infty]$ is coercive, then φ has a bounded minimizing sequence.*

Lemma 11 (see [26]). *For $u \in E_0^{\alpha,p}$, we define the norm*

$$\|u\|_\alpha = \left(\int_0^T |u(t)|^p dt + \int_0^T \left(\left|{}_0^C D_t^\alpha u(t)\right|\right)^p dt\right)^{1/p} \tag{12}$$

is equivalent to the norm $\|u\|_{\alpha,p}$.

3. Main Results

In this section, we further discuss the variational structure of problem (3) and define the functional φ to prove that the critical point of φ is the weak solution of problem (3). In addition, we also give the equivalence between weak solution and the classical solution of this problem. Finally, the existence of classical solution for problem (3) is given by Theorem 16.

For each $i=0,1,2,\cdots,N$, the nonlinear functions F_i satisfy the following assumptions:

(H1). $F_i(t,x)$ is measurable in t for every $x\in R$ and continuously differentiable in x for a.e. $t\in(s_i,t_{i+1}]$, and there exist functions $k_1\in C(R^+,R^+)$, $k_2\in L^1((s_i,t_{i+1}),R^+)$ such that

$$\begin{aligned}|F_i(t,x)|&\le k_1(x)k_2(t),\\ |D_xF_i(t,x)|&\le k_1(|x|)k_2(t).\end{aligned}\tag{13}$$

(H2). There exist constants $\delta\in(0,2)$, $a>0$, and the functions $b_0,b_1\in L^1((s_i,t_i),R^+)$ such that

$$|F_i(t,x)|\le a|x|^2+b_0(t)|x|^\delta+b_1(t),\quad t\in(s_i,t_{i+1}].\tag{14}$$

(H3). There exist constants $c_i,d_i>0$, $\sigma_i\in[0,1)$ $(i=1,2,\cdots,N)$ such that $|I_i(u)|\le c_i+d_i|u|^{\sigma_i}$, for any $u\in R$.

Lemma 12. *A function $u\in E_0^{\alpha,p}$, $\forall v\in E_0^{\alpha,p}$, the following form is an equivalent form of the problem (3):*

$$\begin{aligned}&\int_0^T\phi_p\left({}_0^CD_t^\alpha u(t)\right)\left({}_0^CD_t^\alpha v(t)\right)dt\\&\quad=\sum_{i=0}^N\int_{s_i}^{t_{i+1}}D_xF_i(t,u(t)-u(t_{i+1}))v(t_i)dt\\&\qquad-\sum_{i=0}^N\int_{s_i}^{t_{i+1}}\lambda(t)|u(t)|^{p-2}u(t)v(t)dt-\sum_{i=1}^N(I_iu(t_i)v(t_i)).\end{aligned}\tag{15}$$

Proof. For $v\in E_0^{\alpha,p}$, by (10) have

$$\begin{aligned}&\int_0^T\phi_p\left({}_0^CD_t^\alpha u(t)\right)\left({}_0^CD_t^\alpha v(t)\right)dt\\&\quad=\int_0^T{}_tD_T^{\alpha-1}\phi_p\left({}_0^CD_t^\alpha u(t)\right)v'(t)dt\\&\quad=\sum_{i=0}^N\int_{s_i}^{t_{i+1}}{}_tD_T^{\alpha-1}\phi_p\left({}_0^CD_t^\alpha u(t)\right)v'(t)dt\\&\qquad+\sum_{i=1}^N\int_{t_i}^{s_i}{}_tD_T^{\alpha-1}\phi_p\left({}_0^CD_t^\alpha u(t)\right)v'(t)dt\\&\quad={}_tD_T^{\alpha-1}\phi_p\left({}_0^CD_t^\alpha u(t_1^-)\right)v(t_1)\\&\qquad-\int_{s_0}^{t_1}\frac{d}{dt}\left({}_tD_T^{\alpha-1}\phi_p\left({}_0^CD_t^\alpha u(t)\right)\right)v(t)dt\\&\qquad+\sum_{i=1}^N\left[{}_tD_T^{\alpha-1}\phi_p\left({}_0^CD_t^\alpha u(t)\right)\right]v(t)\Big|_{t_i}^{s_i}\\&\qquad-\sum_{i=1}^N\int_{t_i}^{s_i}\frac{d}{dt}\left({}_tD_T^{\alpha-1}\phi_p\left({}_0^CD_t^\alpha u(t)\right)\right)v(t)dt\\&\qquad+\sum_{i=1}^{N-1}\left[{}_tD_T^{\alpha-1}\phi_p\left({}_0^CD_t^\alpha u(t)\right)\right]v(t)\Big|_{s_i}^{t_{i+1}}\\&\qquad-{}_tD_T^{\alpha-1}\phi_p\left({}_0^CD_t^\alpha u(s_N^+)\right)v(s_N)\\&\qquad-\sum_{i=0}^{N-1}\int_{s_i}^{t_{i+1}}\frac{d}{dt}\left({}_tD_T^{\alpha-1}\phi_p\left({}_0^CD_t^\alpha u(t)\right)\right)v(t)dt\\&\qquad-\int_{s_N}^{T}\frac{d}{dt}\left({}_tD_T^{\alpha-1}\phi_p\left({}_0^CD_t^\alpha u(t)\right)\right)v(t)dt\\&\quad=\int_0^T\left({}_tD_T^{\alpha}\phi_p\left({}_0^CD_t^\alpha u(t)\right)v(t)\right)dt\\&\qquad+\sum_{i=1}^N\left[{}_tD_T^{\alpha-1}\phi_p\left({}_0^CD_t^\alpha u(s_i^-)\right)-{}_tD_T^{\alpha-1}\phi_p\left({}_0^CD_t^\alpha u(s_i^+)\right)\right]v(s_i)\\&\qquad+\sum_{i=1}^N\left[{}_tD_T^{\alpha-1}\phi_p\left({}_0^CD_t^\alpha u(t_i^-)\right)-{}_tD_T^{\alpha-1}\phi_p\left({}_0^CD_t^\alpha u(t_i^+)\right)\right]v(t_i).\end{aligned}\tag{16}$$

We have

$$\begin{aligned}&\int_0^T\left({}_tD_T^{\alpha}\phi_p\left({}_0^CD_t^\alpha u(t)\right)\right)v(t)dt\\&\quad=\int_0^T\phi_p\left({}_0^CD_t^\alpha u(t)\right)\left({}_0^CD_t^\alpha v(t)\right)dt\\&\qquad+\sum_{i=1}^N(I_iu(t_i))v(t_i).\end{aligned}\tag{17}$$

As for problem (3), we have

$$\begin{aligned}&\int_0^T\left({}_tD_T^{\alpha}\phi_p\left({}_0^CD_t^\alpha u(t)\right)\right)v(t)dt\\&\quad=\sum_{i=0}^N\int_{s_i}^{t_{i+1}}\left({}_tD_T^{\alpha}\phi_p\left({}_0^CD_t^\alpha u(t)\right)\right)v(t)dt\\&\qquad+\sum_{i=1}^N\int_{t_i}^{s_i}\left({}_tD_T^{\alpha}\phi_p\left({}_0^CD_t^\alpha u(t)\right)\right)v(t)dt\\&\quad=\sum_{i=0}^N\int_{s_i}^{t_{i+1}}D_xF_i(t,u(t)-u(t_{i+1}))v(t_i)dt\\&\qquad-\sum_{i=0}^N\int_{s_i}^{t_{i+1}}\lambda(t)|u(t)|^{p-2}u(t)v(t)dt\\&\qquad-\sum_{i=1}^N\int_{t_i}^{s_i}\frac{d}{dt}\left({}_tD_T^{\alpha-1}\phi_p\left({}_0^CD_t^\alpha u(t)\right)\right)v(t)dt\\&\quad=\sum_{i=0}^N\int_{s_i}^{t_{i+1}}D_xF_i(t,u(t)-u(t_{i+1}))v(t_i)dt\\&\qquad-\sum_{i=0}^N\int_{s_i}^{t_{i+1}}\lambda(t)|u(t)|^{p-2}u(t)v(t)dt.\end{aligned}\tag{18}$$

Above all, problem (3) has the following equivalent form:

$$\begin{aligned}\int_0^T &\phi_p\left({}_0^C D_t^\alpha u(t)\right)\left({}_0^C D_t^\alpha v(t)\right)dt\\ &= \sum_{i=0}^{N}\int_{s_i}^{t_{i+1}} D_x F_i(t, u(t)-u(t_{i+1}))v(t_i)dt\\ &\quad - \sum_{i=0}^{N}\int_{s_i}^{t_{i+1}} \lambda(t)|u(t)|^{p-2}u(t)v(t)dt - \sum_{i=1}^{N}(I_i u(t_i))v(t_i).\end{aligned} \tag{19}$$

Definition 13. A function $u \in E_0^{\alpha,p}$ is a weak solution of problem (3) if (15) holds for all $v \in E_0^{\alpha,p}$.

Define the function $\varphi : E_0^{\alpha,p} \longrightarrow R$ as

$$\varphi(u) = \frac{1}{p}\|u\|^p - \sum_{i=0}^{N}\phi_i(u) + \sum_{i=1}^{N}\int_0^{u(t_i)} I_i(s)ds, \tag{20}$$

where $\phi_i(u) = \int_{s_i}^{t_{i+1}} F_i(t, u(t)-u(t_{i+1}))dt$.

From (9), we have

$$\begin{aligned}|u(t)-u(t_{i+1})| &\le 2\|u\|_\infty \le \frac{T^{\alpha-(1/p)}}{\Gamma(\alpha)[(\alpha-1)q+1]^{1/q}}\|u\|_\alpha\\ &= 2\Omega\|u\|_\alpha.\end{aligned} \tag{21}$$

Under condition (H1), for a.e. $t \in (s_i, t_{i+1}]$, we have

$$\begin{aligned}\lim_{\varepsilon\to 0}&\frac{1}{\varepsilon}[F_i(t, u(t)-u(t_{i+1})+\varepsilon v(t)) - F_i(t, u(t)-u(t_{i+1})]\\ &= D_x F_i(t, u(t)-u(t_{i+1})v(t).\end{aligned} \tag{22}$$

From (21), for all $\gamma \in (0,1)$,

$$\begin{aligned}|u(t)-u(t_{i+1})+\gamma\varepsilon v(t)| &\le 2\Omega\|u\|_\alpha + \|v\|_\infty\\ &\le 2\Omega\|u\|_\alpha + \Omega\|v\|_\alpha.\end{aligned} \tag{23}$$

According to the mean value theorem and condition (H1), for some $\gamma \in (0,1)$,

$$\begin{aligned}&\left|\frac{1}{\varepsilon}[F_i(t, u(t)-u(t_{i+1})+\varepsilon v(t)) - F_i(t, u(t)-u(t_{i+1}))]\right|\\ &\quad= |D_x F_i(t, u(t)-u(t_{i+1})+\gamma\varepsilon v(t))v(t)|\\ &\quad\le \Omega\|v\|_\alpha \max_{\omega\in[0,2\Omega\|u\|_\alpha+\Omega\|v\|_\alpha)} k_1(\omega)k_2(t).\end{aligned} \tag{24}$$

By the Lebesgue dominated convergence theorem, it is easy to show that ϕ_i has a directional derivative at each point u:

$$\begin{aligned}\left\langle \phi_i'(u), v\right\rangle &= \int_{s_i}^{t_{i+1}} D_x F_i(t, u(t)-u(t_{i+1}))v(t)dt,\\ \left|\left\langle \phi_i'(u), v\right\rangle\right| &\le \int_{s_i}^{t_{i+1}} |D_x F_i(t, u(t)-u(t_{i+1}))||v(t)|dt\\ &\le \Omega\|v\|_\alpha \max_{\omega\in[0,2\Omega\|u\|_\alpha+\Omega\|v\|_\alpha)} k_1(\omega)\int_{s_i}^{t_{i+1}} k_2(t)dt.\end{aligned} \tag{25}$$

Thus, $\phi_i \in (E_0^{\alpha,p})^*$, let $u_k \longrightarrow u$ in $E_0^{\alpha,p}$, then $\{u_k\}$ converges uniformly to u for all $t \in [0,T]$, we have

$$\begin{aligned}\|\phi_i'(u_n) - \phi_i'(u)\|_\alpha \le \Omega\|v\|_\alpha \int_{s_i}^{t_{i+1}} & D_x F_i(t, u_n(t)-u_n(t_{i+1}))\\ &- D_x F_i(t, u(t)-u(t_{i+1}))dt.\end{aligned} \tag{26}$$

Therefore, ϕ_i' is continuous from $E_0^{\alpha,p}$ into $(E_0^{\alpha,p})^*$, $\varphi \in C^1(E_0^{\alpha,p}, R)$ and

$$\begin{aligned}\left\langle \varphi'(u), v\right\rangle &= \int_0^T \phi_p\left({}_0^C D_t^\alpha u(t)\right)\left({}_0^C D_t^\alpha v(t)\right)dt + \sum_{i=1}^{N}(I_i u(t_i))v(t_i)\\ &\quad - \sum_{i=0}^{N}\int_{s_i}^{t_{i+1}} D_x F_i(t, u(t)-u(t_{i+1}))v(t_i)dt\\ &\quad + \sum_{i=0}^{N}\int_{s_i}^{t_{i+1}} \lambda(t)|u(t)|^{p-2}u(t)v(t)dt.\end{aligned} \tag{27}$$

This yields that the critical points of φ are weak solutions of problem (3).

Lemma 14. *The functional $\varphi : E_0^{\alpha,p} \longrightarrow R$ is a weak lower semicontinuous.*

Proof. Let sequence $\{u_k\}_{k=1}^{\infty}$ in $E_0^{\alpha,p}$ be weakly convergent to u in $E_0^{\alpha,p}$, then $\|u\| \le \lim_{k\to\infty} \inf \|u_k\|$.

From Proposition 6, sequence $\{u_k\}_{k=1}^{\infty}$ is convergent uniformly to u in $C([0,T], R)$; thus, we have

$$\begin{aligned}\lim_{k\to\infty}\inf \varphi(u_k) &= \left[\lim_{k\to\infty}\inf \frac{1}{q}\|u\|^p - \sum_{i=0}^{N}\int_{s_i}^{t_{i+1}} F_i\big(t, u_k(t)\right.\\ &\quad \left. - u_k(t_{i+1})\big) + \sum_{i=1}^{N}\int_0^{u_k(t_i)} I_i(s)ds\right]\\ &\ge \frac{1}{q}\|u\|^p - \sum_{i=0}^{N}\int_{s_i}^{t_{i+1}} F_i(t, u(t)-u(t_{i+1}))\\ &\quad + \sum_{i=1}^{N}\int_0^{u_k(t_i)} I_i(s)ds = \varphi(u).\end{aligned} \tag{28}$$

Lemma 15. *$u \in E_0^{\alpha,p}$ is a weak solution of the problem (3), if and only if u is a classical solution of the problem (3).*

Proof. From the definition of classical solutions, without loss of generality, if u is a classical solution of problem (3), then it is a weak solution of (3). Conversely, let $u \in E_0^{\alpha,p}$ be a weak solution of (3); thus, $u(0) = u(T) = 0$ and (15) holds. Now, we prove that u is a classical solution of the problem (3). We take a test function $v \in C_0^\infty(s_i t_{i+1}]$ satisfying $v(t) = 0$, where $t \in [0, s_i) \cup [t_{i+1}, T)$, $i = 0, 1, 2, \cdots, N$. Substituting $v(t)$ into (15) and by Proposition 7, we have

$$\int_{s_i}^{t_{i+1}} \phi_p\left({}_0^C D_t^\alpha u(t)\right)\left({}_0^C D_t^\alpha v(t)\right)dt + \int_{s_i}^{t_{i+1}} \lambda(t)|u(t)|^{p-2}u(t)v(t)dt = \int_{s_i}^{t_{i+1}} D_x F_i(t, u(t) - u(t_{i+1}))v(t_i)dt, \tag{29}$$

$$\int_{s_i}^{t_{i+1}} \phi_p\left({}_0^C D_t^\alpha u(t)\right)\left({}_0^C D_t^\alpha v(t)\right)dt + \int_{s_i}^{t_{i+1}} {}_t D_T^\alpha \phi_p\left({}_0^C D_t^\alpha u(t)\right)v(t)dt < +\infty, \tag{30}$$

that is to say,

$$\begin{aligned} &{}_t D_T^\alpha \phi_p\left({}_0^C D_t^\alpha u(t)\right) + \lambda(t)|u(t)|^{p-2}u(t) \\ &\quad = D_x F_i(t, u(t) - u(t_{i+1})), \quad t \in (s_i, t_{i+1}], i = 0, 1, \cdots, N. \end{aligned} \tag{31}$$

Since $u \in E_0^{\alpha,p} \subset C([0, T])$, we have

$$\int_{s_i}^{t_{i+1}} \left(|u(t)|^p + \left|{}_0^C D_t^\alpha u(t)\right|^p\right)dt < +\infty. \tag{32}$$

Because of $D_x F_i(t, u(t) - u(t_{i+1})) \in C((s_i, t_{i+1}] \times R, R)$, by (31), we have

$${}_t D_T^\alpha \phi_p\left({}_0^C D_t^\alpha u(t)\right) \in AC([s_i, t_{i+1}]). \tag{33}$$

Therefore,

$$\begin{aligned} &{}_t D_T^{\alpha-1} \phi_p\left({}_0^C D_t^\alpha u\right)(s_i^+) = \lim_{t \to s_i^+}\left({}_t D_T^{\alpha-1} \phi_p\left({}_0^C D_t^\alpha u\right)(t)\right), \\ &{}_t D_T^{\alpha-1} \phi_p\left({}_0^C D_t^\alpha u\right)(t_{i+1}^-) = \lim_{t \to t_{i+1}^-}\left({}_t D_T^{\alpha-1} \phi_p\left({}_0^C D_t^\alpha u\right)(t)\right). \end{aligned} \tag{34}$$

In view of (15) and (31), we have

$$\begin{aligned} &\int_0^T \phi_p\left({}_0^C D_t^\alpha u(t)\right)\left({}_0^C D_t^\alpha v(t)\right)dt + \sum_{i=1}^N I_i(u(t_i))v(t_i) \\ &\quad + \sum_{i=1}^N \int_{s_i}^{t_{i+1}} \frac{d}{dt}\left({}_t D_T^{\alpha-1} \phi_p\left({}_0^C D_t^\alpha u(t)\right)\right)v(t)dt = 0, \end{aligned} \tag{35}$$

that is,

$$\begin{aligned} &\sum_{i=1}^N {}_t D_T^{\alpha-1} \phi_p\left({}_0^C D_t^\alpha u(s_i^+)\right)v(s_i^+) - \sum_{i=0}^N {}_t D_T^{\alpha-1} \phi_p\left({}_0^C D_t^\alpha u(t_{i+1}^-)\right)v(t_{i+1}^-) \\ &\quad = \sum_{i=1}^N \int_{t_i}^{s_i} \phi_p\left({}_0^C D_t^\alpha u(t)\right)\left({}_0^C D_t^\alpha v(t)\right)dt + \sum_{i=1}^N (I_i u(t_i))v(t_i). \end{aligned} \tag{36}$$

Without loss of generality, we take the test function $v \in C_0^\infty(t_i, s_i]$ such that $v(t) = 0$, where $t \in [0, t_i) \cup [s_i, T)$, $i = 1, 2, \cdots, N$. Substituting $v(t)$ into (30), we obtain ${}_t D_T^{\alpha-1} \phi_p\left({}_0^C D_t^\alpha u(t)\right)$, $t \in (t_i, s_i]$, $i = 1, 2, 3 \cdots N$, is a constant, a.e.

$$\begin{aligned} &{}_t D_T^{\alpha-1} \phi_p\left({}_0^C D_t^\alpha u(t_i^+)\right) \\ &\quad = {}_t D_T^{\alpha-1} \phi_p\left({}_0^C D_t^\alpha u(s_i^-)\right) \\ &\quad = {}_t D_T^{\alpha-1} \phi_p\left({}_0^C D_t^\alpha u(t)\right), \quad t \in (t_i, s_i], i = 1, 2, \cdots N. \end{aligned} \tag{37}$$

By using (36) and (37), we have

$$\begin{aligned} &\sum_{i=1}^N {}_t D_T^{\alpha-1} \phi_p\left({}_0^C D_t^\alpha u(s_i^+)\right)v(s_i^+) - \sum_{i=0}^N {}_t D_T^{\alpha-1} \phi_p\left({}_0^C D_t^\alpha u(t_{i+1}^-)\right)v(t_{i+1}^-) \\ &\quad = \sum_{i=1}^N {}_t D_T^{\alpha-1} \phi_p\left({}_0^C D_t^\alpha u(t_i^+)\right)v(s_i) + \sum_{i=1}^N (I_i u(t_i))v(t_i). \end{aligned} \tag{38}$$

Hence,

$$\begin{aligned} {}_t D_T^{\alpha-1} \phi_p\left({}_0^C D_t^\alpha u(t_i^+)\right) &= {}_t D_T^{\alpha-1} \phi_p\left({}_0^C D_t^\alpha u(t_i^-)\right) - I_i u(t_i)v(t_i) \\ &= {}_t D_T^{\alpha-1} \phi_p\left({}_0^C D_t^\alpha u(t_i^+)\right)v(s_i) \\ &\quad - \sum_{i=1}^N {}_t D_T^{\alpha-1} \phi_p\left({}_0^C D_t^\alpha u(s_i^+)\right)v(s_i^+). \end{aligned} \tag{39}$$

Then, combining with (37), we have

$$\begin{aligned} &{}_t D_T^{\alpha-1} \phi_p\left({}_0^C D_t^\alpha u\right)(t_i^+) - {}_t D_T^{\alpha-1} \phi_p\left({}_0^C D_t^\alpha u\right)(t_i^-) = I_i u(t_i), \\ &\qquad {}_t D_T^{\alpha-1} \phi_p\left({}_0^C D_t^\alpha u(s_i^+)\right) = {}_t D_T^{\alpha-1} \phi_p\left({}_0^C D_t^\alpha u(s_i^-)\right). \end{aligned} \tag{40}$$

Therefore, u is a classical solution of the problem (3). The proof is completed.

Theorem 16. *Suppose that (H2) and (H3) hold, then problem (3) has at least one classical solution.*

Proof. From Proposition 5, there exist constants A_2, A_3 such that

$$\begin{aligned}\varphi(u) &= \frac{1}{p}\|u\|^p - \sum_{i=0}^{N}\int_{s_i}^{t_{i+1}} F_i(t, u(t)-u(t_{i+1}))dt + \sum_{i=1}^{N}\int_0^{u(t_i)} I_i(s)\\ &\geq \frac{1}{p}\|u\|^p - \sum_{i=0}^{N}\int_{s_i}^{t_{i+1}} b_1(t)dt - \sum_{i=1}^{N} c_1\|u\|_\infty\\ &\quad - \sum_{i=1}^{N}\frac{d_i}{\sigma_i+1}\|u\|_\infty - \sum_{i=0}^{N}\left(\int_{s_i}^{t_{i+1}} a|u(t)-u(t_{i+1})|^2 dt\right.\\ &\quad \left.- \int_{s_i}^{t_{i+1}} b_0(t)|u(t)-u(t_{i+1})|^\delta dt\right)\\ &\geq \frac{1}{p}\|u\|^p - a4\Omega^2\|u\|_\alpha^2\sum_{i=0}^{N}(t_{i+1-si}) - (2\Omega)^\delta\|u\|_\alpha^\delta A_0\\ &\quad - A_1 - \|u\|A_2 - \|u\|_\alpha^{\sigma_i+1}A_3,\end{aligned} \tag{41}$$

where

$$\begin{aligned}A_0 &= \sum_{i=0}^{N}\int_{s_i}^{t_{i+1}} b_0(t)dt,\\ A_1 &= \sum_{i=0}^{N}\int_{s_i}^{t_{i+1}} b_1(t)dt.\end{aligned} \tag{42}$$

Since $p > 2$, we get $\lim_{\|u\|_\alpha\to+\infty}\varphi(u) = +\infty$, then φ is coercive. From Lemma 10 and Lemma 14 that φ satisfied all the conditions of Lemma 9, so φ has a minimum on $E_0^{\alpha,p}$ which is a critical point of φ. Since $I_i(u(t_i)) \neq 0$ $(i = 1, 2, \cdots, N)$, then $u \neq$ constant (u is a critical point of φ).

Therefore, problem (3) has at least one classical solution. The proof is completed.

4. Example

Considering the following p-Laplacian fractional differential equations with instantaneous and noninstantaneous impulses,

$$\begin{cases} {}_tD_T^{3/4}\phi_p\left({}_0^CD_T^{3/4}u(t)\right) + |u|^{p-2}u(t) = D_xF_i(t, u(t)-u(t_{i+1})),\\ t \in (s_i, t_{i+1}], \quad i = 0, 1, 2, \cdots, N,\\ \Delta\left({}_tD_T^{-(1/4)}\phi_p\left({}_0^CD_t^{3/4}u\right)(t_i) = I_i(u(t_i))\right), \quad i = 1, 2, \cdots, N,\\ {}_tD_T^{-(1/4)}\phi_p\left({}_0^CD_t^{3/4}u\right)(t) = {}_tD_T^{\alpha-1}\phi_p\left({}_0^CD_t^{\alpha}u\right)(t_i^+), \quad t \in (t_i, s_i], i = 0, 1, 2, \cdots, N,\\ {}_tD_T^{-(1/4)}\phi_p\left({}_0^CD_t^{3/4}u\right)(s_i^-) = {}_tD_T^{\alpha-1}\phi_p\left({}_0^CD_t^{a}u\right)(s_i^+), \quad i = 1, 2, \cdots, N,\\ u(0) = u(T) = 0, \end{cases} \tag{43}$$

where $\alpha = 3/4$, $T = 2$, $\lambda(t) = 1$, $I_i(u(t_i)) = 1/3 + 1/2 \sin u^{2/3}$, $D_xF_i(t, x) = [1/9(u(t)-u(t_{i+1}))^2 - 3(u(t)-u(t_{i+1})) - 2](1+t)$, we can find $a = 1/3$, $b_0 = 3(1+t)$, $b_1 = 2(1+t)$, $\delta = 1$, $c_i = d_i = 1$, $\sigma_i = 2/3 (i = 1, 2, \cdots, N)$. It is easy to check that (H2) and (H3) hold. Then, by Theorem 16, problem (3) has at least one classical solution.

5. Conclusions

In this paper, we use the variational method to discuss the existence of solutions for Dirichlet boundary value problems with instantaneous and noninstantaneous fractional impulsive differential equations, and the existence results of classical solution for our equations are shown. In addition to extending the linear differential operator to a nonlinear differential operator, it is more important to use the p-Laplacian operator; as far as the author knows, there is not much work to study the solution of the generalized p-Laplacian impulsive fractional system using the variational method. Compared with (2), (3) contains the existence results of (2) when $D_xF_i(t, u(t)-u(t_{i+1})) \equiv f_i(t, u(t))$. Without loss of generality, if $p = 2$, $\alpha = 1$ it evolves into an integer order differential equation which has a certain auxiliary role in solving nonlinear similar problems in the future. At the same time, we also noticed the existence of multiple solutions or the existence of nontrivial solutions to the problem, so we will pay special attention in the following work. Overall, our work summarizes and supplements some of the results in the literature.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interest.

Authors' Contributions

All authors read and approved the final manuscript.

Acknowledgments

This work is supported by the Natural Science Foundation of Xinjiang (2019D01A71), Scientific Research Programs of Colleges in Xinjiang (XJEDU2018Y033), National Natural Science Foundation of China (11961069), Outstanding Young Science and technology personnel Training program of Xinjiang.

References

[1] Y. Chatibi, E. H. El Kinani, and A. Ouhadan, "Lie symmetry analysis of conformable differential equations," *AIMS Mathematics*, vol. 4, no. 4, pp. 1133–1144, 2019.

[2] Y. Chatibi, E. H. El Kinani, and A. Ouhadan, "Lie symmetry analysis and conservation laws for the time fractional Black-Scholes equation," *International Journal of Geometric Methods in Modern Physics*, vol. 17, no. 1, article 2050010, 2020.

[3] R. L. Magin, *Fractional Calculus in Bioengineering*, Begell House Publishers, 2006.

[4] B. Ahmad, A. Alsaedi, S. K. Ntouyas, and J. Tariboon, *Hadamard-Type Fractional Differential Equations, Inclusions and Inequalities*, Springer, Cham, Switzerland, 2017.

[5] F. Mainardi, *Fractional Calculus and Waves in Linear Viscoelasticy*, World Scientific, Singapore, 2010.

[6] Y. Chatibi, E. H. El Kinani, and A. Ouhadan, "On the discrete symmetry analysis of some classical and fractional differential equations," *Mathematical Methods in the Applied Sciences*, pp. 1–11, 2019.

[7] R. Agarwal, S. Hristova, and D. O'Regan, *Non-Instantaneous Impulses in Differential Equations*, Springer, Cham, 2017.

[8] P. Chen, X. Zhang, and Y. Li, "Cauchy problem for fractional non-autonomous evolution equations," *Banach Journal of Mathematical Analysis*, vol. 14, no. 2, pp. 559–584, 2020.

[9] P. Chen, X. Zhang, and Y. Li, "Existence and approximate controllability of fractional evolution equations with nonlocal conditions via resolvent operators," *Fractional Calculus and Applied Analysis*, vol. 23, no. 1, pp. 268–291, 2020.

[10] P. Chen, X. Zhang, and Y. Li, "Approximate controllability of non-autonomous evolution system with nonlocal conditions," *Journal of Dynamical and Control Systems*, vol. 26, no. 1, pp. 1–16, 2020.

[11] P. Chen, X. Zhang, and Y. Li, "A blowup alternative result for fractional nonautonomous evolution equation of Volterra type," *Communications on Pure and Applied Analysis*, vol. 17, no. 5, pp. 1975–1992, 2018.

[12] P. Chen, X. Zhang, and Y. Li, "Non-autonomous parabolic evolution equations with non-instantaneous impulses governed by noncompact evolution families," *Journal of Fixed Point Theory and Applications*, vol. 21, no. 3, article 84, 2019.

[13] P. Chen, X. Zhang, and Y. Li, "Non-autonomous evolution equations of parabolic type with noninstantaneous impulses," *Mediterranean Journal of Mathematics*, vol. 16, no. 5, article 118, 2019.

[14] P. Chen, X. Zhang, and Y. Li, "Fractional non-autonomous evolution equation with nonlocal conditions," *Journal of Pseudo-Differential Operators and Applications*, vol. 10, no. 4, pp. 955–973, 2019.

[15] P. Chen, X. Zhang, and Y. Li, "Iterative method for a new class of evolution equations with noninstantaneous impulses," *Taiwanese Journal of Mathematics*, vol. 21, no. 4, pp. 913–942, 2017.

[16] M. Zuo and X. Hao, "Existence results for impulsive fractional -difference equation with antiperiodic boundary conditions," *Journal of Function Spaces*, vol. 2018, Article ID 3798342, 9 pages, 2018.

[17] P. Li, H. Wang, and Z. Li, "Solutions for impulsive fractional differential equations via variational methods," *Journal of Function Spaces*, vol. 2016, Article ID 2941368, 9 pages, 2016.

[18] G. A. Afrouzi and A. Hadjian, "A variational approach for boundary value problems for impulsive fractional differential equations," *Fractional Calculus and Applied Analysis*, vol. 21, no. 6, pp. 1565–1584, 2018.

[19] E. Hernández and D. O'Regan, "On a new class of abstract impulsive differential equations," *Proceedings of the American Mathematical Society*, vol. 141, no. 5, pp. 1641–1649, 2013.

[20] Y. Tian and M. Zhang, "Variational method to differential equations with instantaneous and noninstantaneous impulses," *Applied Mathematics Letters*, vol. 94, pp. 160–165, 2019.

[21] Y. Zhao, C. Luo, and H. Chen, "Existence results for non-instantaneous impulsive nonlinear fractional differential equation via variational methods," *Bulletin of the Malaysian Mathematical Sciences Society*, vol. 43, no. 3, pp. 2151–2169, 2020.

[22] A. Khaliq and M. U. Rehman, "On variational methods to non-instantaneous impulsive fractional differential equation," *Applied Mathematics Letters*, vol. 83, pp. 95–102, 2018.

[23] L. Bai and J. J. Nieto, "Variational approach to differential equations with not instantaneous impulses," *Applied Mathematics Letters*, vol. 73, pp. 44–48, 2017.

[24] M. Feckan, J. R. Wang, and Y. Zhou, "Periodic solutions for nonlinear evolution equations with noninstantaneous impulses," *Nonautonomous Dynamical Systems*, vol. 1, pp. 93–101, 2014.

[25] W. Zhang and W. Liu, "Variational approach to fractional Dirichlet problem with instantaneous and non-instantaneous impulses," *Applied Mathematics Letters*, vol. 99, article 105993, 2020.

[26] J. Zhou, Y. Deng, and Y. Wang, "Variational approach to p-Laplacian fractional differential equations with instantaneous and non-instantaneous impulses," *Applied Mathematics Letters*, vol. 104, p. 106251, 2020.