Research Article

Eigenvalues of s-Type Operators on Prequasi Normed $C(t, p)$

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We investigate some new topological properties of the multiplication operator on $C(t, p)$ defined by Bilgin (The Punjab University Journal of Mathematics, vol. 30, pp. 67–77, 1997) equipped with the prequasi norm and the prequasi operator ideal formed by this sequence space and s-numbers.

1. Introduction

Summability, multiplication, and ideal operator theorems are very important in mathematical models and have numerous implementations, such as normal series theory, ideal transformations, geometry of Banach spaces, approximation theory, and fixed point theory. For more details, see [1–11]. By $\mathbb{C}^{\mathbb{N}}$, ℓ_{∞} , ℓ_r , and c_0 , we denote the spaces of every, bounded, r-absolutely summable, and convergent to zero sequences of complex numbers. $\mathbb{N}$ indicates the set of nonnegative integers. For a sequence (p_r) with $\inf_r p_r > 0$, Bilgin [12] defined and studied the sequence space

$$C(t, p) = \{x = (x_n) \in \mathbb{C}^{\mathbb{N}} : h(\beta x) < \infty \text{ for some } \beta > 0\}, \quad (1)$$

where $h(x) = \sum_{n=0}^{\infty} \left(\sum_{k=2^{n-1}}^{2^{n+1}-2} (k+1)^{-t} |x_k|/2^n \right)^{p_n}$ and $t \geq 0$.

The space $C(t, p)$ is a Banach space with the Luxemburg norm

$$\|x\| = \inf \left\{ \beta > 0 : h\left(\frac{x}{\beta}\right) \leq 1 \right\}. \quad (2)$$

If $(p_n) \in \ell_{\infty}$, we have

$$C(t, p) = \left\{ (x_i) \in \mathbb{C}^{\mathbb{N}} : \sum_{n=0}^{\infty} \left(\frac{\sum_{k=2^{n-1}}^{2^{n+1}-2} (k+1)^{-t} |x_k|}{2^n} \right)^{p_n} < \infty \right\}. \quad (3)$$

Here and after, we denote $p = (p_n)$, if $t = 0$, then $C(t, p) = \text{ces}(p)$, where $\inf_n p_n > 0$ was defined and investigated by Lim [13]. If $t = 0$ and $p_n = p$, for all $n \in \mathbb{N}$, then $C(t, p) = \text{ces}_p$. Lim [14] defined and determined its dual spaces and characterized some matrix classes. For any pair of Banach spaces T and U , the following notations will be used throughout the article: $\mathcal{B}(T, U)$ (the space of all bounded linear operators from T into U), and if $T = U$, we write $\mathcal{B}(T)$, $(s_r(A))_{r=0}^{\infty}$ [15] (thes-number sequence of operator $A \in \mathcal{B}(T, U)$), $(\alpha_r(A))_{r=0}^{\infty}$ (the approximation number sequence of operator $A \in \mathcal{B}(T, U)$), $\mathcal{F}(T)$ (the space of all finite rank operators on T), $\mathcal{K}(T)$ (the space of all compact operators on T), and $\mathcal{A}(T)$ (the space of all approximable operators on T). For any sequence space E , we will use the notations [16]

$$\begin{aligned}
E^{(s)} &:= \left\{ E^{(s)}(T, U) \right\}, \text{ where } E^{(s)}(T, U) \\
&:= \left\{ A \in \mathcal{B}(T, U) : ((s_i(A)))_{i=0}^\infty \in E \right\}, \\
E^{(\alpha)} &:= \left\{ E^{(\alpha)}(T, U) \right\}, \text{ where } E^{(\alpha)}(T, U) \\
&:= \left\{ A \in \mathcal{B}(T, U) : ((\alpha_i(A)))_{i=0}^\infty \in E \right\}, \\
E^{(\lambda)} &:= \left\{ E^{(\lambda)}(T, U) \right\}, \text{ where} \\
E^{(\lambda)}(T, U) &:= \left\{ A \in \mathcal{B}(T, U) : ((\lambda_n(A)))_{n=0}^\infty \right. \\
&\quad \left. \in E \text{ and } \|A - \lambda_r(A)I\| = 0, \text{ for all } r \in \mathbb{N} \right\}.
\end{aligned} \tag{4}$$

A few of operator ideals in the class of Hilbert spaces or Banach spaces are defined by distinct scalar sequence spaces, such as the ideal of compact operators $\mathcal{K}$ formed by $(d_r(V))$ and c_0 . Pietsch [17] studied the quasi-ideals $(\ell_r)^{(\alpha)}$ for $r \in (0, \infty)$, the ideals of Hilbert Schmidt operators between Hilbert spaces constructed by ℓ_2 , and the ideals of nuclear operators generated by ℓ_1 . He examined that $\bar{\mathcal{F}} = (\ell_r)^{(\alpha)}$ for $r \in [1, \infty)$, where $\bar{\mathcal{F}}$ is the closed class of all finite rank operators, and the class $(\ell_r)^{(\alpha)}$ is a simple Banach and small [18]. The strict inclusion $(\ell_r)^{(\alpha)}(T, U) \subsetneq (\ell_j)^{(\alpha)}(T, U) \subsetneq \mathcal{B}(T, U)$, whenever $j > r > 0$, T and U are infinite dimensional Banach spaces investigated through Makarov and Faried [19]. Faried and Bakery [16] investigated a generalization of the class of operator ideal which is the prequasi operator ideal; they studied several geometric and topological structures of $(\ell_M)^{(s)}$ and $(\text{ces}(r))^{(s)}$. On sequence spaces, Mursaleen and Noman [20, 21] investigated the compact operators on some difference sequence spaces. The multiplication operators on $(\text{ces}(r), \|\cdot\|)$ with the Luxemburg norm $\|\cdot\|$ investigated by Komal et al. [22]. The point of this article is to explain some results of $(C(t, p))_h$ equipped with a prequasi norm h . Firstly, we give the conditions on $(C(t, p))_h$ to be a Banach space. Secondly, we investigate the multiplication operator defined on $(C(t, p))_h$. Finally, some geometric and topological structures of $(C(t, p))_h^{(s)}$ have been studied, such as closed, small, simple Banach, and $(C(t, p))_h^{(s)} = (C(t, p))_h^{(\lambda)}$. A strict inclusion relation of $(C(t, p))_h^{(s)}$ has been studied for different p and t .

2. Definitions and Preliminaries

We will use $e_r = (0, 0, \dots, 1, 0, 0, \dots)$, where 1 shows at the r^{th} place, for all $r \in \mathbb{N}$.

Lemma 1 [17]. *If $E \in \mathcal{B}(T, U)$ and $E \notin \mathcal{A}(T, U)$, then there are $G \in \mathcal{B}(T)$ and $D \in \mathcal{B}(U)$ such that $DEGe_r = e_r$, for all $r \in \mathbb{N}$.*

Definition 2 [17]. A Banach space T is called simple if the algebra $\mathcal{B}(T)$ has one and only one nontrivial closed ideal.

Theorem 3 [17]. *If T is a Banach space with $\dim(T) = \infty$, then*

$$\mathcal{F}(T) \subsetneq \mathcal{A}(T) \subsetneq \mathcal{K}(T) \subsetneq \mathcal{B}(T). \tag{5}$$

Definition 4 [23]. An operator $A \in \mathcal{B}(T)$ is called Fredholm if $\dim((R(A))^c) < \infty$, $\dim(\ker(A)) < \infty$, and $R(\bar{A}) = R(A)$, where $(R(A))^c$ is the complement of range A .

Definition 5 [24]. A class of linear sequence spaces T is called a special space of sequences (sss) if

- (1) $e_r \in T$, for every $r \in \mathbb{N}$
- (2) if $t = (t_r) \in \mathbb{C}^{\mathbb{N}}$, $z = (z_r) \in T$, and $|t_r| \leq |z_r|$, for all $r \in \mathbb{N}$, then $t \in T$, i.e., “ T is solid”
- (3) if $(t_r)_{r=0}^\infty \in T$, then $(t_{[r/2]})_{r=0}^\infty \in T$, where $[r/2]$ denotes the integral part of $r/2$

Definition 6 [24]. A subclass T_h is called a premodular (sss) if there is a function $h : T \rightarrow [0, \infty)$ verifying the conditions

- (i) $h(t) \geq 0$ for each $t \in T$ and $h(t) = 0 \Leftrightarrow t = \theta$, where θ is the zero vector of T
- (ii) there is $d \geq 1$ such that $h(\eta t) \leq d |\eta| h(t)$, for each $t \in T$ and $\eta \in \mathbb{C}$
- (iii) there is $M \geq 1$, $h(t + z) \leq M(h(t) + h(z))$, for all $t, z \in T$
- (iv) if $|t_r| \leq |z_r|$, for every $r \in \mathbb{N}$, then $h((t_r)) \leq h((z_r))$
- (v) there is $M_0 \geq 1$, $h((t_r)) \leq h((t_{[r/2]})) \leq M_0 h((t_r))$
- (vi) for all $t = (t_r)_{r=0}^\infty \in T$, and $\varepsilon > 0$ there is $r_0 \in \mathbb{N}$ such that $h((t_r)_{r=r_0}^\infty) < \varepsilon$
- (vii) there is $\alpha > 0$ such that $h(\eta, 0, 0, 0, \dots) \geq \alpha |\eta| h(1, 0, 0, 0, \dots)$, for all $\eta \in \mathbb{C}$

Definition 7 [24]. Let T be a (sss). If there is a function $h : T \rightarrow [0, \infty)$ verifying the parts (i), (ii), and (iii) of Definition 6. The space T with h is called prequasi normed (sss) and denoted by T_h , which gives a class more general than the quasi normed (sss). If the space T is complete with h , then T_h is called a prequasi Banach (sss).

Theorem 8 [24]. *Every quasinorm (sss) is prequasi norm (sss).*

Theorem 9 [24]. *Every premodular (sss) is prequasi normed (sss).*

A generalization of the usual classes of operator ideal is the class prequasi operator ideal.

Definition 10 [24]. A function $V : \Psi \rightarrow [0, \infty)$ is called a prequasi norm on the class Ψ if it satisfies

- (1) if $A \in \Psi(T, U)$, then $V(A) \geq 0$ and $V(A) = 0 \Leftrightarrow A = 0$
- (2) there is $d \geq 1$ such that $V(\eta A) \leq d |\eta| V(A)$, for every $A \in \Psi(T, U)$ and $\eta \in \mathbb{C}$
- (3) there is $M \geq 1$ such that $V(A_1 + A_2) \leq M[V(A_1) + V(A_2)]$, for every $A_1, A_2 \in \Psi(T, U)$
- (4) there is $\nu \geq 1$ such that if $X \in \mathcal{B}(T_0, T)$, $Y \in \Psi(T, U)$, and $Z \in \mathcal{B}(U, U_0)$, then $V(ZYX) \leq \nu \|Z\| V(Y) \|X\|$, where T_0 and U_0 are normed spaces

Theorem 11 [25]. If E_h is a premodular (sss), then $\mathcal{F}(\bar{T}, U) = E_h^{(s)}(T, U)$, where $V(A) = h(s_r(A))_{r=0}^\infty$.

Theorem 12 [16]. If E_h is a premodular (sss), then $V(B) = h(s_r(B))_{r=0}^\infty$ is a prequasi norm on $E_h^{(s)}$.

The following inequality [26] will be used in the sequel: $|a_r + b_r|^{p_r} \leq H(|a_r|^{p_r} + |b_r|^{p_r})$, where $H = \max \{1, 2^{\sup_r p_r - 1}\}$, $0 \leq p_r \leq \sup_r p_r < \infty$, and $a_r, b_r \in \mathbb{C}$ for each $r \in \mathbb{N}$.

3. Main Results

3.1. Premodular Banach (sss). The conditions on t and p are such that the space $(C(t, p))_h$ is a premodular Banach (sss), where $h(z) = \sum_{r=0}^\infty (\sum_{k=2^r-1}^{2^{r+1}-2} (k+1)^{-t} |z_k|/2^r)^{p_r}$, for all $z \in C(t, p)$.

Theorem 13. Let $(p_r) \in \ell_\infty$ be an increasing with $p_0 > 0$ and $t \geq 0$, then the space $(C(t, p))_h$ is a premodular Banach (sss).

Proof. (1-i) Assume $z, w \in C(t, p)$. From $(p_r) \in \ell_\infty$, we get

$$\begin{aligned} h(z+w) &= \sum_{r=0}^\infty \left(\frac{\sum_{k=2^r-1}^{2^{r+1}-2} (k+1)^{-t} |z_k + w_k|}{2^r} \right)^{p_r} \\ &\leq H \left(\sum_{r=0}^\infty \left(\frac{\sum_{k=2^r-1}^{2^{r+1}-2} (k+1)^{-t} |z_k|}{2^r} \right)^{p_r} \right. \\ &\quad \left. + \sum_{r=0}^\infty \left(\frac{\sum_{k=2^r-1}^{2^{r+1}-2} (k+1)^{-t} |w_k|}{2^r} \right)^{p_r} \right) \\ &= H(h(z) + h(w)) < \infty, \end{aligned} \quad (6)$$

hence $z + w \in C(t, p)$.

(1-ii) If $\eta \in \mathbb{C}$, $z \in C(t, p)$, and from $(p_r) \in \ell_\infty$, one has

$$\begin{aligned} h(\eta z) &= \sum_{r=0}^\infty \left(\frac{\sum_{k=2^r-1}^{2^{r+1}-2} (k+1)^{-t} |\eta z_k|}{2^r} \right)^{p_r} \\ &\leq \sup_r |\eta|^{p_r} \sum_{r=0}^\infty \left(\frac{\sum_{k=2^r-1}^{2^{r+1}-2} (k+1)^{-t} |z_k|}{2^r} \right)^{p_r} \\ &= \sup_r |\eta|^{p_r} h(z) < \infty. \end{aligned} \quad (7)$$

Therefore, $\eta z \in C(t, p)$, from (1-i) and (1-ii), the space $C(t, p)$ is linear. Since $(p_r) \in \ell_\infty$ with $p_0 > 0$, we obtain

$$\begin{aligned} h(e_m) &= \sum_{r=0}^\infty \left(\frac{\sum_{k=2^r-1}^{2^{r+1}-2} (k+1)^{-t} (e_m)_k}{2^r} \right)^{p_r} \\ &= \left(\frac{(m+1)^{-t}}{2^{r_0}} \right)^{p_{r_0}} < \infty, \end{aligned} \quad (8)$$

where $r_0 \in \mathbb{N}$ be such that $2^{r_0} - 1 \leq m \leq 2^{r_0+1} - 2$. Hence $e_m \in C(t, p)$, for all $m \in \mathbb{N}$.

(2) Let $|z_r| \leq |w_r|$, for all $r \in \mathbb{N}$ and $w \in C(t, p)$. Hence,

$$\begin{aligned} h(z) &= \sum_{r=0}^\infty \left(\frac{\sum_{k=2^r-1}^{2^{r+1}-2} (k+1)^{-t} |z_k|}{2^r} \right)^{p_r} \\ &\leq \sum_{r=0}^\infty \left(\frac{\sum_{k=2^r-1}^{2^{r+1}-2} (k+1)^{-t} |w_k|}{2^r} \right)^{p_r} = h(w) < \infty, \end{aligned} \quad (9)$$

hence, $z \in C(t, p)$.

(3) Let $(z_r) \in C(t, p)$, we have

$$\begin{aligned} h(z_{[r/2]}) &= \sum_{r=0}^\infty \left(\frac{\sum_{k=2^r-1}^{2^{r+1}-2} ([k/2] + 1)^{-t} |z_{[k/2]}|}{2^r} \right)^{p_r} \\ &= \sum_{r=0}^\infty \left(\frac{\sum_{k=2^{2r}-1}^{2^{2r+1}-2} ([k/2] + 1)^{-t} |z_{[k/2]}|}{2^{2r}} \right)^{p_{2r}} \\ &\quad + \sum_{r=0}^\infty \left(\frac{\sum_{k=2^{2r+1}-1}^{2^{2r+2}-2} ([k/2] + 1)^{-t} |z_{[k/2]}|}{2^{2r+1}} \right)^{p_{2r+1}} \\ &\leq \sum_{r=0}^\infty \left(\frac{\sum_{k=2^{2r}-1}^{2^{2r+1}-2} ([k/2] + 1)^{-t} |z_{[k/2]}|}{2^{2r}} \right)^{p_r} \\ &\quad + \sum_{r=0}^\infty \left(\frac{\sum_{k=2^{2r+1}-1}^{2^{2r+2}-2} ([k/2] + 1)^{-t} |z_{[k/2]}|}{2^{2r+1}} \right)^{p_r} \\ &\leq \sum_{r=0}^\infty \left(\frac{\sum_{k=2^r-1}^{2^{r+1}-2} 3(2^r)(k+1)^{-t} |z_k|}{2^{2r}} \right)^{p_r} \\ &\quad + \sum_{r=0}^\infty \left(\frac{\sum_{k=2^r-1}^{2^{r+1}-2} 3(2^r)(k+1)^{-t} |z_k|}{2^{2r+1}} \right)^{p_r} \\ &\leq 2(3)^{\sup_r p_r} \sum_{r=0}^\infty \left(\frac{\sum_{k=2^r-1}^{2^{r+1}-2} (k+1)^{-t} |z_k|}{2^r} \right)^{p_r}, \end{aligned} \quad (10)$$

then $(z_{[r/2]}) \in C(t, p)$.

(i) Evidently, $h(z) \geq 0$ and $h(z) = 0 \Leftrightarrow z = \theta$

(ii) There is $d = \max \{1, \sup_r |\eta|^{p_r-1}\} \geq 1$ with $h(\eta z) \leq d |\eta| h(z)$, for each $z \in C(t, p)$ and $\eta \in \mathbb{C}$

- (iii) From condition (1), there is $M \geq 1$ with $h(z+w) \leq M(h(z) + h(w))$ for each $z, w \in C(t, p)$
- (iv) From condition (2), we have $(C(t, p))_h$ which is solid
- (v) From condition (3), we find $M_0 \geq 2(3)^{\sup p_r} \geq 1$
- (vi) Clearly, $\bar{F} = C(t, p)$
- (vii) There is $0 < \alpha \leq |\eta|^{p_0-1}$ with $h(\eta, 0, 0, 0, \dots) \geq \alpha |\eta| h(1, 0, 0, 0, \dots)$, for each $\eta \neq 0$ and $\alpha > 0$, when $\eta = 0$

Therefore, the space $(C(t, p))_h$ is premodular (sss). To prove that $(C(t, p))_h$ is a premodular Banach (sss), suppose $z^r = (z_i^r)_{i=0}^\infty$ is a Cauchy sequence in $(C(t, p))_h$; hence, for every $\varepsilon \in (0, 1)$, there is $r_0 \in \mathbb{N}$ such that for every $r, v \geq r_0$, one has

$$h(z^r - z^v) = \sum_{i=0}^{\infty} \left(\frac{\sum_{j=2^{i-1}}^{2^{i+1}-2} (j+1)^{-t} |z_j^r - z_j^v|}{2^i} \right)^{p_i} < \varepsilon \sup_i p_i. \quad (11)$$

Hence, for $r, v \geq r_0$ and $i \in \mathbb{N}$, we get $|z_i^r - z_i^v| < \varepsilon$. So (z_i^v) is a Cauchy sequence in $\mathbb{C}$ for fixed $i \in \mathbb{N}$; this gives $\lim_{v \rightarrow \infty} z_i^v = z_i^0$ for fixed $i \in \mathbb{N}$. Hence, $h(z^r - z^0) < \varepsilon^{\sup_i p_i}$, for all $r \geq r_0$. Finally, to prove that $z^0 \in C(t, p)$, we have

$$h(z^0) = h(z^0 - z^r + z^r) \leq H(h(z^r - z^0) + h(z^r)) < \infty, \quad (12)$$

so $z^0 \in C(t, p)$. This means that $(C(t, p))_h$ is a premodular Banach (sss).

Corollary 14. If $0 < p < \infty$, then $(ces_p)_h$ is a premodular Banach (sss), where $h(z) = \sum_{i=0}^{\infty} (\sum_{j=2^{i-1}}^{2^{i+1}-2} |z_j|/2^i)^p$ for every $z \in ces_p$.

4. Some Properties of Multiplication Operators on $(C(t, p))_h$

We investigate the necessity and sufficient conditions on t and p such that the multiplication operator defined on the prequasi normed (sss), $(C(t, p))_h$, becomes an element of $\mathcal{B}((C(t, p))_h)$, $\mathcal{A}((C(t, p))_h)$, $\mathcal{K}((C(t, p))_h)$, closed range, and Fredholm operator.

Definition 15. Pick up $\zeta \in \mathbb{C}^{\mathbb{N}} \cap \ell_\infty$, and T_h is a prequasi normed (sss). An operator $A_\zeta : T_h \rightarrow T_h$ is called a multiplication operator, if $A_\zeta z = \zeta z = (\zeta_k z_k)_{k=0}^\infty$, for all $z \in T$. It is called a multiplication operator generated by ζ , if $A_\zeta \in \mathcal{B}(T_h)$ is a multiplication operator.

Theorem 16. If $\zeta \in \mathbb{C}^{\mathbb{N}}$, $(p_n) \in \ell_\infty$ is increasing with $p_0 > 0$ and $t \geq 0$, then

$$\zeta \in \ell_\infty \Leftrightarrow A_\zeta \in \mathcal{B}((C(t, p))_h). \quad (13)$$

Proof. Let $\zeta \in \ell_\infty$. Therefore, there is $D > 0$ such that $|\zeta_r| \leq D$, for every $r \in \mathbb{N}$. Since $(p_r) \in \ell_\infty$ and for $z \in (C(t, p))_h$, we have

$$\begin{aligned} h(A_\zeta z) &= h(\zeta z) = \sum_{i=0}^{\infty} \left(\frac{\sum_{j=2^{i-1}}^{2^{i+1}-2} (j+1)^{-t} |\zeta_j| |z_j|}{2^i} \right)^{p_i} \\ &\leq \sum_{i=0}^{\infty} \left(\frac{\sum_{j=2^{i-1}}^{2^{i+1}-2} (j+1)^{-t} D |z_j|}{2^i} \right)^{p_i} \leq \sup_i (D^{p_i}) h(z), \end{aligned} \quad (14)$$

this implies that $A_\zeta \in \mathcal{B}((C(t, p))_h)$. Conversely, let $A_\zeta \in \mathcal{B}((C(t, p))_h)$, to show that $\zeta \in \ell_\infty$. For, if $\zeta \notin \ell_\infty$, then for all $r \in \mathbb{N}$, there are $i_r \in \mathbb{N}$ such that $\zeta_{i_r} > r$. Hence,

$$\begin{aligned} h(A_\zeta e_{i_r}) &= h(\zeta e_{i_r}) = \sum_{i=0}^{\infty} \left(\frac{\sum_{j=2^{i-1}}^{2^{i+1}-2} (j+1)^{-t} |\zeta_j| (e_{i_r})_j}{2^i} \right)^{p_i} \\ &= \left(\frac{(i_r+1)^{-t} |\zeta_{i_r}|}{2^{i_0}} \right)^{p_{i_0}} > \left(\frac{(i_r+1)^{-t} r}{2^{i_0}} \right)^{p_{i_0}} \\ &= r^{p_{i_0}} h(e_{i_r}), \end{aligned} \quad (15)$$

where $i_0 \in \mathbb{N}$ be such that $2^{i_0} - 1 \leq i_r \leq 2^{i_0+1} - 2$. This shows that $A_\zeta \notin \mathcal{B}((C(t, p))_h)$. So, $\zeta \in \ell_\infty$.

Theorem 17. Assume $\zeta \in \mathbb{C}^{\mathbb{N}}$ and $(C(t, p))_h$ is a prequasi normed (sss). $|\zeta_r| = 1$, for every $r \in \mathbb{N}$ if and only if A_ζ is an isometry.

Proof. If $|\zeta_r| = 1$, for every $r \in \mathbb{N}$. Therefore, we have

$$\begin{aligned} h(A_\zeta z) &= \sum_{i=0}^{\infty} \left(\frac{\sum_{j=2^{i-1}}^{2^{i+1}-2} (j+1)^{-t} |\zeta_j| |z_j|}{2^i} \right)^{p_i} \\ &= \sum_{i=0}^{\infty} \left(\frac{\sum_{j=2^{i-1}}^{2^{i+1}-2} (j+1)^{-t} |z_j|}{2^i} \right)^{p_i} = h(z), \end{aligned} \quad (16)$$

for all $z \in (C(t, p))_h$. Hence, A_ζ is an isometry. Conversely, let A_ζ be an isometry and $|\zeta_r| < 1$, for all $r \in \mathbb{N}$. Then,

$$\begin{aligned} h(A_\zeta z) &= \sum_{i=0}^{\infty} \left(\frac{\sum_{j=2^{i-1}}^{2^{i+1}-2} (j+1)^{-t} |\zeta_j| |z_j|}{2^i} \right)^{p_i} \\ &< \sum_{i=0}^{\infty} \left(\frac{\sum_{j=2^{i-1}}^{2^{i+1}-2} (j+1)^{-t} |z_j|}{2^i} \right)^{p_i} = h(z). \end{aligned} \quad (17)$$

Also if $|\zeta_r| > 1$, then we get $h(\zeta z) > h(z)$. We have a contradiction in the two cases. Therefore, $|\zeta_r| = 1$, for every $r \in \mathbb{N}$.

By $N(D)$, we indicate the cardinality of the set D .

Theorem 18. For $\zeta \in \mathbb{C}^{\mathbb{N}}$, $(p_r) \in \ell_{\infty}$ is increasing with $p_0 > 0$, $t \geq 0$, and $A_{\zeta} \in \mathcal{B}((C(t, p)))_h$. Then,

$$A_{\zeta} \in \mathcal{A}((C(t, p)))_h \Leftrightarrow (\zeta_r)_{r=0}^{\infty} \in c_0. \quad (18)$$

Proof. Let $A_{\zeta} \in \mathcal{A}((C(t, p)))_h$; hence $A_{\zeta} \in \mathcal{K}((C(t, p)))_h$. To show that $(\zeta_r)_{r=0}^{\infty} \in c_0$. Assume $(\zeta_r)_{r=0}^{\infty} \notin c_0$. Therefore, the set $B_{\delta} = \{r \in \mathbb{N} : |\zeta_r| \geq \delta\}$ has $N(B_{\delta}) = \infty$, for $\delta > 0$. Suppose $d_r \in B_{\delta}$, for all $r \in \mathbb{N}$. Therefore, $E = \{e_{d_r} : d_r \in B_{\delta}\} \subseteq \ell_{\infty} \cap (C(t, p))_h$ with $N(E) = \infty$. We have

$$\begin{aligned} h(A_{\zeta}e_{d_r} - A_{\zeta}e_{d_m}) &= h(\zeta e_{d_r} - \zeta e_{d_m}) \\ &= \sum_{i=0}^{\infty} \left(\frac{\sum_{j=2^{i-1}}^{2^{i+1}-2} (j+1)^{-t} |\zeta_j((e_{d_r})_j - (e_{d_m})_j)|}{2^i} \right)^{p_i} \\ &\geq \sum_{i=0}^{\infty} \left(\frac{\sum_{j=2^{i-1}}^{2^{i+1}-2} (j+1)^{-t} |\delta((e_{d_r})_j - (e_{d_m})_j)|}{2^i} \right)^{p_i} \\ &\geq \left(\inf_i \delta^{p_i} \right) \sum_{i=0}^{\infty} \left(\frac{\sum_{j=2^{i-1}}^{2^{i+1}-2} (j+1)^{-t} |(e_{d_r})_j - (e_{d_m})_j|}{2^i} \right)^{p_i} \\ &= \left(\inf_i \delta^{p_i} \right) h(e_{d_r} - e_{d_m}), \end{aligned} \quad (19)$$

for all $d_r, d_m \in B_{\delta}$. This proves $A_{\zeta}(E)$ is not a convergent subsequence. This shows that $A_{\zeta} \notin \mathcal{K}((C(t, p)))_h$; hence, $A_{\zeta} \notin \mathcal{A}((C(t, p)))_h$. This gives a contradiction. So, $\lim_{r \rightarrow \infty} \zeta_r = 0$. Conversely, suppose $\lim_{r \rightarrow \infty} \zeta_r = 0$. Hence, for all $\delta > 0$, we have $N(B_{\delta}) < \infty$. Therefore, for all $\delta > 0$ the space

$$((C(t, p)))_{B_{\delta}} = \{z = (z_r) \in \mathbb{C}^{B_{\delta}} : h(z) < \infty\}, \quad (20)$$

is a finite dimensional. Hence, $A_{\zeta}|_{((C(t, p)))_{B_{\delta}}} \in \mathcal{F}((C(t, p)))_{B_{\delta}}$. Let $r \in \mathbb{N}$ and define $\zeta_r \in \mathbb{C}^{\mathbb{N}}$ as

$$(\zeta_r)_m = \begin{cases} \zeta_m, & m \in B_{1/r}, \\ 0, & \text{otherwise.} \end{cases} \quad (21)$$

Obviously, $A_{\zeta_r} \in \mathcal{F}((C(t, p)))_{B_{1/r}}$ since $\dim((C(t, p))_{B_{1/r}}) < \infty$, for all $r \in \mathbb{N}$. Since $(p_i) \in \ell_{\infty}$ is increasing with $p_0 > 0$, we have

$$\begin{aligned} h((A_{\zeta} - A_{\zeta_r})z) &= \sum_{i=0, i \notin B_{1/r}}^{\infty} \left(\frac{\sum_{j=2^{i-1}}^{2^{i+1}-2} (j+1)^{-t} |(\zeta_j - (\zeta_r)_j)z_j|}{2^i} \right)^{p_i} \\ &\quad + \sum_{i=0, i \in B_{1/r}}^{\infty} \left(\frac{\sum_{j=2^{i-1}}^{2^{i+1}-2} (j+1)^{-t} |(\zeta_j - (\zeta_r)_j)z_j|}{2^i} \right)^{p_i} \\ &= \sum_{i=0, i \notin B_{1/r}}^{\infty} \left(\frac{\sum_{j=2^{i-1}}^{2^{i+1}-2} (j+1)^{-t} |\zeta_j z_j|}{2^i} \right)^{p_i} \\ &\quad + \sum_{i=0, i \in B_{1/r}}^{\infty} \left(\frac{\sum_{j=2^{i-1}}^{2^{i+1}-2} (j+1)^{-t} |\zeta_j z_j|}{2^i} \right)^{p_i} \\ &< \frac{1}{r^{p_0}} \sum_{i=0}^{\infty} \left(\frac{\sum_{j=2^{i-1}}^{2^{i+1}-2} (j+1)^{-t} |z_j|}{2^i} \right)^{p_i} = \frac{1}{r^{p_0}} h(z). \end{aligned} \quad (22)$$

This proves that $\|A_{\zeta} - A_{\zeta_r}\| \leq 1/r^{p_0}$ and $A_{\zeta} = \lim_{r \rightarrow \infty} A_{\zeta_r}$. Therefore, $A_{\zeta} \in \mathcal{A}((C(t, p)))_h$.

According to Theorem 3, it is easy to prove the following theorem.

Theorem 19. For $\zeta \in \mathbb{C}^{\mathbb{N}}$, $(p_r) \in \ell_{\infty}$ is increasing with $p_0 > 0$, $t \geq 0$, and $A_{\zeta} \in \mathcal{B}((C(t, p)))_h$. Then,

$$A_{\zeta} \in \mathcal{K}((C(t, p)))_h \Leftrightarrow (\zeta_r)_{r=0}^{\infty} \in c_0. \quad (23)$$

Corollary 20. If $(p_i) \in \ell_{\infty}$ is increasing with $p_0 > 0$ and $t \geq 0$, we have

$$\mathcal{K}((C(t, p)))_h \subsetneq \mathcal{B}((C(t, p)))_h. \quad (24)$$

Proof. The operator $I = A_{\zeta}$ on $(C(t, p))_h$ is a multiplication operator generated by $\zeta = (1, 1, \dots)$. Therefore, $I \notin \mathcal{K}((C(t, p)))_h$ and $I \in \mathcal{B}((C(t, p)))_h$.

Theorem 21. Suppose $(C(t, p))_h$ is a prequasi Banach (sss) and $A_{\zeta} \in \mathcal{B}((C(t, p)))_h$. Then, there is $\delta > 0$ such that $|\zeta_r| \geq \delta$, for all $r \in (\ker(\zeta))^c$, if and only if, A_{ζ} has a closed range.

Proof. Let the sufficient condition be verified, to show that $R(\bar{A}_{\zeta}) = R(A_{\zeta})$. If f is a limit point of $R(A_{\zeta})$, then there is $A_{\zeta}x_r \in (C(t, p))_h$, for each $r \in \mathbb{N}$ with $\lim_{r \rightarrow \infty} A_{\zeta}x_r = f$. Clearly, the sequence $A_{\zeta}x_r$ is a Cauchy sequence. Since h is nondecreasing, one can see

$$\begin{aligned}
h(A_\zeta x_r - A_\zeta x_m) &= \sum_{i=0}^{\infty} \left(\frac{\sum_{j=2^{i-1}}^{2^{i+1}-2} (j+1)^{-t} |\zeta_j(x_r)_j - \zeta_j(x_m)_j|}{2^i} \right)^{p_i} \\
&= \sum_{i=0, i \in (\ker(\zeta))^c}^{\infty} \left(\frac{\sum_{j=2^{i-1}}^{2^{i+1}-2} (j+1)^{-t} |\zeta_j(x_r)_j - \zeta_j(x_m)_j|}{2^i} \right)^{p_i} \\
&\quad + \sum_{i=0, i \notin (\ker(\zeta))^c}^{\infty} \left(\frac{\sum_{j=2^{i-1}}^{2^{i+1}-2} (j+1)^{-t} |\zeta_j(x_r)_j - \zeta_j(x_m)_j|}{2^i} \right)^{p_i} \\
&\geq \sum_{i=0, i \in (\ker(\zeta))^c}^{\infty} \left(\frac{\sum_{j=2^{i-1}}^{2^{i+1}-2} (j+1)^{-t} |\zeta_j(x_r)_j - \zeta_j(x_m)_j|}{2^i} \right)^{p_i} \\
&= \sum_{i=0}^{\infty} \left(\frac{\sum_{j=2^{i-1}}^{2^{i+1}-2} (j+1)^{-t} |\zeta_j(y_r)_j - \zeta_j(y_m)_j|}{2^i} \right)^{p_i} \\
&> \sum_{i=0}^{\infty} \left(\frac{\sum_{j=2^{i-1}}^{2^{i+1}-2} (j+1)^{-t} |\varepsilon(y_r)_j - \varepsilon(y_m)_j|}{2^i} \right)^{p_i} \\
&= h(\varepsilon(y_r - y_m)),
\end{aligned} \tag{25}$$

where

$$(y_r)_k = \begin{cases} (x_r)_k, & k \in (\ker(\zeta))^c, \\ 0, & k \notin (\ker(\zeta))^c. \end{cases} \tag{26}$$

This proves that (y_r) is a Cauchy sequence in $(C(t, p))_h$. But $(C(t, p))_h$ is complete. Therefore, there exists $x \in (C(t, p))_h$ such that $\lim_{r \rightarrow \infty} y_r = x$. In view of the continuity of A_ζ , hence $\lim_{r \rightarrow \infty} A_\zeta y_r = A_\zeta x$. But $\lim_{n \rightarrow \infty} A_\zeta x_n = \lim_{n \rightarrow \infty} A_\zeta y_n = f$. Therefore, $A_\zeta x = f$. Hence $f \in R(A_\zeta)$. Therefore, $R(\bar{A}_\zeta) = R(A_\zeta)$. Let the necessary condition be satisfied. Therefore, there is $\varepsilon > 0$ such that $h(A_\zeta x) \geq h(\varepsilon x)$, for all $x \in ((C(t, p))_h)_{(\ker(\zeta))^c}$. Let $D = \{k \in (\ker(\zeta))^c : |\zeta_k| < \varepsilon\}$. If $D \neq \emptyset$, then for $n_0 \in D$, since h is nondecreasing, we have

$$\begin{aligned}
h(A_\zeta e_{n_0}) &= \sum_{i=0}^{\infty} \left(\frac{\sum_{j=2^{i-1}}^{2^{i+1}-2} (j+1)^{-t} |\zeta_j| |(e_{n_0})_j|}{2^i} \right)^{p_i} \\
&< \sum_{i=0}^{\infty} \left(\frac{\sum_{j=2^{i-1}}^{2^{i+1}-2} (j+1)^{-t} \varepsilon |(e_{n_0})_j|}{2^i} \right)^{p_i} = h(\varepsilon e_{n_0}),
\end{aligned} \tag{27}$$

which gives a contradiction. So, $D = \emptyset$ with $|\zeta_r| \geq \varepsilon$, for every $r \in (\ker(\zeta))^c$.

Theorem 22. Pick up $\zeta \in \mathbb{C}^{\mathbb{N}}$, and $(C(t, p))_h$ is a prequasi Banach (sss). There is $d > 0$ and $D > 0$ with $d < \zeta_r < D$, for every $r \in \mathbb{N}$, if and only if, $A_\zeta \in \mathcal{B}((C(t, p))_h)$ is invertible.

Proof. Let the condition be true. Define $\eta \in \mathbb{C}^{\mathbb{N}}$ by $\eta_n = 1/\zeta_n$. Then, by Theorem 16, A_ζ and T_η are bounded linear operators. Also $A_\zeta \cdot A_\eta = A_\eta \cdot A_\zeta = I$. Hence, A_ζ is the inverse of A_η . Conversely, let A_ζ be invertible. Hence, $R(A_\zeta) = ((C(t, p))_h)_{\mathbb{N}}$.

Therefore, $R(\bar{A}_\zeta) = R(A_\zeta)$. Hence, by Theorem 21, there is $d > 0$ with $|\zeta_r| \geq d$, for every $r \in (\ker(\zeta))^c$. Now, $\ker(\zeta) = \emptyset$; otherwise, $\zeta_{r_0} = 0$; for some $r_0 \in \mathbb{N}$, we have $e_{r_0} \in \ker(A_\zeta)$; this implies a contradiction, since $\ker(A_\zeta) = \emptyset$. So, $|\zeta_r| \geq d$, for every $r \in \mathbb{N}$. Since $A_\zeta \in \mathcal{B}((C(t, p))_h)$, from Theorem 16, there is $D > 0$ with $|\zeta_r| \leq D$, for every $r \in \mathbb{N}$. Therefore, we get that $d \leq |\zeta_r| \leq D$, for each $r \in \mathbb{N}$.

Theorem 23. For a prequasi Banach (sss) $(C(t, p))_h$. The operator A_ζ is Fredholm, if and only if, (i) $N(\ker(\zeta)) < \infty$ and (ii) $|\zeta_r| \geq \varepsilon$, for each $r \in (\ker(\zeta))^c$.

Proof. Let the sufficient condition be true and $N(\ker(\zeta)) = \infty$, so $e_r \in \ker(A_\zeta)$, for every $r \in \ker(\zeta)$. We have $\dim(\ker(A_\zeta)) = \infty$, since e_n 's are linearly independent. This implies a contradiction. Hence, $N(\ker(\zeta)) < \infty$. From Theorem 21, the condition (ii) follows. Next, if the necessary conditions are true, from Theorem 21, the condition (ii) gives that $R(\bar{A}_\zeta) = R(A_\zeta)$. By condition (i), we have $\dim(\ker(A_\zeta)) < \infty$ and $\dim(R((A_\zeta)^c)) < \infty$. This implies that A_ζ is Fredholm.

5. Banach and Closed Prequasi Ideal

We explain the sufficient conditions on t and p such that $((C(t, p))_h^{(s)}, V)$ is Banach and a closed prequasi ideal, where $h(x) = \sum_{i=0}^{\infty} \left(\sum_{j=2^{i-1}}^{2^{i+1}-2} (j+1)^{-t} |x_j|/2^i \right)^{p_i}$, for all $x \in C(t, p)$ and $V(A) = h((s_n(A))_{n=0}^{\infty})$.

Theorem 24. Let T, U be Banach spaces, $(p_r) \in \ell_\infty$ be increasing with $p_0 > 0$ and $t \geq 0$, then $((C(t, p))_h^{(s)}, V)$ is a prequasi Banach ideal.

Proof. Let the sufficient conditions be satisfied. From Theorem 13, the space $(C(t, p))_h$ is a premodular (sss). Therefore, by Theorem 12, the function $V(A) = h((s_n(A))_{n=0}^{\infty})$ is a prequasi norm on $(C(t, p))_h^{(s)}$. Let (A_i) be a Cauchy sequence in $(C(t, p))_h^{(s)}(T, U)$, since $\mathcal{B}(T, U) \supseteq (C(t, p))_h^{(s)}(T, U)$; we get

$$\begin{aligned}
V(A_i - A_j) &= \sum_{n=0}^{\infty} \left(\frac{\sum_{m=2^{n-1}}^{2^{n+1}-2} (m+1)^{-t} s_m(A_i - A_j)}{2^n} \right)^{p_n} \\
&\geq \|A_i - A_j\|^{p_0}.
\end{aligned} \tag{28}$$

Hence, $(A_i)_{i \in \mathbb{N}}$ is a Cauchy sequence in the Banach space $\mathcal{B}(T, U)$. Therefore, there is $A \in \mathcal{B}(T, U)$ such that $\lim_{i \rightarrow \infty} \|A_i - A\| = 0$, since $(s_r(A_i))_{r=0}^{\infty} \in (C(t, p))_h$ for every $i \in \mathbb{N}$.

By using parts (ii), (iii), and (iv) of Definition (6), one can see

$$\begin{aligned}
 V(A) &= \sum_{i=0}^{\infty} \left(\frac{\sum_{j=2^{i-1}}^{2^{i+1}-2} (j+1)^{-t} s_j(A)}{2^i} \right)^{p_i} \\
 &= \sum_{i=0}^{\infty} \left(\frac{\sum_{j=2^{i-1}}^{2^{i+1}-2} (j+1)^{-t} s_j(A - A_m + A_m)}{2^i} \right)^{p_i} \\
 &\leq M \sum_{i=0}^{\infty} \left(\frac{\sum_{j=2^{i-1}}^{2^{i+1}-2} (j+1)^{-t} s_{[j/2]}(A - A_m)}{2^i} \right)^{p_i} \\
 &\quad + M \sum_{i=0}^{\infty} \left(\frac{\sum_{j=2^{i-1}}^{2^{i+1}-2} (j+1)^{-t} s_{[j/2]}(A_m)}{2^i} \right)^{p_i} \\
 &\leq M \sum_{i=0}^{\infty} \left(\frac{\sum_{j=2^{i-1}}^{2^{i+1}-2} (j+1)^{-t} \|A_m - A\|}{2^i} \right)^{p_i} \\
 &\quad + M M_0 \sum_{i=0}^{\infty} \left(\frac{\sum_{j=2^{i-1}}^{2^{i+1}-2} (j+1)^{-t} s_j(A_m)}{2^i} \right)^{p_i} < \varepsilon.
 \end{aligned} \tag{29}$$

Therefore, $(s_r(A))_{r=0}^{\infty} \in (C(t, p))_h$; hence, $A \in (C(t, p))_h^{(s)}(T, U)$.

Theorem 25. Let T, U be normed spaces and $(p_r) \in \ell_{\infty}$ be increasing with $p_0 > 0$ and $t \geq 0$, then $((C(t, p))_h^{(s)}, V)$ is a prequasi closed ideal.

Proof. Let the sufficient conditions be satisfied. From Theorem 13, the space $(C(t, p))_h$ is a premodular (sss). Therefore, by Theorem 12, the function $V(A) = h((s_n(A))_{n=0}^{\infty})$ is a prequasi norm on $(C(t, p))_h^{(s)}$. Assume $A_m \in (C(t, p))_h^{(s)}(T, U)$ for every $m \in \mathbb{N}$ and $\lim_{m \rightarrow \infty} V(A_m - A) = 0$, since $\mathcal{B}(T, U) \supseteq (C(t, p))_h^{(s)}(T, U)$, we have

$$\begin{aligned}
 V(A - A_j) &= \sum_{n=0}^{\infty} \left(\frac{\sum_{m=2^{n-1}}^{2^{n+1}-2} (m+1)^{-t} s_m(A - A_j)}{2^n} \right)^{p_n} \\
 &\geq \|A - A_j\|^{p_0}.
 \end{aligned} \tag{30}$$

Hence, $(A_j)_{j \in \mathbb{N}}$ is a convergent sequence in $\mathcal{B}(T, U)$. So, $(s_r(A_j))_{r=0}^{\infty} \in (C(t, p))_h$ for each $j \in \mathbb{N}$. By using parts (ii), (iii), and (iv) of Definition (6), we have

$$\begin{aligned}
 V(A) &= \sum_{i=0}^{\infty} \left(\frac{\sum_{j=2^{i-1}}^{2^{i+1}-2} (j+1)^{-t} s_j(A)}{2^i} \right)^{p_i} \\
 &= \sum_{i=0}^{\infty} \left(\frac{\sum_{j=2^{i-1}}^{2^{i+1}-2} (j+1)^{-t} s_j(A - A_m + A_m)}{2^i} \right)^{p_i} \\
 &\leq M \sum_{i=0}^{\infty} \left(\frac{\sum_{j=2^{i-1}}^{2^{i+1}-2} (j+1)^{-t} s_{[j/2]}(A - A_m)}{2^i} \right)^{p_i}
 \end{aligned}$$

$$\begin{aligned}
 &+ M \sum_{i=0}^{\infty} \left(\frac{\sum_{j=2^{i-1}}^{2^{i+1}-2} (j+1)^{-t} s_{[j/2]}(A_m)}{2^i} \right)^{p_i} \\
 &\leq M \sum_{i=0}^{\infty} \left(\frac{\sum_{j=2^{i-1}}^{2^{i+1}-2} (j+1)^{-t} \|A_m - A\|}{2^i} \right)^{p_i} \\
 &\quad + M M_0 \sum_{i=0}^{\infty} \left(\frac{\sum_{j=2^{i-1}}^{2^{i+1}-2} (j+1)^{-t} s_j(A_m)}{2^i} \right)^{p_i} < \varepsilon.
 \end{aligned} \tag{31}$$

Therefore, $(s_r(A))_{r=0}^{\infty} \in (C(t, p))_h$; hence, $A \in (C(t, p))_h^{(s)}(T, U)$.

6. Smallness and Simplicity of the Prequasi Ideal

We examine here some inclusion relations and topological structures of $(C(t, p))_h^{(s)}$.

Theorem 26. Let T or U be finite dimensional Banach spaces, $t_q > t_p > 0$ and increasing sequences $(p_n) \in \ell_{\infty}$, $(q_n) \in \ell_{\infty}$ with $0 < p_r < q_r$, for all $r \in \mathbb{N}$, then

$$(C(t_p, p))_h^{(s)}(T, U) \mathbb{P}(C(t_q, q))_h^{(s)}(T, U) \mathcal{UB}(T, U). \tag{32}$$

Proof. Let the conditions be satisfied, if $A \in (C(t_p, p))_h^{(s)}(T, U)$, then $(s_j(A)) \in C(t_p, p)$. We have

$$\begin{aligned}
 &\sum_{i=0}^{\infty} \left(\frac{\sum_{j=2^{i-1}}^{2^{i+1}-2} (j+1)^{-t_q} s_j(A)}{2^i} \right)^{q_i} \\
 &< \sum_{i=0}^{\infty} \left(\frac{\sum_{j=2^{i-1}}^{2^{i+1}-2} (j+1)^{-t_p} s_j(A)}{2^i} \right)^{p_i} < \infty,
 \end{aligned} \tag{33}$$

hence, $A \in (C(t_q, q))_h^{(s)}(T, U)$. Next, if we take $(s_j(A))_{j=0}^{\infty}$ such that $\sum_{j=2^{i-1}}^{2^{i+1}-2} (j+1)^{-t_p} s_j(A) = 2^i(i+1)^{-(1/p_i)}$, one can find $A \in \mathcal{B}(T, U)$ with

$$\begin{aligned}
 \sum_{i=0}^{\infty} \left(\frac{\sum_{j=2^{i-1}}^{2^{i+1}-2} (j+1)^{-t_p} s_j(A)}{2^i} \right)^{p_i} &= \sum_{i=0}^{\infty} \frac{1}{i+1} = \infty, \\
 \sum_{i=0}^{\infty} \left(\frac{\sum_{j=2^{i-1}}^{2^{i+1}-2} (j+1)^{-t_q} s_j(A)}{2^i} \right)^{q_i} &\leq \sum_{i=0}^{\infty} \left(\frac{\sum_{j=2^{i-1}}^{2^{i+1}-2} (j+1)^{-t_p} s_j(A)}{2^i} \right)^{q_i} \\
 &= \sum_{i=0}^{\infty} \left(\frac{1}{i+1} \right)^{q_i/p_i} < \infty.
 \end{aligned} \tag{34}$$

Hence, $A \notin (C(t_p, p))_h^{(s)}(T, U)$ and $A \in (C(t_q, q))_h^{(s)}(T, U)$. Clearly, $(C(t_q, q))_h^{(s)}(T, U) \subset \mathcal{B}(T, U)$. Secondly, by choosing $(s_j(A))_{j=0}^{\infty}$ such that $\sum_{j=2^{i-1}}^{2^{i+1}-2} (j+1)^{-t_q} s_j(A) = 2^i(i+1)^{-(1/q_i)}$.

Hence, there is $A \in \mathcal{B}(T, U)$ with $A \notin (C(t_q, q))_h^{(s)}(T, U)$. This completes the proof.

We study in the next theorem that $(C(t, p))_h^{(s)}$ is not small.

Theorem 27. *If T and U are infinite dimensional Banach spaces, $(p_r) \in \ell_\infty$ is increasing with $p_0 > 0$ and $t > 0$, then $(C(t, p))_h^{(s)}$ is not small.*

Proof. Let the conditions be satisfied. Therefore, $(C(t, p))_h^{(s)}$ is a prequasi operator ideal. For all $A \in \mathcal{B}(T, U)$, there is $C > 0$ such that $\|A\| \leq C$; we have

$$\begin{aligned} h(A) &= \sum_{i=0}^{\infty} \left(\frac{\sum_{j=2^{i-1}}^{2^{i+1}-2} (j+1)^{-t} s_j(IA)}{2^i} \right)^{p_i} \\ &\leq \sup_n C^{p_n} \sum_{i=0}^{\infty} \left(\frac{\sum_{j=2^{i-1}}^{2^{i+1}-2} (j+1)^{-t} s_j(I)}{2^i} \right)^{p_i} \\ &\leq \sup_n C^{p_n} \sum_{i=0}^{\infty} \frac{1}{2^{itp_0}} < \infty. \end{aligned} \quad (35)$$

Hence, $A \in (C(t, p))_h^{(s)}(T, U)$ which gives $(C(t, p))_h^{(s)}(T, U) = \mathcal{B}(T, U)$. This finishes the proof.

We investigate here the simplicity of $(C(t, p))_h^{(s)}$.

Theorem 28. *For any finite dimensional Banach spaces T or U , $t_q > t_p > 0$ and increasing sequences $(p_r) \in \ell_\infty$, $(q_r) \in \ell_\infty$ with $0 < p_r < q_r$, for every $r \in \mathbb{N}$, then*

$$\begin{aligned} \mathcal{B}\left((C(t_q, q))_h^{(s)}, (C(t_p, p))_h^{(s)}\right) \\ = \mathcal{A}\left((C(t_q, q))_h^{(s)}, (C(t_p, p))_h^{(s)}\right). \end{aligned} \quad (36)$$

Proof. Let there is $D \in \mathcal{B}((C(t_q, q))_h^{(s)}, (C(t_p, p))_h^{(s)})$ with $D \notin \mathcal{A}((C(t_q, q))_h^{(s)}, (C(t_p, p))_h^{(s)})$. By Lemma 1, we have $A \in \mathcal{B}((C(t_q, q))_h^{(s)})$ and $B \in \mathcal{B}((C(t_p, p))_h^{(s)})$ with $\text{BDAI}_k = I_k$. Then, it follows, for all $k \in \mathbb{N}$ that

$$\begin{aligned} \|I_k\|_{(C(t_p, p))_h^{(s)}} &= \sum_{i=0}^{\infty} \left(\frac{\sum_{j=2^{i-1}}^{2^{i+1}-2} (j+1)^{-t_p} s_j(I_k)}{2^i} \right)^{p_i} \\ &\leq \|\text{BDA}\| \|I_k\|_{(C(t_q, q))_h^{(s)}} \\ &\leq \sum_{i=0}^{\infty} \left(\frac{\sum_{j=2^{i-1}}^{2^{i+1}-2} (j+1)^{-t_q} s_j(I_k)}{2^i} \right)^{q_i}. \end{aligned} \quad (37)$$

By Theorem 26, we get a contradiction. Therefore, $D \in \mathcal{A}((C(t_q, q))_h^{(s)}, (C(t_p, p))_h^{(s)})$.

Theorem 29. *For any finite dimensional Banach spaces T or U . If $(p_r) \in \ell_\infty$ is increasing with $p_0 > 1$, then $(C(t, p))_h^{(s)}$ is simple.*

Proof. Assume there is $D \in \mathcal{K}((C(t, p))_h^{(s)})$ with $D \notin \mathcal{A}((C(t, p))_h^{(s)})$. By Lemma 1, we have $A, B \in \mathcal{B}((C(t, p))_h^{(s)})$ with $\text{BDAI}_k = I_k$. This means that $I_{(C(t, p))_h^{(s)}} \in \mathcal{K}((C(t, p))_h^{(s)})$. Consequently, $\mathcal{B}(S_{(C(t, p))_h^{(s)}}) = \mathcal{K}((C(t, p))_h^{(s)})$. Therefore, $\mathcal{A}((C(t, p))_h^{(s)})$ is the only nontrivial closed ideal in $\mathcal{B}((C(t, p))_h^{(s)})$.

We study the approximation of the prequasi ideal $(C(t, p))_h^{(s)}$.

Theorem 30. *For any finite dimensional Banach spaces T or U . If $(p_r) \in \ell_\infty$ is increasing with $p_0 > 0$, then $\mathcal{F}(\bar{T}, U) = (C(t, p))_h^{(s)}(T, U)$.*

Proof. Since $(C(t, p))_h$ is a premodular (sss), then from Theorem 11, we have $\mathcal{F}(\bar{T}, U) = (C(t, p))_h^{(s)}(T, U)$. Conversely, since $I_4 \in S_{(C(1,1))_h}$ but $p_0 = 0$. This gives a counter example.

7. Spectrum of s-Type Operators

We explain the spectrum of the prequasi ideal $(C(t, p))_h^{(s)}$.

Theorem 31. *For any finite dimensional Banach spaces T or U . If $(p_r) \in \ell_\infty$ is increasing with $p_0 > 0$, then*

$$(C(t, p))_h^{(\lambda)}(T, U) = (C(t, p))_h^{(s)}(T, U). \quad (38)$$

Proof. Let $A \in (C(t, p))_h^{(\lambda)}(T, U)$, then $(\lambda_r(A))_{r=0}^\infty \in C(t, p)$ and $\|A - \lambda_n(A)I\| = 0$, for every $r \in \mathbb{N}$. Therefore, $A = \lambda_r(A)I$, for every $r \in \mathbb{N}$, then $s_r(A) = s_r(\lambda_r(A)I) = |\lambda_r(A)|$, for every $r \in \mathbb{N}$. Hence $(s_r(A))_{r=0}^\infty \in C(t, p)$, so $A \in (C(t, p))_h^{(s)}(T, U)$. Therefore, the sequence $(s_r(A))_{r=0}^\infty$ is the eigenvalues of A . Secondly, let the sufficient conditions be verified and $A \in (C(t, p))_h^{(s)}(T, U)$. Therefore, $(s_r(A))_{r=0}^\infty \in C(t, p)$. Hence, we have

$$\sum_{i=0}^{\infty} \left(\frac{\sum_{j=2^{i-1}}^{2^{i+1}-2} (j+1)^{-t} s_j(A)}{2^i} \right)^{p_i} \geq \sum_{i=0}^{\infty} [s_i(A)]^{p_i}. \quad (39)$$

Hence, $(s_i(A))_{i=0}^\infty \in \ell_{(p_i)}$, so $\lim_{i \rightarrow \infty} s_i(A) = 0$. Assume $\|A - s_i(A)I\|^{-1}$ exists, for every $i \in \mathbb{N}$. Therefore, $\|A - s_i(A)I\|^{-1}$ exists and is bounded, for every $i \in \mathbb{N}$. So, $\lim_{i \rightarrow \infty} \|A - s_i(A)I\|^{-1} = \|A\|^{-1}$ exists and is bounded. From the prequasi operator ideal of $((C(t, p))_h^{(s)}, V)$, we obtain

$$I = AA^{-1} \in (C(t, p))_h^{(s)}(T, U) \Rightarrow (s_i(I))_{i=0}^\infty \in C(t, p) \Rightarrow \lim_{i \rightarrow \infty} s_i(I) = 0. \quad (40)$$

We have a contradiction, since $\lim_{i \rightarrow \infty} s_i(I) = 1$. Therefore, $\|A - s_i(A)I\| = 0$, for every $i \in \mathbb{N}$. This gives $A \in (C(t, p))_h^{(\lambda)}(T, U)$.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Ethical Approval

This article does not contain any studies with human participants or animals performed by any of the authors.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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References

- [1] R. K. Singh and A. Kumar, "Multiplication operators and composition operators with closed ranges," *Bulletin of the Australian Mathematical Society*, vol. 16, no. 2, pp. 247–252, 1977.
- [2] M. B. Abrahamse, "Multiplication operators," in *Hilbert Space Operators*, vol. 693, pp. 17–36, Springer Verlag, 1978.
- [3] K. Raj, S. K. Sharma, and A. Kumar, "Multiplication operators on Musielak-Orlicz spaces of Bochner type," *Journal of Advanced Studies in Topology*, vol. 3, no. 4, pp. 1–7, 2012.
- [4] A. Sharma, K. Raj, and S. K. Sharma, "Products of multiplication composition and differentiation operators from H^∞ to weighted Bloch spaces," *Indian Journal of Mathematics*, vol. 54, pp. 159–179, 2012.
- [5] R. K. Singh and J. S. Manhas, *Composition Operators on Function Spaces*, North Holland, North Holland, 1993.
- [6] H. Takagi and K. Yokouchi, "Multiplication and composition operators between two L_p -spaces," *Contemporary Mathematics*, vol. 232, pp. 321–338, 1999.
- [7] E. E. Kara and M. İlkhani, "On a new class of s -type operators," *Konuralp Journal of Mathematics*, vol. 3, no. 1, pp. 1–11, 2015.
- [8] P. Z. Alp and E. E. Kara, "A new class of operator ideals and approximation numbers," *New Trends in Mathematical Science*, vol. 4, no. 6, pp. 8–15, 2018.
- [9] A. P. Zengin and E. E. Kara, "Some equivalent quasinorms on $L_{p,E}$," *Facta Universitatis-Series Mathematics and Informatics*, vol. 33, no. 5, pp. 739–746, 2018.
- [10] P. Z. Alp and E. E. Kara, "The new class $L_{z,p,E}$ of s -type operators," *AIMS Mathematics*, vol. 4, no. 3, pp. 779–791, 2019.
- [11] M. İlkhani, S. Demiriz, and E. E. Kara, "Multiplication operators on Cesàro second order function spaces," *Positivity*, vol. 24, no. 3, pp. 605–614, 2020.
- [12] T. Bilgin, "The sequence space $C(s, p)$ and related matrix transformations," *The Punjab University Journal of Mathematics*, vol. 30, pp. 67–77, 1997.
- [13] K. P. Lim, "Matrix transformation on certain sequence space," *Tamkang Journal of Mathematics*, vol. 8, pp. 213–220, 1977.
- [14] K. P. Lim, "Matrix transformation in the Cesaro sequence spaces," *Kyungpook Mathematical Journal*, vol. 14, pp. 221–227, 1974.
- [15] A. Pietsch, *Eigenvalues and s -Numbers*, Cambridge University Press, New York, NY, USA, 1986.
- [16] N. Faried and A. A. Bakery, "Small operator ideals formed by s numbers on generalized Cesàro and Orlicz sequence spaces," *Journal of Inequalities and Applications*, vol. 2018, no. 1, 2018.
- [17] A. Pietsch, *Operator Ideals*, no. 81j:47001, article MR582655, 1980 North-Holland Publishing Company, Amsterdam-New York-Oxford, 1980.
- [18] A. Pietsch, "Small ideals of operators," *Studia Mathematica*, vol. 51, no. 3, pp. 265–267, 1974.
- [19] B. M. Makarov and N. Faried, *Some Properties of Operator Ideals Constructed by s Numbers (in Russian)*, *Theory of Operators in Functional Spaces*, Siberian Division of the Russian Academy of Sciences, Novosibirsk, Russia, 1977.
- [20] M. Mursaleen and A. K. Noman, "Compactness by the Hausdorff measure of noncompactness," *Nonlinear Analysis*, vol. 73, no. 8, pp. 2541–2557, 2010.
- [21] M. Mursaleen and A. K. Noman, "Compactness of matrix operators on some new difference sequence spaces," *Linear Algebra and its Applications*, vol. 436, no. 1, pp. 41–52, 2012.
- [22] B. S. Komal, S. Pandoh, and K. Raj, "Multiplication operators on Cesàro sequence spaces," *Demonstratio Mathematica*, vol. 49, no. 4, 2016.
- [23] T. Mrowka, *A Brief Introduction to Linear Analysis: Fredholm Operators, Geometry of Manifolds*, Massachusetts Institute of Technology: MIT OpenCourseWare, 2004.
- [24] A. A. Bakery and M. M. Mohammed, "Small pre-quasi Banach operator ideals of type Orlicz-Cesàro mean sequence spaces," *Journal of Function Spaces*, vol. 2019, Article ID 7265010, 9 pages, 2019.
- [25] A. A. Bakery and M. M. Mohammed, "Some properties of pre-quasi operator ideal of type generalized Cesàro sequence space defined by weighted means," *Open Mathematics*, vol. 17, no. 1, pp. 1703–1715, 2019.
- [26] B. Altay and F. Başar, "Generalization of the sequence space $\ell(p)$ derived by weighted mean," *Journal of Mathematical Analysis and Applications*, vol. 330, no. 1, pp. 174–185, 2007.