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Research Article

Nonexistence Results for Some Classes of Nonlinear Fractional Differential Inequalities

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We study the nonexistence of global solutions for new classes of nonlinear fractional differential inequalities. Namely, sufficient conditions are provided so that the considered problems admit no global solutions. The proofs of our results are based on the test function method and some integral estimates.

1. Introduction

We first consider the problem

$$\begin{cases} {}^C D_0^{1+\alpha} u(t) + {}^C D_0^{1+\beta} u(t) \geq \left| {}^C D_0^{\gamma} u(t) \right|^p, & t > 0, \\ \left(u(0), u'(0) \right) = (u_0, u_1), \end{cases} \tag{1}$$

where $p > 1$, $\alpha, \beta, \gamma \in (0, 1)$, ${}^C D_0^{\kappa}$, $\kappa \in \{1 + \alpha, 1 + \beta, \gamma\}$ is the Caputo fractional derivative of order κ, $u_0 \in \mathbb{R}$, and $u_1 \geq 0$. Namely, we are interested in providing sufficient conditions for which problem (1) admits no global solution. Next, we study the same question for the inhomogeneous problem

$$\begin{cases} {}^C D_0^{1|+\alpha} u(t) + {}^C D_0^{1|+\beta} u(t) \geq \left| {}^C D_0^{\gamma} u(t) \right|^p + f(t), & t > 0, \\ \left(u(0), u'(0) \right) = (u_0, u_1), \end{cases} \tag{2}$$

where $p, q > 1$, $\alpha, \beta, \gamma \in (0, 1)$, $u_0 \in \mathbb{R}$, $u_1 \geq 0$, $f \in L^1_{loc}([0,\infty))$, $f \geq 0$, and $f \not\equiv 0$.

Due to the importance of fractional calculus in applications (see e.g. [1–5]), in the past few decades, there has been a growing interest in the study of fractional differential equations. In particular, from the theoretical point of view, the existence of solutions for different classes of fractional differential equations was investigated in many contributions (see e.g. [6–12] and the references therein).

For the issue of nonexistence of solutions for fractional differential equations and inequalities, we refer to [13–22] and the references therein. In particular, in [17], Laskri and Tatar studied the problem

$$\begin{cases} D_0^{\alpha} u(t) \geq t^{\gamma} |y(t)|^p, & t > 0, \\ I_0^{1-\alpha} u(t)\big|_{t=0} = b, \end{cases} \tag{3}$$

where $p > 1$, $0 < \alpha < 1$, $\gamma > -\alpha$, and $b \geq 0$, D_0^{α} is the Riemann-Liouville fractional derivative of order α, and $I_0^{1-\alpha}$ is the left-sided Riemann-Liouville fractional integral of order $1 - \alpha$. It was shown that, if $p \leq (\gamma + 1/1 - \alpha)$, then problem (3) does not admit nontrivial global solution. In [16], Kassim et al. studied the problem

$$\begin{cases} {}^C D_0^{\alpha} u(t) + {}^C D_0^{\beta} u(t) \geq t^{\gamma} |y(t)|^p, & t > 0, \\ u^{(i)}(0) = b_i, & i = 0, 1, \cdots, n-1, \end{cases} \tag{4}$$

where $p > 1$, $n \geq 1$ is an integer, $n-1 < \beta \leq \alpha < n$, and $b_i \geq 0$. It was shown that, if

$$p(1-\beta)-1 < \gamma < p-1, \tag{5}$$

then problem (4) does not admit nontrivial global solution. In [15], Furati and Kirane investigated the system of nonlinear fractional differential equations

$$\begin{cases} u'(t)+{}^{C}D_0^{\alpha}u(t) = |v(t)|^q, & t>0, \\ v'(t)+{}^{C}D_0^{\beta}v(t) = |u(t)|^p, & t>0, \end{cases} \tag{6}$$

subject to the initial conditions

$$(u(0), v(0)) = (u_0, v_0), \tag{7}$$

where $0 < \alpha, \beta < 1$, $p, q > 1$, and $u_0, v_0 > 0$. It was shown that, if

$$1 - \frac{1}{pq} \leq \max\left\{\alpha + \frac{\beta}{p}, \beta + \frac{\alpha}{q}\right\}, \tag{8}$$

then solutions to system (6) subject to (7) blow up in a finite time.

For the issue of nonexistence of global solutions for fractional in time evolution equations, we refer to [6, 23–25] and the references therein.

On the other hand, to the best of our knowledge, the nonexistence of global solutions for problems of types (1) and (2) was not yet investigated.

Before stating our main results, let us mention what we mean by global solutions to problems (1) and (2).

Definition 1. A function $u \in AC^2([0,\infty))$ is said to be a global solution to problem (1), if u satisfies

$${}^{C}D_0^{1+\alpha}u(t)+{}^{C}D_0^{1+\beta}u(t) \geq \left|{}^{C}D_0^{\gamma}u(t)\right|^p, \tag{9}$$

for almost every where $t > 0$, and

$$\left(u(0), u'(0)\right) = (u_0, u_1). \tag{10}$$

Definition 2. A function $u \in AC^2([0,\infty))$ is said to be a global solution to problem (2), if u satisfies

$${}^{C}D_0^{1+\alpha}u(t)+{}^{C}D_0^{1+\beta}u(t) \geq \left|{}^{C}D_0^{\gamma}u(t)\right|^p + f(t), \tag{11}$$

for almost every where $t > 0$, and

$$\left(u(0), u'(0)\right) = (u_0, u_1). \tag{12}$$

We first consider problem (1). We discuss separately the cases $u_1 > 0$ and $u_1 = 0$.

Theorem 3. *Let $\alpha, \beta, \gamma \in (0,1)$, and $u_0 \in \mathbb{R}$. If $u_1 > 0$, then for all $p > 1$, problem (1) admits no global solution.*

Theorem 4. *Let $\alpha, \beta, \gamma \in (0,1)$, $\alpha \leq \beta$, $u_0 \in \mathbb{R}$, and $u_1 = 0$.*

(i) If $\gamma \leq \alpha$, then for all $p > 1$, the only global solution to problem (1) is $u \equiv u_0$

(ii) If $\gamma > \alpha$, then for all

$$1 < p < \frac{1}{\gamma - \alpha}, \tag{13}$$

the only global solution to problem (1) is $u \equiv u_0$.

Next, we consider problem (2).

Theorem 5. *Let $\alpha, \beta, \gamma \in (0,1)$, $u_0 \in \mathbb{R}$, $f \in L^1_{loc}([0,\infty))$, $f \geq 0$, and $f \not\equiv 0$. If $u_1 > 0$, then for all $p > 1$, problem (2) admits no global solution.*

Theorem 6. *Let $p > 1$, $\alpha, \beta, \gamma \in (0,1)$, $\alpha \leq \beta$, $u_0 \in \mathbb{R}$, $u_1 = 0$, $f \in L^1_{loc}([0,\infty))$, $f \geq 0$, and $f \not\equiv 0$. If*

$$\limsup_{T\to+\infty} T^{(((\alpha-\gamma)p+1)/(p-1))} \int_0^{T/2} f(t)\,dt = +\infty, \tag{14}$$

then problem (2) admits no global solution.

We discuss below some special cases of Theorem 6.

Corollary 7. *Let $\alpha, \beta, \gamma \in (0,1)$, $\alpha \leq \beta$, $u_0 \in \mathbb{R}$, and $u_1 = 0$. Let*

$$f(t) = e^{at}, \quad t > 0, \tag{15}$$

where $a \in \mathbb{R}$ and $a \neq 0$.

(i) If $a > 0$, then for all $p > 1$, problem (2) admits no global solution

(ii) If $a < 0$ and $\gamma \leq \alpha$, then for all $p > 1$, problem (2) admits no global solution

(iii) If $a < 0$ and $\gamma > \alpha$, then for all

$$1 < p < \frac{1}{\gamma - \alpha}, \tag{16}$$

problem (2) admits no global solution.

Corollary 8. *Let $\alpha, \beta, \gamma \in (0,1)$, $\alpha \leq \beta$, $u_0 \in \mathbb{R}$, and $u_1 = 0$. Let*

$$f(t) = t^{\sigma}, \quad t > 0, \tag{17}$$

where $\sigma > -1$.

(i) *If $\sigma \geq 0$, then for all $p > 1$, problem (2) admits no global solution*

(ii) *Let $-1 < \sigma < 0$*

(a) *If $\gamma \leq \alpha$, then for all $p > 1$, problem (2) admits no global solution*

(b) *If $\gamma > \alpha$ and $\gamma - \alpha - 1 \leq \sigma < 0$, then for all $p > 1$, problem (2) admits no global solution*

(c) *If $\gamma > \alpha$ and $-1 < \sigma < \gamma - \alpha - 1$, then for all*

$$1 < p < \frac{\sigma}{\sigma - \gamma + \alpha + 1}, \tag{18}$$

problem (2) admits no global solution.

The rest of the paper is organized as follows. In Section 2, we recall briefly some standard notions on fractional calculus and prove some properties. Section 3 is devoted to the Proofs of Theorems 3, 4, 5, and 6 and Corollaries 7 and 8.

2. Some Preliminaries

We denote by $AC([0,\infty))$ the space of absolutely continuous functions on $[0,\infty)$. Given an integer $n \geq 2$, we denote by $AC^n([0,\infty))$ the space of functions f which have continuous derivatives up to order $n-1$ on $[0,\infty)$ such that $f^{(n-1)} \in AC([0,\infty))$. Here, $f^{(n-1)}$ denotes the derivative of order $n-1$ of f.

Let $T > 0$ be fixed. Given $\rho > 0$ and $f \in L^1(0,T)$, the left-sided Riemann-Liouville fractional integral of order ρ of f is defined by

$$\left(I_0^\rho f\right)(t) = \frac{1}{\Gamma(\rho)} \int_0^t (t-s)^{\rho-1} f(s)\, ds, \tag{19}$$

for almost everywhere $0 \leq t \leq T$. Here, Γ denotes the Gamma function. The right-sided Riemann-Liouville fractional integral of order ρ of f is defined by

$$\left(I_T^\rho f\right)(t) = \frac{1}{\Gamma(\rho)} \int_t^T (s-t)^{\rho-1} f(s)\, ds, \tag{20}$$

for almost everywhere $0 \leq t \leq T$. Notice that, if $f \in C([0,T)]$, then $I_0^\rho f$ is defined for all $0 < t \leq T$. Moreover, one has $\lim_{t\to 0^+} (I_0^\rho f)(t) = 0$. Similarly, if $f \in C([0,T)]$, then $I_T^\rho f$ is defined for all $0 \leq t < T$. Moreover, one has $\lim_{t\to T^-} (I_T^\rho f)(t) = 0$.

Lemma 9 (see [5]). *Let $\rho, \kappa > 0$ and $f \in L^\tau(0,T)$, where $1 \leq \tau \leq \infty$. Then*

$$I_0^\rho (I_0^\kappa f)(t) = I_0^\kappa \left(I_0^\rho f\right)(t) = \left(I_0^{\rho+\kappa} f\right)(t), \tag{21}$$

for almsot everywhere $0 \leq t \leq T$.

Lemma 10 (see [5]). *Let $\rho > 0$, $\tau, \mu \geq 1$, and $(1/\tau) + (1/\mu) \leq 1 + \rho$ ($\tau \neq 1$ and $\mu \neq 1$ in the case $(1/\tau) + (1/\mu) = 1 + \rho$). If $f \in L^\tau(0,T)$ and $g \in L^\mu(0,T)$, then*

$$\int_0^T \left(I_0^\rho f\right)(t) g(t)\, dt = \int_0^T \left(I_T^\rho g\right)(t) f(t)\, dt. \tag{22}$$

Let $n - 1 < \rho < n$ and $f \in AC^n([0,\infty))$, where $n \geq 1$ is an integer. The (left-sided) Caputo fractional derivative of order ρ of f is defined by

$$\left({}^C D_0^\rho f\right)(t) = \left(I_0^{n-\rho} f^{(n)}\right)(t), \tag{23}$$

for almost everywhere $t > 0$. Here, for $n = 1$, $AC^1([0,\infty)) = AC([0,\infty))$.

For $\lambda \gg 1$ (λ is large enough), we define the function

$$\xi(t) = T^{-\lambda}(T-t)^\lambda, \quad 0 \leq t \leq T. \tag{24}$$

Lemma 11. *Let $\rho > 0$ and $0 < \kappa < 1$. Then*

$$\left(I_T^\rho \xi\right)(t) = \frac{\Gamma(\lambda+1)}{\Gamma(\rho+\lambda+1)} T^{-\lambda}(T-t)^{\lambda+\rho}, \quad 0 \leq t < T, \tag{25}$$

$$\left(I_T^\rho \xi\right)'(t) = \frac{-\Gamma(\lambda+1)}{\Gamma(\rho+\lambda)} T^{-\lambda}(T-t)^{\lambda+\rho-1}, \quad 0 \leq t < T, \tag{26}$$

$$\left(I_T^\rho \xi\right)''(t) = \frac{\Gamma(\lambda+1)}{\Gamma(\rho+\lambda-1)} T^{-\lambda}(T-t)^{\lambda+\rho-2}, \quad 0 \leq t < T, \tag{27}$$

$$I_T^\rho \left[\left(I_T^{1-\kappa} \xi\right)''\right](t) = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+\rho-\kappa)} T^{-\lambda}(T-t)^{\lambda+\rho-1-\kappa}, \quad 0 \leq t < T. \tag{28}$$

Proof. We prove only (25). Namely, differentiating (25), (26) follows. Similarly, differentiating (26), (27) follows. Moreover, taking $\rho = 1 - \kappa$ in (27) and using a similar calculation as in the proof of (25), (28) follows.

For $t \in [0,T)$, one has

$$\begin{aligned}\left(I_T^\rho \xi\right)(t) &= \frac{T^{-\lambda}}{\Gamma(\rho)} \int_t^T (s-t)^{\rho-1}(T-s)^\lambda\, ds \\ &= \frac{T^{-\lambda}}{\Gamma(\rho)} \int_t^T (s-t)^{\rho-1}((T-t)-(s-t))^\lambda\, ds \\ &= \frac{T^{-\lambda}(T-t)^\lambda}{\Gamma(\rho)} \int_t^T (s-t)^{\rho-1}\left(1 - \frac{s-t}{T-t}\right)^\lambda ds.\end{aligned} \tag{29}$$

Using the change of variable $z = (s - t/T - t)$, one obtains

$$\begin{aligned}\left(I_T^{\rho}\xi\right)(t) &= \frac{T^{-\lambda}(T-t)^{\lambda+\rho}}{\Gamma(\rho)}\int_0^1 z^{\rho-1}(1-z)^{(\lambda+1)-1}\,dz \\ &= \frac{T^{-\lambda}(T-t)^{\lambda+\rho}}{\Gamma(\rho)}B(\rho,\lambda+1),\end{aligned} \tag{30}$$

where $B(\cdot,\cdot)$ is the beta function. Using the property

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}, \quad a,b>0, \tag{31}$$

one obtains (25).

3. Proofs

The proofs of our results are based on the test function method (see e.g. [26])and some integral estimates.

Proof of Theorem 3. Let us suppose that $u \in AC^2([0,\infty))$ is a global solution to (1). For $T>0$, multiplying the differential inequality in (1) by ξ, where ξ is the function defined by (24), and integrating over $(0,T)$, one obtains

$$\int_0^T \left|{}^C D_0^{\gamma}u(t)\right|^p\xi(t)\,dt \le \int_0^T {}^C D_0^{1+\alpha}u(t)\xi(t)\,dt + \int_0^T {}^C D_0^{1+\beta}u(t)\xi(t)\,dt. \tag{32}$$

Without restriction of the generality, we may suppose that

$$0<\alpha\le\beta<1. \tag{33}$$

On the other hand, using Lemma 10, one obtains

$$\begin{aligned}\int_0^T {}^C D_0^{1+\alpha}u(t)\xi(t)\,dt &= \int_0^T \left(I_0^{1-\alpha}u''\right)(t)\xi(t)\,dt \\ &= \int_0^T u''(t)\left(I_T^{1-\alpha}\xi\right)(t)\,dt.\end{aligned} \tag{34}$$

Using an integration by parts, the initial conditions and (25), it holds that

$$\begin{aligned}\int_0^T {}^C D_0^{1+\alpha}u(t)\xi(t)\,dt &= \left[u'(t)\left(I_T^{1-\alpha}\xi\right)(t)\right]_0^T - \int_0^T u'(t)\left(I_T^{1-\alpha}\xi\right)'(t)\,dt \\ &= -u_1\left(I_T^{1-\alpha}\xi\right)(0) - \int_0^T u'(t)\left(I_T^{1-\alpha}\xi\right)'(t)\,dt.\end{aligned} \tag{35}$$

On the other hand, by Lemma 9 and using the initial conditions, one obtains

$$\begin{aligned}u'(t) = (u(t)-u_0)' &= \left(\int_0^t u'(s)\,ds\right) = \left(I_0^1 u'\right)'(t) \\ &= \left[I_0^{\gamma}\left(I_0^{1-\gamma}u'\right)\right]'(t).\end{aligned} \tag{36}$$

Therefore, by (35), one obtains

$$\begin{aligned}\int_0^T {}^C D_0^{1+\alpha}u(t)\xi(t)\,dt = &-u_1\left(I_T^{1-\alpha}\xi\right)(0) \\ &- \int_0^T \left[I_0^{\gamma}\left(I_0^{1-\gamma}u'\right)\right](t)\left(I_T^{1-\alpha}\xi\right)'(t)\,dt.\end{aligned} \tag{37}$$

Using an integration by parts, the initial conditions, (26) and Lemma 10, it holds that

$$\begin{aligned}\int_0^T {}^C D_0^{1+\alpha}u(t)\xi(t)\,dt &= -u_1\left(I_T^{1-\alpha}\xi\right)(0) - \left[(u(t)-u_0)\left(I_T^{1-\alpha}\xi\right)'(t)\right]_0^T \\ &\quad - \int_0^T I_0^{\gamma}\left(I_0^{1-\gamma}u'\right)(t)\left(I_T^{1-\alpha}\xi\right)''(t)\,dt \\ &= -u_1\left(I_T^{1-\alpha}\xi\right)(0) \\ &\quad - \int_0^T \left(I_0^{1-\gamma}u'\right)(t)I_T^{\gamma}\left[\left(I_T^{1-\alpha}\xi\right)''\right](t)\,dt \\ &= -u_1\left(I_T^{1-\alpha}\xi\right)(0) - \int_0^T {}^C D_0^{\gamma}u(t)I_T^{\gamma}\left[\left(I_T^{1-\alpha}\xi\right)''\right](t)\,dt.\end{aligned} \tag{38}$$

Similarly, one has

$$\begin{aligned}\int_0^T {}^C D_0^{1+\beta}u(t)\xi(t)\,dt = &-u_1\left(I_T^{1-\beta}\xi\right)(0) \\ &- \int_0^T {}^C D_0^{\gamma}u(t)I_T^{\gamma}\left[\left(I_T^{1-\beta}\xi\right)''\right](t)\,dt.\end{aligned} \tag{39}$$

Next, using (32), (38), and (39), one obtains

$$\begin{aligned}&\int_0^T \left|{}^C D_0^{\gamma}u(t)\right|^p\xi(t)\,dt + u_1\left(\left(I_T^{1-\alpha}\xi\right)(0) + \left(I_T^{1-\beta}\xi\right)(0)\right) \\ &\le \int_0^T \left|{}^C D_0^{\gamma}u(t)\right|\left|I_T^{\gamma}\left[\left(I_T^{1-\alpha}\xi\right)''\right](t)\right|dt + \int_0^T \left|{}^C D_0^{\gamma}u(t)\right|\left|I_T^{\gamma}\left[\left(I_T^{1-\beta}\xi\right)''\right](t)\right|dt.\end{aligned} \tag{40}$$

On the other hand, using ε-Young inequality with $0<\varepsilon<(1/2)$, one obtains

$$\begin{aligned}\int_0^T \left|{}^C D_0^{\gamma}u(t)\right|\left|I_T^{\gamma}\left[\left(I_T^{1-\alpha}\xi\right)''\right](t)\right|dt &\le \varepsilon\int_0^T \left|{}^C D_0^{\gamma}u(t)\right|^p\xi(t)\,dt \\ &\quad + C(\varepsilon,p)\int_0^T \xi(t)^{(-1/(p-1))}\left|I_T^{\gamma}\left[\left(I_T^{1-\alpha}\xi\right)''\right](t)\right|^{(p/(p-1))}dt,\end{aligned} \tag{41}$$

where $C(\varepsilon, p)$ is a positive real number that depends only on ε and p. Similarly, one has

$$\int_0^T \left| {}^C D_0^\gamma u(t) \right| \left| I_T^\gamma \left[\left(I_T^{1-\beta} \xi \right)'' \right](t) \right| dt \le \varepsilon \int_0^T \left| {}^C D_0^\gamma u(t) \right|^p \xi(t)\, dt + C(\varepsilon, p) \int_0^T \xi(t)^{(-1/(p-1))} \left| I_T^\gamma \left[\left(I_T^{1-\beta} \xi \right)'' \right](t) \right|^{(p/(p-1))} dt. \tag{42}$$

Hence, it follows from (40), (41), and (42) that

$$(1-2\varepsilon) \int_0^T \left| {}^C D_0^\gamma u(t) \right|^p \xi(t)\, dt + u_1 \left(\left(I_T^{1-\alpha} \xi \right)(0) + \left(I_T^{1-\beta} \xi \right)(0) \right) \le C(\varepsilon, p) \left(\int_0^T \xi(t)^{(-1/(p-1))} \left| I_T^\gamma \left[\left(I_T^{1-\alpha} \xi \right)'' \right](t) \right|^{(p/(p-1))} dt + \int_0^T \xi(t)^{(-1/(p-1))} \left| I_T^\gamma \left[\left(I_T^{1-\beta} \xi \right)'' \right](t) \right|^{(p/(p-1))} dt \right). \tag{43}$$

Since $0 < \varepsilon < (1/2)$, one deduces from (43) that

$$u_1 \left(\left(I_T^{1-\alpha} \xi \right)(0) + \left(I_T^{1-\beta} \xi \right)(0) \right) \le C(\varepsilon, p) \left(\int_0^T \xi(t)^{(-1/(p-1))} \left| I_T^\gamma \left[\left(I_T^{1-\alpha} \xi \right)'' \right](t) \right|^{(p/(p-1))} dt + \int_0^T \xi(t)^{(-1/(p-1))} \left| I_T^\gamma \left[\left(I_T^{1-\beta} \xi \right)'' \right](t) \right|^{(p/(p-1))} dt \right). \tag{44}$$

On the other hand, by (25), one has

$$\left(I_T^{1-\alpha} \xi \right)(0) = \frac{\Gamma(\lambda+1)}{\Gamma(2+\lambda-\alpha)} T^{1-\alpha}, \tag{45}$$

and

$$\left(I_T^{1-\beta} \xi \right)(0) = \frac{\Gamma(\lambda+1)}{\Gamma(2+\lambda-\beta)} T^{1-\beta}, \tag{46}$$

which yield

$$u_1 \left(\left(I_T^{1-\alpha} \xi \right)(0) + \left(I_T^{1-\beta} \xi \right)(0) \right) = \Gamma(\lambda+1) u_1 T^{1-\alpha} \left(\frac{1}{\Gamma(2+\lambda-\alpha)} + \frac{T^{\alpha-\beta}}{\Gamma(2+\lambda-\beta)} \right). \tag{47}$$

Since $u_1 > 0$, one deduces that

$$u_1 \left(\left(I_T^{1-\alpha} \xi \right)(0) + \left(I_T^{1-\beta} \xi \right)(0) \right) \ge C_1 u_1 T^{1-\alpha}, \tag{48}$$

where $C_1 = (\Gamma(\lambda+1)/\Gamma(2+\lambda-\alpha)) > 0$. Next, using (28) with $\rho = \gamma$ and $\kappa = \alpha$, one obtains

$$\begin{aligned} &\int_0^T \xi(t)^{(-1/(p-1))} \left| I_T^\gamma \left[\left(I_T^{1-\alpha} \xi \right)'' \right](t) \right|^{(p/(p-1))} dt \\ &= \left(\frac{\Gamma(\lambda+1)}{\Gamma(\lambda+\gamma-\alpha)} \right)^{(p/(p-1))} T^{-\lambda} \int_0^T (T-t)^{\lambda+(((\gamma-1-\alpha)p)/(p-1))} dt \\ &= \left(\frac{\Gamma(\lambda+1)}{\Gamma(\lambda+\gamma-\alpha)} \right)^{(p/(p-1))} T^{(((\gamma-1-\alpha)p)/(p-1))} \int_0^T \left(1 - \frac{t}{T} \right)^{\lambda+(((\gamma-1-\alpha)p)/(p-1))} dt \\ &= \left(\frac{\Gamma(\lambda+1)}{\Gamma(\lambda+\gamma-\alpha)} \right)^{(p/(p-1))} T^{(((\gamma-1-\alpha)p)/(p-1))+1} \int_0^1 (1-s)^{\lambda+(((\gamma-1-\alpha)p)/(p-1))} ds, \end{aligned} \tag{49}$$

which yields

$$\int_0^T \xi(t)^{(-1/(p-1))} \left| I_T^\gamma \left[\left(I_T^{1-\alpha} \xi \right)'' \right](t) \right|^{(p/(p-1))} dt = C_2 T^{(((\gamma-1-\alpha)p)/(p-1))+1}, \tag{50}$$

where

$$C_2 = \left(\frac{\Gamma(\lambda+1)}{\Gamma(\lambda+\gamma-\alpha)} \right)^{(p/(p-1))} \int_0^1 (1-s)^{\lambda+(((\gamma-1-\alpha)p)/(p-1))} ds > 0. \tag{51}$$

Similarly, one has

$$\int_0^T \xi(t)^{(-1/(p-1))} \left| I_T^\gamma \left[\left(I_T^{1-\beta} \xi \right)'' \right](t) \right|^{(p/(p-1))} dt = C_3 T^{(((\gamma-1-\beta)p)/(p-1))+1}, \tag{52}$$

where

$$C_3 = \left(\frac{\Gamma(\lambda+1)}{\Gamma(\lambda+\gamma-\beta)} \right)^{(p/(p-1))} \int_0^1 (1-s)^{\lambda+(((\gamma-1-\beta)p)/(p-1))} ds > 0. \tag{53}$$

Therefore, it follows from (44), (48), (50), and (52) that

$$C_1 u_1 T^{1-\alpha} \le C(\varepsilon, p) \left(C_2 T^{(((\gamma-1-\alpha)p)/(p-1))+1} + C_3 T^{(((\gamma-1-\beta)p)/(p-1))+1} \right), \tag{54}$$

which yields

$$u_1 \le \frac{C(\varepsilon, p)}{C_1} T^{\alpha+(((\gamma-1-\alpha)p)/(p-1))} \left(C_2 + C_3 T^{(((\alpha-\beta)p)/(p-1))} \right), T > 0. \tag{55}$$

Notice that for all $p > 1$, one has

$$\begin{cases} \alpha + \dfrac{(\gamma-1-\alpha)p}{p-1} < 0, \\ \dfrac{(\alpha-\beta)p}{p-1} \le 0 \,(\text{from}(33)). \end{cases} \tag{56}$$

Hence, using (56) and passing to the limit as $T \longrightarrow +\infty$ in (55), one obtains $u_1 \leq 0$, which contradicts the fact that $u_1 > 0$. Therefore, one deduces that for all $p > 1$, problem (1) admits no global solution.

Proof of Theorem 4. Let $u_1 = 0$. First, one observes that in this case $u \equiv u_1$ is a global solution to (1). Suppose now that $u \in AC^2([0,\infty))$ is a global solution to (1). Taking $u_1 = 0$ in (43), one obtains

$$\begin{aligned}\int_0^T &\left| {}^C D_0^\gamma u(t)\right|^p \xi(t) dt \\ &\leq C'(\varepsilon, p)\left(\int_0^T \xi(t)^{(-1/(p-1))}\left|I_T^\gamma\left[\left(I_T^{1-\alpha}\xi\right)''\right](t)\right|^{(p/(p-1))} dt \right.\\ &\left.\quad + \int_0^T \xi(t)^{(-1/(p-1))}\left|I_T^\gamma\left[\left(I_T^{1-\beta}\xi\right)''\right](t)\right|^{(p/(p-1))} dt\right),\end{aligned} \tag{57}$$

for all $T > 0$, where $C'(\varepsilon, p) = (C(\varepsilon, p)/1 - 2\varepsilon)$ and $0 < \varepsilon < (1/2)$. Next, using (24) and the estimates (50) and (52), one obtains

$$\begin{aligned}\int_0^T &\left| {}^C D_0^\gamma u(t)\right|^p \left(1 - \frac{t}{T}\right)^\lambda dt \\ &\leq C'(\varepsilon, p) T^{(((\gamma-1-\alpha)p)/(p-1))+1}\left(C_2 + C_3 T^{(((\alpha-\beta)p)/(p-1))}\right).\end{aligned} \tag{58}$$

Notice that since $\alpha \leq \beta$, one has

$$\frac{(\alpha-\beta)p}{p-1} \leq 0. \tag{59}$$

Moreover, if $\gamma \leq \alpha$, then

$$\frac{(\gamma-1-\alpha)p}{p-1} + 1 < 0. \tag{60}$$

Hence, passing to the infimum limit as $T \longrightarrow +\infty$ in (58) and using Fatou's lemma, one obtains

$$\int_0^\infty \left| {}^C D_0^\gamma u(t)\right|^p dt = 0, \tag{61}$$

which yields

$${}^C D_0^\gamma u(t) = 0, \tag{62}$$

for almost everywhere $t > 0$. Then, using the initial conditions and Lemma 9, one deduces that

$$I_0^\gamma\left({}^C D_0^\gamma u\right)(t) = I_0^\gamma\left(I_0^{1-\gamma} u'\right)(t) = \left(I_0^1 u'\right)(t) = u(t) - u_0 = 0, \tag{63}$$

for almost everywhere $t > 0$. Since u is continuous ($u \in AC^2([0,\infty))$), it holds that $u(t) = u_0$ for all $t \geq 0$. This proves part (i) of Theorem 4.

Suppose now that $\gamma > \alpha$. In this case, if $1 < p < (1/\gamma - \alpha)$, then (60) holds. Hence, proceeding as above, one obtains $u(t) = u_0$ for all $t \geq 0$, which proves part (ii) of Theorem 4.

Proof of Theorem 5. It is sufficient to observe that any global solution to problem (2) is a global solution to problem (1). Hence, using Theorem 3, one deduces that problem (2) admits no global solution.

Proof of Theorem 6. Let us suppose that $u \in AC^2([0,\infty))$ is a global solution to problem (2). Proceeding as in the Proof of Theorem 4 and using that $u_1 = 0$, one obtains

$$\begin{aligned}\int_0^T f(t)\xi(t)\, dt &\leq C\left(\int_0^T \xi(t)^{(-1/(p-1))}\left|I_T^\gamma\left[\left(I_T^{1-\alpha}\xi\right)''\right](t)\right|^{(p/(p-1))} dt \right.\\ &\left.\quad + \int_0^T \xi(t)^{(-1/(p-1))}\left|I_T^\gamma\left[\left(I_T^{1-\beta}\xi\right)''\right](t)\right|^{(p/(p-1))} dt\right),\end{aligned} \tag{64}$$

for all $T > 0$, where $C > 0$ is a constant (independent on T). On the other hand, by (24), one has

$$\int_0^T f(t)\xi(t)\, dt = \int_0^T T^{-\lambda}(T-t)^\lambda f(t)\, dt \geq 2^{-\lambda}\int_0^{T/2} f(t)\, dt. \tag{65}$$

Moreover, by (50) and (52), one has

$$\begin{aligned}\int_0^T &\xi(t)^{(-1/(p-1))}\left|I_T^\gamma\left[\left(I_T^{1-\alpha}\xi\right)''\right](t)\right|^{(p/(p-1))} dt \\ &+ \int_0^T \xi(t)^{(-1/(p-1))}\left|I_T^\gamma\left[\left(I_T^{1-\beta}\xi\right)''\right](t)\right|^{(p/(p-1))} dt \\ &\leq C_2 T^{(((\gamma-1-\alpha)p)/(p-1))+1} \\ &\quad + C_3 T^{(((\gamma-1-\beta)p)/(p-1))+1 = T^{(((\gamma-\alpha)p-1)/(p-1))}\left(C_2 + C_3 T^{(((\alpha-\beta)p)/(p-1))}\right)}.\end{aligned} \tag{66}$$

Next, it follows from (64), (65), and (66) that

$$2^{-\lambda}\int_0^{T/2} f(t)\, dt \leq C T^{(((\gamma-\alpha)p-1)/(p-1))}\left(C_2 + C_3 T^{(((\alpha-\beta)p)/(p-1))}\right), \tag{67}$$

which yields

$$T^{((1+(\alpha-\gamma)p)/(p-1))}\int_0^{T/2} f(t)\, dt \leq 2^\lambda C\left(C_2 + C_3 T^{(((\alpha-\beta)p)/(p-1))}\right). \tag{68}$$

Finally, passing to the supremum limit as $T \longrightarrow +\infty$ in (68), using (14) and the fact $\alpha \leq \beta$, a contradiction follows.

Proof of Corollary 7. For all $a \neq 0$, one has

$$\int_0^{T/2} f(t)dt = \frac{1}{a}\left(e^{aT/2} - 1\right). \quad (69)$$

If $a > 0$, then

$$\int_0^{T/2} f(t)dt \sim \frac{1}{a}e^{aT/2}, \quad \text{as } T \longrightarrow +\infty, \quad (70)$$

which yields

$$\begin{aligned}\lim_{T\to+\infty} & T^{(((\alpha-\gamma)p+1)/(p-1))}\int_0^{T/2} f(t)\,dt \\ &= \lim_{T\to+\infty}\frac{1}{a}T^{(((\alpha-\gamma)p+1)/(p-1))}e^{aT/2} = +\infty.\end{aligned} \quad (71)$$

Hence, by Theorem 6, one deduces that for all $p > 1$, problem (2) admits no global solution, which proves part (i).

If $a < 0$, then

$$\int_0^{T/2} f(t)dt \sim \frac{-1}{a}, \quad \text{as } T \longrightarrow +\infty, \quad (72)$$

which yields

$$\lim_{T\to+\infty} T^{(((\alpha-\gamma)p+1)/(p-1))}\int_0^{T/2} f(t)\,dt = \lim_{T\to+\infty}\frac{-1}{a}T^{(((\alpha-\gamma)p+1)/(p-1))}. \quad (73)$$

Therefore, if $\alpha \geq \gamma$, one has

$$(\alpha-\gamma)p + 1 > 0, \quad p > 1, \quad (74)$$

which yields

$$\lim_{T\to+\infty} T^{(((\alpha-\gamma)p+1)/(p-1))}\int_0^{T/2} f(t)\,dt = +\infty, \quad p > 1. \quad (75)$$

Hence, by Theorem 6, one deduces that for all $p > 1$, problem (2) admits no global solution, which proves part (ii). On the other hand, if $\gamma > \alpha$, one has

$$(\alpha-\gamma)p + 1 > 0, \quad 1 < p < \frac{1}{\gamma-\alpha}, \quad (76)$$

which yields

$$\lim_{T\to+\infty} T^{(((\alpha-\gamma)p+1)/(p-1))}\int_0^{T/2} f(t)\,dt = +\infty, \quad 1 < p < \frac{1}{\gamma-\alpha}. \quad (77)$$

Hence, by Theorem 6, one deduces that for all $1 < p < (1/\gamma - \alpha)$, problem (2) admits no global solution, which proves part (iii).

Proof of Corollary 8. For all $\sigma > -1$, one has

$$\int_0^{T/2} f(t)\,dt = \frac{1}{2^{\sigma+1}(\sigma+1)}\left(T^{\sigma+1} - 2^{\sigma+1}\right), \quad (78)$$

which yields

$$\int_0^{T/2} f(t)dt \sim \frac{T^{\sigma+1}}{2^{\sigma+1}(\sigma+1)}, \quad \text{as } T \longrightarrow +\infty. \quad (79)$$

Hence,

$$\begin{aligned}\lim_{T\to+\infty} & T^{(((\alpha-\gamma)p+1)/(p-1))}\int_0^{T/2} f(t)\,dt \\ &= \lim_{T\to+\infty}\frac{1}{2^{\sigma+1}(\sigma+1)}T^{(((\alpha-\gamma)p+1)/(p-1))+\sigma+1}.\end{aligned} \quad (80)$$

Notice that

$$\frac{(\alpha-\gamma)p+1}{p-1} + \sigma + 1 > 0 \Leftrightarrow (\alpha + 1 - \gamma + \sigma)p > \sigma. \quad (81)$$

Hence, by Theorem 6, one deduces that for all $p > 1$ satisfying

$$(\alpha + 1 - \gamma + \sigma)p > \sigma, \quad (82)$$

problem (2) admits no global solution.

Consider the case $\sigma \geq 0$. In this case, for all $p > 1$, one has

$$(\alpha + 1 - \gamma + \sigma)p > \sigma p \geq \sigma. \quad (83)$$

Then (82) is satisfied for all $p > 1$. Hence, one deduces that for all $p > 1$, problem (2) admits no global solution, which proves part (i).

Suppose now that $-1 < \sigma < 0$. If $\gamma \leq \alpha$, then for all $p > 1$, one has

$$(\alpha + 1 - \gamma + \sigma)p \geq (\sigma + 1)p > 0 > \sigma. \quad (84)$$

Then, (82) is satisfied for all $p > 1$. Hence, one deduces that for all $p > 1$, problem (2) admits no global solution, which proves part (ii)(a). On the other hand, if $\gamma > \alpha$ and $-1 < \gamma - \alpha - 1 \leq \sigma < 0$, then

$$(\alpha + 1 - \gamma + \sigma)p \geq 0 > \sigma. \quad (85)$$

Hence, (82) is satisfied for all $p > 1$. Therefore, for all $p > 1$, problem (2) admits no global solution, which proves part (ii)(b). Finally, if $\gamma > \alpha$ and $-1 < \sigma < \gamma - \alpha - 1$, then for all $p > 1$, (82) is equivalent to

$$1 < p < \frac{\sigma}{\alpha + 1 - \gamma + \sigma}. \quad (86)$$

Hence, for all p satisfying the above condition, problem (2) admits no global solution, which proves part (ii)(c).

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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