Research Article

Existence of Solutions to Elliptic Problem with Convection Term and Lower-Order Term

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In this paper, we establish the existence of solutions to the following noncoercivity Dirichlet problem

$$\begin{cases} -\operatorname{div}(M(x)\nabla u) + |u|^{p-1}u = -\operatorname{div}(uE(x)) + f(x), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N (N > 2)$ is a bounded smooth domain with $0 \in \Omega$, f belongs to the Lebesgue space $L^m(\Omega)$ with $m \geq 1, p > 0$. The main innovation point of this paper is the combined effects of the convection terms and lower-order terms in elliptic equations.

1. Introduction

The main purpose of this paper is to prove the existence of solutions to the following elliptic boundary value problem

$$\begin{cases} -\operatorname{div}(M(x)\nabla u) + |u|^{p-1}u = -\operatorname{div}(uE(x)) + f(x), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \quad (1)$$

where Ω is a bounded smooth subset of $\mathbb{R}^N (N > 2)$ with $0 \in \Omega$, and $M : \Omega \rightarrow \mathbb{R}^{N^2}$ is a bounded measurable matrix, which satisfies the following conditions: there are two positive constants α and β , such that, for a.e. $x \in \Omega$ and every $\xi \in \mathbb{R}^N$,

$$\alpha|\xi|^2 \leq M(x)\xi \cdot \xi, \quad |M(x)| \leq \beta. \quad (2)$$

Moreover, $E(x)$ is a measurable vector field, which satisfies

$$|E| \leq \frac{A}{|x|}, \quad A \in \mathbb{R}^+, \quad (3)$$

$f(x)$ is a measurable function which satisfies

$$f \in L^m(\Omega) (m \geq 1), \quad (4)$$

where $L^m(\Omega)$ represents the Lebesgue space.

When the problem (1) without the lower-order terms, it becomes the following elliptic Dirichlet problem

$$\begin{cases} -\operatorname{div}(M(x)\nabla u) = -\operatorname{div}(uE(x)) + f(x), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \quad (5)$$

Problem (5) was studied by Boccardo in a series of papers. More precisely, in [1], the existence of solutions u to problem (5) was proved provided $E \in (L^N(\Omega))^N$:

- (i) If $f \in L^m(\Omega)$ with $2N/N + 2 \leq m < N/2$, then $u \in W_0^{1,2}(\Omega) \cap L^{m^{**}}(\Omega)$, where $m^{**} = Nm/N - 2m$
- (ii) If $f \in L^m(\Omega)$ with $1 < m < 2N/N + 2$, then $u \in W_0^{1,m^*}(\Omega)$, where $m^* = Nm/N - m$
- (iii) If $f \in L^m(\Omega)$ with $m > N/2$, then $u \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$
- (iv) If $f \in L^1(\Omega)$, then $u \in W_0^{1,q}(\Omega)$, where $q < N/N - 1$

When $f \in L^m(\Omega)$ and vector field E satisfies (3), which does not belong to $(L^N(\Omega))^N$, Boccardo [2] shown that

- (i) If $A < \alpha(N - 2m)/m$ and $2N/N + 2 \leq m < N/2$, then $u \in W_0^{1,2}(\Omega) \cap L^{m^{**}}(\Omega)$
- (ii) If $A < \alpha(N - 2m)/m$ and $1 < m < 2N/N + 2$, then $u \in W_0^{1,m^*}(\Omega)$
- (iii) If $A < \alpha(N - 2)$ and $m = 1$, then $\nabla u \in (M^{N/N-1}(\Omega))^N$ and $u \in W_0^{1,q}(\Omega)$, where $q < N/N - 1$

Furthermore, in the same paper, the existence of entropy solutions to problem (5) also be considered provided $E \in (L^2(\Omega))^N$ and $f \in L^1(\Omega)$.

Recently, continuation of [2], Boccardo and Orsina [3] studied the existence of distributional solution $u \in W_0^{1,q}(\Omega)$ to problem (5) with $q < N\alpha/A + \alpha$ provided $\alpha(N - 2) \leq A < \alpha(N - 1)$ and $f \in L^1(\Omega)$. Moreover, u verifies a prior estimation:

$$\left(\int_{\Omega} |\nabla u|^q \right)^{1/q} \leq C_E \|f\|_{L^1(\Omega)}. \quad (6)$$

The constant C_E lies on E , α , and Ω . Some other results about noncoercivity elliptic problems see [4–16] and reference therein.

It is well known that the presence of lower-order term will improve the regularity properties of the solutions. When $p = 1$ and $M(x)$ satisfying (2), Boccardo [2] shown that the existence of entropy solution u of (1) provided $E \in (L^2(\Omega))^N$ and $f \in L^1(\Omega)$.

Moreover, u verifies the following estimations:

$$\begin{aligned} \int_{\Omega} |u| &\leq \int_{\Omega} |f|, \\ \int_{\Omega} |\nabla \log(1 + |u|)|^2 &\leq \frac{1}{\alpha^2} \int_{\Omega} |E|^2 + \frac{2}{\alpha} \int_{\Omega} |f|, \\ \frac{\alpha}{2} \int_{\Omega} |\nabla T_k(u)|^2 &\leq \frac{k^2}{2\alpha} \int_{\Omega} |E|^2 + k \int_{\Omega} |f|. \end{aligned} \quad (7)$$

Furthermore, there is a weak solution $u \in W_0^{1,2}(\Omega) \cap L^{p+1}(\Omega)$ of (1) provided (2) holds, $p > N + 2/N - 2$, $f \in L^{p+1/p}(\Omega)$, and $E \in (L^{2(p+1)/p-1}(\Omega))^N$.

For some other results about elliptic problems with lower-order terms, see [17–26] and reference therein.

With motivation from the results of the above-cited papers, the main goal of this paper is to further study the regularity of solutions to problem (1) with $f \in L^m(\Omega)$ ($m \geq 1$). The main features of this paper are the presence of the convection term $\operatorname{div}(uE(x))$, which leads to the noncoercivity of $-\operatorname{div}(M(x)\nabla u) + \operatorname{div}(uE(x))$ in $W_0^{1,2}(\Omega)$. Therefore, in order to overcome the coercivity difficulty, we use truncation technique and consider the corresponding approximate Dirichlet problem, see (19) for more details.

The main results are the following:

Theorem 1. Suppose that Ω is a bounded smooth domain of $\mathbb{R}^N$ ($N > 2$) with $0 \in \Omega$ and (2)–(4) hold.

- (a) There is a weak solution $u \in W_0^{1,2}(\Omega) \cap L^{p+1}(\Omega)$ to problem (1) provided $A < \alpha(N - 2)/2$ and $f \in L^m(\Omega)$ with $1 \leq m \leq 1 + (1/p)$
- (b) There is a weak solution $u \in L^{m_p}(\Omega)$ to problem (1) provided $A < \alpha(N - 2)/p(m - 1) + 1$ and $f \in L^m(\Omega)$ with $m > p + 1/p$. Furthermore,

$$\|u\|_{L^{m_p}(\Omega)} \leq C \|f\|_{L^m(\Omega)}, \quad (8)$$

where

$$m_p = \frac{(p(m - 1) + 1)N}{N - 2}. \quad (9)$$

Remark 2. A point worth emphasizing is that our results further refine the conclusions of [2]. More precisely, under different assumptions on E , we give the existence of solutions to problem (1) with $f \in L^m(\Omega)$ for $1 \leq m \leq p + 1/p$ and $m > p + 1/p$, respectively, rather than $f \in L^{p+1/p}(\Omega)$.

Remark 3. Obviously,

$$m_p = \frac{(p(m - 1) + 1)N}{N - 2} \longrightarrow +\infty \text{ as } p \longrightarrow +\infty, \quad (10)$$

which shows the regularizing effect of the lower-order term for the regularity properties of the solutions to problem (1).

Remark 4. It is clear that

$$A < \frac{\alpha(N - 2)}{p(m - 1) + 1} \longrightarrow 0 \text{ as } p \longrightarrow +\infty, \quad (11)$$

which shows that the lower-order term intensifies the requirement on E .

The paper is organized as follows. In Section 2, we give some definitions and lemmas. In Section 3, the Proof of Theorem 1 is given.

2. Useful Tools and Function Setting

In order to prove Theorem 1, the following basic definitions and lemmas are needed. First of all, we give the definitions of weak solution to (1).

Definition 5. We say that $u \in W_0^{1,2}(\Omega)$ is a weak solution to problem (1), if $\int_{\Omega} |u|^{p-1} u dx < \infty$ and

$$\int_{\Omega} M(x) \nabla u \nabla v + \int_{\Omega} |u|^{p-1} uv = \int_{\Omega} u E(x) \nabla v + \int_{\Omega} f v, \quad (12)$$

for every $v \in W_0^{1,2}(\Omega)$.

The following is the definition of the truncation function.

Definition 6. For $\forall k \geq 0, s \in \mathbb{R}$, the truncation function defined by

$$T_k(s) = \max \{-k, \min \{k, s\}\}, \quad G_k(s) = s - T_k(s). \quad (13)$$

Now, let us briefly recall the Sobolev's embedding theorem.

Lemma 7. Assume that $p = 2$, then there is a normal number S , such that, for $\forall u \in C_0^{\infty}(\mathbb{R}^N)$ satisfies

$$\|u\|_{2^*} \leq S \|\nabla u\|_2, \quad (14)$$

where

$$2^* = \frac{2N}{N-2}. \quad (15)$$

The following Hölder inequality plays an important role in this paper.

Lemma 8 (Hölder inequality). Assume that $1 < p, q < \infty$, $(1/p) + (1/q) = 1$. Then, if $u \in L^p(\Omega)$, $v \in L^q(\Omega)$, we have

$$\int_{\Omega} |uv| dx \leq \|u\|_{L^p(\Omega)} \|v\|_{L^q(\Omega)}. \quad (16)$$

The results of the Hardy inequality and its generalization can be founded in [27–29]. In this paper, we use the following Hardy inequality repeatedly.

Lemma 9 (see [3]). For $\forall v(x) \in W_0^{1,2}(\Omega)$, we have

$$H \left(\int_{\Omega} \frac{|v|^2}{|x|^2} \right)^{1/2} \leq \left(\int_{\Omega} |\nabla v|^2 \right)^{1/2}, \quad (17)$$

where

$$H = \frac{N-2}{2}. \quad (18)$$

3. Proof of Main Theorem

In this part, we are going to give the Proof of Theorem 1 in a similar way as [2, 17–21, 23, 26]. In order to do this, first of all, we consider the following approximate problem:

$$\begin{cases} -\operatorname{div} (M(x) \nabla u_n) + |u_n|^{p-1} u_n = -\operatorname{div} \left(\frac{u_n}{1 + (1/n)|u_n|} \frac{E(x)}{1 + (1/n)|E(x)|} \right) + f_n(x), & x \in \Omega, \\ u_n(x) = 0, & x \in \partial\Omega, \end{cases} \quad (19)$$

where

$$f_n(x) = \frac{f(x)}{1 + (1/n)|f(x)|}. \quad (20)$$

Let us start with the following conclusions.

First of all, the following lemma gives an information on the summability of $|u|^{p-1} u$.

Lemma 10. Let $f \in L^m(\Omega)$, $m \geq 1$. Then, for every $n \in \mathbb{N}$, there exists a solution $u_n \in W_0^{1,2}(\Omega)$ to (19) such that

$$\int_{\Omega} |u_n|^{pm} \leq \int_{\Omega} |f|^m. \quad (21)$$

Proof. In order to get the estimates (21), we will consider the following two cases separately.

Case $m > 1$. Select $\phi = |u_n|^{p(m-1)-1} u_n$ as a test function in (19), by (3), we have

$$\begin{aligned} & \int_{\Omega} M(x) \nabla u_n \cdot \nabla |u_n|^{p(m-1)-1} u_n + \int_{\Omega} |u_n|^{pm} \\ &= \int_{\Omega} \frac{u_n}{1 + (1/n)|u_n|} \frac{E(x)}{1 + (1/n)|E(x)|} \cdot \nabla |u_n|^{p(m-1)-1} u_n \\ &+ \int_{\Omega} f_n(x) |u_n|^{p(m-1)-1} u_n \leq A \int_{\Omega} \frac{u_n}{|x|} |\nabla |u_n|^{p(m-1)-1} u_n| \\ &+ \int_{\Omega} f_n(x) |u_n|^{p(m-1)-1} u_n. \end{aligned} \quad (22)$$

According to (2) and (22), we get

$$\begin{aligned}
 & \alpha p(m-1) \int_{\Omega} |\nabla u_n|^2 |u_n|^{p(m-1)-1} + \int_{\Omega} |u_n|^{pm} \\
 & \leq Ap(m-1) \int_{\Omega} \frac{u_n}{|x|} |\nabla u_n| |u_n|^{p(m-1)-1} + \int_{\Omega} f_n(x) |u_n|^{p(m-1)-1} u_n \\
 & = \frac{2Ap(m-1)}{p(m-1)+1} \int_{\Omega} \frac{|u_n|^{p(m-1)+1/2}}{|x|} |\nabla |u_n|^{p(m-1)+1/2}| \\
 & \quad + \int_{\Omega} f_n(x) |u_n|^{p(m-1)-1} u_n.
 \end{aligned} \tag{23}$$

Using the Hölder inequality and the Hardy inequality for the first term on the right of (23), we obtain

$$\begin{aligned}
 & \frac{2Ap(m-1)}{p(m-1)+1} \int_{\Omega} \frac{|u_n|^{p(m-1)+1/2}}{|x|} |\nabla |u_n|^{p(m-1)+1/2}| \\
 & \leq \frac{2Ap(m-1)}{[p(m-1)+1]H} \int_{\Omega} |\nabla |u_n|^{p(m-1)+1/2}|^2.
 \end{aligned} \tag{24}$$

Since

$$\begin{aligned}
 & \alpha p(m-1) \int_{\Omega} |\nabla u_n|^2 |u_n|^{p(m-1)-1} \\
 & = \frac{4\alpha p(m-1)}{[p(m-1)+1]^2} \int_{\Omega} |\nabla |u_n|^{p(m-1)+1/2}|^2.
 \end{aligned} \tag{25}$$

Thus, taking into account (23)–(25), we arrive at

$$\begin{aligned}
 & \left[\frac{4\alpha p(m-1)}{[p(m-1)+1]^2} - \frac{2Ap(m-1)}{[p(m-1)+1]H} \right] \int_{\Omega} |\nabla |u_n|^{p(m-1)+1/2}|^2 \\
 & \quad + \int_{\Omega} |u_n|^{pm} \leq \int_{\Omega} f_n(x) |u_n|^{p(m-1)-1} u_n.
 \end{aligned} \tag{26}$$

This fact leads to

$$\int_{\Omega} |u_n|^{pm} \leq \int_{\Omega} f_n(x) |u_n|^{p(m-1)-1} u_n, \tag{27}$$

provided

$$A < \frac{\alpha(N-2)}{p(m-1)+1}, \tag{28}$$

where $H = N - 2/2$.

Applying the Hölder inequality on the right-hand side of (27), we get

$$\int_{\Omega} f_n(x) |u_n|^{p(m-1)-1} u_n \leq \|f\|_{L^m(\Omega)} \left(\int_{\Omega} |u_n|^{pm} \right)^{1/m'}, \tag{29}$$

where $m' = m/m - 1$, which together with (27), implies that (21) holds.

When $m = 1$, taking $T_k(u_n)/k$ as a test function in (19), according to (3), we have

$$\begin{aligned}
 & \int_{\Omega} M(x) \nabla u_n \cdot \nabla \frac{T_k(u_n)}{k} + \int_{\Omega} |u_n|^{p-1} u_n \frac{T_k(u_n)}{k} \\
 & = \int_{\Omega} \frac{u_n}{1 + (1/n)|u_n|} \frac{E(x)}{1 + (1/n)|E(x)|} \\
 & \quad \cdot \nabla \frac{T_k(u_n)}{k} + \int_{\Omega} f_n(x) \frac{T_k(u_n)}{k} \\
 & \leq A \int_{\Omega} \frac{u_n}{|x|} \left| \nabla \frac{T_k(u_n)}{k} \right| + \int_{\Omega} f_n(x) \frac{T_k(u_n)}{k}.
 \end{aligned} \tag{30}$$

For the first term on the left-hand side of (30), we have

$$\int_{\Omega} M(x) \nabla u_n \cdot \nabla \frac{T_k(u_n)}{k} = \frac{1}{k} \int_{|u_n| \leq k} M(x) \nabla u_n \cdot \nabla u_n. \tag{31}$$

For the first term on the right-hand side of (30), using the Hölder inequality and the Hardy inequality, we get

$$A \int_{\Omega} \frac{u_n}{|x|} \left| \nabla \frac{T_k(u_n)}{k} \right| = \frac{A}{k} \int_{|u_n| \leq k} \frac{u_n}{|x|} |\nabla u_n| \leq \frac{A}{Hk} \int_{|u_n| \leq k} |\nabla u_n|^2. \tag{32}$$

Combining (30)–(31) with (2), we have

$$\left(\frac{\alpha}{k} - \frac{A}{Hk} \right) \int_{|u_n| \leq k} |\nabla u_n|^2 + \int_{\Omega} |u_n|^p \frac{T_k(u_n)}{k} \leq \int_{\Omega} f_n(x) \frac{T_k(u_n)}{k}. \tag{33}$$

Since $A < \alpha(N-2)/2$, we obtain

$$\int_{|u_n| \leq k} |u_n|^p \frac{T_k(u_n)}{k} \leq \int_{|u_n| \leq k} f_n(x) \frac{T_k(u_n)}{k} \leq \int_{\Omega} f_n(x) \leq \int_{\Omega} |f|. \tag{34}$$

Fatou lemma implies, for $k \rightarrow \infty$, the expression of (21) holds.

Next, we will prove the following existence result.

Lemma 11. Assume (2), (3) hold with $A < \alpha(N-2)/2$, $f \in L^m(\Omega)$ with $1 \leq m \leq 1 + (1/p)$. Then, there is a weak solution $u \in W_0^{1,2}(\Omega) \cap L^{p+1}(\Omega)$ to problem (1).

Proof. Set $f_n(x) = f(x)/1 + (1/n)|f(x)|$ obviously, $f_n(x) \rightarrow f(x)$ in $L^1(\Omega)$ as $n \rightarrow \infty$. Let $\phi = T_k(u_n)$ as a test function in (19), using the Hölder inequality and the Hardy inequality, we get

$$\begin{aligned}
& \int_{\Omega} M(x) \nabla u_n \cdot \nabla T_k(u_n) + \int_{\Omega} |u_n|^{p-1} u_n T_k(u_n) \\
&= \int_{\Omega} \frac{u_n}{1 + (1/n)|u_n|} \frac{E(x)}{1 + (1/n)|E(x)|} \\
&\quad \cdot \nabla T_k(u_n) + \int_{\Omega} f_n(x) T_k(u_n) \\
&\leq A \int_{\Omega} \frac{u_n}{|x|} |\nabla T_k(u_n)| + k \int_{\Omega} |f(x)| \\
&= A \int_{\Omega} \frac{T_k(u_n)}{|x|} |\nabla T_k(u_n)| + k \int_{\Omega} |f(x)| \\
&\leq \frac{A}{H} \int_{\Omega} |\nabla T_k(u_n)|^2 + k \int_{\Omega} |f(x)|.
\end{aligned} \tag{35}$$

Moreover, by (2) and a direct calculation, we get

$$\begin{aligned}
\alpha \int_{\Omega} |\nabla T_k(u_n)|^2 + \int_{\Omega} T_k^{p+1}(u_n) &\leq \int_{\Omega} M(x) \nabla T_k(u_n) \cdot \nabla T_k(u_n) \\
&\quad + \int_{\Omega} |u_n|^p T_k(u_n).
\end{aligned} \tag{36}$$

Since $T_k(u_n) \in W_0^{1,2}(\Omega)$, by Lemma 10, we get $m = 1 + (1/p)$. Together (35) with (36), it follows that

$$\left(\alpha - \frac{A}{H} \right) \int_{\Omega} |\nabla T_k(u_n)|^2 + \int_{\Omega} T_k^{p+1}(u_n) \leq C(\alpha, N, f, k). \tag{37}$$

We deduce that $T_k(u_n)$ is uniformly bounded in $W_0^{1,2}(\Omega) \cap L^{p+1}(\Omega)$. Then, we pass to the limit in the approximation problem (19); up to a subsequence, there is a function $u \in W_0^{1,2}(\Omega) \cap L^{p+1}(\Omega)$.

Now, we want to prove that $|u_n|^p \rightarrow |u|^p$. Let $\psi_i(\sigma)$ be defined by

$$\psi_i(\sigma) = \begin{cases} 1, & \sigma \geq t, \\ 0, & |\sigma| < t, \\ -1, & \sigma \leq -t. \end{cases} \tag{38}$$

Selecting $\phi = \psi_i(u_n)$ as a test function in (19), we have

$$\int_{\Omega} |u_n|^p \psi_i(u_n) \leq \int_{\Omega} f \psi_i(u_n), \tag{39}$$

which implies that

$$\int_{\{u_n > t\} \cap \Omega} |u_n|^p \leq \int_{\{u_n > t\} \cap \Omega} f. \tag{40}$$

Let $E \subset \Omega$ is measurable. For any $t > 0$, we get

$$\int_E |u_n|^p \leq t^p |E| + \int_{E \cap \{u_n > t\}} |u_n|^p \leq t^p |E| + \int_{\{u_n > t\}} |f|. \tag{41}$$

The above fact and $f \in L^1(\Omega)$ allow us to say that, for any given $\varepsilon > 0$, there is t_ε satisfies

$$\int_{\{u_n > t_\varepsilon\}} |f| \leq \varepsilon. \tag{42}$$

Thus,

$$\int_E |u_n|^p \leq t_\varepsilon^p |E| + \varepsilon. \tag{43}$$

Therefore,

$$\lim_{|E| \rightarrow 0} \int_E |u_n|^p \leq \varepsilon. \tag{44}$$

Thus, we prove that $\lim_{|E| \rightarrow 0} \int_E |u_n|^p = 0$. Vitali theorem implies that $|u_n|^p \rightarrow |u|^p$ in $L^1(\Omega)$. In other words,

$$\lim_{n \rightarrow \infty} \int_{\Omega} |u_n|^p = \int_{\Omega} |u|^p. \tag{45}$$

Finally, let us show the second part of Theorem 1, that is, the existence of solution to problem (1) in the case where $f \in L^m(\Omega)$ with $m > p + 1/p$.

Lemma 12. Assume that $f \in L^m(\Omega)$ with $m > p + 1/p$, and

$$A \leq \frac{\alpha(N-2)}{p(m-1)+1}. \tag{46}$$

Then, there is a weak solution u to problem (1) that such that

$$\|u\|_{L^{m_p}(\Omega)} \leq C \|f\|_{L^m(\Omega)}, \tag{47}$$

where

$$m_p = \frac{(p(m-1)+1)N}{N-2}. \tag{48}$$

Proof. Define $\beta = p(m-1) > 1$, it satisfies $p + \beta = \beta m'$. Applying $\phi = |u_n|^{\beta-1} u_n$ as a test function in (19), by (3), we get

$$\begin{aligned}
& \int_{\Omega} M(x) \nabla u_n \cdot \nabla |u_n|^{\beta-1} u_n + \int_{\Omega} |u_n|^{p+\beta} \\
&= \int_{\Omega} \frac{u_n}{1 + (1/n)|u_n|} \frac{E(x)}{1 + (1/n)|E(x)|} \\
&\quad \cdot \nabla |u_n|^{\beta-1} u_n + \int_{\Omega} f_n(x) |u_n|^{\beta-1} u_n \\
&\leq A \int_{\Omega} \frac{u_n}{|x|} |\nabla |u_n|^{\beta-1} u_n| + \int_{\Omega} f_n(x) |u_n|^{\beta-1} u_n \\
&= A \beta \int_{\Omega} \frac{|u_n|^\beta}{|x|} |\nabla u_n| + \int_{\Omega} f_n(x) |u_n|^{\beta-1} u_n.
\end{aligned} \tag{49}$$

By (2), (49) can become

$$\begin{aligned} & \alpha\beta \int_{\Omega} |\nabla u_n|^2 |u_n|^{\beta-1} + \int_{\Omega} |u_n|^{p+\beta} \\ & \leq A\beta \int_{\Omega} \frac{|u_n|^\beta}{|x|} |\nabla u_n| + \int_{\Omega} f_n(x) |u_n|^{\beta-1} u_n. \end{aligned} \quad (50)$$

Since

$$\alpha\beta \int_{\Omega} |\nabla u_n|^2 |u_n|^{\beta-1} = \frac{4\alpha\beta}{(\beta+1)^2} \int_{\Omega} |\nabla |u_n|^{\beta+1/2}|^2. \quad (51)$$

Applying the Hölder inequality and the Hardy inequality for the first term on the right-hand side of (50), it can be rewritten as

$$\begin{aligned} & \frac{4\alpha\beta}{(\beta+1)^2} \int_{\Omega} |\nabla |u_n|^{\beta+1/2}|^2 + \int_{\Omega} |u_n|^{p+\beta} \\ & \leq \frac{2A\beta}{(\beta+1)H} \int_{\Omega} |\nabla |u_n|^{\beta+1/2}|^2 + \int_{\Omega} f_n(x) |u_n|^{\beta-1} u_n. \end{aligned} \quad (52)$$

We conclude that

$$\begin{aligned} & \left(\frac{4\alpha\beta}{(\beta+1)^2} - \frac{2A\beta}{(\beta+1)H} \right) \int_{\Omega} |\nabla |u_n|^{\beta+1/2}|^2 + \int_{\Omega} |u_n|^{p+\beta} \\ & \leq \int_{\Omega} f_n(x) |u_n|^{\beta-1} u_n \leq \|f\|_{L^m(\Omega)} \left(\int_{\Omega} |u_n|^{\beta m'} \right)^{1/m'}. \end{aligned} \quad (53)$$

With this choice of β , by Lemma 10, we arrive at

$$\int_{\Omega} |\nabla |u_n|^{\beta+1/2}|^2 \leq C \|f\|_{L^m(\Omega)}. \quad (54)$$

According to the Sobolev embedding theorem, we obtain

$$\left(\int_{\Omega} |u_n|^{(\beta+1)2^*/2} \right)^{1/2^*} \leq C \|f\|_{L^m(\Omega)}. \quad (55)$$

Moreover,

$$\frac{(\beta+1)2^*}{2} = \frac{N(p(m-1)+1)}{N-2} = m_p, \quad (56)$$

where $2^* = 2N/N-2$.

As a consequence, there exists a function $|u| \in L^{(\beta+1)2^*/2}(\Omega)$.

Similarly, we can prove $|u_n|^p \rightarrow |u|^p$. Then, we have proved the existence result.

Proof of Theorem 1. We can combine Lemmas 11–12 to get Theorem 1.

Data Availability

No data were used to support this study.

Conflicts of Interest

All authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally to this work.

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