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Research Article

On Durrmeyer Type λ-Bernstein Operators via (p, q)-Calculus

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In the present paper, Durrmeyer type λ-Bernstein operators via (p, q)-calculus are constructed, the first and second moments and central moments of these operators are estimated, a Korovkin type approximation theorem is established, and the estimates on the rate of convergence by using the modulus of continuity of second order and Steklov mean are studied, a convergence theorem for the Lipschitz continuous functions is also obtained. Finally, some numerical examples are given to show that these operators we defined converge faster in some λ cases than Durrmeyer type (p, q)-Bernstein operators.

1. Introduction

In 2016, Mursaleen et al. [1] proposed the following (p, q)-analogue of Bernstein operators:

$$B_{n,p,q}(f;x) = \sum_{k=0}^{n} b_{n,k}(x;p,q) f\left(\frac{[k]_{p,q}}{p^{k-n}[n]_{p,q}}\right), x \in [0,1], \quad (1)$$

where $b_{n,k}(x;p,q)(k=0,1,\cdots,n)$ are (p, q)-Bernstein basis functions and defined as

$$b_{n,k}(x;p,q) = \frac{1}{p^{n(n-1)/2}} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{k(k-1)/2} x^k (1 \ominus x)_{p,q}^{n-k}, x \in [0,1]. \quad (2)$$

They also introduced and studied some important approximation properties of the Stancu type of operators (1) in [2]. After their construction, there are more and more papers on the study of (p, q)-analogue of Bernstein type operators, we mention some of them as [3–11], we also mention some other positive linear operators as [12–19].

Very recently, Cai et al. [20] proposed the following λ-Bernstein operators based on (p, q)-integers as

$$B_{n,p,q}^{\lambda}(f;x) = \sum_{k=0}^{n} b_{n,k}^{\lambda}(x;p,q) f\left(\frac{[k]_{p,q}}{p^{k-n}[n]_{p,q}}\right), x \in [0,1], \quad (3)$$

where

$$\begin{cases} b_{n,0}^{\lambda}(x;p,q) = b_{n,0}(x;p,q) - \dfrac{\lambda}{p^{1-n}[n]_{p,q}+1} b_{n+1,1}(x;p,q), \\ b_{n,k}^{\lambda}(x;p,q) = b_{n,k}(x;p,q) + \lambda\left(\dfrac{p^{1-n}[n]_{p,q} - 2p^{1-k}[k]_{p,q}+1}{p^{2-2n}[n]_{p,q}^2-1} b_{n+1,k}(x;p,q) - \dfrac{p^{1-n}[n]_{p,q} - 2qp^{-k}[k]_{p,q}-1}{p^{2-2n}[n]_{p,q}^2-1} b_{n+1,k+1}(x;p,q)\right), (k=1,2,\cdots,n-1) \\ b_{n,n}^{\lambda}(x;p,q) = b_{n,n}(x;p,q) - \dfrac{\lambda}{p^{1-n}[n]_{p,q}+1} b_{n+1,n}(x;p,q), \end{cases} \tag{4}$$

$b_{n,k}(x;p,q)(k=0,1,\cdots,n)$ are defined in (2), $\lambda \in [-1,1]$, $n \geq 2$, $x \in [0,1]$, and $0 < q < p \leq 1$. They also constructed the (p,q)-analogue of Kantorovich type λ-Bernstein operators and investigated their A-statistical convergence properties in [21].

Inspired by above research, based on (3), we introduce Durrmeyer type λ-Bernstein operators via (p,q)-calculus as

$$D_{n,p,q}^{\lambda}(f;x) = [n+1]_{p,q} p^{-n} \sum_{k=0}^{n} \frac{p^k}{q^k} b_{n,k}^{\lambda}(x;p,q) \cdot \int_0^1 b_{n,k}(qt;p,q) f(t) d_{p,q}t, \tag{5}$$

where $x \in [0,1]$, $0 < q < p \leq 1$, $b_{n,k}^{\lambda}(x;p,q)(k=0,1,\cdots,n)$ are defined in (4) and $b_{n,k}(\cdot;p,q)(k=0,1,\cdots,n)$ are defined in (2). Apparently, when $\lambda = 0$, operators $D_{n,p,q}^{0}(f;x)$ become to $D_n^{(p,q)}(f;x)$ which are defined as follows in [11],

$$D_n^{(p,q)}(f;x) = [n+1]_{p,q} p^{-n^2} \sum_{k=0}^{n} b_{n,k}^{(p,q)}(x) \left(\frac{q}{p}\right)^{-k} \cdot \int_0^1 b_{n,k}^{(p,q)}(qt) f(t) d_{p,q}t, \tag{6}$$

where $b_{n,k}^{(p,q)}(x) = p^{n(n-1)/2} b_{n,k}(x;p,q)$. We will show that operators $D_{n,p,q}^{\lambda}(f;x)$ converge to f faster than operators $D_n^{(p,q)}(f;x)$ in some cases of λ by some numerical examples in Section 4, say, we have more modeling flexibility when adding the parameter λ.

We first mention some definitions based on (p,q)-integers; details can be found in [22–26]. For any fixed real number $p > 0$ and $q > 0$, the (p,q)-integers $[n]_{p,q}$ are defined by

$$[n]_{p,q} = p^{n-1} + p^{n-2}q + p^{n-3}q^2 + \cdots + pq^{n-2} + q^{n-1} = \begin{cases} \dfrac{p^n - q^n}{p-q}, & p \neq q \neq 1; \\ \dfrac{1-q^n}{1-q}, & p = 1; \\ n, & p = q = 1. \end{cases} \tag{7}$$

(p,q)-factorial and (p,q)-binomial coefficients are defined as follows:

$$[n]_{p,q}! = \begin{cases} [n]_{p,q}[n-1]_{p,q} \cdots [1]_{p,q}, & n = 1,2,\cdots; \\ 1, & n = 0, \end{cases} \quad \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \frac{[n]_{p,q}!}{[k]_{p,q}![n-k]_{p,q}!}. \tag{8}$$

The (p,q)-power basis $(x \oplus t)_{p,q}^n$ and $(x \ominus t)_{p,q}^n$ are defined by

$$\begin{aligned} (x \oplus t)_{p,q}^n &= (x+t)(px+qt)(p^2x+q^2t) \cdots (p^{n-1}x+q^{n-1}t), \\ (x \ominus t)_{p,q}^n &= (x-t)(px-qt)(p^2x-q^2t) \cdots (p^{n-1}x-q^{n-1}t). \end{aligned} \tag{9}$$

Let $f : [0,a] \to \mathbb{R}$, then (p,q)-integration of a funciton f is defined by

$$\int_0^a f(x) d_{p,q}x = (p-q)a \sum_{k=0}^{\infty} \frac{q^k}{p^{k+1}} f\left(\frac{q^k}{p^{k+1}} \mathrm{a}\right), \left|\frac{p}{q}\right| > 1. \tag{10}$$

This paper is mainly organized as follows: in Section 2, we estimate some moments and central moments of $D_{n,p,q}^{\lambda}(f;x)$ in order to obtain our main results; in Section 3, we study a Korovkin type approximation theorem and estimate the rate of convergence of $D_{n,p,q}^{\lambda}(f)$ to f by using the second order modulus of smoothness, Peetre's K-functional, Steklov mean function, and Lipschitz class function; in Section 4, we give some numerical experiments to verify our theoretical results; in the final section, a conclusion is given.

2. Some Lemmas

Before giving our main results, we need the following lemmas.

Lemma 1. *Let $0 < q < p \le 1$, we have*

$$\int_0^1 b_{n,k}(qt\,;p,q)d_{p,q}t = \left(\frac{q}{p}\right)^k \frac{p^n}{[n+1]_{p,q}}\,; \tag{11}$$

$$\int_0^1 t b_{n,k}(qt\,;p,q)d_{p,q}t = \left(\frac{q}{p}\right)^k p^{-k}\frac{p^{2n}[k+1]_{p,q}}{[n+1]_{p,q}[n+2]_{p,q}}\,; \tag{12}$$

$$\int_0^1 t^2 b_{n,k}(qt\,;p,q)d_{p,q}t = \left(\frac{q}{p}\right)^k p^{-2k}\frac{p^{3n}[k+1]_{p,q}[k+2]_{p,q}}{[n+1]_{p,q}[n+2]_{p,q}[n+3]_{p,q}}. \tag{13}$$

Proof. According to the following equations of Lemma 1 in [11]

$$\int_0^1 b_{n,k}^{(p,q)}(qt)t^s d_{p,q}t = \left(\frac{q}{p}\right)^k p^{-ks}p^{n(n+2s+1)/2}\frac{[n]_{p,q}![k+s]_{p,q}!}{[k]_{p,q}![n+s+1]_{p,q}!},\ s=0,1,2,3\cdots, \tag{14}$$

and the fact that $b_{n,k}^{(p,q)}(x) = p^{n(n-1)/2}b_{n,k}(x\,;p,q)$, we get the proof of Lemma 1 easily.

Lemma 2. *Let $e_k(t) = t^k (k=0,1,2)$, $\lambda \in [-1,1]$, $x \in [0,1]$, and $0 < q < p \le 1$, then for the operators $D_{n,p,q}^{\lambda}(f\,;x)$, we have*

$$D_{n,p,q}^{\lambda}(e_0\,;x) = 1\,; \tag{15}$$

$$\begin{aligned}
D_{n,p,q}^{\lambda}(e_1\,;x) = {} & \frac{x}{q} + \frac{p^n\left(q-[2]_{p,q}x\right)}{q[n+2]_{p,q}} \\
& + \frac{2\lambda q[n]_{p,q}[n+1]_{p,q}x^2\left(1-x^{n-1}\right)(1-q/p)}{p^n[n+2]_{p,q}\left(p^{2-2n}[n]_{p,q}^2-1\right)} \\
& + \frac{\lambda p^n\left(1-x^{n+1}\right)}{[n+2]_{p,q}\left(p^{1-n}[n]_{p,q}-1\right)} \\
& - \frac{\lambda[n+1]_{p,q}x(1-x^n)(1-q/p)}{[n+2]_{p,q}\left(p^{1-n}[n]_{p,q}-1\right)} \\
& - \frac{2\lambda q[n+1]_{p,q}x(1-x^n)}{p[n+2]_{p,q}\left(p^{2-2n}[n]_{p,q}^2-1\right)} \\
& - \frac{\lambda(1\ominus x)_{p,q}^{n+1}}{p^{n(n-1)/2}[n+2]_{p,q}\left(p^{1-n}[n]_{p,q}-1\right)}\,;
\end{aligned} \tag{16}$$

$$\begin{aligned}
D_{n,p,q}^{\lambda}(e_2\,;x) = {} & \frac{q^3[n]_{p,q}^2x^2}{[n+2]_{p,q}[n+3]_{p,q}} \\
& + \frac{p^{n-1}q[n]_{p,q}x\left([2]_{p,q}^2-q^2x\right)+[2]_{p,q}p^{2n}}{[n+2]_{p,q}[n+3]_{p,q}} \\
& + \frac{\lambda q^2\left(1-q^2/p^2\right)[n]_{p,q}[n+1]_{p,q}x^2}{\left(p^{1-n}[n]_{p,q}+1\right)[n+2]_{p,q}[n+3]_{p,q}} \\
& \cdot\left[\frac{2q^2[n-1]_{p,q}x\left(1-x^{n-2}\right)}{p[n]_{p,q}-p^n}-\left(1-x^{n-1}\right)\right] \\
& + \frac{2\lambda q[n]_{p,q}[n+1]_{p,q}x^2\left(1-x^{n-1}\right)\left(p-[3]_{p,q}q^2/p^3\right)}{\left(p^{2-2n}[n]_{p,q}^2-1\right)[n+2]_{p,q}[n+3]_{p,q}} \\
& + \frac{[2]_{p,q}\lambda p^{2n}\left(1-x^{n+1}\right)}{\left(p^{1-n}[n]_{p,q}-1\right)[n+2]_{p,q}[n+3]_{p,q}} \\
& + \frac{\lambda p^{n-2}q[n+1]_{p,q}x(1-x^n)}{[n+2]_{p,q}[n+3]_{p,q}} \\
& \cdot\left[\frac{p^2+q^2}{p^{1-n}[n]_{p,q}+1}+\frac{p(1-p/q)(p+2q)}{p^{1-n}[n]_{p,q}-1}\right] \\
& - \frac{[2]_{p,q}\lambda p^n(1\ominus x)_{p,q}^{n+1}}{p^{n(n-1)/2}\left(p^{1-n}[n]_{p,q}-1\right)[n+2]_{p,q}[n+3]_{p,q}} \\
& - \frac{8\lambda p^{n-1}q^2[n+1]_{p,q}x(1-x^n)}{\left(p^{2-2n}[n]_{p,q}^2-1\right)[n+2]_{p,q}[n+3]_{p,q}} \\
& + \frac{2\lambda p^{n-1}q(2q-p)[n+1]_{p,q}x(1!x)_{p,q}^n}{p^{n(n-1)/2}\left(p^{2-2n}[n]_{p,q}^2-1\right)[n+2]_{p,q}[n+3]_{p,q}}.
\end{aligned} \tag{17}$$

Proof. By (5), (11), and Lemma 2 of [20], we have

$$\begin{aligned}
D_{n,p,q}^{\lambda}(1\,;x) &= [n+1]_{p,q}p^{-n}\sum_{k=0}^{n}\frac{p^k}{q^k}b_{n,k}^{\lambda}(x\,;p,q)\int_0^1 b_{n,k}(qt\,;p,q)d_{p,q} \\
&= [n+1]_{p,q}p^{-n}\sum_{k=0}^{n}\frac{p^k}{q^k}b_{n,k}^{\lambda}(x\,;p,q)\frac{q^k}{p^k}\frac{p^n}{[n+1]_{p,q}} \\
&= \sum_{k=0}^{n} b_{n,k}^{\lambda}(x\,;p,q) = 1.
\end{aligned} \tag{18}$$

Next, by the fact that $[k+1]_{p,q} = \mathrm{p}^k + q[k]_{p,q}$, (5) and (12), we get

$$\begin{aligned}
D_{n,p,q}^{\lambda}(t\,;x) &= [n+1]_{p,q}p^{-n}\sum_{k=0}^{n}\frac{p^k}{q^k}b_{n,k}^{\lambda}(x\,;p,q)\int_0^1 b_{n,k}(qt\,;p,q)t d_{p,q}t \\
&= [n+1]_{p,q}p^{-n}\sum_{k=0}^{n}\frac{p^k}{q^k}b_{n,k}^{\lambda}(x\,;p,q)\frac{q^k}{p^k}p^{-k}\frac{p^{2n}[k+1]_{p,q}}{[n+1]_{p,q}[n+2]_{p,q}} \\
&= \sum_{k=0}^{n} b_{n,k}^{\lambda}(x\,;p,q)\frac{p^n[k+1]_{p,q}}{p^k[n+2]_{p,q}} \\
&= \frac{q[n]_{p,q}}{[n+2]_{p,q}}B_{n,p,q}^{\lambda}(t\,;x) + \frac{p^n}{[n+2]_{p,q}}B_{n,p,q}^{\lambda}(1\,;x).
\end{aligned} \tag{19}$$

Then, the desired of (16) can be obtained by Lemma 2 and Lemma 3 of [20] and easy computations. Finally, by (5) and (13), we have

$$\begin{aligned} D_{n,p,q}^{\lambda}(t^2;x) &= [n+1]_{p,q}p^{-n}\sum_{k=0}^{n}\frac{p^k}{q^k}b_{n,k}^{\lambda}(x;p,q)\int_0^1 b_{n,k}(qt;p,q)t^2d_{p,q}t \\ &= [n+1]_{p,q}p^{-n}\sum_{k=0}^{n}\frac{p^k}{q^k}b_{n,k}^{\lambda}(x;p,q)\frac{q^k}{p^k}p^{-2k} \\ &\quad\cdot\frac{p^{3n}[k+1]_{p,q}[k+2]_{p,q}}{[n+1]_{p,q}[n+2]_{p,q}[n+3]_{p,q}} \\ &= \sum_{k=0}^{n}b_{n,k}^{\lambda}(x;p,q)\frac{p^{2n}[k+1]_{p,q}[k+2]_{p,q}}{p^{2k}[n+2]_{p,q}[n+3]_{p,q}}. \end{aligned} \tag{20}$$

Using $[k+1]_{p,q}[k+2]_{p,q} = q^3[k]_{p,q}^2 + p^kq(2q+p)[k]_{p,q} + [2]_{p,q}p^{2k}$, we obtain

$$\begin{aligned} D_{n,p,q}^{\lambda}(t^2;x) &= \frac{q^3[n]_{p,q}^2}{[n+2]_{p,q}[n+3]_{p,q}}B_{n,p,q}^{\lambda}(t^2;x) \\ &\quad+\frac{p^nq(p+2q)[n]_{p,q}}{[n+2]_{p,q}[n+3]_{p,q}}B_{n,p,q}^{\lambda}(t;x) \\ &\quad+\frac{[2]_{p,q}p^{2n}}{[n+2]_{p,q}[n+3]_{p,q}}. \end{aligned} \tag{21}$$

We can get (17) by Lemma 2–4 of [20] and some computations. Lemma 2 is proved.

Lemma 3. *Let $\Phi_x^k(t) = (t-x)^k (k=1,2)$, $\lambda\in[-1,1]$, $x\in[0,1]$, and $0<q<p\le 1$, then we have*

$$\begin{aligned} D_{n,p,q}^{\lambda}(\Phi_x^1;x) &= \frac{1-q}{q}x + \frac{p^n\left(q-[2]_{p,q}x\right)}{q[n+2]_{p,q}} \\ &\quad+\frac{2\lambda q[n]_{p,q}[n+1]_{p,q}x^2\left(1-x^{n-1}\right)(1-q/p)}{p^n[n+2]_{p,q}\left(p^{2-2n}[n]_{p,q}^2-1\right)} \\ &\quad+\frac{\lambda p^n\left(1-x^{n+1}\right)}{[n+2]_{p,q}\left(p^{1-n}[n]_{p,q}-1\right)} \\ &\quad-\frac{\lambda[n+1]_{p,q}x(1-x^n)(1-q/p)}{[n+2]_{p,q}\left(p^{1-n}[n]_{p,q}-1\right)} \\ &\quad-\frac{2\lambda q[n+1]_{p,q}x(1-x^n)}{p[n+2]_{p,q}\left(p^{2-2n}[n]_{p,q}^2-1\right)} \\ &\quad-\frac{\lambda(1\ominus x)_{p,q}^{n+1}}{p^{n(n-1)/2}[n+2]_{p,q}\left(p^{1-n}[n]_{p,q}-1\right)} := \Omega_{n,p,q}^{\lambda}(x); \end{aligned} \tag{22}$$

$$\begin{aligned} &\le \frac{1-q}{q} + \frac{1}{[n+2]_{p,q}} + \frac{2}{p^{1-n}[n]_{p,q}-1} \\ &\quad+\frac{3}{p^{n(n-1)/2}[n+2]_{p,q}\left(p^{1-n}[n]_{p,q}-1\right)} \\ &:= \Theta(n;p,q); \end{aligned} \tag{23}$$

$$\begin{aligned} D_{n,p,q}^{\lambda}(\Phi_x^2;x) &\le \frac{(1-q)^2}{q^2} + \frac{8}{q[n+2]_{p,q}} + \frac{4}{p^{1-n}[n]_{p,q}-1} \\ &\quad+\frac{1}{[n+3]_{p,q}} + \frac{8}{q^2[n+2]_{p,q}[n+3]_{p,q}} \\ &\quad+\frac{11}{p^{n(n-1)/2}[n+2]_{p,q}\left(p^{1-n}[n]_{p,q}-1\right)} \\ &\quad+\frac{12}{p^{n(n-1)/2}[n+2]_{p,q}[n+3]_{p,q}\left(p^{1-n}[n]_{p,q}-1\right)} \\ &:= \Psi(n;p,q). \end{aligned} \tag{24}$$

Proof. We can obtain (22) easily by (15) and (16). For $\lambda\in[0,1]$, we have

$$\begin{aligned} D_{n,p,q}^{\lambda}(\Phi_x^1;x) &\le \frac{1-q}{q} + \frac{p^n}{[n+2]_{p,q}} \\ &\quad+\frac{2q[n]_{p,q}[n+1]_{p,q}(1-q/p)}{p^n[n+2]_{p,q}\left(p^{2-2n}[n]_{p,q}^2-1\right)} \\ &\quad+\frac{p^n}{[n+2]_{p,q}\left(p^{1-n}[n]_{p,q}-1\right)}. \end{aligned} \tag{25}$$

For $\lambda\in[-1,0]$, we have

$$\begin{aligned} D_{n,p,q}^{\lambda}(\Phi_x^1;x) &\le \frac{1-q}{q} + \frac{p^n}{[n+2]_{p,q}} \\ &\quad+\frac{[n+1]_{p,q}(1-q/p)}{[n+2]_{p,q}\left(p^{1-n}[n]_{p,q}-1\right)} \\ &\quad+\frac{2[n+1]_{p,q}}{[n+2]_{p,q}\left(p^{2-2n}[n]_{p,q}^2-1\right)} \\ &\quad+\frac{1}{p^{n(n-1)/2}[n+2]_{p,q}\left(p^{1-n}[n]_{p,q}-1\right)}. \end{aligned} \tag{26}$$

On one hand, since

$$\frac{[n]_{p,q}}{p^n\left(p^{1-n}[n]_{p,q}+1\right)} = \frac{[n]_{p,q}}{p[n]_{p,q}+p^n} \le \frac{[n]_{p,q}}{p[n]_{p,q}+q^n} \le \frac{[n]_{p,q}}{[n+1]_{p,q}} \le 1, \tag{27}$$

we get

$$\frac{2q[n]_{p,q}[n+1]_{p,q}(1-q/p)}{p^n[n+2]_{p,q}\left(p^{2-2n}[n]_{p,q}^2-1\right)} \le \frac{2q[n+1]_{p,q}(1-q/p)}{[n+2]_{p,q}\left(p^{1-n}[n]_{p,q}-1\right)}. \tag{28}$$

On the other hand, we have

$$\begin{aligned}\frac{[n+1]_{p,q}}{[n+2]_{p,q}\left(p^{2-2n}[n]_{p,q}^2-1\right)} &\le \frac{p^n}{[n+2]_{p,q}\left(p^{1-n}[n]_{p,q}-1\right)}\\ &\le \frac{1}{p^{n(n-1)/2}[n+2]_{p,q}\left(p^{1-n}[n]_{p,q}-1\right)}\end{aligned} \tag{29}$$

with the fact that $[n+1]_{p,q}/p^{1-n}[n]_{p,q}+1=p^n[n+1]_{p,q}/p[n]_{p,q}+p^n\le p^n$. Combing (25)–(29), we have

$$\begin{aligned}D_{n,p,q}^{\lambda}\left(\Phi_x^1;x\right) &\le \frac{1-q}{q}+\frac{p^n}{[n+2]_{p,q}}+\frac{2[n+1]_{p,q}(1-q/p)}{[n+2]_{p,q}\left(p^{1-n}[n]_{p,q}-1\right)}\\ &\quad+\frac{3}{p^{n(n-1)/2}[n+2]_{p,q}\left(p^{1-n}[n]_{p,q}-1\right)}\\ &\le \frac{1-q}{q}+\frac{1}{[n+2]_{p,q}}+\frac{2}{p^{1-n}[n]_{p,q}-1}\\ &\quad+\frac{3}{p^{n(n-1)/2}[n+2]_{p,q}\left(p^{1-n}[n]_{p,q}-1\right)}.\end{aligned} \tag{30}$$

Thus, the desired result of (23) is proved. Finally, by Lemma 2 and the linear property of $D_{n,p,q}^{\lambda}(f)$, we have

$$\begin{aligned}D_{n,p,q}^{\lambda}\left(\Phi_x^2;x\right) &= D_{n,p,q}^{\lambda}(e_2;x)-2xD_{n,p,q}^{\lambda}(e_1;x)+x^2\\ &\le \left(\frac{q^3[n]_{p,q}^2}{[n+2]_{p,q}[n+3]_{p,q}}-\frac{2}{q}+1\right)x^2\\ &\quad+\frac{p^{n-1}q[2]_{p,q}^2[n]_{p,q}x}{[n+2]_{p,q}[n+3]_{p,q}}+\frac{[2]_{p,q}p^{2n}}{[n+2]_{p,q}[n+3]_{p,q}}\\ &\quad+\frac{2[2]_{p,q}p^nx^2}{q[n+2]_{p,q}}+\Lambda_{n,p,q}^{\lambda}(x)\\ &\le \frac{(1-q)^2}{q^2}+\frac{q[2]_{p,q}^2+2[2]_{p,q}}{q[n+2]_{p,q}}\\ &\quad+\frac{[2]_{p,q}\left(q^2+[3]_{p,q}\right)}{q^2[n+2]_{p,q}[n+3]_{p,q}}+\Lambda_{n,p,q}^{\lambda}(x),\end{aligned} \tag{31}$$

where $\Lambda_{n,p,q}^{\lambda}(x)$ is some function related to λ, $[n]_{p,q}$ and x, and we will estimate it in two cases. For $\lambda\in[0,1]$, we have

$$\begin{aligned}\Lambda_{n,p,q}^{\lambda}(x) &\le \frac{4}{p^{1-n}[n]_{p,q}-1}+\frac{[2]_{p,q}}{[n+2]_{p,q}[n+3]_{p,q}}\\ &\quad+\frac{4}{p[n+2]_{p,q}\left(p^{1-n}[n]_{p,q}-1\right)}\\ &\quad+\frac{2}{p^{n(n-1)/2}[n+2]_{p,q}\left(p^{1-n}[n]_{p,q}-1\right)}\\ &\quad+\frac{[2]_{p,q}}{\left(p^{1-n}[n]_{p,q}-1\right)[n+2]_{p,q}[n+3]_{p,q}}\\ &\quad+\frac{2}{p^{n(n-1)/2}\left(p^{1-n}[n]_{p,q}-1\right)[n+2]_{p,q}[n+3]_{p,q}}.\end{aligned} \tag{32}$$

For $\lambda\in[-1,0]$, we have

$$\begin{aligned}\Lambda_{n,p,q}^{\lambda}(x) &\le \frac{4}{p^{1-n}[n]_{p,q}-1}+\frac{1}{[n+3]_{p,q}}\\ &\quad+\frac{2[3]_{p,q}+2}{[n+2]_{p,q}\left(p^{1-n}[n]_{p,q}-1\right)}\\ &\quad+\frac{3}{q\left(p^{1-n}[n]_{p,q}-1\right)[n+2]_{p,q}[n+3]_{p,q}}\\ &\quad+\frac{8}{\left(p^{1-n}[n]_{p,q}-1\right)[n+2]_{p,q}[n+3]_{p,q}}\\ &\quad+\frac{2+[2]_{p,q}}{p^{n(n-1)/2}\left(p^{1-n}[n]_{p,q}-1\right)[n+2]_{p,q}[n+3]_{p,q}}.\end{aligned} \tag{33}$$

From the above two equations (32) and (33), we obtain

$$\begin{aligned}\Lambda_{n,p,q}^{\lambda}(x) &\le \frac{4}{p^{1-n}[n]_{p,q}-1}+\frac{1}{[n+3]_{p,q}}\\ &\quad+\frac{2[3]_{p,q}+2}{p[\mathrm{n}+2]_{p,q}\left(p^{1-n}[n]_{p,q}-1\right)}\\ &\quad+\frac{3}{p^{n(n-1)/2}[n+2]_{p,q}\left(p^{1-n}[n]_{p,q}-1\right)}\\ &\quad+\frac{8}{\left(p^{1-n}[n]_{p,q}-1\right)[n+2]_{p,q}[n+3]_{p,q}}\\ &\quad+\frac{2+[2]_{p,q}}{p^{n(n-1)/2}\left(p^{1-n}[n]_{p,q}-1\right)[n+2]_{p,q}[n+3]_{p,q}}.\end{aligned} \tag{34}$$

Combing (31), (33), and (34), we get

$$D_{n,p,q}^{\lambda}\left(\Phi_x^2;x\right) \le \frac{(1-q)^2}{q^2} + \frac{8}{q[n+2]_{p,q}} + \frac{4}{p^{1-n}[n]_{p,q}-1} + \frac{1}{[n+3]_{p,q}} + \frac{8}{q^2[n+2]_{p,q}[n+3]_{p,q}} + \frac{11}{p^{n(n-1)/2}[n+2]_{p,q}\left(p^{1-n}[n]_{p,q}-1\right)} + \frac{12}{p^{n(n-1)/2}[n+2]_{p,q}[n+3]_{p,q}\left(p^{1-n}[n]_{p,q}-1\right)}. \tag{35}$$

Thus, we arrive at (24). Lemma 3 is proved.

Lemma 4. *(See [6]).*

Let sequences $q := \{q_n\} = \{1-\alpha_n\}$, $p := \{p_n\} = \{1-\beta_n\}$ such that $0 \le \beta_n < \alpha_n < 1$, $\alpha_n \to 0$, $\beta_n \to 0$ as $n \to \infty$. The following statements are true

(A) If $\lim_{n\to\infty} e^{n(\beta_n-\alpha_n)} = 1$ and $e^{n\beta_n}/n \to 0$, then $[n]_{p_n,q_n} \to \infty$.

(B) If $\overline{\lim}_{n\to\infty} e^{n(\beta_n-\alpha_n)} < 1$ and $e^{n\beta_n}(\alpha_n-\beta_n) \to 0$, then $[n]_{p_n,q_n} \to \infty$.

(C) If $\underline{\lim}_{n\to\infty} e^{n(\beta_n-\alpha_n)} < 1$, $\overline{\lim}_{n\to\infty} e^{n(\beta_n-\alpha_n)} = 1$ and $\max\{e^{n\beta_n}/n, e^{n\beta_n}(\alpha_n-\beta_n)\} \to 0$, then $[n]_{p_n,q_n} \to \infty$.

3. Rate of Convergence

In the sequel, let sequences $q := \{q_n\}$ and $p := \{p_n\}$ satisfy the conditions of Lemma 4. We first give a Korovkin type approximation theorem for $D_{n,p,q}^{\lambda}(f)$.

Theorem 5. *Let f be a continuous function on $[0,1]$, $\lambda \in [-1,1]$ and $n > 1$, then $D_{n,p,q}^{\lambda}(f;x)$ converge uniformly to f on $[0,1]$.*

Proof. Since the hypothesis of sequences p and q, we know that $[n+i]_{p,q} \to \infty (i = 1,2,3)$ as $n \to \infty$. It is easy to get $D_{n,p,q}^{\lambda}(e_k;x) \to x^k (k = 0,1,2)$ combining the relation $[n+i]_{p,q} = [i]_{p,q}p^n + q^i[n]_{p,q} (i = 0,1,2)$. Therefore, we obtain the desired result due to the well-known Korovkin theorem (see [27], pp. 8-9).

Let f be a continuous function on $[0,1]$ and endowed with the norm $\|f\| = \sup_{x\in[0,1]}|f(x)|$. Peetre's K-functional is defined by

$$K_2(f;\delta) = \inf_{g\in C^2}\left\{\|f-g\| + \delta\|g''\|\right\}, \tag{36}$$

where $\delta > 0$ and $C^2 = \{g \in C[0,1]: g', g'' \in C[0,1]\}$. The second order modulus of smoothness is defined as

$$\omega_2(f;\delta) = \sup_{0<h\le\delta}\sup_{x,x+h,x+2h\in[0,1]}|f(x+2h) - 2f(x+h) + f(x)|. \tag{37}$$

We know that there is a relationship between $K_2(f;\delta)$ and $\omega_2(f;\sqrt{\delta})$, that is

$$K_2(f;\delta) \le C\omega_2\left(f;\sqrt{\delta}\right), \tag{38}$$

where C is a positive constant. The modulus of continuity is denoted by

$$\omega(f;\delta) = \sup_{0<h\le\delta}\sup_{x,x+h\in[0,1]}|f(x+h) - f(x)|. \tag{39}$$

Then, the rate of convergence of $D_{n,p,q}^{\lambda}(f)$ to f is given as follows.

Theorem 6. *Let f be a continuous function on $[0,1]$, $\lambda \in [-1,1]$, and $n > 1$, we have*

$$\left|D_{n,p,q}^{\lambda}(f;x) - f(x)\right| \le C\omega_2\left(f;\frac{1}{2}\sqrt{(\Theta(n;p,q))^2 + \Phi(n;p,q)}\right) + \omega(f;\Theta(n;p,q)), \tag{40}$$

where C is a positive constant, $\Theta(n;p,q)$ and $\Psi(n;p,q)$ are defined in (23) and (24).

Proof. Let us define auxiliary operators $\widehat{D}_{n,p,q}^{\lambda}(f;x)$ which preserve linear functions as

$$\widehat{D}_{n,p,q}^{\lambda}(f;x) = D_{n,p,q}^{\lambda}(f;x) - f\left(x + \Omega_{n,p,q}^{\lambda}(x)\right) + f(x), \tag{41}$$

where $\Omega_{n,p,q}^{\lambda}(x)$ is defined in (22). Obviously,

$$\widehat{D}_{n,p,q}^{\lambda}(t-x;x) = 0. \tag{42}$$

Set $g \in C^2$, by Taylor's expansion, we have

$$g(t) = g(x) + g'(t-x) + \int_x^t (t-u)g''(u)du, x, t \in [0,1]. \tag{43}$$

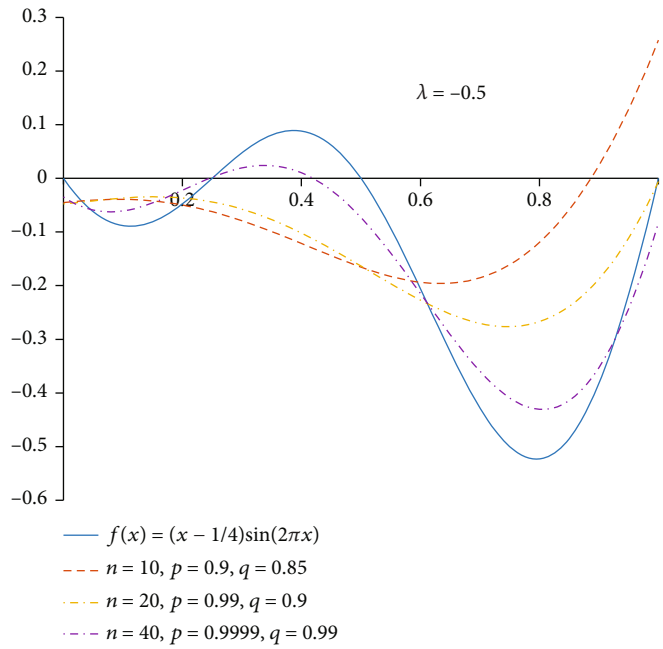

FIGURE 1: The convergence of $D_{n,p,q}^{-0.5}(f\,;x)$ to $f(x)$.

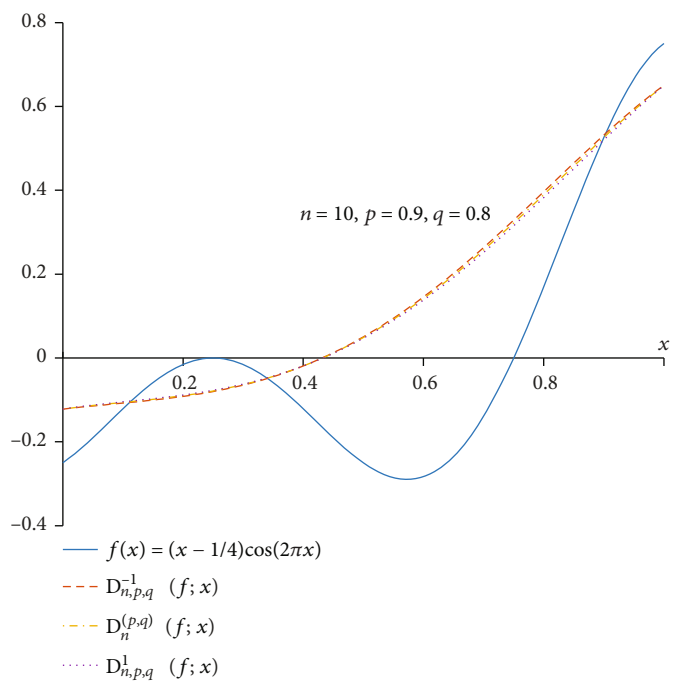

FIGURE 2: The approximation graphs of $D_n^{(p,q)}(f\,;x)$, $D_{n,p,q}^{-1}(f\,;x)$, and $D_{n,p,q}^{1}(f\,;x)$.

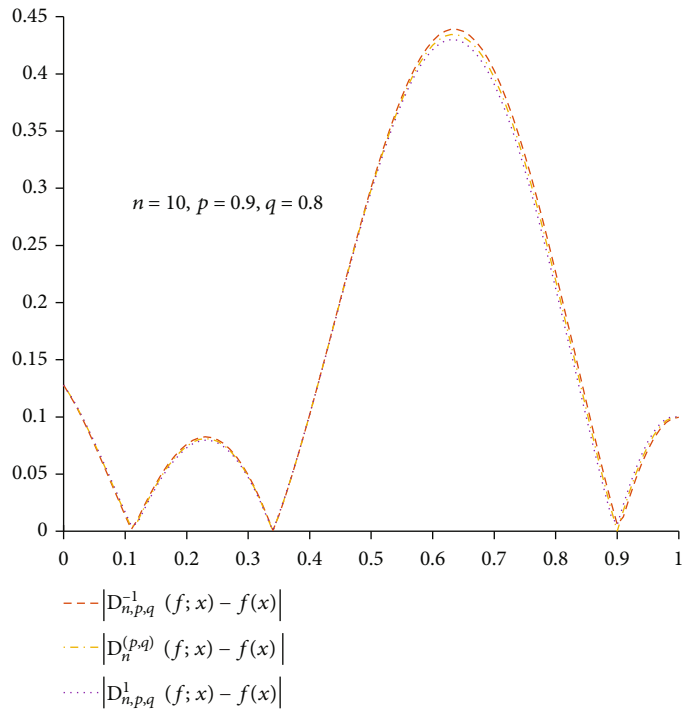

Figure 3: The error graphs of $|D_n^{(p,q)}(f;x)-f(x)|$, $|D_{n,p,q}^{-1}(f;x)-f(x)|$, and $|D_{n,p,q}^{1}(f;x)-f(x)|$.

Table 1: The maximum error of $|D_n^{(p,q)}(f;x)-f(x)|$, $|D_{n,p,q}^{-1}(f;x)-f(x)|$ and $|D_{n,p,q}^{1}(f;x)-f(x)|$ with $p=1-1/n$, $q=1-2/n$.

n	10	20	30	40	50	70	100
$\max\left(\left\|D_{n,p,q}^{-1}(f;x)-f(x)\right\|\right)$	0.4390	0.2868	0.2110	0.1672	0.1385	0.1033	0.0749
$\max\left(\left\|D_{n}^{(p,q)}(f;x)-f(x)\right\|\right)$	0.4345	0.2848	0.2101	0.1667	0.1382	0.1031	0.0748
$\max\left(\left\|D_{n,p,q}^{1}(f;x)-f(x)\right\|\right)$	0.4301	0.2827	0.2091	0.1661	0.1379	0.1030	0.0747

Applying $\widehat{D}_{n,p,q}^{\lambda}(g;x)$ to (43) and by (42), we obtain

$$\widehat{D}_{n,p,q}^{\lambda}(g;x)-g(x)=\widehat{D}_{n,p,q}^{\lambda}\left(\int_x^t (t-u)g''(u)du;x\right). \quad (44)$$

Thus, by (41), we have

$$\begin{aligned}\left|\widehat{D}_{n,p,q}^{\lambda}(g;x)-g(x)\right| &\le \left|D_{n,p,q}^{\lambda}\left(\int_x^t (t-u)g''(u)du;x\right)\right| \\ &\quad+\left|\int_x^{x+\Omega_{n,p,q}^{\lambda}(x)}\left(x+\Omega_{n,p,q}^{\lambda}(x)-u\right)g''(u)du\right| \\ &\le \left[D_{n,p,q}^{\lambda}(\Phi_x^2;x)+\left(\Omega_{n,p,q}^{\lambda}(x)\right)^2\right]\|g''\| \\ &\le \left[(\Theta(n;p,q))^2+\Phi(n;p,q)\right]\|g''\|,\end{aligned} \quad (45)$$

where $\Theta(n;p,q)$ and $\Psi(n;p,q)$ are defined in (23) and (24). According to (41), (5), and (15), we have

$$\left|\widehat{D}_{n,p,q}^{\lambda}(f;x)\right| \le \left|D_{n,p,q}^{\lambda}(f;x)\right|+2\|f\| \le 3\|f\|. \quad (46)$$

Using (41), (45), and (46), we get

$$\begin{aligned}\left|D_{n,p,q}^{\lambda}(f;x)-f(x)\right| &\le \left|\widehat{D}_{n,p,q}^{\lambda}(f-g;x)-(f-g)(x)\right| \\ &\quad+\left|\widehat{D}_{n,p,q}^{\lambda}(g;x)-g(x)\right| \\ &\quad+\left|f\left(x+\Omega_{n,p,q}^{\lambda}(x)\right)-f(x)\right| \\ &\le 4\|f-g\|+\left[(\Theta(n;p,q))^2\right. \\ &\quad\left.+\Phi(n;p,q)\right]\|g''\|+\omega(f;\Theta(n;p,q)).\end{aligned} \quad (47)$$

Taking the infimum on the right hand side over all $g \in C^2$, we have

$$\left|D_{n,p,q}^{\lambda}(f;x)-f(x)\right| \le 4K_2\left(f;\frac{1}{4}\left[(\Theta(n;p,q))^2+\Phi(n;p,q)\right]\right) + \omega(f;\Theta(n;p,q)). \tag{48}$$

Therefore, we obtain

$$\left|D_{n,p,q}^{\lambda}(f;x)-f(x)\right| \le C\omega_2\left(f;\frac{1}{2}\sqrt{(\Theta(n;\mathrm{p},q))^2+\Phi(n;p,q)}\right) + \omega(f;\Theta(n;p,q)). \tag{49}$$

where C is a positive constant, $\Theta(n;p,q)$ and $\Psi(n;p,q)$ are defined in (23) and (24). Theorem 6 is proved.

Let f be a continuous function on $[0,1]$, the Steklov mean function is defined as

One can write

$$f_h(x)-f(x)=\frac{4}{h^2}\int_0^{h/2}\int_0^{h/2}[2f(x+u+v) - f(x+2(u+v))-f(x)]dudv \tag{50}$$

by the fact that f_h is continuous on $[0,1]$. It is obvious that

$$||f_h-f|| \le \tilde{\omega}_2(f,h), \tag{51}$$

where $\tilde{\omega}_2(f,\delta)=\sup_{x,u,v\ge 0}\sup_{|u-v|\le\delta}|f(x+2u)-2f(x+u+v)+f(x+2v)|$. If f is continuous on $[0,1]$, so are f_h', f_h'' and

$$||f_h'|| \le \frac{5}{h}\tilde{\omega}(f,h), ||f_h''|| \le \frac{9}{h^2}\tilde{\omega}_2(f,h), \tag{52}$$

where $\tilde{\omega}(f,h)=\sup_{x,u,v\ge 0,|u-v|<h}|f(x+u)-f(x+v)|$. Details can be found in [28].

Now, we apply Steklov mean to prove the following theorem.

Theorem 7. *Let f be a continuous function on $[0,1]$, $\lambda\in[-1,1]$, and $n>1$, we have*

$$\left|D_{n,p,q}^{\lambda}(f;x)-f(x)\right| \le 5\tilde{\omega}(f,\Theta(n;p,q)) + \frac{13}{2}\tilde{\omega}_2\left(f,\sqrt{\Phi(n;p,q)}\right), \tag{53}$$

where $\Theta(n;p,q)$ and $\Psi(n;p,q)$ are defined in (23) and (24).

Proof. Since

$$\left|D_{n,p,q}^{\lambda}(f;x)-f(x)\right| \le D_{n,p,q}^{\lambda}(|f-f_h|;x) + \left|D_{n,p,q}^{\lambda}(f_h-f_h(x);x)\right| + |f_h(x)-f(x)|. \tag{54}$$

By (5), (15), and (51), we have

$$D_{n,p,q}^{\lambda}(|f-f_h|;x) \le \left|\left|D_{n,p,q}^{\lambda}(f-f_h)\right|\right| \le ||f-f_h|| \le \tilde{\omega}_2(f,h). \tag{55}$$

By Taylor's expansion, Lemma 3 and (52), we obtain

$$\begin{aligned}\left|D_{n,p,q}^{\lambda}(f_h-f_h(x);x)\right| &\le |f_h'(x)|D_{n,p,q}^{\lambda}(\Phi_x^1;x) + \frac{1}{2}||f_h''||D_{n,p,q}^{\lambda}(\Phi_x^2;x)\\ &\le \frac{5\Theta(n;p,q)}{h_1}\tilde{\omega}(f,h_1) + \frac{9\Psi(n;p,q)}{2h_2^2}\tilde{\omega}_2(f,h_2).\end{aligned} \tag{56}$$

Therefore, by (54)–(56), we have

$$\begin{aligned}\left|D_{n,p,q}^{\lambda}(f;x)-f(x)\right| &\le \frac{5\Theta(n;p,q)}{h_1}\tilde{\omega}(f,h_1) + \left(\frac{9\Psi(n;p,q)}{2h_2^2}+2\right)\tilde{\omega}_2(f,h_2)\\ &= 5\tilde{\omega}(f,\Theta(n;p,q)) + \frac{13}{2}\tilde{\omega}_2\left(f,\sqrt{\Phi(n;p,q)}\right),\end{aligned} \tag{57}$$

by choosing $h_1=\Theta(n;p,q)$, $h_2=\sqrt{\Psi(n;p,q)}$. Theorem 7 is proved.

Finally, we study the rate of convergence of $D_{n,p,q}^{\lambda}(f)$ with the help of functions of Lipschitz class $\mathrm{Lip}_M(\xi)$, where M is a positive constant, $0<\xi\le 1$. A function f belongs to $\mathrm{Lip}_M(\xi)$ if

$$|f(t)-f(x)| \le M|t-x|^{\xi}(t,x\in[0,1]). \tag{58}$$

We have the following theorem.

Theorem 8. *Let $f\in Lip_M(\xi)$, $\lambda\in[-1,1]$, and $n>1$, we have*

$$\left|D_{n,p,q}^{\lambda}(f;x)-f(x)\right| \le M(\Psi(n;p,q))^{\xi/2}, \tag{59}$$

where $\Psi(n;p,q)$ is defined in (24).

Proof. Since $f \in \mathrm{Lip}_M(\xi)$ and $D^{\lambda}_{n,p,q}(f)$ are linear positive operators, using Hölder's inequality, we have

$$\begin{aligned}\left|D^{\lambda}_{n,p,q}(f;x)-f(x)\right| &\le D^{\lambda}_{n,p,q}(|f(t)-f(x)|;x)\\ &\le MD^{\lambda}_{n,p,q}\left(|t-x|^{\xi};x\right)\\ &= M[n+1]_{p,q}p^{-n}\sum_{k=0}^{n}\left[\frac{p^k}{q^k}b^{\lambda}_{n,k}(x;p,q)\right]^{2-\xi/2}\\ &\quad\cdot\left[\frac{p^k}{q^k}b^{\lambda}_{n,k}(x;p,q)\right]^{\xi/2}\int_0^1 b_{n,k}(qt;p,q)|t-x|^{\xi}d_{p,q}t\\ &\le M\left[[n+1]_{p,q}p^{-n}\sum_{k=0}^{n}\frac{p^k}{q^k}b^{\lambda}_{n,k}(x;p,q)\right.\\ &\quad\left.\int_0^1 b_{n,k}(qt;p,q)d_{p,q}t\right]^{2-\xi/2}\\ &\quad\left[[n+1]_{p,q}p^{-n}\sum_{k=0}^{n}\frac{p^k}{q^k}b^{\lambda}_{n,k}(x;p,q)\right.\\ &\quad\left.\int_0^1 b_{n,k}(qt;p,q)(t-x)^2d_{p,q}t\right]^{\xi/2}\\ &= MD^{\lambda}_{n,p,q}\left((t-x)^2;x\right)^{\xi/2}.\end{aligned} \tag{60}$$

Thus, Theorem 8 can be obtained by (24).

4. Numerical Examples

In this section, we give several numerical examples to show the convergence of $D^{\lambda}_{n,p,q}(f;x)$ and $D^{(p,q)}_{n}(f;x)$ to $f(x)$ with different values of parameters.

Example 9. Let $f(x)=(x-1/4)\sin(2\pi x)$ and $\lambda=-0.5$. The graphs of $f(x)$ and $D^{\lambda}_{n,p,q}(f;x)$ with different values of parameters $(n=10, p=0.9, q=0.85; n=20, p=0.99, q=0.9; n=40, p=0.9999, q=0.99)$ are shown in Figure 1.

Example 10. Let $f(x)=(x-1/4)\cos(2\pi x)$ and $n=10, p=0.9, q=0.8$. The graphs of $f(x)$, $D^{(p,q)}_{n}(f;x)$ and $D^{\lambda}_{n,p,q}(f;x)$ for $\lambda=-1$ and $\lambda=1$ are given in Figure 2. The error graphs of $|D^{(p,q)}_{n}(f;x)-f(x)|$ and $|D^{\lambda}_{n,p,q}(f;x)-f(x)|$ for $\lambda=-1$ and $\lambda=1$ are given in Figure 3. Moreover, in Table 1, there are given the maximum errors of $|D^{(p,q)}_{n}(f;x)-f(x)|$, $|D^{-1}_{n,p,q}(f;x)-f(x)|$, and $|D^{1}_{n,p,q}(f;x)-f(x)|$ with different values of parameters, where $p=1-1/n, q=1-2/n$.

5. Conclusion

In the present paper, we proposed a class of Durrmeyer type λ-Bernstein operators based on (p, q)-calculus. Due to the parameter λ, we have more flexibility in modeling. We studied the Korovkin type theorem, the estimated rate of convergence by using Peetres K-functional, the modulus of continuity of second order and Steklov mean; we also obtained a convergence theorem for the Lipschitz continuous functions. To make things more intuitive, we also give some numerical examples.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

We declare that there is no conflict of interest.

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