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Research Article

Best Proximity Coincidence Point Results for (α, D)-Proximal Generalized Geraghty Mappings in JS-Metric Spaces

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We introduce a type of Geraghty contractions in a JS-metric space X, called (α, D)-proximal generalized Geraghty mappings. By using the triangular-(α, D)-proximal admissible property, we obtain the existence and uniqueness theorem of best proximity coincidence points for these mappings together with some corollaries and illustrative examples. As an application, we give a best proximity coincidence point result in X endowed with a binary relation.

1. Introduction and Preliminaries

Let $T : A \to B$ be a map where A and B are two nonempty subsets of a metric space X. It is known that if T is a nonself-map, the equation $Tx = x$ does not always have a solution, and it clearly has no solution when A and B are disjoint. However, it is possible to determine an approximate solution x^* such that the error is $d(x^*, Tx^*) = d(A, B)$. Such point x^* is called a best proximity point of T. The best proximity point theorem was first studied in [1]. Then, there has been a wide range of research in this framework. Many researchers have studied and generalized the result in many aspects (for example, see [2–15]). For some recent articles regarding these points, see [16, 17] where Geraghty type mappings were studied and [18] where cyclic and noncyclic nonexpansive mappings were considered.

One of the well-known generalizations of the Banach contraction principle is the result given by Geraghty [19] which enriches the principle by considering the class of mappings $\theta : [0,\infty) \to [0, 1)$ such that $t_n \to 0$ when $\theta(t_n) \to 1$. By including 1 in the ranges of those mappings θ, Ayari [20] provided a new result on the existence and uniqueness of the best proximity point for α-proximal Geraghty mappings.

The concept of the best proximity coincidence point, which is an extension of a best proximity point problem, was mentioned in [21] (see also [22]) where some results of mappings in generalized metric spaces were presented. A point a is called a best proximity coincidence point of the pair (g, T), where g is a self-map on A, if $d(ga, Ta) = d(A, B)$. Clearly, if g is the identity map, then each best proximity coincidence point of the pair (g, T) is a best proximity point for T.

A large number of results concerning these point problems in various metric spaces have been investigated since then. Hussain and the coauthors contributed several interesting results and generalizations in [23–25], including the recent article [26] where best proximity point results for Suzuki-Edelstein proximal contractions were studied. (See also, [27–31] for his work.)

Zhang and Su [32] weakened the P-property, called the weak P-property, and improved a best proximity point theorem for Geraghty nonself-contractions. In 2018, Komal et al. [33] obtained best proximity coincidence point theorems for α-Geraghty contractions (g, T) in metric spaces by using the weak P-property where g is an isometry.

The concept of generalized metric spaces (or JS-metric spaces) was introduced in [34] in 2015. It is a generalization of standard metric spaces covering many topological structures.

Let X be a nonempty set, and let $D : X \times X \to [0,\infty]$ be a function. For each $x \in X$, we set

$$C(D, X, x) = \left\{ \{x_n\} \subseteq X : \lim_{n\to\infty} D(x_n, x) = 0 \right\}. \quad (1)$$

Definition 1 (see [34]). *A function $D : X \times X \to [0,\infty]$ is called a generalized metric on X if it satisfies the following conditions.*

(D_1)For any $x, y \in X$, $D(x, y) = 0$ implies $x = y$.
(D_2)For any $x, y \in X$, $D(x, y) = D(y, x)$.
(D_3)There exists a constant $C_X > 0$ such that

$$D(x, y) \leq C_x \limsup_{n\to\infty} D(x_n, y), \quad (2)$$

whenever $x, y \in X$ and $\{x_n\} \in C(D, X, x)$.

In this case, we say that (X, D) is a generalized metric space. It is, however, usually called a JS-metric space.

Remark 2. We note that, in general, results of best proximity points using the weak P -property in usual metric spaces might not be attained in the setting of JS -metric spaces. For example, $D(x, x)$ is not necessarily equal to 0, and $D(x_n, y_n)$ might not converge to $D(x, y)$ when $x_n \to x$ and $y_n \to y$.

Let $X := (X, D)$ be a JS-metric space. We now discuss the convergence and the continuity in these spaces.

Definition 3 (see [34]). *Let $\{x_n\}$ be a sequence in X. The sequence $\{x_n\}$ is said to D -converge to $x \in X$ if $\{x_n\} \in C(D, X, x)$. Moreover, $\{x_n\}$ is called a D -Cauchy sequence if $\lim_{m,n\to\infty} D(x_n, x_m) = 0$. Finally, (X, D) is said to be D -complete if each D -Cauchy sequence in X is a D -convergent sequence in X.*

Proposition 4 (see [34]). *For any $x, y \in X$, if $\{x_n\} \in C(D, X, x) \cap C(D, X, y)$, then $x = y$.*

Definition 5 (see [34]). *A function $f : X \to X$ is said to be D -continuous at a point $x_0 \in X$ if for any $\{x_n\} \in C(D, X, x_0)$, $\{f x_n\} \in C(D, X, f x_0)$. In addition, f is said to be D -continuous on X if it is D -continuous at each point in X.*

The concept of α-admissible mapping was introduced by Samet et al. [35] in 2012. The notion of triangular α-admissible mappings was then given by Karapinar [36]. Recently, Khemphet [37] extended the concept as follows.

Definition 6 (see [37]). *Let (X, D) be a generalized metric space, and let f and g be self-mappings on X . Given that $\alpha : X \times X \to [0,\infty)$ is a function, f is said to be triangular- (α, D) -admissible w.r.t. g if, for all $x, y, z \in X$, the following conditions hold.*

(i) If $\alpha(gx, gy) \geq 1$, then $\alpha(fx, fy) \geq 1$ and $D(gx, gy) < \infty$.

(ii) If $\alpha(x, z) \geq 1$ and $\alpha(z, y) \geq 1$, then $\alpha(x, y) \geq 1$.

In this article, we introduce a type of Geraghty contractions which will be called (α, D)-proximal generalized Geraghty mappings. These maps are motivated by the work of Khemphet [37]. Using the weak P-property in the setting of JS-metric space, we establish a result on the existence and uniqueness of the best proximity coincidence point for these mappings. Examples showing the validity of the main result and some corollaries are listed. Finally, by applying our main result, we obtain a best proximity coincidence point result in X endowed with a binary relation. Note that some other results of best proximity points in X endowed with binary relations can be deduced from our result.

2. Main Results

Throughout this article, let $X := (X, D)$ be a JS-metric space, and let A and B be nonempty disjoint subsets of X. Also, we require the following notations:

$$\begin{aligned} D(A, B) &:= \inf \{D(a, b): a \in A, b \in B\}, \\ A_0 &:= \{a \in A : \text{there exists } b \in B \text{ such that } D(a, b) = D(A, B)\}, \\ B_0 &:= \{b \in B : \text{there exists } a \in A \text{ such that } D(a, b) = D(A, B)\}. \end{aligned} \quad (3)$$

Clearly, if one of A_0 and B_0 is nonempty, then so is the other.

Definition 7 (see [21]). *Let $T : A \to B$ and $S : A \to A$ be mappings. An element $x^* \in A$ is said to be a best proximity coincidence point of the pair (S, T) if $D(Sx^*, Tx^*) = D(A, B)$. The set of all best proximity coincidence points of the pair (S, T) is denoted by $BC(S, T)$.*

Definition 8 (see [32]). *Suppose that A_0 is nonempty. The pair (A, B) is said to have the weak P -property if and only if $D(x_1, y_1) = D(x_2, y_2) = D(A, B)$ implies $D(x_1, x_2) \leq D(y_1, y_2)$, where $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$.*

Definition 9. Let $T : A \to B$ and $S : A \to A$ be mappings. The pair (S, T) is said to be triangular- (α, D) -proximal admissible if the following conditions hold.

(i) If $\alpha(Sx_1, Sx_2) \geq 1$ and $D(Su_1, Tx_1) = D(Su_2, Tx_2) = D(A, B)$, then $\alpha(Su_1, Su_2) \geq 1$ and $D(Su_1, Su_2) < \infty$ for all $x_1, x_2, u_1, u_2 \in A$.

(ii) If $\alpha(x, z) \geq 1$ and $\alpha(z, y) \geq 1$, then $\alpha(x, y) \geq 1$, for all $x, y, z \in X$.

We consider the class of mappings Θ which is a slight generalization of the well-known class of $[0, 1)$-valued functions introduced by Geraghty [19]:

$$\Theta := \{\theta : [0,\infty] \to [0, 1]: \theta(t_n) \to 1 \text{ implies } t_n \to 0\}. \quad (4)$$

Now, we introduce a class of our contractions as follows.

Definition 10. Let $T : A \to B$ and $S : A \to A$ be mappings. Given that $\alpha : X \times X \to [0,\infty)$ is a function, the pair (S, T) is said to be an (α, D) -proximal generalized Geraghty mapping if the following conditions hold.

(i) (S, T) is triangular-(α, D)-proximal admissible.

(ii) There is $\theta \in \Theta$ such that for all $x, y, u, v \in A$, if $D(Su, Tx) = D(Sv, Ty) = D(A, B)$ and $\alpha(Sx, Sy) \geq 1$, then

$$\alpha(Sx, Sy)D(Tx, Ty) \leq \theta(M(x, y, u, v))M(x, y, u, v), \quad (5)$$

where $M(x, y, u, v) = \max\{D(Sx, Sy), D(Sx, Su), D(Sy, Sv)\}$.

We first give a useful lemma.

Lemma 11. *Let $\alpha : X \times X \to [0,\infty)$ be a function. Let $T : A \to B$ and $S : A \to A$ be two mappings such that (S, T) is an (α, D) -proximal generalized Geraghty mapping, and let (A, B) have the weak P -property. If $\alpha(Sx, Sy) \geq 1$ for all $x, y \in BC(S, T)$, then $Sx = Sy$.*

Proof. Let $x, y \in BC(S, T)$, we have that

$$D(Sx, Tx) = D(Sy, Ty) = D(A, B). \quad (6)$$

From the assumption, $\alpha(Sx, Sx) \geq 1$, $\alpha(Sy, Sy) \geq 1$, and $\alpha(Sx, Sy) \geq 1$. Since $\alpha(Sx, Sy) \geq 1$, (S, T) is triangular-(α, D)-proximal admissible and (6), we have that $D(Sx, Sy) < \infty$. Also, since $D(Sx, Tx) = D(Sx, Tx) = D(A, B)$, $\alpha(Sx, Sx) \geq 1$ and (S, T) is triangular-(α, D)-proximal admissible, then $D(Sx, Sx) < \infty$. Similarly, we can show that $D(Sy, Sy) < \infty$.

Note that

$$\begin{aligned} M(x, x, x, x) &= \max\{D(Sx, Sx), D(Sx, Sx), D(Sx, Sx)\} \\ &= D(Sx, Sx) < \infty. \end{aligned} \quad (7)$$

Since (S, T) is an (α, D)-proximal generalized Geraghty mapping, and (A, B) has the weak P-property,

$$\begin{aligned} D(Sx, Sx) &\leq \alpha(Sx, Sx)D(Sx, Sx) \leq \alpha(Sx, Sx)D(Tx, Tx) \\ &\leq \theta(D(Sx, Sx))D(Sx, Sx), \end{aligned} \quad (8)$$

for some $\theta \in \Theta$. From the property of θ, we can conclude that $D(Sx, Sx) = 0$. Similarly, we also have that $D(Sy, Sy) = 0$. Then,

$$\begin{aligned} M(x, y, x, y) &:= \max\{D(Sx, Sy), D(Sx, Sx), D(Sy, Sy)\} \\ &= D(Sx, Sy) < \infty. \end{aligned} \quad (9)$$

Since $\alpha(Sx, Sy) \geq 1$, we have that

$$\begin{aligned} D(Sx, Sy) &\leq \alpha(Sx, Sy)D(Sx, Sy) \leq \alpha(Sx, Sy)D(Tx, Ty) \\ &\leq \theta(M(x, y, x, y))M(x, y, x, y) \\ &= \theta(D(Sx, Sy))D(Sx, Sy), \end{aligned} \quad (10)$$

for some $\theta \in \Theta$. Thus, $D(Sx, Sy) = 0$ which implies that $Sx = Sy$.

Theorem 12. *Let $\emptyset \neq A_0 \subseteq S(A_0)$, and let $(S(A_0), D)$ be D -complete. Given that $\alpha : X \times X \to [0,\infty)$ is a function, and let $T : A \to B$ and $S : A \to A$ be mappings such that (S, T) is an (α, D) -proximal generalized Geraghty mapping. Suppose that the following conditions hold.*

(i) $T(A_0) \subseteq B_0$ and the pair (A, B) has the weak P -property.

(ii) There exist $x, y \in A_0$ such that $D(Sx, Ty) = D(A, B)$, $\alpha(Sy, Sx) \geq 1$ and $D(Sy, Sx) < \infty$.

(iii) For $\{Sx_n\} \in C(D, S(A_0), Sx^*)$ such that $\alpha(Sx_n, Sx_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, there is a subsequence $\{Sx_{n_k}\}$ with $\alpha(Sx_{n_k}, Sx^*) \geq 1$ for all $k \in \mathbb{N}$.

Then, there exists $x^* \in A_0$ such that $D(Sx^*, Tx^*) = D(A, B)$. Moreover, if $\alpha(Sx^*, Sy^*) \geq 1$ for all $x^*, y^* \in BC(S, T)$ and S is injective, then (S, T) has a unique best proximity coincidence point.

Proof. From (ii), there exist $x_0, x_1 \in A_0$ such that

$$D(Sx_1, Tx_0) = D(A, B), \alpha(Sx_0, Sx_1) \geq 1, D(Sx_0, Sx_1) < \infty. \quad (11)$$

Since $T(A_0) \subseteq B_0$, $A_0 \subseteq S(A_0)$, and (S, T) is triangular-(α, D)-proximal admissible, there exists $x_2 \in A_0$ such that

$$D(Sx_2, Tx_1) = D(A, B), \alpha(Sx_1, Sx_2) \geq 1, D(Sx_1, Sx_2) < \infty. \quad (12)$$

Continuing in this way, we obtain a sequence $\{Sx_n\} \subseteq S(A_0)$ such that for all $n \in \mathbb{N}$,

$$\begin{aligned} D(Sx_n, Tx_{n-1}) = D(A, B) = D(Sx_{n+1}, Tx_n), \alpha(Sx_{n-1}, Sx_n) \\ \geq 1, D(Sx_{n-1}, Sx_n) < \infty. \end{aligned} \quad (13)$$

Using the weak P-property to (13), for n and $n + 1$, we have that

$$D(Sx_n, Sx_{n+1}) \leq D(Tx_{n-1}, Tx_n) \quad \text{for all } n \in \mathbb{N}. \tag{14}$$

If there exists $n_0 \in \mathbb{N} \cup \{0\}$ such that $Sx_{n_0} = Sx_{n_0+1}$, then from (13),

$$D\left(Sx_{n_0+1}, Tx_{n_0}\right) = D\left(Sx_{n_0}, Tx_{n_0}\right) = D(A, B). \tag{15}$$

Now suppose that $Sx_n \neq Sx_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. By the definition of D, $D(Sx_n, Sx_{n+1}) \neq 0$. We will first show that $\lim_{n\to\infty} D(Sx_{n-1}, Sx_n) = 0$. Let $n \in \mathbb{N}$. Since (S, T) is an (α, D)-proximal generalized Geraghty mapping together with (13) and (14), we obtain that

$$\begin{aligned} D(Sx_n, Sx_{n+1}) &\leq D(Tx_{n-1}, Tx_n) \\ &\leq \alpha(Sx_{n-1}, Sx_n) D(Tx_{n-1}, Tx_n) \\ &\leq \theta(M(x_{n-1}, x_n, x_n, x_{n+1})) M(x_{n-1}, x_n, x_n, x_{n+1}) \\ &\leq M(x_{n-1}, x_n, x_n, x_{n+1}), \end{aligned} \tag{16}$$

where

$$M(x_{n-1}, x_n, x_n, x_{n+1}) = \max \{D(Sx_{n-1}, Sx_n), D(Sx_n, Sx_{n+1})\}. \tag{17}$$

If $M(x_{n-1}, x_n, x_n, x_{n+1}) = D(Sx_n, Sx_{n+1})$, then by (16),

$$\begin{aligned} D(Sx_n, Sx_{n+1}) &\leq \theta(D(Sx_n, Sx_{n+1})) D(Sx_n, Sx_{n+1}) \\ &\leq D(Sx_n, Sx_{n+1}). \end{aligned} \tag{18}$$

Since $D(Sx_n, Sx_{n+1}) > 0$ for all $n \geq 0$,

$$1 \leq \theta(D(Sx_n, Sx_{n+1})) \leq 1, \tag{19}$$

and thus,

$$\lim_{n\to\infty} \theta(D(Sx_n, Sx_{n+1})) = 1. \tag{20}$$

By the definition of θ, $\lim_{n\to\infty} D(Sx_n, Sx_{n+1}) = 0$.

If $M(x_{n-1}, x_n, x_n, x_{n+1}) = D(Sx_{n-1}, Sx_n)$, we again have that

$$D(Sx_n, Sx_{n+1}) \leq \theta(D(Sx_{n-1}, Sx_n)) D(Sx_{n-1}, Sx_n) \leq D(Sx_{n-1}, Sx_n). \tag{21}$$

Since n is arbitrary, $\{D(Sx_n, Sx_{n+1})\}$ is nonnegative and nonincreasing. Therefore, $\{D(Sx_n, Sx_{n+1})\}$ converges to $s \geq 0$. Suppose on the contrary that $s > 0$. From (21),

$$\frac{D(Sx_n, Sx_{n+1})}{D(Sx_{n-1}, Sx_n)} \leq \theta(D(Sx_{n-1}, Sx_n)) \leq 1. \tag{22}$$

It follows that $\lim_{n\to\infty} \theta(D(Sx_{n-1}, Sx_n)) = 1$. Since $\theta \in \Theta$, we have that $\lim_{n\to\infty} D(Sx_{n-1}, Sx_n) = 0$ which is a contradiction. Thus, s must be 0 and that

$$\lim_{n\to\infty} D(Sx_{n-1}, Sx_n) = 0. \tag{23}$$

Next, we shall show that $\{Sx_n\}$ is a D-Cauchy sequence. Suppose that this is not the case. Then, there exists $\varepsilon > 0$ such that for any $k \in \mathbb{N}$, there are subsequences $\{Sx_{n_k}\}$ and $\{Sx_{m_k}\}$ of $\{Sx_n\}$ satisfying $D(Sx_{n_k}, Sx_{m_k}) \geq \varepsilon$ for $m_k \geq n_k \geq k$.

Since (S, T) is triangular-(α, D)-proximal admissible, it is easy to see that

$$\alpha(Sx_n, Sx_m) \geq 1 \text{ and } D(Sx_n, Sx_m) < \infty \text{ when } m \geq n \text{ for all } m, n \in \mathbb{N}. \tag{24}$$

It follows from (13) and (24) that for any $k \in \mathbb{N}$,

$$\begin{aligned} \alpha\left(Sx_{n_k-1}, Sx_{m_k-1}\right) \geq 1 \text{ and } & D\left(Sx_{n_k}, Tx_{n_k-1}\right) \\ &= D(A, B) = D\left(Sx_{m_k}, Tx_{m_k-1}\right). \end{aligned} \tag{25}$$

Since (S, T) is an (α, D)-proximal generalized Geraghty mapping and (A, B) has the weak P-property, we obtain that

$$\begin{aligned} D(Sx_{n_k}, Sx_{m_k}) &\leq D(Tx_{n_k-1}, Tx_{m_k-1}) \\ &\leq \alpha\left(Sx_{n_k-1}, Sx_{m_k-1}\right) D\left(Tx_{n_k-1}, Tx_{m_k-1}\right) \\ &\leq \theta\left(M\left(x_{n_k-1}, x_{m_k-1}, x_{n_k}, x_{m_k}\right)\right) M\left(x_{n_k-1}, x_{m_k-1}, x_{n_k}, x_{m_k}\right), \end{aligned} \tag{26}$$

where

$$\begin{aligned} & M\left(x_{n_k-1}, x_{m_k-1}, x_{n_k}, x_{m_k}\right) \\ &= \max \left\{D\left(Sx_{n_k-1}, Sx_{m_k-1}\right), D\left(Sx_{n_k-1}, Sx_{n_k}\right), D\left(Sx_{m_k-1}, Sx_{m_k}\right)\right\}. \end{aligned} \tag{27}$$

If $M(x_{n_k-1}, x_{m_k-1}, x_{n_k}, x_{m_k})$ is either $D(Sx_{n_k-1}, Sx_{n_k})$ or $D(Sx_{m_k-1}, Sx_{m_k})$, then, by (23), $\lim_{k\to\infty} D(Sx_{n_k}, Sx_{m_k}) = 0$. This contradicts the assumption that $\{Sx_n\}$ is not D-Cauchy. Thus, $M(x_{n_k-1}, x_{m_k-1}, x_{n_k}, x_{m_k}) = D(Sx_{n_k-1}, Sx_{m_k-1})$.

As a consequence,

$$D\left(Sx_{n_k}, Sx_{m_k}\right) \leq \theta\left(D\left(Sx_{n_k-1}, Sx_{m_k-1}\right)\right) D\left(Sx_{n_k-1}, Sx_{m_k-1}\right). \tag{28}$$

By repeating the same steps, it follows that

$$D\left(Sx_{n_k-i}, Sx_{m_k-i}\right) \leq \theta\left(D\left(Sx_{n_k-i-1}, Sx_{m_k-i-1}\right)\right) D\left(Sx_{n_k-i-1}, Sx_{m_k-i-1}\right), \tag{29}$$

where $i = 0, 1, 2, \cdots, n_k - 1$. Therefore,

$$D\left(Sx_{n_k}, Sx_{m_k}\right) \leq \prod_{i=1}^{n_k} \theta\left(D\left(Sx_{n_k-i}, Sx_{m_k-i}\right)\right) D\left(Sx_0, Sx_{m_k-n_k}\right). \tag{30}$$

Let $i_k \in \{1, 2, \cdots, n_k\}$ such that

$$\theta\left(D\left(Sx_{n_k-i_k}, Sx_{m_k-i_k}\right)\right) = \max\left\{\theta\left(D\left(Sx_{n_k-i}, Sx_{m_k-i}\right)\right): 1 \le i \le n_k\right\}. \tag{31}$$

Define

$$\eta = \limsup_{k\to\infty}\left\{\theta\left(D\left(Sx_{n_k-i_k}, Sx_{m_k-i_k}\right)\right)\right\}. \tag{32}$$

If $\eta < 1$, $\lim_{k\to\infty} D(Sx_{n_k}, Sx_{m_k}) = 0$ which is impossible. Thus, $\eta = 1$. Without loss of generality, we may assume that $\lim_{k\to\infty}\theta(D(Sx_{n_k-i_k}, Sx_{n_k+m_k-i_k})) = 1$. By the definition of θ, $\lim_{k\to\infty} D(Sx_{n_k-i_k}, Sx_{n_k+m_k-i_k}) = 0$. Then, there exists $k_0 \in \mathbb{N}$ such that

$$D\left(Sx_{n_{k_0}-i_{k_0}}, Sx_{n_{k_0}+m_{k_0}-i_{k_0}}\right) < \frac{\varepsilon}{2}. \tag{33}$$

Now,

$$\varepsilon \le D\left(Sx_{n_{k_0}}, Sx_{m_{k_0}}\right) \le \prod_{j=1}^{i_{k_0}}\theta\left(D\left(Sx_{n_{k_0}-j}, Sx_{m_{k_0}-j}\right)\right)D\left(Sx_{n_{k_0}-i_{k_0}}, Sx_{m_{k_0}-i_{k_0}}\right) < \frac{\varepsilon}{2}, \tag{34}$$

which is a contradiction. Therefore, $\{Sx_n\}$ is a D-Cauchy sequence.

Since $(S(A_0), D)$ is D-complete, there exists $x^* \in A_0$ such that

$$\lim_{n\to\infty} D(Sx_n, Sx^*) = 0. \tag{35}$$

Equivalently,

$$\{Sx_n\} \in C(D, S(A_0), Sx^*). \tag{36}$$

Since $A_0 \subseteq S(A_0)$ and $T(A_0) \subseteq B_0$, it follows that there exists $a \in A_0$ such that

$$D(Sa, Tx^*) = D(A, B). \tag{37}$$

By (13) and (iii), there is a subsequence $\{Sx_{n_k}\}$ of $\{Sx_n\}$ such that $\alpha(Sx_{n_k}, x^*) \ge 1$ for all $k \in \mathbb{N}$. From (13), we have that

$$D\left(Sx_{n_k+1}, Tx_{n_k}\right) = D(A, B) \quad \text{for all } k \in \mathbb{N}. \tag{38}$$

By the weak P-property, (37) and (38), we obtain that $D(Sx_{n_k+1}, Sa) \le D(Tx_{n_k}, Tx^*)$.

Since $\alpha(Sx_{n_k}, x^*) \ge 1$ and (S, T) is an (α, D)-proximal generalized Geraghty mapping,

$$\begin{aligned} D\left(Sx_{n_k+1}, Sa\right) \le D\left(Tx_{n_k}, Tx^*\right) &\le \alpha\left(Sx_{n_k}, Sx^*\right)D\left(Tx_{n_k}, Tx^*\right) \\ &\le \theta\left(M\left(x_{n_k}, x^*, x_{n_k+1}, a\right)\right)M\left(x_{n_k}, x^*, x_{n_k+1}, a\right) \\ &\le M\left(x_{n_k}, x^*, x_{n_k+1}, a\right), \quad \text{for all } k \ge 1, \end{aligned} \tag{39}$$

where

$$M\left(x_{n_k}, x^*, x_{n_k+1}, a\right) = \max\left\{D\left(Sx_{n_k}, Sx^*\right), D\left(Sx_{n_k}, Sx_{n_k+1}\right), D(Sx^*, Sa)\right\}. \tag{40}$$

By (23) and (35), we immediately have that

$$\lim_{k\to\infty} M\left(x_{n_k}, x^*, x_{n_k+1}, a\right) = D(Sx^*, Sa) \ge 0. \tag{41}$$

If $D(Sx^*, Sa) > 0$, by letting $k \to \infty$ in (39),

$$1 \le \lim_{k\to\infty}\theta\left(M\left(x_{n_k}, x^*, x_{n_k+1}, a\right)\right) \le 1. \tag{42}$$

We subsequently have that

$$\lim_{k\to\infty}\theta\left(M\left(x_{n_k}, x^*, x_{n_k+1}, a\right)\right) = 1. \tag{43}$$

By the property of θ,

$$\lim_{k\to\infty} M\left(x_{n_k}, x^*, x_{n_k+1}, a\right) = D(Sx^*, Sa) = 0, \tag{44}$$

which is a contradiction. It follows that $D(Sx^*, Sa)$ must be equal to 0, and thus $Sx^* = Sa$. Therefore, from (37), there exists $x^* \in A$ such that

$$D(Sx^*, Tx^*) = D(A, B). \tag{45}$$

Suppose further that $x^*, y^* \in BC(S, T)$ and $\alpha(x^*, y^*) \ge 1$. By Lemma 11, $Sx^* = Sy^*$. Since S is injective, $x^* = y^*$. The proof is now completed.

Example 13. Let $X = [-3, 3]$ be equipped with the JS-metric D given by

$$D(x, y) = \begin{cases} |x| + |y|, & x \ne 0 \text{ and } y \ne 0, \\ \left|\frac{x}{2}\right|, & y = 0, \\ \left|\frac{y}{2}\right|, & x = 0. \end{cases} \tag{46}$$

Choose $A = [-2, 0]$ and $B = [0, 1]$. Let $T : A \to B$ be a mapping defined by

$$T(x) = -\frac{x}{3}, \quad \text{for all } x \in A, \tag{47}$$

and let a mapping $S : A \to A$ be defined by

$$S(x) = \frac{x}{2}, \quad \text{for all } x \in A. \tag{48}$$

It is not difficult to see that $D(A, B) = 0$ and (A, B) has the weak P-property. Next, define the map $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 1, & \text{if } x \ne 0 \text{ or } y = 0, \\ 0, & \text{otherwise,} \end{cases} \tag{49}$$

for all $x, y \in X$. Since $A_0 = \{0\} = B_0$, then $T(A_0) = \{0\} \subseteq B_0 = \{0\}$ and $A_0 = \{0\} \subseteq S(A_0) = \{0\}$. Also, there is $0 \in A_0$ satisfying

$$D(S(0), T(0)) = D(0,0) = 0 = D(A,B), \alpha(0,0) \geq 1. \quad (50)$$

We will first show that (S, T) is triangular-(α, D)-proximal admissible.

Let $x_1, x_2, u_1, u_2 \in A$ such that $\alpha(Sx_1, Sx_2) \geq 1$ and

$$D(Su_1, Tx_1) = D(Su_2, Tx_2) = D(A,B). \quad (51)$$

Then, $Sx_1 \neq 0$ or $Sx_2 = 0$ and

$$D\left(\frac{u_1}{2}, -\frac{x_1}{3}\right) = D\left(\frac{u_2}{2}, -\frac{x_2}{3}\right) = 0. \quad (52)$$

Assume that $\alpha(Su_1, Su_2) \neq 1$, then $u_1/2 = Su_1 = 0$ and $u_2/2 = Su_2 \neq 0$.

Since $\alpha(Sx_1, Sx_2) \geq 1$, we consider the following two cases.

Case 1. If $Sx_2 \neq 0$, then $Sx_1 \neq 0$, and thus,

$$\left|-\frac{x_1}{6}\right| = D\left(0, -\frac{x_1}{3}\right) = 0. \quad (53)$$

Then $x_1 = 0$. This implies that $Sx_1 = 0$ which is impossible.

Case 2. If $Sx_2 = 0 = x_2/2$, then

$$D\left(Su_2, -\frac{x_2}{3}\right) = D\left(\frac{u_2}{2}, 0\right) = \left|\frac{u_2}{4}\right| = 0. \quad (54)$$

This implies that $u_2 = 0$ and $Su_2 = 0$ which is impossible. Thus, $\alpha(Su_1, Su_2) \geq 1$.

Next, assume that $\alpha(x, z) \geq 1$ and $\alpha(z, y) \geq 1$. Then, we can see that $y = 0$ if $z = 0$, and $x \neq 0$ if $z \neq 0$. Hence, $x \neq 0$ or $y = 0$, and thus, $\alpha(x, y) \geq 1$. This means that (S, T) is triangular-(α, D)-proximal admissible.

We note that there is a map $\theta \in \Theta$ defined by $\theta(t) = 2/3$.

Now, for x, y satisfying $\alpha(Sx, Sy) \geq 1$, we have that $Sx \neq 0$ or $Sy = 0$. We consider the following two cases.

Case 1. If $Sy = 0$, then $y = 0$ and

$$\begin{aligned}\alpha(Sx, Sy)D(Tx, Ty) &= \alpha(Sx, 0)D(Tx, T(0)) = D\left(-\frac{x}{3}, 0\right) \\ &= \left|-\frac{x}{6}\right| = \frac{2}{3}\left|\frac{x}{4}\right| = \frac{2}{3}D(Sx, Sy) \\ &\leq \theta(M(x, y, u, v))M(x, y, u, v).\end{aligned} \quad (55)$$

Case 2. If $Sy \neq 0$, then $Sx \neq 0$, and thus, $x \neq 0$ and $y \neq 0$. We obtain that

$$\begin{aligned}\alpha(x, y)D(Tx, Ty) &= D(Tx, Ty) = D\left(-\frac{x}{3}, -\frac{y}{3}\right) = \left|-\frac{x}{3} - \frac{y}{3}\right| \\ &\leq \frac{2}{3}|x| + |y| = \frac{2}{3}D(Sx, Sy) \\ &\leq \theta(M(x, y, u, v))M(x, y, u, v).\end{aligned} \quad (56)$$

Therefore, (S, T) is an (α, D)-proximal generalized Geraghty mapping.

Finally, we will show that assumption (iii) in Theorem 12 holds. Let $a \in A_0$ and $\{Sx_n\} \in C(D, S(A_0), Sa)$ such that $\alpha(Sx_n, Sx_{n+1}) \geq 1$ for all $n \in \mathbb{N}$. Then,

$$Sx_n \neq 0 \text{ or } Sx_{n+1} = 0 \text{ for each } n \in \mathbb{N}. \quad (57)$$

If $Sx_n \neq 0$ for all $n \in \mathbb{N}$, then $\alpha(Sx_n, Sa) \geq 1$ for all $n \in \mathbb{N}$. Assume that there exists $n_0 \in \mathbb{N}$ such that $Sx_{n_0} = 0$. By (57), $Sx_k = 0$ for all $k \geq n_0$. Suppose that $Sa \neq 0$. Then,

$$D(Sx_k, Sa) = D(0, a) = \left|\frac{a}{2}\right| \neq 0 \quad \text{for all } k \geq n_0. \quad (58)$$

This contradicts with the fact that $\{Sx_n\} \in C(D, S(A_0), Sa)$. Thus, $Sa = 0$ and so $\alpha(Sx_n, Sa) \geq 1$. We also have that $(S(A_0), D)$ is D-complete. Therefore, by Theorem 12, (S, T) has a best proximity coincidence point, which is 0.

Example 14. Let $X = \mathbb{R}^2$ be equipped with the JS -metric D given by

$$D((x_1, y_1), (x_2, y_2)) = \begin{cases} |x_1 - x_2| + |y_1 - y_2|, & (x_1, x_2) \neq (0,0), (y_1, y_2) \neq (0,0), \\ |x_1 - x_2|, & (y_1, y_2) = (0,0), \\ \dfrac{|y_1 - y_2|}{2}, & (x_1, x_2) = (0,0). \end{cases} \quad (59)$$

We consider the disjoint subsets A and B of X given by $A = \{(-1, y) ; 0 \leq y \leq 1\}$ and $B = \{(1, y) ; 0 \leq y \leq 1\}$. We can check that $D(A, B) = 2$ and the pair (A, B) has the weak P-property.

Let $T : A \to B$ be a map defined by

$$T(-1, y) = (1, \ln(1 + y)), \quad \text{for all } (-1, y) \in A, \quad (60)$$

and let $S : A \to A$ be a map defined by

$$S(-1, y) = (-1, y), \quad \text{for all } (-1, y) \in A. \quad (61)$$

Then, we consider a map $\alpha : X \times X \to [0, \infty)$ given by

$$\alpha((x_1, y_1), (x_2, y_2)) = \begin{cases} 1, & \text{if } x_1 \leq x_2, y_1 \geq y_2, \\ 0, & \text{otherwise}, \end{cases} \quad (62)$$

for all $x = (x_1, y_1), y = (x_2, y_2) \in X$.

Next, we will show that (S, T) is triangular-(α, D)-proximal admissible. Let $x, y, u, v \in A$ such that $x = (-1, \hat{x})$, $y = (-1, \hat{y})$, $u = (-1, \hat{u})$, and $v = (-1, \hat{v})$ satisfying $\alpha(Sx, Sy) \geq 1$ and

$$D(Su, Tx) = D(Sv, Ty) = D(A, B). \tag{63}$$

Consequently, $\hat{x} \geq \hat{y}$ and

$$D((-1, \hat{u}), (-1, \ln(1+\hat{x}))) = D((-1, \hat{v}), (-1, \ln(1+\hat{y}))) = 2. \tag{64}$$

It follows that $\hat{u} = \ln(1+\hat{x}))$ and $\hat{v} = \ln(1+\hat{y})$. Since $\hat{x} \geq \hat{y}$, then $\hat{u} \geq \hat{v}$, and thus, $\alpha(Su, Sv) \geq 1$.

Assume that $\alpha(x, y) \geq 1$ and $\alpha(y, u) \geq 1$. Then, we can see that $\hat{x} \geq \hat{y}$ and $\hat{y} \geq \hat{u}$. Therefore, $\hat{x} \geq \hat{u}$, and thus, $\alpha(x, u) \geq 1$. This means that (S, T) is triangular-(α, D)-proximal admissible.

We choose a map $\theta \in \Theta$ which is defined by

$$\theta(t) = \begin{cases} 1, & t = 0, \\ \dfrac{\ln(1+t)}{t}, & t > 0. \end{cases} \tag{65}$$

Let $x, y, u, v \in A$ such that $x = (-1, \hat{x}), y = (-1, \hat{y}), u = (-1, \hat{u})$, and $v = (-1, \hat{v})$ satisfying $\alpha(Sx, Sy) \geq 1$. If $x = y$, then we are done. Suppose that $x \neq y$. It follows that $D(x, y) > 0$, and so, $M(x, y, u, v) > 0$. Thus,

$$\begin{aligned} \alpha(Sx, Sy)D(Tx, Ty) &= D(Tx, Ty) \\ &= D((1, \ln(1+\hat{x})), (1, \ln(1+\hat{y}))) \\ &= |\ln(1+\hat{x}) - \ln(1+\hat{y})| \\ &= \left| \ln\left(\frac{1+\hat{y}+\hat{x}-\hat{y}}{1+\hat{y}}\right) \right| \leq \ln(1+|\hat{x}-\hat{y}|) \\ &= \ln(1+D(x, y)) \leq \ln(1+M(x, y, u, v)) \\ &= \left[\frac{\ln(1+M(x, y, u, v))}{M(x, y, u, v)}\right] M(x, y, u, v) \\ &= \theta(M(x, y, u, v))M(x, y, u, v). \end{aligned} \tag{66}$$

Therefore, (S, T) is an (α, D)-proximal generalized Geraghty mapping.

Since $A_0 = A = \{(-1, y); 0 \leq y \leq 1\}$ and $B_0 = B = \{(1, y); 0 \leq y \leq 1\}$,

$$\begin{aligned} T(A_0) &= \{(1, y); 0 \leq y \leq \ln 2\} \subseteq B_0, \\ A_0 &= \{(-1, 0)\} \subseteq S(A_0) = A_0. \end{aligned} \tag{67}$$

Also, $(S(A_0), D)$ is D-complete, and there is $(-1, 0) \in A_0$ satisfying

$$\begin{aligned} D(S(-1, 0), T(-1, 0)) &= D((-1, 0), (1, 0)) = 2 = D(A, B), \\ \alpha(S(-1, 0), T(-1, 0)) &= \alpha((-1, 0), (1, 0)) \geq 1. \end{aligned} \tag{68}$$

We have left to that show assumption (iii) in Theorem 12 holds. Let $a = (-1, \hat{a}) \in A_0$ and $\{Sx_n\} \in C(D, S(A_0), Sa)$ such that $\alpha(Sx_n, Sx_{n+1}) = \alpha((-1, y_n), (-1, y_{n+1}) \geq 1$ for all $n \in \mathbb{N}$. Then, $y_n \geq y_{n+1}$ for all n. Since $S(A_0) = A_0 = \{(-1, y); 0 \leq y \leq 1\}$ and $\{y_n\}$ is nonincreasing which $\{Sx_n\} \in C(D, S(A_0), Sa) = C(D, A_0, a)$. It follows that $y_n \geq \hat{a}$ for all $n \in \mathbb{N}$. Then, $\alpha(Sx_n, Sa) \geq 1$ for all $n \in \mathbb{N}$. Therefore, by Theorem 12, (S, T) has a best proximity coincidence point.

Next, we present a corollary of our result. The following definition is required.

Definition 15. Let $T : A \to B$ and $S : A \to A$ be mappings. Let $\alpha : X \times X \to [0,\infty)$ be a function. Then, the pair (S, T) is said to be an (α, D) -proximal mapping if the following conditions hold.

(i) The pair (S, T) is triangular-(α, D)-proximal admissible.

(ii) There exists $k \in [0, 1)$ such that for all $x, y, u, v \in X$, if $D(Su, Tx) = D(Sv, Ty) = D(A, B)$ and $\alpha(Sx, Sy) \geq 1$, then

$$D(Tx, Ty) \leq kD(x, y). \tag{69}$$

By putting $\theta(t) = k$, where $k \in [0, 1)$ in Theorem 12, we have the following result.

Corollary 16. *Let $A_0 \subseteq S(A_0)$ and $(S(A_0), D)$ be D -complete. Given that $\alpha : X \times X \to [0,\infty)$ is a function, and let $T : A \to B$ and $S : A \to A$ be mappings such that (S, T) is an (α, D) -proximal mapping. Suppose that the following conditions hold.*

(i) $T(A_0) \subseteq B_0$ and the pair (A, B) has the weak P -property.

(ii) There exist $x, y \in A_0$ such that $D(Sx, Ty) = D(A, B)$ and $\alpha(Sy, Sx) \geq 1$ and $D(Sy, Sx) < \infty$.

(iii) For $\{Sx_n\} \in C(D, S(A_0), Sx^*)$, if $\alpha(Sx_n, Sx_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, then there is a subsequence $\{Sx_{n_k}\}$ with $\alpha(Sx_{n_k}, Sx^*) \geq 1$ for all $k \in \mathbb{N}$.

Then, there exists $x^* \in A$ such that $D(Sx^*, Tx^*) = D(A, B)$. Moreover, if $\alpha(Sx^*, Sy^*) \geq 1$ for all $x^*, y^* \in BC(S, T)$, then (S, T) has a unique best proximity coincidence point.

3. Consequence

We will apply our result on the best proximity coincidence point on a JS-metric space endowed with a binary relation R.

Let $T : A \to B$ and $S : A \to A$ be mappings. The pair (S, T) is said to be (R, D)-proximally comparative if $SxRSy$ and $D(Su_1, Tx) = D(Su_2, Ty) = D(A, B) \Rightarrow Su_1RSu_2$ and $D(Su_1, Su_2) < \infty$ for all $x, y, u_1, u_2 \in A$.

Definition 17. Let $T : A \to B$ and $S : A \to A$ be mappings. The pair (S, T) is said to be an (R, D) -proximally comparative generalized Geraghty mapping if the following hold.

(1) The pair (S, T) is (R, D)-proximally comparative.

(2) There exists $\theta \in \Theta$ such that for all $x, y, u, v \in A$, if $D(Su, Tx) = D(Sv, Ty) = D(A, B)$ and $SxRSy$, then

$$D(Tx, Ty) \leq \theta(M(x, y, u, v))M(x, y, u, v), \quad (70)$$

where $M(x, y, u, v) = \max \{D(Sx, Sy), D(Sx, Su), D(Sy, Sv)\}$.

Corollary 18. *Let X be endowed with a symmetric, transitive binary relation R. Let $T : A \to B$ and $S : A \to A$ be mappings such that $\emptyset \neq A_0 \subseteq S(A_0)$ and $(S(A_0), D)$ be D -complete. If (S, T) is an (R, D) -proximally comparative generalized Geraghty mapping and the following conditions hold:*

(i) $T(A_0) \subseteq B_0$ and the pair (A, B) has the weak P -property;

(ii) there exist $x, y \in A_0$ such that $D(Sx, Ty) = D(A, B)$ and $SyRSx$ and $D(Sy, Sx) < \infty$;

(iii) for $\{Sx_n\} \in C(D, S(A_0), Sx^*)$, if $Sx_n RSx_{n+1}$ for all $n \in \mathbb{N}$, then there is a subsequence $\{Sx_{n_k}\}$ with $Sx_{n_k} R Sx^*$ for all $k \in \mathbb{N}$,

then there exists $x^* \in A_0$ such that $D(Sx^*, Tx^*) = D(A, B)$. Moreover, if $Sx^* RSy^*$ for all $x^*, y^* \in BC(S, T)$ and S is injective, then (S, T) has a unique best proximity coincidence point.

Proof. Define

$$\alpha(x, y) = \begin{cases} 1, & \text{if } xRy, \\ 0, & \text{otherwise}, \end{cases} \quad (71)$$

for all $x, y \in X$. We can see that the hypotheses of Theorem 12 hold which imply that there is $x^* \in A$ such that $D(Sx^*, Tx^*) = D(A, B)$. Let $x^*, y^* \in BC(S, T)$. Then, $Sx^* RS y^*$ which implies that $\alpha(Sx^*, Sy^*) \geq 1$. Again, by Theorem 12, $x^* = y^*$.

4. Conclusion and Open Questions

We have introduced new classes of Geraghty's type mappings called (α, D)-proximal generalized Geraghty mappings. Then, we investigated some conditions for this type of mappings to have a best proximity coincidence point in JS-metric spaces using the weak P-property. The question is whether one can extend Theorem 12 to the framework of common best proximity point in a JS-metric space X. Can we also extend the result when X is other generalized metric spaces?

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors have no conflict of interests regarding the publication of this paper.

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