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Research Article

On Fuzzy Fixed Points and an Application to Ordinary Fuzzy Differential Equations

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The aim of this paper is to obtain the common fuzz fixed points of α-fuzzy mappings satisfying generalized almost (Y, Λ) -contraction in complete metric spaces. Our results are extensions and improvements of the several well-known recent and classical results in literature. We give an example for supporting these results. As an application, we apply our obtained results to study the existence of a solution for a second order nonlinear boundary value problem.

1. Introduction and Preliminaries

The fixed point result of Banach [1] is an interesting tool that ensures the existence and uniqueness of a fixed point of self-mappings defined on metric spaces. Later, various extensions and generalizations of Banach's theorem appeared by defining a variety of contractive type conditions for self and non-self-mappings on different spaces (see [2–27]).

In [17, 18], Berinde discussed many contraction type mappings. He initiated the notion of almost contractions.

Definition 1. A self-mapping $\widehat{S}$ on a metric space (in short MS) (ω, ∂) is named as an almost contraction, if there are $\lambda \in [0, 1)$ and $L \geq 0$ so that for all $r, \widehat{J} \in \omega$,

$$\partial\left(\widehat{S}(r), \widehat{S}(\widehat{J})\right) < \lambda\partial\left(r, \widehat{j}\right) + L\partial\left(\widehat{j}, \widehat{S}\left(\widehat{j}\right)\right). \tag{1}$$

This almost contraction has been generalized as follows.

Definition 2. A self-mapping $\widehat{S}$ on a MS (ω, ∂) is called a generalized almost contraction, if there are $\lambda \in [0, 1)$and $L \geq 0$ so that for all $r, \widehat{J} \in \omega$,

$$\partial\left(\widehat{S}(r), \widehat{S}(\widehat{J})\right) < \lambda\partial\left(r, \widehat{J}\right) + L \min\left\{\partial\left(\upsilon, \widehat{S}\upsilon\right), \partial\left(\iota, \widehat{S}\iota\right), \partial\left(\upsilon, \widehat{S}\iota\right), \partial\left(\iota, \widehat{S}\upsilon\right)\right\}. \tag{2}$$

Wardowski [28] presented the notion of F-contractions and established a related fixed point theorem.

Definition 3 (see [28]). A self-mapping $\widehat{S}$ on a MS (ω, ∂) said to be an F -contraction if there are $F \in F$ and $\tau > 0$ such that

$$\begin{aligned}\forall r, \widehat{J} \in \omega, \partial\left(\widehat{S}(r), \widehat{S}(\widehat{J})\right) > 0 \Rightarrow \tau + F\left(\partial\left(\widehat{S}(r), \widehat{S}(\widehat{J})\right)\right) \\ \leq F\left(\partial\left(r, \widehat{J}\right)\right),\end{aligned} \tag{3}$$

where F is the set of functions $F : (0, \infty) \longrightarrow (-\infty, \infty)$ such that

(F1) For all $0 < r < \widehat{J}$, $F(r) < F(\widehat{J})$,
(F2) For each sequence $\{t_n\}_{n=1}^{\infty}$ of positive numbers,

$$\lim_{n\to\infty} F(t) = -\infty \text{ifandonlyif} \lim_{n\to\infty} t_n = 0 \tag{4}$$

(F3) There is $k \in (0, 1)$ so that $\lim_{t\to 0^+} t^k F(t) = 0$,

Later on, Altun et al. [22] modified Definition 3 by adding the condition (F4):

(F4) $F(\inf(P)) = \inf(F(P))$, for all $P \subset (0,\infty)$ with $\inf(P) > 0$.

Denote by F^* the set of functions $F : (0,\infty) \longrightarrow (-\infty,\infty)$ verifying the conditions $(F1)$-$(F4)$.

Theorem 4 (see [28]). *Let $\widehat{S} : \omega \longrightarrow \omega$ be an F-contraction on a complete MS. Then, T admits a unique fixed point $s^* \in \omega$, and for each $r \in \omega$, the sequence $\{T^j r\}_{j\in\mathbb{N}}$ is convergent to r^*.*

Definition 5 (see [4]). A self-mapping $\widehat{S}$ *on a* MS (ω, ∂) is said to be a θ-contraction, if there are $\ell \in (0,1)$ and $\theta \in \Theta$ so that

$$r, \widehat{J} \in \omega, \partial\left(\widehat{S}(r), \widehat{S}(\widehat{J})\right) \neq 0 \Rightarrow \theta\left(\partial\left(\widehat{S}(r), \widehat{S}(\widehat{J})\right)\right) \leq [\theta(\partial(r, J\wedge))]^{\ell}, \tag{5}$$

where Θ is the set of functions $\theta : (0,\infty) \longrightarrow (1,\infty)$ so that

($\Theta 1$) θ is nondecreasing,

($\Theta 2$) For each sequence $\{t_n\} \subset (0,\infty)$,

$$\lim_{n\to\infty} \theta(t_n) = 1 \text{ if and only if } \lim_{n\to\infty} t_n = 0^+, \tag{6}$$

($\Theta 3$) There are $r \in (0,1)$ and $\hbar \in (0,\infty]$ such that $\lim_{t\to 0^+}(\theta(t) - 1/t^r) = \hbar$,

($\Theta 4$) θ is continuous.

The following result was established by Jleli and Samet [4].

Theorem 6 (see [4]). *Each θ-contraction mapping on a complete MS (ω, ∂) possesses a unique fixed point.*

As in [22], Hancer et al. [12] took the following:

($\Theta 5$) $\theta(\inf Q) = \inf \theta(Q)$, for all $Q \subset (0,\infty)$ with $\inf Q > 0$.

Denote by Θ^* the set of functions $Q : (0,\infty) \longrightarrow (1,\infty)$ so that ($\Theta 1$), ($\Theta 2$), ($\Theta 3$), and ($\Theta 5$) hold.

As in [9], we denote by Ξ the set of functions $\theta : (0,\infty) \longrightarrow (1,\infty)$ so that

($\Theta 1$)$'$ θ is nondecreasing,

($\Theta 2$)$'$ $\inf_{t\in(0,\infty)} \theta(t) = 1$,

($\Theta 3$)$'$ θ is continuous.

Theorem 7 (see [9]). *Let $\widehat{S}$ be a self-mapping on a MS (ω, ∂). We have the equivalence of the following:*

(i) $\widehat{S}$ is a θ-contraction with $\theta \in \Xi$; ,

(ii) $\widehat{S}$ is a F-contraction with $F \in F$.

Very recently, Liu et al. [9] introduced the notion of (Y, Λ)-type contractions and established new fixed point theorems for such kind of mappings in complete MS.

Definition 8. A self-mapping $\widehat{S}$ on a MS (ω, ∂) is said to be a (Y, Λ)-type contraction, if there are a comparison function Y *and* $\phi \in \Phi$ such that for all $r, \widehat{J} \in \omega$,

$$\partial\left(\widehat{S}(r), \widehat{S}(\widehat{J})\right) > 0 \Rightarrow \Lambda\left(\partial\left(\widehat{S}(r), \widehat{S}(\widehat{J})\right)\right) \leq Y\left[\Lambda\left(M\left(r, \widehat{J}\right)\right)\right], \tag{7}$$

where

$$M\left(r, \widehat{J}\right) = \max\left\{\partial\left(r, \widehat{J}\right), \partial\left(r, \widehat{S}(r)\right), \partial\left(\widehat{J}, \widehat{S}(\widehat{J})\right), \frac{\partial\left(r, \widehat{S}(\widehat{J})\right) + \partial\left(\widehat{J}, \widehat{S}(r)\right)}{2}\right\}, \tag{8}$$

and Φ is the set of functions $\Lambda : (0,\infty) \longrightarrow (0,\infty)$ so that

($\Phi 1$) Λ is nondecreasing,

($\Phi 2$) For each sequence $\{t_n\} \subset (0,\infty)$,

$$\lim_{n\to\infty} \Lambda(t_n) = 0 \text{ iff } \lim_{n\to\infty} t_n = 0, \tag{9}$$

($\Phi 3$) Λ is continuous.

As in [2], a function $Y : (0,\infty) \longrightarrow (0,\infty)$ is named as a comparison function if

(1) Y is increasing, that is, $t_1 < t_2 \Rightarrow Y(t_1) \leq Y(t_2)$,

(2) $\lim_{n\to\infty}^{n} Y(t) = 0$ for all $t > 0$, where Y^n stands for the nth iterate of Y.

Denote by Ψ the set of comparison functions.

Lemma 9 (see [9]). *Let $\Lambda : (0,\infty) \longrightarrow (0,\infty)$ be a nondecreasing and continuous function with $\inf_{t\in(0,\infty)} \Lambda(t) = 0$, and $\{t_k\}_k$ be a sequence in $(0,\infty)$. Then,*

$$\lim_{k\to\infty} \Lambda(t_k) = 0 \quad \text{iff} \lim_{k\to\infty} t_k = 0. \tag{10}$$

Example 10 (see [2]). Consider the following comparison functions:

(1) $Y(t) = \lambda t$ with $0 < \lambda < 1$,

$$Y(t) = \begin{cases} \dfrac{t}{2}, & 0 < t < 1, \\ \dfrac{t}{3}, & 1 \leq t, \end{cases} \tag{11}$$

$$Y(t) = \frac{t}{t+1}.$$

Example 11 (see [9]). For all $t \in (0,\infty)$, consider

$$\Lambda_1(\eta) = t,$$
$$\Lambda_2(\eta) = \sqrt{t\sqrt{t}}, \tag{12}$$
$$\Lambda_3(\eta) = te^{t}.$$

Here, $\Lambda_1, \Lambda_2, \Lambda_3 \in \Phi$.

On the other hand, using the notion of a fuzzy set, Heilpern [29] initiated the family of fuzzy mappings, which generalized the concept of set-valued mappings, and presented a fixed point result for fuzzy contraction mappings in the context of metric linear spaces. Mention that the theorem of Heilpern [29] is considered as a fuzzy extension of the BCP. Later on, several researchers worked on fixed point results involving fuzzy mappings, see ([30–46]).

Let $CB(\omega)$ be the set family of bounded and closed subsets of a MS (ω, ∂). For $r \in \omega$ and $A, B \in CB(\omega)$, set

$$\partial(r, B) = \inf_{\widehat{J} \in B} \partial\left(r, \widehat{J}\right). \tag{13}$$

Given $H : CB(\omega) \times CB(\omega) \longrightarrow [0, 1)$ as

$$H(A, B) = \max\left\{\sup_{r\in A}\partial(r, B), \sup_{\widehat{J}\in B}\partial\left(\widehat{J}, A\right)\right\}. \tag{14}$$

H is a metric on $CB(\omega)$. It is named as the Pompeiu-Hausdorff metric induced by ∂. A fuzzy set in ω is a function with domain ω and values in $[0, 1]$. Denote by I^ω the set fuzzy sets in ω. Let A be a fuzzy set and $r \in \omega$, then the function values $A(r)$ is named as the grade of membership of r in A. The α-level set of A is denoted by $[A]_\alpha$ and is defined by

$$[A]_\alpha = \{\omega : \Lambda(r) \geq \alpha\} \text{if} \alpha \in (0, 1] \text{and} [A]_0 = \{\omega : \overline{\Lambda(r) > 0}\}. \tag{15}$$

Here, $\bar{B}$ denotes the closure of the set B. Let $\mathfrak{I}(\omega)$ be the collection of all fuzzy sets in ω. For $A, B \in \mathfrak{I}(\omega)$, $A \subset B$ means $A(r) \leq B(r)$ for each $r \in \omega$. The fuzzy set $\chi_{\{r\}}$ is denoted by $\{r\}$ unless and until it is stated, where $\chi_{\{r\}}$ is the characteristic function of the crisp set A. If there is $\alpha \in [0, 1]$ so that $[A]_\alpha, [B]_\alpha \in CB(\omega)$, then consider

$$P_\alpha(A, B) = \inf_{r\in[A]_\alpha, \widehat{J}\in[B]_\alpha} \partial\left(r, \widehat{J}\right), D_\alpha(A, B) = H\left([A]_\alpha, [B]_\alpha\right). \tag{16}$$

If $[A]_\alpha, [B]_\alpha \in CB(\omega)$ for each $\alpha \in [0, 1]$, then take

$$P(A, B) = \sup_\alpha P_\alpha(A, B), \partial_\infty(A, B) = \sup_\alpha D_\alpha(A, B). \tag{17}$$

We write $p(r, B)$ instead of $p(\{r\}, B)$. A fuzzy set A in a metric linear space V is said to be an approximate quantity if and only if $[A]_\alpha$ is compact and convex in V for each $\alpha \in [0, 1]$ and $\sup_{r\in V} A(r) = 1$. The set of approximate quantities in V is denoted by $W(V)$. Let ω be an arbitrary set and Y be a MS. A mapping $\widehat{S}$ is called fuzzy mapping (in short, FM) if $\widehat{S}$ is a mapping from ω into $\mathfrak{I}(Y)$. A fuzzy mapping $\widehat{S}$ is a fuzzy subset on $\omega \times Y$ with the membership function $\widehat{S}(r)(\widehat{J})$. The function $\widehat{S}(r)(\widehat{J})$ is the grade of membership of $\widehat{J}$ in $\widehat{S}(r)$.

Definition 12. Let $\widehat{S}, \Gamma$ be *FM* from ω into $\mathfrak{I}(Y)$. An element σ in ω is said to be an α-fuzzy fixed point of Γ if there is $\alpha \in [0, 1]$ so that $\sigma \in [\Gamma u]_\alpha$. $\sigma \in \omega$ is named as a common α-fuzzy fixed point of $\widehat{S}$ and Γ if there is $\alpha \in [0, 1]$ so that $\sigma \in [\widehat{S}u]_\alpha \cap [\Gamma\sigma]_\alpha$. $\alpha = 1$ is called a common fixed point of fuzzy mappings.

In this manuscript, we ensure the existence of some common α-fuzzy fixed points for fuzzy mappings for almost (Y, Λ)-contractions in the class of complete MS. Our theorems generalize some known results in literature.

Now, we need the following.

Lemma 13 (see [16]). *Let (ω, ∂) be a MS and $A, B \in CB(\omega)$, then for every $\eta \in A$,*

$$\partial(\eta, B) \leq H(A, B). \tag{18}$$

Lemma 14 (see [30]). *Let V be a metric linear space and $\widehat{S} : V \longrightarrow W(V)$ be a fuzzy mapping and $\rho_0 \in V$. Then, there is $\rho_1 \in V$ so that $\{\rho_1\} \subset \widehat{S}(\rho_0)$.*

Lemma 15 (see [47]). *Let (ω, ∂) be a metric space, $r^* \in \omega$ and $\widehat{S}, \Gamma : \omega \longrightarrow \mathfrak{I}(\omega)$ be FM such that $\widehat{S}(r)$ is a nonempty compact set for each $r \in \omega$. Then, $r^* \in \widehat{S}(r^*)$ if and only if $\widehat{S}(r^*)(r^*) \geq \widehat{S}(r^*)(r)$ for each $r \in \omega$.*

2. Main Results

From now on, (ω, ∂) is assumed to be a complete MS.

Theorem 16. *Let $\widehat{S}, \Gamma : \omega \longrightarrow \mathfrak{I}(\omega)$ be FM and for each $r \in \omega$, and there exist $\alpha_{\widehat{S}}(r), \alpha_\Gamma(r) \in (0, 1]$ such that $[\widehat{S}(r)]_{\alpha_{\widehat{S}}(r)}$ and $[\Gamma(\widehat{J})]_{\alpha_\Gamma(r)}$ are nonempty, closed, and bounded subsets of ω. Assume that there exist $Y \in \Psi$, $\Lambda \in \Phi$, and $L \geq 0$ such that for all $r, \widehat{J} \in \omega$, $H([\widehat{S}(r)]_{\alpha_{\widehat{S}}(r)}, [\Gamma(\widehat{J})]_{\alpha_\Gamma(\widehat{J})}) > 0$ implies*

$$\Lambda\left(H\left(\left[\widehat{S}(r)\right]_{\alpha_{\widehat{S}}(r)}, \left[\Gamma\left(\widehat{J}\right)\right]_{\alpha_\Gamma\left(\widehat{J}\right)}\right)\right) \leq Y\left(\Lambda\left(M\left(r, \widehat{J}\right)\right)\right) + LE\left(r, \widehat{J}\right), \tag{19}$$

where

$$M\left(r, \widehat{J}\right) = \max\left\{\begin{array}{c} \partial\left(r, \widehat{J}\right), \partial\left(r, \left[\widehat{S}(r)\right]_{\alpha_{\widehat{S}}(r)}\right), \\ \partial\left(\widehat{J}, \left[\Gamma\left(\widehat{J}\right)\right]_{\alpha_\Gamma\left(\widehat{J}\right)}\right), \dfrac{\partial\left(r, \left[\Gamma\left(\widehat{J}\right)\right]_{\alpha_\Gamma\left(\widehat{J}\right)}\right) + \partial\left(\widehat{J}, \left[\widehat{S}(r)\right]_{\alpha_{\widehat{S}}(r)}\right)}{2} \end{array}\right\}, \tag{20}$$

$$E\left(r, \widehat{J}\right) = \min\left\{\begin{array}{c} \partial\left(r, \left[\widehat{S}(r)\right]_{\alpha_{\widehat{S}}(r)}\right), \partial\left(\widehat{J}, \left[\Gamma\left(\widehat{J}\right)\right]_{\alpha_\Gamma\left(\widehat{J}\right)}\right) \\ , \partial\left(r, \left[\Gamma\left(\widehat{J}\right)\right]_{\alpha_\Gamma\left(\widehat{J}\right)}\right), \partial\left(\widehat{J}, \left[\widehat{S}(r)\right]_{\alpha_{\widehat{S}}(r)}\right) \end{array}\right\}. \tag{21}$$

If Y is continuous, then there exists some $u \in \omega$ such that $u \in [\widehat{S}(u)]_{\alpha_{\widehat{S}}(u)} \cap [\Gamma(u)]_{\alpha_\Gamma(u)}$.

Proof. Let $r_0 \in \omega$. By hypotheses, there exists $\alpha_{\widehat{S}}(r_0) \in (0,1]$ such that $[\widehat{S}(r_0)]_{\alpha_{\widehat{S}}(r_0)}$ is a nonempty, closed, and bounded subset of ω. For convenience, we denote $\alpha_{\widehat{S}}(r_0)$ by α_1. Let $r_1 \in [\widehat{S}(r_0)]_{\alpha_{\widehat{S}}(x_0)}$, then there exists $\alpha_\Gamma(r_1) \in (0,1]$ such that $[\Gamma(r_1)]_{\alpha_\Gamma(r_1)}$ is a nonempty, closed, and bounded subset of ω . Since Λ is nondecreasing, we have used (19) and Lemma 13

$$\begin{aligned}\Lambda\left(\partial\left(r_1, [\Gamma(r_1)]_{\alpha_\Gamma(r_1)}\right)\right) &\le \Lambda\left(H\left(\left[\widehat{S}(r_0)\right]_{\alpha_{\widehat{S}}(r_0)}, [\Gamma(r_1)]_{\alpha_\Gamma(r_1)}\right)\right)\\ &\le Y(\Lambda(M(r_0,r_1))) + LE(r_0,r_1),\end{aligned} \tag{22}$$

where

$$M(r_0,r_1) = \max\left\{\begin{array}{c}\partial(r_0,r_1), \partial\left(r_0, \left[\widehat{S}(r_0)\right]_{\alpha_{\widehat{S}}(r_0)}\right), \partial\left(r_1, [\Gamma(r_1)]_{\alpha_\Gamma(r_1)}\right),\\ \dfrac{\partial\left(r_0, [\Gamma(r_1)]_{\alpha_\Gamma(r_1)}\right) + \partial\left(r_1, \left[\widehat{S}(r_0)\right]_{\alpha_{\widehat{S}}(r_0)}\right)}{2}\end{array}\right\}, \tag{23}$$

$$E(r_0,r_1) = \min\left\{\begin{array}{c}\partial\left(r_0, \left[\widehat{S}(r_0)\right]_{\alpha_{\widehat{S}}(r_0)}\right), \partial\left(r_1, [\Gamma(r_1)]_{\alpha_\Gamma(r_1)}\right),\\ \partial\left(r_0, [\Gamma(r_1)]_{\alpha_\Gamma(r_1)}\right), \partial\left(r_1, \left[\widehat{S}(r_0)\right]_{\alpha_{\widehat{S}}(r_0)}\right)\end{array}\right\}. \tag{24}$$

By (Φ3), we have

$$\Lambda\left(\partial\left(r_1, [\Gamma(r_1)]_{\alpha_\Gamma(r_1)}\right)\right) = \inf_{\widehat{J} \in [\Gamma(r_1)]_{\alpha_\Gamma(r_1)}} \Lambda\left(\partial\left(r_1, \widehat{J}\right)\right). \tag{25}$$

Thus, there exists $r_1 \in [\Gamma(r_1)]_{\alpha_\Gamma(r_1)}$ such that

$$\Lambda\left(\partial\left(r_1, [\Gamma(r_1)]_{\alpha_\Gamma(r_1)}\right)\right) = \Lambda(\partial(r_1,r_2)). \tag{26}$$

Then from (22), we have

$$\Lambda(\partial(r_1,r_2)) \le Y(\Lambda(M(r_0,r_1))) + LE(r_0,r_1), \tag{27}$$

where

$$\begin{aligned}M(r_0,r_1) &= \max\left\{\begin{array}{c}\partial(r_0,r_1), \partial(r_0,r_1), \partial(r_1,r_2),\\ \dfrac{\partial(r_0,r_2) + \partial(r_1,r_1)}{2}\end{array}\right\}\\ &= \max\{\partial(r_0,r_1), \partial(r_1,r_2)\},\end{aligned} \tag{28}$$

$$E(r_0,r_1) = \min\left\{\begin{array}{c}\partial(r_0,r_1), \partial(r_1,r_2),\\ \partial(x_0,x_2), \partial(r_1,r_1)\end{array}\right\} = 0. \tag{29}$$

If $\max\{\partial(r_0,r_1), (r_1,r_2)\} = \partial(r_1,r_2)$, then from (27), we have

$$\Lambda(\partial(r_1,r_2)) \le Y(\Lambda(\partial(r_1,r_2))) < \Lambda(\partial(r_1,r_2)), \tag{30}$$

which is a contradiction. Thus, $\max\{\partial(r_0,r_1), (r_1,r_2)\} = \partial(r_0,r_1)$. By (27), we get that

$$\Lambda(\partial(r_1,r_2)) \le Y(\Lambda(\partial(r_0,r_1))). \tag{31}$$

Next, there exists $\alpha_{\widehat{S}}(r_2) \in (0,1]$ such that $[\widehat{S}(r_2)]_{\alpha_{\widehat{S}}(r_2)}$ is a nonempty, closed, and bounded subset of ω. By Lemma 13, using (19) and the fact that Λ is nondecreasing, we have

$$\begin{aligned}\Lambda\left(\partial\left(r_2, \left[\widehat{S}(r_2)\right]_{\alpha_{\widehat{S}}(r_2)}\right)\right) &\le \Lambda\left(H\left(\left[\widehat{S}(r_2)\right]_{\alpha_{\widehat{S}}(r_2)}, [\Gamma(r_1)]_{\alpha_\Gamma(r_1)}\right)\right)\\ &\le Y(\Lambda(M(r_1,r_2))) + LE(r_1,r_2),\end{aligned} \tag{32}$$

where

$$M(r_1,r_2) = \max\left\{\begin{array}{c}\partial(r_1,r_2), \partial\left(r_2, \left[\widehat{S}(r_2)\right]_{\alpha_{\widehat{S}}(r_2)}\right), \partial\left(r_1, [\Gamma(r_1)]_{\alpha_\Gamma(r_1)}\right),\\ \dfrac{\partial\left(r_2, [\Gamma(r_1)]_{\alpha_\Gamma(r_1)}\right) + \partial\left(r_1, \left[\widehat{S}(r_2)\right]_{\alpha_{\widehat{S}}(r_2)}\right)}{2}\end{array}\right\}, \tag{33}$$

$$E(r_1,r_2) = \min\left\{\begin{array}{c}\partial\left(r_2, \left[\widehat{S}(r_2)\right]_{\alpha_{\widehat{S}}(r_2)}\right), \partial\left(r_1, [\Gamma(r_1)]_{\alpha_\Gamma(r_1)}\right),\\ \partial\left(r_2, [\Gamma(r_1)]_{\alpha_\Gamma(r_1)}\right), \partial\left(r_1, \left[\widehat{S}(r_2)\right]_{\alpha_{\widehat{S}}(r_2)}\right)\end{array}\right\}. \tag{34}$$

From (Φ4), we have

$$\Lambda\left(\partial\left(r_2, \left[\widehat{S}(r_2)\right]_{\alpha_{\widehat{S}}(r_2)}\right)\right) = \inf_{\widehat{J}_1 \in \left[\widehat{S}(r_2)\right]_{\alpha_{\widehat{S}}(r_2)}} \Lambda\left(\partial\left(r_2, \widehat{J}_1\right)\right). \tag{35}$$

Thus, there exists $r_3 \in [\widehat{S}(r_2)]_{\alpha_{\widehat{S}}(r_2)}$ such that

$$\Lambda\left(\partial\left(r_2, \left[\widehat{S}(r_2)\right]_{\alpha_{\widehat{S}}(r_2)}\right)\right) = \Lambda(\partial(r_2,r_3)). \tag{36}$$

Then from (32), we have

$$\Lambda(\partial(r_2,r_3)) \le Y(\Lambda(M(r_1,r_2))) + LE(r_1,r_2), \tag{37}$$

where

$$\begin{aligned}M(r_1,r_2) &= \max\left\{\begin{array}{c}\partial(r_1,r_2), \partial(r_2,r_3), d(r_1,r_2),\\ \dfrac{\partial(r_1,r3) + \partial(r_2,r_2)}{2}\end{array}\right\}\\ &= \max\{\partial(r_1,r_2), \partial(r_2,r_3)\},\end{aligned} \tag{38}$$

$$E(r_1,r_2) = \min\left\{\begin{array}{c}\partial(r_2,r_3), \partial(r_1,r_2),\\ \partial(r_2,r_2), \partial(r_1,r_3)\end{array}\right\} = 0. \tag{39}$$

If $\max\{\partial(r_1,r_2), (r_2,r_3)\} = \partial(r_2,r_3)$, then from (37), we have

$$\Lambda(\partial(r_2,r_3)) \le Y(\Lambda(\partial(r_2,r_3))) < \Lambda(\partial(r_2,r_3)), \tag{40}$$

which is a contradiction. Thus, $\max\{(r_1, r_2), (r_2, r_3)\} = \partial(r_1, r_2)$. By (37), we get that

$$\Lambda(\partial(r_2, r_3)) \le Y(\Lambda(\partial(r_1, r_2))). \tag{41}$$

By continuing this process, we construct a sequence $\{r_n\}$ in ω such that $r_{2n+1} \in [\widehat{S}(r_{2n})]_{\alpha_{\widehat{S}}(r_{2n})}$ and $r_{2n+2} \in [\Gamma(r_{2n+1})]_{\alpha_\Gamma(r_{2n+1})}$

$$\Lambda(\partial(r_{2n+1}, r_{2n+2})) \le Y(\Lambda(\partial(r_{2n}, r_{2n+1}))), \text{for all } n \in \mathbb{N}. \tag{42}$$

$$\Lambda(\partial(r_{2n+2}, r_{2n+3})) \le Y(\Lambda(\partial(r_{2n+1}, r_{2n+2}))), \text{for all } n \in \mathbb{N}. \tag{43}$$

Thus, from (42) and (43), we have

$$\Lambda(\partial(r_n, r_{n+1})) \le Y(\Lambda(\partial(r_{n-1}, r_n))), \text{for all } n \in \mathbb{N}. \tag{44}$$

This implies that

$$\begin{aligned}\Lambda(\partial(r_n, r_{n+1})) &\le Y(\Lambda(\partial(r_{n-1}, r_n))) \le Y^2(\Lambda(\partial(r_{n-2}, r_{n-1}))) \\ &\le \cdots \le Y^n(\Lambda(\partial(r_0, r_1))).\end{aligned} \tag{45}$$

Letting $n \longrightarrow \infty$, we get

$$0 \le \lim_{n\to\infty} \Lambda(\partial(r_n, r_{n+1})) \le \lim_{n\to\infty} Y^n(\Lambda(\partial(r_0, r_1))) = 0. \tag{46}$$

It yields that

$$\lim_{n\to\infty} \Lambda(\partial(r_n, r_{n+1})) = 0. \tag{47}$$

This together with $(\Phi 2)$ and Lemma 9 gives that

$$\lim_{n\to\infty} \partial(r_n, r_{n+1}) = 0. \tag{48}$$

Now, we will prove that $\{r_n\}$ is a Cauchy sequence. Arguing by contradiction, we assume that there are $\varepsilon > 0$ and sequences $\{p_n\}_{n=1}^{\infty}$ and $\{q_n\}_{n=1}^{\infty}$ of integers so that for all $n \in \mathbb{N}, p_n > q_n > n$ with $d(r_{p(n)}, r_{q(n)}) \ge \varepsilon$,

$$\partial\left(r_{p(n)-1}, r_{q(n)}\right) < \varepsilon. \tag{49}$$

Thus,

$$\begin{aligned}\varepsilon \le \partial\left(r_{p(n)}, r_{q(n)}\right) &\le \partial\left(r_{p(n)}, r_{p(n)-1}\right) + \partial\left(r_{p(n)-1}, r_{q(n)}\right) \\ &\le \partial\left(r_{p(n)}, r_{p(n)-1}\right) + \varepsilon.\end{aligned} \tag{50}$$

Letting $n \longrightarrow \infty$ in (2.13), we get

$$\lim_{n\to\infty} \partial\left(r_{p(n)}, r_{q(n)}\right) = \varepsilon. \tag{51}$$

Again,

$$\begin{aligned}\partial\left(r_{p(n)}, r_{q(n)}\right) &\le \partial\left(r_{p(n)}, r_{p(n)+1}\right) + \partial\left(r_{p(n)+1}, r_{q(n)+1}\right) \\ &\quad + \partial\left(r_{q(n)+1}, r_{q(n)}\right),\end{aligned} \tag{52}$$

$$\begin{aligned}\partial\left(r_{p(n)+1}, r_{q(n)+1}\right) &\le \partial\left(r_{p(n)+1}, r_{p(n)}\right) + \partial\left(r_{p(n)}, r_{q(n)}\right) \\ &\quad + \partial\left(r_{q(n)}, r_{q(n)+1}\right).\end{aligned} \tag{53}$$

Taking $n \longrightarrow \infty$ in (51) and (52), we get

$$\lim_{n\to\infty} \partial\left(r_{p(n)+1}, r_{q(n)+1}\right) = \varepsilon. \tag{54}$$

From (48) and (51), we can choose an integer $n_0 \ge 1$ so that by (19), we get

$$\begin{aligned}&\Lambda\left(\partial\left(r_{p(n)+1}, r_{q(n)+1}\right)\right) \\ &\quad \le \Lambda\left(H\left(\left[\widehat{S}(r)_{p(n)}\right]_{\alpha_{\widehat{S}}\left(r_{p(n)}\right)}, \left[\Gamma(r)_{q(n)}\right]_{\alpha_\Gamma\left(r_{q(n)}\right)}\right)\right) \\ &\quad \le Y\left(\Lambda\left(M\left(r_{p(n)}, r_{q(n)}\right)\right)\right) + LE\left(r_{p(n)}, r_{q(n)}\right),\end{aligned} \tag{55}$$

where

$$\begin{aligned}M\left(r_{p(n)}, r_{q(n)}\right) &= \max\left\{\begin{array}{c}\partial\left(r_{p(n)}, r_{q(n)}\right), \partial\left(r_{p(n)}, \left[\widehat{S}(r)_{p(n)}\right]_{\alpha_{\widehat{S}}\left(r_{p(n)}\right)}\right), \partial\left(r_{q(n)}, \left[\Gamma(r)_{q(n)}\right]_{\alpha_\Gamma\left(r_{q(n)}\right)}\right), \\ \dfrac{\partial\left(r_{p(n)}, \left[\Gamma(r)_{q(n)}\right]_{\alpha_\Gamma\left(r_{q(n)}\right)}\right) + \partial\left(r_{q(n)}, \left[\widehat{S}(r)_{p(n)}\right]_{\alpha_{\widehat{S}}\left(r_{p(n)}\right)}\right)}{2}\end{array}\right\} \\ &\le \max\left\{\begin{array}{c}\partial\left(r_{p(n)}, r_{q(n)}\right), \partial\left(r_{p(n)}, r_{p(n)+1}\right), \partial\left(r_{q(n)}, r_{q(n)+1}\right), \\ \dfrac{\partial\left(r_{p(n)}, r_{q(n)+1}\right) + \partial\left(r_{q(n)}, r_{p(n)+1}\right)}{2}\end{array}\right\},\end{aligned} \tag{56}$$

$$\begin{aligned}E\left(r_{p(n)}, r_{q(n)}\right) &= \min\left\{\begin{array}{c}\partial\left(r_{p(n)}, \left[\widehat{S}(r)_{p(n)}\right]_{\alpha_{\widehat{S}}\left(r_{p(n)}\right)}\right), \partial\left(r_{q(n)}, \left[\Gamma(r)_{q(n)}\right]_{\alpha_\Gamma\left(r_{q(n)}\right)}\right), \\ \partial\left(r_{p(n)}, \left[\Gamma(r)_{q(n)}\right]_{\alpha_\Gamma\left(r_{q(n)}\right)}\right), \partial\left(r_{q(n)}, \left[\widehat{S}(r)_{p(n)}\right]_{\alpha_{\widehat{S}}\left(r_{p(n)}\right)}\right)\end{array}\right\} \\ &\le \min\left\{\begin{array}{c}\partial\left(r_{p(n)}, r_{p(n)+1}\right), \partial\left(r_{q(n)}, r_{q(n)+1}\right), \\ \partial\left(r_{p(n)}, r_{q(n)+1}\right), \partial\left(r_{q(n)}, r_{p(n)+1}\right)\end{array}\right\} = 0.\end{aligned} \tag{57}$$

Letting $n \longrightarrow \infty$ in the above inequality, since Y and Λ are continuous and by using (48), (49), (51), and (54), we get

$$\begin{aligned}\Lambda(\varepsilon) &= \lim_{n\to\infty} \Lambda\left(\partial\left(r_{p(n)+1}, r_{q(n)+1}\right)\right) \\ &\le \lim_{n\to\infty} Y\left(\Lambda\left(M\left(r_{p(n)}, r_{q(n)}\right)\right)\right) \\ &= Y(\Lambda(\varepsilon)) < \Lambda(\varepsilon).\end{aligned} \tag{58}$$

It is a contradiction, so $\{r_n\}$ is Cauchy. Since ω is complete, $\{r_n\}$ converges to $r^* \in \omega$, i.e., $\lim_{n\to\infty} d(r_n, r^*) = 0$. We claim that $r^* \in [\Gamma(r^*)]_{\alpha_\Gamma(x^*)}$. Assume that $r^* \notin [\Gamma(r^*)]_{\alpha_\Gamma(x^*)}$ (that is, $\partial(r^*, [\Gamma(r^*)]_{\alpha_\Gamma(r^*)}) > 0$), then there are $n_0 \in \mathbb{N}$ and a subsequence $\{r_{n_k}\}$ of $\{r_n\}$ so that $\partial(r_{2n_k+1}, [\Gamma(r^*)]_{\alpha_\Gamma(r^*)}) > 0$, for all $n_k \geq n_0$. Since $\partial(r_{2n_k+1}, [\Gamma(r^*)]_{\alpha_\Gamma(r^*)}) > 0$, for all $n_k \geq n_0$, so by $(\Phi 1)$, we have

$$\begin{aligned}\Lambda\left(\partial\left(r_{2n_k+1}, [\Gamma(r^*)]_{\alpha_\Gamma(r^*)}\right)\right) &\leq \Lambda\left(H\left(\left[\widehat{S}(r_{2n_k})\right]_{\alpha_{\widehat{S}}(r_{2n_k})}, [\Gamma(r^*)]_{\alpha_\Gamma(r^*)}\right)\right) \\ &\leq Y\left(\Lambda\left(M\left(r_{2n_k}, r^*\right)\right)\right) + LE\left(r_{2n_k}, r^*\right),\end{aligned} \tag{59}$$

where

$$\begin{aligned}M\left(r_{2n_k}, r^*\right) &= \max\left\{\begin{array}{c}\partial\left(r_{2n_k}, r^*\right), \partial\left(r_{2n_k}, \left[\widehat{S}(r_{2n_k})\right]_{\alpha_{\widehat{S}}(r_{2n_k})}\right), \partial\left(r^*, [\Gamma(r^*)]_{\alpha_\Gamma(r^*)}\right), \\ \dfrac{\partial\left(r_{2n_k}, [\Gamma(r^*)]_{\alpha_\Gamma(r^*)}\right) + \partial\left(r^*, \left[\widehat{S}(r_{2n_k})\right]_{\alpha_{\widehat{S}}(r_{2n_k})}\right)}{2}\end{array}\right\} \\ &\leq \max\left\{\begin{array}{c}\partial\left(r_{2n_k}, r^*\right), \partial\left(r_{2n_k}, r_{2n_k+1}\right), \partial\left(r^*, [\Gamma(r^*)]_{\alpha_\Gamma(r^*)}\right), \\ \dfrac{\partial\left(r_{2n_k}, [\Gamma(r^*)]_{\alpha_\Gamma(r^*)}\right) + \partial\left(r^*, r_{2n_k+1}\right)}{2}\end{array}\right\},\end{aligned} \tag{60}$$

$$\begin{aligned}E\left(r_{2n_k}, r^*\right) &= \min\left\{\begin{array}{c}\partial\left(r_{2n_k}, \left[\widehat{S}(r_{2n_k})\right]_{\alpha_{\widehat{S}}(r_{2n_k})}\right), \partial\left(r^*, [\Gamma(r^*)]_{\alpha_\Gamma(r^*)}\right), \\ \partial\left(r_{2n_k}, [\Gamma(r^*)]_{\alpha_\Gamma(r^*)}\right), \partial\left(r^*, \left[\widehat{S}(r_{2n_k})\right]_{\alpha_{\widehat{S}}(r_{2n_k})}\right)\end{array}\right\} \\ &\leq \min\left\{\begin{array}{c}\partial\left(r_{2n_k}, r_{2n_k+1}\right), \partial\left(r^*, [\Gamma(r^*)]_{\alpha_\Gamma(r^*)}\right), \\ \partial\left(r_{2n_k}, [\Gamma(r^*)]_{\alpha_\Gamma(r^*)}\right), \partial\left(r^*, r_{2n_k+1}\right)\end{array}\right\}.\end{aligned} \tag{61}$$

Letting $k \longrightarrow \infty$ and using the continuity of Λ and Y, we have

$$\begin{aligned}\Lambda\left(\partial\left(r^*, [\Gamma(r^*)]_{\alpha_\Gamma(r^*)}\right)\right) &\leq Y\left(\Lambda\left(\partial\left(r^*, [\Gamma(r^*)]_{\alpha_\Gamma(r^*)}\right)\right)\right) \\ &\quad + 0 < \Lambda\left(\partial\left(r^*, [\Gamma(r^*)]_{\alpha_\Gamma(r^*)}\right)\right),\end{aligned} \tag{62}$$

which is a contradiction. Hence, $\partial(r^*, [\Gamma(r^*)]_{\alpha_\Gamma(r^*)}) = 0$, and $r^* \in [\Gamma(r^*)]_{\alpha_\Gamma(r^*)}$. Similarly, one can easily prove that $r^* \in [\widehat{S}(r^*)]_{\alpha_{\widehat{S}}(r^*)}$. Therefore, $r^* \in [\widehat{S}(r^*)]_{\alpha_{\widehat{S}}(r^*)} \cap [\Gamma(r^*)]_{\Gamma(r^*)}$.

Example 17. Let $\omega = [0, 1]$ endowed with the metric $\partial(r, \widehat{J}) = |r - \widehat{J}|$. Consider $\Lambda, Y : (0, \infty) \longrightarrow (0, \infty)$ as $\Lambda(t) = t$ and $Y(t) = 99t/100$. Here, $\Lambda, Y \in \Phi^*$ and Y are continuous. For $\alpha \in (0, 1]$, given $\widehat{S}, \Gamma : \omega \longrightarrow \mathfrak{I}(\omega)$ *by*

$$\widehat{S}(r)(t) = \begin{cases}\alpha, & 0 \leq t \leq \dfrac{r}{60} \\ \dfrac{\alpha}{2}, & \dfrac{r}{60} \leq t \leq \dfrac{r}{40} \\ \dfrac{\alpha}{3}, & \dfrac{r}{40} \leq t \leq \dfrac{r}{20} \\ \dfrac{\alpha}{5}, & \dfrac{r}{20} \leq t \leq 1\end{cases}, \tag{63}$$

$$\Gamma(r)(t) = \begin{cases}\alpha, & 0 \leq t \leq \dfrac{r}{15} \\ \dfrac{\alpha}{3}, & \dfrac{r}{15} \leq t \leq \dfrac{r}{10} \\ \dfrac{\alpha}{4}, & \dfrac{r}{10} \leq t \leq \dfrac{r}{5} \\ \dfrac{\alpha}{7}, & \dfrac{r}{5} \leq t \leq 1\end{cases}, \tag{64}$$

such that

$$\begin{aligned}\left[\widehat{S}(r)\right]_\alpha &= \left[0, \frac{r}{60}\right], \\ [\Gamma(r)]_\alpha &= \left[0, \frac{r}{15}\right].\end{aligned} \tag{65}$$

For $r, \widehat{J} \in \omega$ with $H([\widehat{S}(r)]_{\alpha_{\widehat{S}}(r)}, [\Gamma(\widehat{J})]_{\alpha_\Gamma(\widehat{J})}) > 0$, we have

$$\Lambda\left(H\left(\left[\widehat{S}(r)\right]_{\alpha_{\widehat{S}}(r)}, \left[\Gamma(\widehat{J})\right]_{\alpha_\Gamma(\widehat{J})}\right)\right) \leq Y\left(\Lambda\left(M\left(r, \widehat{J}\right)\right)\right) + LE\left(r, \widehat{J}\right). \tag{66}$$

where

$$M\left(r, \widehat{J}\right) = \max\left\{\begin{array}{c}\partial\left(r, \widehat{J}\right), \partial\left(r, \left[\widehat{S}(r)\right]_{\alpha_{\widehat{S}}(r)}\right), \\ \partial\left(\widehat{J}, \left[\Gamma(\widehat{J})\right]_{\alpha_\Gamma(\widehat{J})}\right), \dfrac{\partial\left(r, \left[\Gamma(\widehat{J})\right]_{\alpha_\Gamma(\widehat{J})}\right) + \partial\left(\widehat{J}, \left[\widehat{S}(r)\right]_{\alpha_{\widehat{S}}(r)}\right)}{2}\end{array}\right\}, \tag{67}$$

$$E\left(r, \widehat{J}\right) = \min\left\{\begin{array}{c}\partial\left(r, \left[\widehat{S}(r)\right]_{\alpha_{\widehat{S}}(r)}\right), \partial\left(\widehat{J}, \left[\Gamma(\widehat{J})\right]_{\alpha_\Gamma(\widehat{J})}\right) \\ , \partial\left(r, \left[\Gamma(\widehat{J})\right]_{\alpha_\Gamma(\widehat{J})}\right), \partial\left(\widehat{J}, \left[\widehat{S}(r)\right]_{\alpha_{\widehat{S}}(r)}\right)\end{array}\right\}. \tag{68}$$

Hence, all the conditions of Theorem 16 (with $L = 1$) are satisfied, and $0 \in [S0]_\alpha \cap [T0]_\alpha$.

Corollary 18. *Let $\widehat{S}, \Gamma$ be FM from ω into $\mathfrak{I}(\omega)$, and for each $r \in \omega$, there exist $\alpha_{\widehat{S}}(r)$, $\alpha_\Gamma(r) \in (0, 1]$ such that $[\widehat{S}(r)]_{\alpha_{\widehat{S}}(r)}$ and $[\Gamma(\widehat{J})]_{\alpha_\Gamma(\widehat{J})}$ are nonempty, closed, and bounded subsets of ω. Assume that there are $Y \in \Psi$ and $\Lambda \in \Phi$ such that for all $r, \widehat{J} \in \omega$,*

$$
\begin{aligned}
&H\left(\left[\widehat{S}(r)\right]_{\alpha_{\widehat{S}}(r)},\left[\Gamma(\widehat{J})\right]_{\alpha_{\Gamma}(\widehat{J})}\right)\\
&\quad>0 \Rightarrow \Lambda\left(H\left(\left[\widehat{S}(r)\right]_{\alpha_{\widehat{S}}(r)},\left[\Gamma(\widehat{J})\right]_{\alpha_{\Gamma}(\widehat{J})}\right)\right)\\
&\quad\leq Y\left(\Lambda\left(M\left(r,\widehat{J}\right)\right)\right),
\end{aligned}
\tag{69}
$$

where $M(r,\widehat{J})$ is defined by (20). If Y is continuous, then there is $u\in\omega$ such that $u\in[\widehat{S}(u)]_{\alpha_{\widehat{S}}(u)}\cap[\Gamma(u)]_{\alpha_{\Gamma}(u)}$.

Proof. Set $L=0$ in Theorem 16.

Corollary 19. *Let $\widehat{S}$ be a FM from ω into $\mathfrak{I}(\omega)$, and for each $r\in\omega$, there exist $\alpha_{\widehat{S}}(r)\in(0,1]$ such that $[\widehat{S}(r)]_{\alpha_{\widehat{S}}(r)}$ is nonempty, closed, and bounded subsets of ω. Assume that there are $Y\in\Psi$, $\Lambda\in\Phi$, and $L\geq 0$ such that for all $r,\widehat{J}\in\omega$,*

$$
\begin{aligned}
&H\left(\left[\widehat{S}(r)\right]_{\alpha_{\widehat{S}}(r)},\left[\widehat{S}(\widehat{J})\right]_{\alpha_{\widehat{S}}(\widehat{J})}\right)\\
&\quad>0 \Rightarrow \Lambda\left(H\left(\left[\widehat{S}(r)\right]_{\alpha_{\widehat{S}}(r)},\left[\widehat{S}(\widehat{J})\right]_{\alpha_{\widehat{S}}(\widehat{J})}\right)\right)\\
&\quad\leq Y\left(\Lambda\left(M\left(r,\widehat{J}\right)\right)\right)+LE\left(r,\widehat{J}\right),
\end{aligned}
\tag{70}
$$

where

$$
M\left(r,\widehat{J}\right)=\max\left\{\partial\left(r,\widehat{J}\right),D\left(r,\left[\widehat{S}(r)\right]_{\alpha_{\widehat{S}}(r)}\right),D\left(\widehat{J},\left[\widehat{S}(\widehat{J})\right]_{\alpha_{\widehat{S}}(\widehat{J})}\right),\frac{D\left(r,\left[\widehat{S}(\widehat{J})\right]_{\alpha_{\widehat{S}}(\widehat{J})}\right)+D\left(\widehat{J},\left[\widehat{S}(r)\right]_{\alpha_{\widehat{S}}(r)}\right)}{2}\right\},
\tag{71}
$$

$$
E\left(r,\widehat{J}\right)=\min\left\{D\left(r,\left[\widehat{S}(r)\right]_{\alpha_{\widehat{S}}(r)}\right),D\left(\widehat{J},\left[\widehat{S}(\widehat{J})\right]_{\alpha_{\widehat{S}}(\widehat{J})}\right),D\left(r,\left[\widehat{S}(\widehat{J})\right]_{\alpha_{\widehat{S}}(\widehat{J})}\right),D\left(\widehat{J},\left[\widehat{S}(r)\right]_{\alpha_{\widehat{S}}(r)}\right)\right\}.
\tag{72}
$$

If Y is continuous, then there is $u\in\omega$ so that $u\in[\widehat{S}(u)]_{\alpha_{\widehat{S}}(u)}$.

Proof. Take $\widehat{S}=\Gamma$ in Theorem 16.

Now, we consider multivalued mappings.

Theorem 20. *Let $F,G:\omega\longrightarrow CB(\omega)$ be multivalued mappings. Assume that there are $Y\in\Psi$, $\Lambda\in\Phi$, and $L\geq 0$ such that for all $r,\widehat{J}\in\omega$,*

$$
\begin{aligned}
H\left(F(r),G(\widehat{J})\right)>0 &\Rightarrow \Lambda\left(H\left(F(r),G(\widehat{J})\right)\right)\\
&\leq Y\left(\Lambda\left(M\left(r,\widehat{J}\right)\right)\right)+LE\left(r,\widehat{J}\right),
\end{aligned}
\tag{73}
$$

where

$$
M\left(r,\widehat{J}\right)=\max\left\{\partial\left(r,\widehat{J}\right),\partial(r,F(r)),\partial\left(\widehat{J},G(\widehat{J})\right),\frac{\partial\left(r,G(\widehat{J})\right)+\partial\left(\widehat{J},F(r)\right)}{2}\right\},
\tag{74}
$$

$$
E\left(r,\widehat{J}\right)=\min\left\{\partial(r,F(r)),\partial\left(\widehat{J},G(\widehat{J})\right),\partial\left(r,G(\widehat{J})\right),\partial\left(\widehat{J},F(r)\right)\right\}.
\tag{75}
$$

If Y is continuous, then there is $u\in\omega$ such that $u\in F(u)\cap G(u)$.

Proof. Consider a mapping $\alpha:\omega\longrightarrow(0,1]$ and a pair of fuzzy mappings $\widehat{S},\Gamma:\omega\longrightarrow\sqsubseteq(\omega)$ defined by

$$
\widehat{S}(r)(t)=\begin{cases}\alpha(r), & \text{if } t\in F(r)\\ 0, & \text{if } t\notin F(r),\end{cases}\quad\text{and}\quad\Gamma(r)(t)=\begin{cases}\alpha(r), & \text{if } t\in G(r)\\ 0, & \text{if } t\notin G(r).\end{cases}
\tag{76}
$$

Then,

$$
\left[\widehat{S}(r)\right]_{\alpha(r)}=\left\{t:\widehat{S}(r)(t)\geq\alpha(r)\right\}=F(r),
\tag{77}
$$

$$
[\Gamma(r)]_{\alpha(r)}=\{t:\Gamma(r)(t)\geq\alpha(r)\}=G(r).
\tag{78}
$$

Thus, by Theorem 16, there is $u\in\omega$ so that $u\in[\widehat{S}(u)]_{\alpha_{\widehat{S}}(u)}\cap[\Gamma(u)]_{\alpha_{\Gamma}(u)}=F(u)\cap G(u)$.

Theorem 21. *Let $\widehat{S},\Gamma:\omega\longrightarrow W(\omega)$ be FM. Assume that there are $Y\in\Psi$, $\Lambda\in\Phi$, and $L\geq 0$ such that for all $r,\widehat{J}\in\omega$,*

$$
\begin{aligned}
\partial_{\infty}\left(\widehat{S}(r),\Gamma(\widehat{J})\right)>0 &\Rightarrow \Lambda\left(\partial_{\infty}\left(\widehat{S}(r),\Gamma(\widehat{J})\right)\right)\\
&\leq Y\left(\Lambda\left(M\left(r,\widehat{J}\right)\right)\right)+LE\left(r,\widehat{J}\right),
\end{aligned}
\tag{79}
$$

where

$$
M\left(r,\widehat{J}\right)=\max\left\{p\left(r,\widehat{J}\right),p\left(r,\widehat{S}(r)\right),p\left(\widehat{J},\Gamma(\widehat{J})\right),\frac{p\left(r,\Gamma(\widehat{J})\right)+p\left(\widehat{J},\widehat{S}(r)\right)}{2}\right\},
\tag{80}
$$

$$
E\left(r,\widehat{J}\right)=\min\left\{p\left(r,\widehat{S}(r)\right),p\left(\widehat{J},\Gamma(\widehat{J})\right),p\left(r,\Gamma(\widehat{J})\right),p\left(\widehat{J},\widehat{S}(r)\right)\right\}.
\tag{81}
$$

If Y is continuous, then there is $u\in\omega$ so that $\{u\}\subset\widehat{S}(u)$ and $\{u\}\subset\Gamma(u)$.

Proof. Let $r\in\omega$; then by Lemma 14, there is $\widehat{J}\in\omega$ so that $\widehat{J}\in[\widehat{S}(r)]_1$. Similarly, we can find $e\in\omega$ such that $e\in[\Gamma(r)]_1$. It follows that for each $r\in\omega$, $[\widehat{S}(r)]_{\alpha(r)}$ and $[\Gamma(r)]_{\alpha(r)}$ are nonempty, closed, and bounded subsets of ω. As $\alpha(r)=\alpha(\widehat{J})=1$, by the definition of a ∂_{∞}-metric for fuzzy sets, we have

$$
H\left(\left[\widehat{S}(r)\right]_{\alpha(r)},\left[\Gamma(\widehat{J})\right]_{\alpha(\widehat{J})}\right)\leq\partial_{\infty}\left(\widehat{S}(r),\Gamma(\widehat{J})\right),\text{for all } r,\widehat{J}\in\omega.
\tag{82}
$$

Since ϕ is nondecreasing, we have

$$
\begin{aligned}
&\Lambda\left(H\left(\left[\widehat{S}(r)\right]_{\alpha(r)},\left[\Gamma(\widehat{J})\right]_{\alpha(\widehat{J})}\right)\right)\\
&\quad\leq\Lambda\left(\partial_{\infty}\left(\widehat{S}(r),\Gamma(\widehat{J})\right)\right)\leq Y\left(\Lambda\left(M\left(r,\widehat{J}\right)\right)\right)+LE\left(r,\widehat{J}\right),
\end{aligned}
\tag{83}
$$

where

$$M(r,\hat{J}) = \max\left\{p(r,\hat{J}), p(r,\hat{S}(r)), p(\hat{J},\Gamma(\hat{J})), \frac{p(r,\Gamma(\hat{J})) + p(\hat{J},\hat{S}(r))}{2}\right\}, \tag{84}$$

$$E(r,\hat{J}) = \min\{p(r,\hat{S}(r)), p(\hat{J},\Gamma(\hat{J})), p(r,\Gamma(\hat{J})), p(\hat{J},\hat{S}(r))\}, \tag{85}$$

for all $r, \hat{J} \in \omega$. Since $[\hat{S}(r)]_1 \subseteq [\hat{S}(r)]_{\alpha(r)}$, we have

$$\partial_\infty\left(r, [\hat{S}(r)]_{\alpha(r)}\right) \le \partial(r, [\hat{S}(r)]_1) \text{ for each } \alpha \in (0,1]. \tag{86}$$

It yields that

$$P(r,\hat{S}(r)) \le \partial(r, [\hat{S}(r)]_1). \tag{87}$$

Similarly,

$$p(r,\Gamma(r)) \le \partial(x, [\Gamma(r)]_1). \tag{88}$$

It yields that for all $r, \hat{J} \in \omega$,

$$\Lambda(H([\hat{S}(r)]_1, [\Gamma(r)]_1)) \le Y(\Lambda(M(r,\hat{J}))) + LE(r,\hat{J}), \tag{89}$$

where

$$M(r,\hat{J}) = \max\left\{\partial(r,\hat{J}), \partial(r,[\hat{S}(r)]_1), \partial(\hat{J},[\Gamma(r)]_1), \frac{\partial(r,[\Gamma(\hat{J})]_1) + \partial(\hat{J},[\hat{S}(r)]_1)}{2}\right\}, \tag{90}$$

$$E(r,\hat{J}) = \min\{\partial(r,[\hat{S}(r)]_1), \partial(\hat{J},[\Gamma(\hat{J})]_1), \partial(r,[\Gamma(\hat{J})]_1), \partial(\hat{J},[\hat{S}(r)]_1)\}. \tag{91}$$

By Theorem 16, we obtain $u \in \omega$ such that $[\hat{S}(u)]_1 \cap [\Gamma(u)]_1$, that is, $\{u\} \subset \hat{S}(u)$ and $\{u\} \subset \Gamma(u)$.

Corollary 22. *Let (ϕ, ∂) be a complete MS, and $\hat{S}, \Gamma : \omega \longrightarrow W(\omega)$ be FM. Assume that there exist $Y \in \Psi$ and $\Lambda \in \Phi$ such that for all $r, \hat{J} \in \omega$,*

$$\partial_\infty(\hat{S}(r), \Gamma(\hat{J})) > 0 \Rightarrow \Lambda(\partial_\infty(\hat{S}(r), \Gamma(\hat{J}))) \le Y(\Lambda(M(r,\hat{J}))), \tag{92}$$

where $M(r,\hat{J})$ is defined by (80). If Y is continuous, then there is $u \in \omega$ so that $\{u\} \subset \hat{S}(u)$ and $\{u\} \subset \Gamma(u)$.

Proof. Take $L = 0$ in Theorem 21.

We denote by $\hat{S}$ (for details, see [43, 44]) the setvalued mapping induced by a FM $\hat{S} : \omega \longrightarrow F(\omega)$, i.e.,

$$\hat{S}(r) = \left\{\hat{J} : \hat{S}(\hat{J})(t) = \max_{t\in\omega} \hat{S}(r)(t)\right\}. \tag{93}$$

Corollary 23. *Let $\hat{S}, \Gamma : \omega \longrightarrow \mathfrak{J}(\omega)$ be FM such that for all $r \in \omega$, $\hat{S}(r)$ and $\hat{\Gamma}(r)$ are nonempty, closed, and bounded subsets of ω. Assume that there exist $Y \in \Psi$, $\Lambda \in \Phi$, and $L \ge 0$ such that for all $r, \hat{J} \in \omega$,*

$$\begin{aligned} H(\hat{S}(r), \hat{\Gamma}(\hat{J})) > 0 \Rightarrow \Lambda(H(\hat{S}(r), \hat{\Gamma}(\hat{J}))) \\ \le Y(\Lambda(M(r,\hat{J}))) + LE(r,\hat{J}), \end{aligned} \tag{94}$$

where

$$M(r,\hat{J}) = \max\left\{\partial(r,\hat{J}), \partial(r,\hat{S}(r)), \partial(\hat{J},\hat{\Gamma}(\hat{J})), \frac{\partial(r,\hat{\Gamma}(\hat{J})) + \partial(\hat{J},\hat{S}(r))}{2}\right\}, \tag{95}$$

$$E(r,\hat{J}) = \min\left\{\partial(r,\hat{S}(r)), \partial(\hat{J},\hat{\Gamma}(\hat{J})), \partial(r,\hat{\Gamma}(\hat{J})), \partial(\hat{J},\hat{S}(r))\right\}. \tag{96}$$

If Y is continuous, then there is $r^* \in \omega$ so that $\hat{S}(r^*)(r^*) \ge \hat{S}(r^*)(r)$ and $\Gamma(r^*)(r^*) \ge \Gamma(r^*)(r)$, for all $r \in \omega$.

Proof. By Theorem 16, there is $r^* \in \omega$ so that $r^* \in \hat{S}(r^*) \cap \hat{\Gamma}(r^*)$. Then by Lemma 15, we have

$$\begin{aligned} \hat{S}(r^*)(r^*) &\ge \hat{S}(r^*)(r), \\ \Gamma(r^*)(r^*) &\ge \Gamma(r^*)(r) \text{ for all } r \in \omega. \end{aligned} \tag{97}$$

Corollary 24. *Let $\hat{S}, \Gamma : \omega \longrightarrow \mathfrak{J}(\omega)$ be FM such that for all $r \in \omega$, $\hat{S}(r)$ and $\hat{\Gamma}(r)$ are nonempty, closed, and bounded subsets of ω. Assume that there are $Y \in \Psi$, $\Lambda \in \Phi$, and $L \ge 0$ so that for all $r, \hat{J} \in \omega$,*

$$H(\hat{S}(r), \hat{\Gamma}(\hat{J})) > 0 \Rightarrow \Lambda(H(\hat{S}(r), \hat{\Gamma}(\hat{J}))) \le Y(\Lambda(M(r,\hat{J}))), \tag{98}$$

where

$$M(r,\hat{J}) = \max\left\{\partial(r,\hat{J}), \partial(r,\hat{S}(r)), \partial(\hat{J},\hat{\Gamma}(\hat{J})), \frac{\partial(r,\hat{\Gamma}(\hat{J})) + \partial(\hat{J},\hat{S}(r))}{2}\right\}. \tag{99}$$

If Y is continuous, then there is $r^* \in \omega$ such that $(r^*)(r^*) \ge \hat{S}(r^*)(r)$ and $\Gamma(r^*)(r^*) \ge \Gamma(r^*)(r)$, for all $r \in \omega$.

Proof. Take $L = 0$ in Corollary 23.

3. Some Consequences

Corollary 25. *Let $\hat{S}, \Gamma : \omega \longrightarrow \mathfrak{J}(\omega)$ be FM, and for each $r \in \omega$, there exist $\alpha_{\hat{S}}(r)$, $\alpha_\Gamma(r) \in (0,1]$ such that $[\hat{S}(r)]_{\alpha_{\hat{S}}(r)}$ and $[\Gamma(\hat{J})]_{\alpha_\Gamma(r)}$ are nonempty, closed, and bounded subsets of ω. Assume that there are $\lambda \in (0,1)$ and $L \ge 0$ such that for all $r, \hat{J} \in \omega$,*

$$H\left(\left[\widehat{S}(r)\right]_{\alpha_{\widehat{S}}(r)},\left[\Gamma(\widehat{J})\right]_{\alpha_{\Gamma}(r)}\right)\leq\lambda M(r,\widehat{J})+LE(r,\widehat{J}),\quad(100)$$

where $M(r,\widehat{J})$ and $E(r,\widehat{J})$ are defined by (20) and (21), respectively. Then, there is $u\in\omega$ such that $u\in[\widehat{S}(u)]_{\alpha_{\widehat{S}}(u)}\cap[\Gamma(u)]_{\alpha_{\Gamma}(u)}$.

Proof. Take $Y(t)=\lambda t$ and $\Lambda(t)=t$ in Theorem 16.

Corollary 26. *Let $\widehat{S},\Gamma:\omega\longrightarrow\mathfrak{J}(\omega)$ be FM, and for each $r\in\omega$, there are $\alpha_{\widehat{S}}(r)$, $\alpha_{\Gamma}(r)\in(0,1]$, such that $[\widehat{S}(r)]_{\alpha_{\widehat{S}}(r)}$ and $[\Gamma(\widehat{J})]_{\alpha_{\Gamma}(r)}$ are nonempty, closed, and bounded subsets of ω. Assume that there are $\theta\in\Theta^*$ and $k\in(0,1)$ such that for all $r,\widehat{J}\in\omega$,$H([\widehat{S}(r)]_{\alpha_{\widehat{S}}(r)},[\Gamma(\widehat{J})]_{\alpha_{\Gamma}(r)})>0$ implies*

$$\theta\left(H\left(\left[\widehat{S}(r)\right]_{\alpha_{\widehat{S}}(r)},\left[\Gamma(\widehat{J})\right]_{\alpha_{\Gamma}(r)}\right)\right)\leq\left[\theta\left(M(r,\widehat{J})\right)\right]^{k},\quad(101)$$

where $M(r,\widehat{J})$ is defined by (20). Then, there is $u\in\omega$ such that $u\in[\widehat{S}(u)]_{\alpha_{\widehat{S}}(u)}\cap[\Gamma(u)]_{\alpha_{\Gamma}(u)}$.

Proof. Take $Y(t):=(\ln k)t$, $\Lambda(t)=\ln t$, and $L=0$ in Theorem 16.

Corollary 27. *Let $\widehat{S},\Gamma:\omega\longrightarrow\mathfrak{J}(\omega)$ be FM, and for each $r\in\omega$, there exist $\alpha_{\widehat{S}}(r)$, $\alpha_{\Gamma}(r)\in(0,1]$, such that $[\widehat{S}(r)]_{\alpha_{\widehat{S}}(r)}$ and $[\Gamma(\widehat{J})]_{\alpha_{\Gamma}(r)}$ are nonempty, closed, and bounded subsets of ω. Assume that there are $F\in\sqsubseteq^*$ and $\tau>0$ such that for all $r,\widehat{J}\in\omega$,$H([\widehat{S}(r)]_{\alpha_{\widehat{S}}(r)},[\Gamma(\widehat{J})]_{\alpha_{\Gamma}(r)})>0$ implies*

$$\tau+F\left(H\left(\left[\widehat{S}(r)\right]_{\alpha_{\widehat{S}}(r)},\left[\Gamma(\widehat{J})\right]_{\alpha_{\Gamma}(r)}\right)\right)\leq F(r,\widehat{J}),\quad(102)$$

where $M(r,\widehat{J})$ is defined by (20). Then, there is $u\in X$ so that $u\in[\widehat{S}(u)]_{\alpha_{\widehat{S}}(u)}\cap[\Gamma(u)]_{\alpha_{\Gamma}(u)}$.

Proof. Take $Y(t)=e^{-\tau}t$, $\Lambda(t)=e^{t}$, and $L=0$ in Theorem 16.

Corollary 28. *Let $\widehat{S},\Gamma:\omega\longrightarrow\mathfrak{J}(\omega)$ be FM, and for each $r\in\omega$, there exist $\alpha_{\widehat{S}}(r)$, $\alpha_{\Gamma}(r)\in(0,1]$, such that $[\widehat{S}(r)]_{\alpha_{\widehat{S}}(r)}$ and $[\Gamma(\widehat{J})]_{\alpha_{\Gamma}(r)}$ nonempty, closed, and bounded subsets of ω. Assume that there is $L\geq0$ such that for all $r,\widehat{J}\in\omega$ implies*

$$H\left(\left[\widehat{S}(r)\right]_{\alpha_{\widehat{S}}(r)},\left[\Gamma(\widehat{J})\right]_{\alpha_{\Gamma}(r)}\right)\leq\beta\left(M(r,\widehat{J})\right)M(r,\widehat{J})+LE(r,\widehat{J}),\quad(103)$$

where $M(r,\widehat{J})$ and $E(r,\widehat{J})$ are defined by (20) and (21), respectively, and $\beta:[0,'\infty)\longrightarrow[0,\infty)$ is such that $\lim_{r\to t^+}\beta(r)<1$ for each $t\in(0,\infty)$. Then, there is $u\in\omega$ such that $u\in[\widehat{S}(u)]_{\alpha_{\widehat{S}}(u)}\cap[\Gamma(u)]_{\alpha_{\Gamma}(u)}$.

Proof. It follows from Theorem 16 by taking $Y(t):=\beta(t)t$ and $Y(t)=t$.

4. Application to an Ordinary Fuzzy Differential Equation

In this section, we apply our obtained results to study the existence of a solution for the second order nonlinear boundary value problem:

$$\begin{cases}r''(t)=K\left(t,r(t),r'(t)\right), & t\in[0,a],\ a>0\\ r(t_1)=r_1, & t_1\in[0,a]\\ r(t_2)=r_2, & t_2\in[0,a]\end{cases},\quad(104)$$

where $K:[0,a]\times W(\omega)\times W(\omega)\longrightarrow W(\omega)$ is a continuous function. This problem is equivalent to the integral equation ([45, 46, 48]):

$$r(t)=\int_{t_1}^{t_2}G(t,s)K\left(s,r(s),r'(s)\right)ds+\beta(t),\ t\in[0,a],\quad(105)$$

where Green's function G is given by

$$G(t,s)=\begin{cases}\dfrac{(t_2-t)(s-t)}{t_2-t_1}, & \text{if } t_1\leq s\leq t\leq t_2,\\ \dfrac{(t_2-s)(t-t_1)}{t_2-t_1}, & \text{if } t_1\leq t\leq s\leq t_2,\end{cases},\quad(106)$$

and $\beta(t)$ satisfies $\beta''=0$, $\beta(t_1)=r_1$, and $\beta(t_2)=r_2$. Let us recall some properties of $G(t,s)$. Particularly,

$$\begin{aligned}&\int_{t_1}^{t_2}|G(t,s)|ds\leq\frac{(t_2-t_1)^2}{8},\\ &\int_{t_1}^{t_2}|G_t(t,s)|ds\leq\frac{t_2-t_1}{2}.\end{aligned}\quad(107)$$

Our investigation is based on the existence of a common fixed point for a pair of integral operators given as follows:

$$\widehat{S}(r)(t)=\int_{t_1}^{t_2}G(t,s)K_1\left(s,r(s),r'(s)\right)ds+\beta(t),\ t\in[0,a],\quad(108)$$

$$\Gamma(r)(t)=\int_{t_1}^{t_2}G(t,s)K_2\left(s,r(s),r'(s)\right)ds+\beta(t),\ t\in[0,a],\quad(109)$$

where $K_1,K_2\in C([0,a]\times W(\omega)\times W(\omega),W(\omega))$, $x\in C^1([0,a],W(\omega))$, and $\beta\in C([0,a],W(\omega))$.

Theorem 29. *Suppose that*

(a) *$K_1,K_2:[0,a]\times W(\omega)\times W(\omega)\longrightarrow W(\omega)$ are increasing in their second and third variables,*

(b) *There is $r_0 \in C^1([0, a], W(\omega))$ such that for all $t \in [0, a]$, we have*

$$r_0(t) = \int_{t_1}^{t_2} G(t, s) K_1\left(s, r_0(s), r_0'(s)\right) ds + \beta(t), t_1, t_2 \in [0, a], \tag{110}$$

(c) *There are $\gamma, \delta > 0$ such that for all $t \in [0, a]$, we have*

$$\begin{aligned} &\left|K_1\left(t, r(t), r'(t)\right) - K_2\left(t, \widehat{J}(t), \widehat{J}'(t)\right)\right| \\ &\quad \le \gamma\left|r(t) - \widehat{J}(t)\right| + \delta\left|r'(t) - \widehat{J}'(t)\right|, \end{aligned} \tag{111}$$

for all $r, \widehat{J} \in C^1([0, a], W(\omega))$, with $K_1(., ., .) \neq K_2(., ., .)$,

(d) *For $\gamma, \delta > 0$ and $t_1, t_2 \in [0, a]$, we have*

$$\gamma \frac{(t_2 - t_1)^2}{8} + \delta \frac{(t_2 - t_1)}{2} < \frac{9}{10}, \tag{112}$$

(e) *if $r, \widehat{J} \in C^1([0, a], W(\omega))$ is comparable, then every $u \in (\Gamma_1 r)_1$ and every $v \in (\Gamma_1 \widehat{J})_1$ are comparable.*

Then, the pair of nonlinear integral equations

$$r(t) = \int_{t_1}^{t_2} G(t, s) K_1\left(s, r(s), r'(s)\right) ds + \beta(t), t \in [0, a], \tag{113}$$

$$r(t) = \int_{t_1}^{t_2} G(t, s) K_2\left(s, r(s), r'(s)\right) ds + \beta(t), t \in [0, a], \tag{114}$$

has a common solution in $C^1([t_1, t_2], W(\omega))$.

Proof. Consider $C = C^1([t_1, t_2], W(\omega))$ with the metric

$$\partial_\infty\left(r, \widehat{J}\right) = \max_{t_1 \le t \le t_2} \left\{\gamma\left|r(t) - \widehat{J}(t)\right| + \delta\left|r'(t) - \widehat{J}'(t)\right|\right\}. \tag{115}$$

Note that (C, ∂_∞) is a complete linear MS. Let $\widehat{S}, \Gamma : C \longrightarrow C$ be two integral operators defined by (108) and (109). Clearly, $\widehat{S}$ and Γ are well defined since K_1, K_2 and β are continuous functions. Now, r^* is a solution of (113) and (114) if and only if r^* is a common fixed point of $\widehat{S}$ and Γ. By hypothesis (a), $\widehat{S}, \Gamma$ is increasing. Next, for all $r, \widehat{J} \in C$ with $K_1(., ., .) \neq K_2(., ., .)$, by hypothesis (c), we have successively

$$\begin{aligned} &\left|\widehat{S}(r)(t) - \Gamma\left(\widehat{J}\right)(t)\right| \le \int_{t_1}^{t_2} |G(t, s)|\left|K_1\left(s, r(s), r'(s)\right) - K_2\left(s, \widehat{J}(s), J\wedge'(s)\right)\right| \\ &\quad ds \le \partial_\infty\left(r, \widehat{J}\right) \int_{t_1}^{t_2} |G(t, s)| ds \le \frac{(t_2 - t_1)^2}{8} \partial_\infty\left(r, \widehat{J}\right), \end{aligned} \tag{116}$$

$$\begin{aligned} &\left|\left(\widehat{S}(r)\right)'(t) - \left(\Gamma\left(\widehat{J}\right)\right)'(t)\right| \le \int_{t_1}^{t_2} |G_t(t, s)|\left|K_1\left(s, r(s), r'(s)\right) - K_2\left(s, \widehat{J}(s), y'(s)\right)\right| \\ &\quad ds \le \partial_\infty\left(r, \widehat{J}\right) \int_{t_1}^{t_2} |G_t(t, s)| ds \le \frac{(t_2 - t_1)}{2} \partial_\infty\left(r, \widehat{J}\right). \end{aligned} \tag{117}$$

From (116) and (117), we easily obtain

$$\begin{aligned} \partial_\infty\left(\widehat{S}(r), \Gamma\left(\widehat{J}\right)\right) &\le \left(\gamma \frac{(t_2 - t_1)^2}{8} + \delta \frac{(t_2 - t_1)}{2}\right) \partial_\infty\left(r, \widehat{J}\right) \\ &< \frac{9}{10} \partial_\infty\left(r, \widehat{J}\right). \end{aligned} \tag{118}$$

It implies that

$$e^{\partial_\infty\left(S\wedge(r), \Gamma\left(\widehat{J}\right)\right)} < e^{9/10 \partial_\infty\left(r, \widehat{J}\right)} < e^{\partial_\infty\left(r, \widehat{J}\right)}. \tag{119}$$

Therefore,

$$\partial_\infty\left(\widehat{S}(r), \Gamma\left(\widehat{J}\right)\right) e^{\partial_\infty\left(\widehat{S}(r), \Gamma\left(\widehat{J}\right)\right)} < \frac{9}{10} \partial_\infty\left(r, \widehat{J}\right) e^{\partial_\infty\left(r, \widehat{J}\right)}. \tag{120}$$

Let $\Lambda, Y : (0, \infty) \longrightarrow (0, \infty)$ be defined by

$$\begin{aligned} \Lambda(t) &= t e^t, t > 0, \\ Y(t) &= \frac{9t}{10}, t > 0, \end{aligned} \tag{121}$$

respectively.

Thus, we have

$$\Lambda\left(\partial_\infty\left(\widehat{S}(r), \Gamma\left(\widehat{J}\right)\right)\right) \le Y\left(\Lambda\left(\partial_\infty\left(r, \widehat{J}\right)\right)\right) \le Y\left(\Lambda\left(M\left(r, \widehat{J}\right)\right)\right). \tag{122}$$

Therefore, by Corollary 28, $\widehat{S}$ and Γ have a common fixed point $r^* \in C$, that is, r^* is a common solution of (113) and (114). As an immediate consequence of Theorem 29, in the case of $\widehat{S} = \Gamma$, we find that the integral equation (105) has a solution in C, and hence, the second order nonlinear boundary value problem (104) has a solution.

5. Conclusion

In the present work, we introduced a new concept of fuzzy mappings in complete metric spaces. Also, we derived the existence of α-fuzzy common fixed points for two fuzzy mappings under generalized almost (Y, Λ)-contractions in complete metric spaces. We also gave an illustrative example to support our main results. We further showed some relations between multivalued mappings and fuzzy mappings, which can be utilized to ensure the existence of a common fixed point for multivalued mappings. Finally, we applied our main results to provide a solution for a second order nonlinear boundary value problem.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare no conflict of interest.

Authors' Contributions

All authors contributed equally in writing this article. All authors read and approved the final manuscript.

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