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Research Article

Hölder Regularity of Quasiminimizers to Generalized Orlicz Functional on the Heisenberg Group

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In this paper, we apply De Giorgi-Moser iteration to establish the Hölder regularity of quasiminimizers to generalized Orlicz functional on the Heisenberg group by using the Riesz potential, maximal function, Calderón-Zygmund decomposition, and covering Lemma on the context of the Heisenberg Group. The functional includes the p-Laplace functional on the Heisenberg group which has been studied and the variable exponential functional and the double phase growth functional on the Heisenberg group that have not been studied.

1. Introduction

In this paper, we concern the generalized Orlicz functional

$$\int_{\Omega} \Phi(x, |\nabla_H u|) dx. \tag{1}$$

on the Heisenberg group, where

$$\Phi(x, |\nabla_H u|) \sim \varphi(x, |\nabla_H u|) \in \Phi_w(\Omega) \tag{2}$$

(we say that $f \sim g$ if and only if there exist c_0 and $c_1 > 0$ such that $c_0 f \le g \le c_1 f$), $\nabla_H u = (X_1 u, X_2 u, \cdots, X_{2n} u)$, $\Phi_w(\Omega)$, is the generalized Orlicz space (see Section 2 for details). Write

$$\varphi_B^+(\tau) := \sup_{x\in B} \varphi(x,\tau) \text{ and } \varphi_B^-(\tau) := \inf_{x\in B} \varphi(x,\tau) \tag{3}$$

and abbreviate $\varphi^{\pm} = \varphi^{\pm}_{\Omega}$. Here, we assume that

(A1) There exists $\beta \in (0,1)$, such that $\varphi^+(\beta) \le 1 \le \varphi^-(1)$.

(A2) There exists $\beta \in (0,1)$, such that for every ball $B \subset \Omega$,

$$\varphi_B^+(\beta\tau) \le \varphi_B^-(\tau) \text{ for } \tau \in \left[1, (\varphi_B^-)^{-1}\left(\frac{1}{|B|}\right)\right]. \tag{4}$$

(A2)$'$ There exists $\beta \in (0,1)$, such that for every ball $B \subset \Omega$,

$$\varphi_B^+(\beta\tau) \le \varphi_B^-(\tau) \text{ for } \tau \in \left[1, \frac{1}{diamB}\right]. \tag{5}$$

(A3) There exist $\gamma^- > 1$ and $\lambda \ge 1$, such that $\varphi(x,\tau)/\tau^{\gamma^-}$ is λ-almost increasing (λ-almost increasing means that there exists a constant $\lambda \ge 1$ such that $\varphi(x,s)/s^{\gamma^-} \le \lambda(\varphi(x,\tau)/\tau^{\gamma^-})$ for all $s \le \tau$) with respect to $\tau > 0$.

(A4) There exist $\gamma^+ > 1$ and $\lambda \ge 1$, such that $\varphi(x,\tau)/\tau^{\gamma^+}$ is λ-almost decreasing with respect to $\tau > 0$.

If $\varphi(x,\tau)/\tau^{\gamma^-}$ is increasing, then assumption (A3) can be written by (A3)$'$. If $\varphi(x,\tau)/\tau^{\gamma^+}$ is decreasing, then assumption (A4) can be written by (A4)$'$. Note that $\gamma^+ \ge \gamma^-$, and that all these assumptions are invariant to equivalent generalized N-functions and scaling (see (100) below).

It is clear that the Euler-Lagrange equation corresponding to functional (1) is

$$div_H\left(\frac{\varphi'(x, |\nabla_H u|)}{|\nabla_H u|}\nabla_H u\right) = 0, \tag{6}$$

where $\varphi'(x,\tau)$ denotes the derivative of $\varphi(x,\tau)$ with respect to τ. If for any $v \in HW_0^{1,\varphi}(\Omega)$, it holds

$$\int_\Omega \frac{\varphi'(x,|\nabla_H u|)}{|\nabla_H u|}\nabla_H u\cdot\nabla_H v dx = 0, \qquad (7)$$

then we say that $u \in HW^{1,\varphi}(\Omega)$ is a weak solution of (6). When functional $\Phi(x,|\nabla_H u|)$ in (1) satisfies $\Phi(x,\tau) \sim (\Lambda+\tau^2)^{p-2/2}\tau$, the regularity of weak solutions to the corresponding Euler-Lagrange equation has been studied by many scholars. While $0<\Lambda<1$, for p not being far from 2, Manfredi and Mingione in [1] got the Hölder continuity of the ordinary gradient of weak solutions and derived smoothness of weak solutions by using the method in [2]; for $1<p<4$, the second order differentiability of weak solutions was deduced by Domokos in [3], which generalized the results in Marchi [4]. While $0\le\Lambda<1$, for p not being far from 2, Domokos and Manfredi in [5] used the Calderón-Zygmund theory on the Heisenberg group to study regularity of weak solutions; for $2\le p<4$, Mingione, Zatorska-Goldstein, and Zhong in [6] concluded the $C^{1,\alpha}$ regularity of weak solutions by using a double-bootstrap method, energy estimates, and interpolation inequalities; for $1<p<\infty$, Zhong in [7] got the $C^{1,\alpha}$ regularity of weak solutions by using the energy estimate, the Moser iteration, and the oscillation estimate; Zhang and Niu in [8] proved the Γ^α regularity of the gradient of weak solutions as $\Phi(x,\tau)\sim\varphi(\tau)$, where $\varphi(\tau)$ belongs to the Orlicz space including the function $\varphi(\tau)=(\Lambda+\tau^2)^{p-2/2}\tau$. We observe that $\Phi(x,\tau)$ discussed before depends only on the second variable τ. For the more general case depending on two variables x and τ, there is no relevant result on the Heisenberg group. In the Euclidean space, there are many results about the variable exponential case (i.e., $\Phi(x,\tau)\sim\tau^{p(x)}$), the (p,q)-growth case (i.e., $c_1(\tau^p-1)\le\Phi(x,\tau)\le c_2(\tau^q+1)$, $q>p$), and the double phase case (i.e., $\Phi(x,\tau)=\tau^p+a(x)\tau^q$), such as [9–14]. Harjulehto, Hästö, and Klen considered the functional $\int_\Omega\Phi(x,|\nabla u|)dx$ including the above three cases and proved the existence of quasiminimizers in [15]. Whereafter, Harjulehto, Hästö, and Toivanen in [16] obtained the Hölder regularity of quasiminimizers by using the De Giorgi-Moser iteration and some tools in harmonic analysis.

In this paper, we consider the Hölder regularity of the quasiminimizers of the functional (1) inspired by [16]. The main difference is that we need to use the Sobolev inequality, the Riesz potential, and the maximal function on the Heisenberg group. In addition, to derive the regularity, we prove a covering Lemma by the Calderón-Zygmund decomposition on the Heisenberg group.

Before stating the main results, we give the following definition.

Definition 1. Let $\varphi\in\Phi_w(\Omega)$ (the generalized orlicz space, see next section). We say that $u\in HW_{loc}^{1,\varphi}(\Omega)$ is a local quasiminimizer of (1), if there exists a constant $K\ge1$ such that

$$\int_{\{v\neq0\}}\varphi(x,|\nabla_H u|)dx\le K\int_{\{v\neq0\}}\varphi(x,|\nabla_H(u+v)|)dx, \qquad (8)$$

for any $v\in HW_{loc}^{1,\varphi}(\Omega)$ with $sptv\subset\Omega$.

Now, we state the main results:

Theorem 2 (Harnack inequality). *Assume that $\varphi\in\Phi_w(\Omega)$ satisfies (A1), (A2), (A2)$'$, (A3), and (A4), If $u\in HW_{loc}^{1,\varphi}(\Omega)$ is a nonnegative local quasiminimizer of (1), then for a compact set $K\subset\subset\Omega$, there exists R_0 as shown in Lemma 27 below such that*

$$\operatorname*{esssup}_{Q_R} u\le c\left(\operatorname*{ess\,inf}_{Q_R} u+R\right), \qquad (9)$$

for all $R\in(0,R_0)$ and cubes $Q_{6R}\subset\subset\Omega$ with centered in K, where c depends only on the parameters of (A1), (A2), (A2)$'$, (A3) and (A4), $\wp$ and $\|u\|_{L^\infty(Q_{2R})}$.

Theorem 3 (Hölder continuity). *Let $\varphi\in\Phi_w(\Omega)$ satisfy (A1), (A2), (A2)$'$, (A3), and (A4). If $u\in HW_{loc}^{1,\varphi}(\Omega)$ is a local quasiminimizer of (1), then $u\in C_{loc}^\alpha(\Omega)$ for some $\alpha>0$.*

This paper is organized as follows. In Section 2, we first give the definitions and related knowledge of the Heisenberg Group, then introduce the generalized N-function and its related properties. Some definitions of function spaces and some known Lemmas are given. In Section 3, we use the De Giorgi-Moser iteration to obtain the local boundedness of the quasiminimizer. As the result of the third section, when the radius approaches 0, the constant will blow up; so in Section 4, the upper bound of the solution is improved, but the solution is needed to bounded. In Section 5, on the basis of the results obtained in Section 4, we first prove a covering Lemma by the Calderón-Zygmund decomposition and then use it to obtain the Harnack inequality and Hölder continuity.

The common generalized Orlicz functions [see [15]] involves

$$\begin{aligned}\varphi_1(x,\tau)&=\tau^{p(x)}\log(1+\tau),\\ \varphi_2(x,\tau)&=\tau^p+a(x)\tau^q,\\ \varphi_3(x,\tau)&=\tau^{p(x)},\\ \varphi_4(x,\tau)&=\frac{1}{p(x)}\tau^{p(x)}.\end{aligned} \qquad (10)$$

Then, $\Phi(x,|\nabla_H u|)$ in (1) can have the concrete relations

$$\Phi(x,|\nabla_H u|)\sim\varphi_1(x,|\nabla_H u|),$$
$$\Phi(x,|\nabla_H u|)\sim\varphi_2(x,|\nabla_H u|)(\text{the double phase case}),$$
$$\Phi(x,|\nabla_H u|)\sim\varphi_3(x,|\nabla_H u|)(\text{the variable exponential case}),$$
$$\Phi(x,|\nabla_H u|)\sim\varphi_4(x,|\nabla_H u|)=\frac{1}{p(x)}|\nabla_H u|^{p(x)}. \tag{11}$$

In this paper, we always denote a positive constant by c which may vary from line to line, $x=(x_1,\cdots,x_{2n},t)=(x',t)$. We assume that $\Omega\subset\mathbb{H}^n$ is a bounded domain, Q is a cube whose side length is R in the x' direction, R^2 in the t direction, and its edge is parallel to the coordinate axis and denote $\operatorname{diam}Q:=((2n)^2+1)^{1/4}R>(2n)^{1/2}R$. Let cQ be a concentric cube whose side length is c times Q the in x' direction and c^2 times Q in the t direction. For $f\in L^1(\Omega)$, we denote $\langle f\rangle_\Omega=1/|\Omega|\int_\Omega f(x)dx$.

2. Preliminaries

In this section, we first recall the related knowledge of the Heisenberg Group, then introduce the definition of the generalized N-function and some properties related to it. Finally, some function spaces and lemmas are given.

2.1. Heisenberg Group $\mathbb{H}^n$. The Euclidean space $\mathbb{R}^{2n+1}$, $n\ge 1$ with the group multiplication

$$x\circ y=\left(x_1+y_1,x_2+y_2,\cdots,x_{2n}+y_{2n},t+s+\frac{1}{2}\sum_{i=1}^{n}(x_iy_{n+i}-x_{n+i}y_i)\right), \tag{12}$$

where $x=(x_1,x_2,\cdots,x_{2n},t)$, $y=(y_1,y_2,\cdots,y_{2n},s)\in\mathbb{R}^{2n+1}$ leads to the Heisenberg group $\mathbb{H}^n$. The scaling on $\mathbb{H}^n$ is defined as

$$\delta x=\left(\delta x_1,\delta x_2,\cdots,\delta x_{2n},\delta^2 t\right). \tag{13}$$

The left invariant vector fields on $\mathbb{H}^n$ are of the form

$$X_i=\partial_{x_i}-\frac{x_{n+i}}{2}\partial_t,\ X_{n+i}=\partial_{x_{n+i}}+\frac{x_i}{2}\partial_t,\ 1\le i\le n, \tag{14}$$

and a nontrivial commutator on $\mathbb{H}^n$ is

$$T=\partial_t=[X_i,X_{n+i}]=X_iX_{n+i}-X_{n+i}X_i,\ 1\le i\le n. \tag{15}$$

We call that $X_1,X_2,\cdots,X_{2n}$ are the horizontal vector fields on $\mathbb{H}^n$ and T the vertical vector field. Denote the horizontal gradient of a smooth function u on $\mathbb{H}^n$ by

$$\nabla_H u=(X_1u,X_2u,\cdots,X_{2n}u). \tag{16}$$

The homogeneous dimension of $\mathbb{H}^n$ is $\wp=2n+2$. The Haar measure in $\mathbb{H}^n$ is equivalent to the Lebesgue measure in $\mathbb{R}^{2n+1}$. We denote the Lebesgue measure of a measurable set $E\subset\mathbb{H}^n$ by $|E|$. The Carnot-Carathèodary metric (CC-metric) between two points in $\mathbb{H}^n$ is the shortest length of the horizontal curve joining them, denoted by d. The ball defined by the CC-metric is

$$B_R(x)=\{y\in\mathbb{H}^n: d(y,x)<R\}. \tag{17}$$

One has

$$|B_R(x)|=R^{\wp}|B_1(x)|. \tag{18}$$

For $x=(x_1,x_2,\cdots,x_{2n},t)$, its module is defined by

$$\|x\|_{\mathbb{H}^n}=\left(\left(\sum_{i=1}^{2n}x_i^2\right)^2+t^2\right)^{\frac{1}{4}}. \tag{19}$$

The CC-metric d is equivalent to the Korànyi metric

$$d(x,y)=\left\|x^{-1}y\right\|_{\mathbb{H}^n}. \tag{20}$$

2.2. Generalized N-Function and Its Related Properties

Definition 4 (generalized N-function). A real valued function $\varphi(x,\tau)$ defined on $\Omega\times[0,+\infty)$ is said to be a generalized N-function, and if $\varphi(x,\tau)$ is a Lebesgue measurable with respect to x, the derivative $\varphi'(x,\tau)$ *of* $\varphi(x,\tau)$ with respect to τ exists, and $\varphi'(x,\tau)$ is right continuous, nondecreasing, and satisfies $\varphi'(x,0)=0$ and $\varphi'(x,\tau)>0\ (\tau>0)$.

If for any $\tau\ge 0$ and $x\in\Omega$, there exists $L\ge 1$ such that

$$\psi\left(x,\frac{\tau}{L}\right)\le\varphi(x,\tau)\le\psi(x,L\tau), \tag{21}$$

then we say functions φ and ψ are equivalent denoted by $\varphi\simeq\psi$. If for any $\tau\ge 0$, there exists $c_1>0$ such that

$$\varphi(x,2\tau)\le c_1\varphi(x,\tau)\,; \tag{22}$$

then, we say that $\varphi(x,\tau)$ satisfies the strong Δ_2-condition and denotes the minimum constant c_1 by $\Delta_2(\varphi)$. Since

$$\varphi(x,\tau)\le\varphi(x,2\tau), \tag{23}$$

the strong Δ_2-condition is equivalent to $\varphi(x,\tau)\sim\varphi(x,2\tau)$. Obviously, if φ satisfies the strong Δ_2-condition, then $\sim\Leftrightarrow\simeq$. For a family of generalized N-functions, we define

$$\Delta_2(\{\varphi_\lambda\}):=\sup_\lambda\Delta_2(\varphi_\lambda). \tag{24}$$

Let

$$\left(\varphi'\right)^{-1}(x,\tau):=\sup\left\{v\in[0,+\infty):\varphi'(x,v)\le\tau\right\},\quad\text{for }\tau\ge 0. \tag{25}$$

If $\varphi'=\varphi'(x,\tau)$ is strictly increasing with respect to τ, then $(\varphi')^{-1}$ is the inverse function of φ'. Writing

$$\varphi^*(x,\tau) := \int_0^\tau \left(\varphi'\right)^{-1}(x,\vartheta)d\vartheta, \tau \geq 0, \tag{26}$$

then φ^* is also a generalized N-function and satisfies

$$(\varphi^*)'(x,\tau) = \left(\varphi'\right)^{-1}(x,\tau),\ \tau \geq 0. \tag{27}$$

Note that φ^* is the complementary function of φ and $(\varphi^*)^* = \varphi$. For any $\delta > 0$, there exists c_δ depending only on $\Delta_2(\{\varphi, \varphi^*\})$, such that

$$\tau v \leq \delta\varphi(x,\tau) + c_\delta\varphi^*(x,v) \tag{28}$$

for any $\tau, v \geq 0$, and this inequality is called Young's inequality ([16]). For some $a, b > 0$ and any $\tau \geq 0$, we denote

$$\rho(x,\tau) = a\varphi(x,b\tau), \tag{29}$$

then

$$\rho^*(x,\tau) = a\varphi^*\left(x, \frac{\tau}{ab}\right). \tag{30}$$

If φ and ρ are generalized N-functions and satisfy $\varphi(x,\tau) \leq \rho(x,\tau)$ for $\tau \geq 0$, then for any $\tau \geq 0$, it holds

$$\rho^*(x,\tau) \leq \varphi^*(x,\tau). \tag{31}$$

2.3. Some Function Spaces and Lemmas. We denote the real valued measurable function space by $L^0(\Omega)$. If the generalized N-function $\varphi(x,\tau)$ satisfies the strong Δ_2-condition on $\mathbb{H}^n$, then

$$L^\varphi(\mathbb{H}^n) := \left\{ f \in L^0(\mathbb{H}^n) : \int_{\mathbb{H}^n} \varphi(x, |f(x)|)dx < \infty \right\} \tag{32}$$

is a Banach space with (Luxemburg) norm

$$\|f\|_{L^\varphi(\mathbb{H}^n)} := \inf\left\{ \lambda > 0 : \int_{\mathbb{H}^n} \varphi\left(x, \frac{|f(x)|}{\lambda}\right)dx \leq 1 \right\}. \tag{33}$$

We call that it is a generalized Orlicz space or Musielak-Orlicz space denoting by $\Phi_w(\mathbb{H}^n)$. For $\Omega \subset \mathbb{H}^n$, the generalized Orlicz-Sobolev space $HW^{1,\varphi}(\Omega)$ is defined as

$$HW^{1,\varphi}(\Omega) := \{u : u, \nabla_H u \in L^\varphi(\Omega)\}, \tag{34}$$

and the local generalized Orlicz-Sobolev space $HW^{1,\varphi}_{\text{loc}}(\Omega)$ as

$$HW^{1,\varphi}_{\text{loc}}(\Omega) := \left\{ u \mid u \in HW^{1,\varphi}\left(\Omega'\right), \text{for any } \Omega' \subset\subset \Omega \right\}. \tag{35}$$

The space $HW_0{}^{1,\varphi}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $HW^{1,\varphi}(\Omega)$.

If $\Omega \subset \mathbb{H}^n$ is bounded and $\varphi \in \Phi_w(\Omega)$ satisfies the assumptions (A1) and (A3), then $L^\varphi(\Omega) \hookrightarrow L^{\gamma^-}(\Omega)$, $HW^{1,\varphi}(\Omega) \hookrightarrow HW^{1,\gamma^-}(\Omega)$, $HW_0^{1,\varphi}(\Omega) \hookrightarrow HW_0^{1,\gamma^-}(\Omega)$. For their proofs, one can refer to Lemmas 4.4, 6.2, and 6.9 in [15] with some evident changes.

We now describe their proofs (Lemmas 5–11) that are similar to ones in [16] with some suitable revisions.

Lemma 5. *If $\varphi \in \Phi_w(\Omega)$, then there exists a generalized N-function ψ increasing strictly such that $\psi \sim \varphi$, and so, ψ is a bijection.*

Lemma 6. *Let $\varphi \in \Phi_w(\Omega)$, then*

(1) The strong Δ_2-condition is equivalent to (A4).

(2) If $\varphi(x,\tau)$ is convex with respect to τ, then the strong Δ_2-condition is equivalent to (A4)$'$.

Lemma 7. *The assumption (A1) implies $\varphi^-(x,1) \sim 1$.*

Lemma 8. *Let $\varphi \in \Phi_w(\Omega)$ be a bijection with respect to τ or satisfy the strong Δ_2-condition. Then,*

(1) The assumption (A4) is equivalent to that $\varphi^{-1}(x,\tau)/\tau^{1/\gamma^+}$ is almost increasing uniformally in Ω

(2) The assumption (A3) is equivalent to that $\varphi^{-1}(x,\tau)/\tau^{1/\gamma^-}$ is almost decreasing uniformally in Ω

If $\gamma^+ \leq \wp$, then we can get (A2)$'$ from (A1), (A2), and (A4).

Lemma 9. *Let $\varphi \in \Phi_w(\Omega)$ satisfy the assumptions (A1), (A2), and (A4). If $\gamma^+ \leq \wp$, then $\varphi(x,\tau)$ satisfies (A2)$'$.*

The proof of Lemma 9 is similar to that of Lemma 12 in [16], and a simple distinct is that we should use the fact $|B_R| = R^\wp |B_1|$ on the Heisenberg group.

Lemma 10. *Let $\varphi \in \Phi_w(\Omega)$ satisfy the assumption (A2) or (A2)$'$. Then, there exists $\beta \in (0,1)$ such that for any $\sqrt{\wp}Q \subset \Omega$, we have*

$$\varphi^+_Q(\beta\tau) \leq \varphi^-_Q(\tau) \tag{36}$$

for $\tau \in [1, M]$, where $M := (\varphi^-_Q)^{-1}(1/|\sqrt{\wp}Q|)$ under (A2) and $M := 1/\sqrt{\wp}\text{diam}Q$ under (A2)$'$.

The proof of Lemma 10 is similar to that of Lemma 13 in [16], but we should employ the statement in the process that on the Heisenberg Group, if B is the smallest ball containing Q, then $Q \subset B \subset \sqrt{\wp}Q$.

Lemma 11. *Let $\varphi \in \Phi_w(\Omega)$ satisfy the assumption (A2) and be a bijection. Then, there exists $\beta \in (0,1)$ such that for any cube Q with $\sqrt{\wp}Q \subset \Omega$ and $|\sqrt{\wp}Q| \leq 1$, we have*

$$\beta\varphi^{-1}(y,\tau) \le \varphi^{-1}(x,\tau) \tag{37}$$

for any $\tau \in [1, (1/|\sqrt{\wp}Q|)]$ *and* $x, y \in Q$.

For the following lemma, one can refer to [17].

Lemma 12 [17]. *If* $\varphi \in \Phi_w(\Omega)$, *then* φ_B^- *satisfies the Jensen type inequality*

$$\varphi_B^-\left(\frac{1}{2|B|}\int_B f\,dx\right) \le \frac{1}{|B|}\int_B \varphi_B^-(f)\,dx. \tag{38}$$

In the generalized Orlicz space, the Hölder inequality with the constant 2 holds, see ([13], Lemma 9). It is stated that for any $f, g \in L^{\varphi}(\Omega)$, it follows

$$\int_\Omega |fg|\,dx \le 2\|f\|_{\varphi}\|g\|_{\varphi^*}. \tag{39}$$

Because the Heisenberg Group is a special case of Carnot groups, the conclusions on Carnot groups are also true on $\mathbb{H}^n$. We write some conclusions in monograph ([18], p276-280) on $\mathbb{H}^n$. For $0 < \alpha < \wp$, $f : \mathbb{H}^n \to \mathbb{R}$, we formally define the Riesz potential operator I_α as

$$I_\alpha(f)(x) = \int_\Omega \frac{f(y)}{(d(x,y))^{\wp-\alpha}}\,dy, \tag{40}$$

where $d(x, y)$ denotes $d(y^{-1}\circ x)$. We also call that I_α is the fractional integral of order α, and I_1 is abreviated to I.

Lemma 13 (Hardy-Littlewood-Sobolev inequality, [18]). *Let* $1 < \alpha < \wp$, $1 < p < \wp/\alpha$, $q > p$, *and*

$$\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{\wp}. \tag{41}$$

Then, there exists a positive constant $c = c(\alpha, p)$ *such that for every* $f \in L^p(\mathbb{H}^n)$, *we have*

$$\|I_\alpha(f)\|_q \le c\|f\|_p. \tag{42}$$

For a function $f \in L^p(\mathbb{H}^n, \mathbb{C})$, $1 < p < \infty$, *we define the maximal function as*

$$M(f)(x) := \sup_{r>0} \frac{1}{B(x,r)}\int_{B(x,r)} |f(y)|\,dy,\ x \in \mathbb{H}^n. \tag{43}$$

One has the statement (maximal function theorem): if $1 < p < \infty$, *then there exists a positive constant* $c = c(p)$ *such that for every* $f \in L^p(\mathbb{H}^n, \mathbb{C})$, *we have (see [18])*

$$\|M(f)\|_p \le c\|f\|_p. \tag{44}$$

Lemma 14 (Sobolev-Stein embedding, (, p280)). *Let* $1 < p < \wp$. *Then, there exists a constant* $c = c(p)$ *such that*

$$\|u\|_q \le c\|\nabla_H u\|_p \tag{45}$$

for every $u \in C_0^\infty(\mathbb{H}^n, \mathbb{R})$, *where*

$$\frac{1}{q} = \frac{1}{p} - \frac{1}{\wp}\quad \left(i.e.\, q = \frac{\wp p}{\wp - p}\right). \tag{46}$$

Proof. For $u \in C_0^\infty(\mathbb{H}^n, \mathbb{R})$, the representation formula (5.16) in [18] yields

$$u(x) = -\int_{\mathbb{H}^n} \Gamma(x^{-1}\circ y)Lu(y)\,dy, \tag{47}$$

where $L = \sum_{j=1}^{2n} X_j^2$, $X_j^* = -X_j$. By the integrating by parts, we get

$$u(x) = \int_{\mathbb{H}^n} (\nabla_H\Gamma)(x^{-1}\circ y)\nabla_H u(y)\,dy. \tag{48}$$

In addition, out of the origin, one sees

$$\nabla_H\Gamma = \beta_d\nabla_H(d^{2-\wp}) = (2-\wp)\beta_d d^{1-\wp}\nabla_H d. \tag{49}$$

Because $\nabla_H d$ is smooth in $\mathbb{H}^n \setminus \{0\}$ and $|\nabla_H d| < 1$, we obtain that for a constant $c > 0$,

$$|\nabla_H\Gamma| \le c d^{1-\wp}. \tag{50}$$

Using (48), it yields

$$|u(x)| \le c\int_{\mathbb{H}^n} |\nabla_H u(y)|d(x,y)^{1-\wp}\,dy = cI_1(|\nabla_H u|)(x). \tag{51}$$

Then, by Lemma 13, we gain

$$\|u\|_q \le c\|I_1(|\nabla_H u|)\|_q \le c\|\nabla_H u\|_p, \tag{52}$$

where

$$\frac{1}{q} = \frac{1}{p} - \frac{1}{\wp}\quad \left(i.e.\, q = \frac{\wp p}{\wp - p}\right). \tag{53}$$

This ends the proof.

Noting that $C_0^\infty(B_r)$ is dense in $HW_0^{1,q}(B_r)$, we have the following result from Lemma 14.

Lemma 15 (Sobolev inequality). *Let* $1 \le q < \wp = 2n + 2$, $B_r \subset \mathbb{H}^n$. *For any* $u \in HW_0^{1,q}(B_r)$, *it follows*

$$\left(\frac{1}{|B_r|}\int_{B_r} |u|^{\frac{\wp q}{\wp - q}}\,dx\right)^{\frac{\wp - q}{\wp q}} \le cr\left(\frac{1}{|B_r|}\int_{B_r} |\nabla_H u|^q\,dx\right)^{\frac{1}{q}}, \tag{54}$$

where $c = c(\wp, q) > 0$.

Since $C^\infty(\Omega)$ *is not dense in* $HW^{1,\varphi}(\Omega)$, *we need to prove.*

Lemma 16. *Let $\Omega \subset \mathbb{H}^n$, $\varphi \in \Phi_w(\Omega)$, satisfy (A1), (A2), and (A4). If $v \in HW^{1,\varphi}(\Omega)$ with $sptv \subset \Omega$, then $v \in HW_0^{1,\varphi}(\Omega)$.*

Proof. Because $sptv \subset \Omega$ and $sptv$ are closed, we can find $\Omega' \subset \Omega$ which is bounded, quasiconvex, and contains $sptv$. Values outside Ω' do not affect the claim, so we verify the claim in Ω'. In order to simplify the notation, we might assume that Ω is bounded and quasiconvex . Owing to

$$\nabla_H(v * \eta) = (\nabla_H v) * \eta, \text{for } \eta \in C_0^\infty(\Omega) \tag{55}$$

similarly to the proofs of theorems 6.6 and 6.5 in [15], we know that $C^\infty(\Omega) \cap HW^{1,\varphi}(\Omega)$ is dense in $HW^{1,\varphi}(\Omega)$; then there is a sequence of $\{v_i\} \in C^\infty(\Omega) \cap HW^{1,\varphi}(\Omega)$ convergencing to v. We take a cut-off function $\eta \in C_0^\infty(\Omega)$ with $0 \le \eta \le 1$ and $\eta \equiv 1$ in $sptv$ and see $\{\eta v_i\} \in C_0^\infty(\Omega)$. Since

$$\|v - \eta v_i\|_{HW^{1,\varphi}(\Omega)} = \|\eta(v - v_i)\|_{HW^{1,\varphi}(\Omega)} \le \|v - v_i\|_{HW^{1,\varphi}(\Omega)} \longrightarrow 0, \tag{56}$$

it follows

$$v \in HW_0^{1,\varphi}(\Omega). \tag{57}$$

Lemma 17. *Let $Q \subset \mathbb{H}^n$ with $|Q|$ being finite. If $\varphi \in \Phi_w(\Omega)$, then for all $v \in HW_0^{1,\varphi}(\Omega)$, we have*

$$\int_Q \varphi(|v|)^{\frac{\wp}{\wp-1}} dx \le c\left(\int_Q \varphi(|\nabla_H v|)dx\right)^{\frac{\wp}{\wp-1}}, \tag{58}$$

where c depends only on $\wp$ and $|Q|$.

The proof is similar to the proof of Lemma 24 in [16], and it only needs to change the classical Sobolev inequality in the Euclidean space into the Sobolev inequality (54) on $\mathbb{H}^n$.

Lemma 18. *Suppose that $\varphi \in \Phi_w(3Q)$ satisfies (A1), (A2), and (A3), then there exists $\beta \in (0,1)$ such that for all $f_1 \in L^\varphi(3Q)$ with*

$$\|f_1\|_{L^\varphi(3Q)} \le 1,\ sptf_1 \subset Q \times \{\{0\} \cup [1,\infty)\}, \tag{59}$$

we have

$$\int_Q \varphi(x, \beta M f_1)dx \le c\int_Q \varphi(x, f_1)dx, \tag{60}$$

where c depends only on $\wp$ and the parameters of (A1), (A2), and (A3).

The proof is similar to the proof of Lemma 23 in [16].

3. Local Boundedness

Unless otherwise specified, we will use the following notations. Suppose that $0 \in \Omega \subset \mathbb{H}^n$, $0 < R < R_0 \le 1/2$,

$$Q_R \coloneqq Q(0,R) \tag{61}$$

is a cube whose side length is R in the x' direction, R^2 in the t direction, and its edge is parallel to the coordinate axis with centered 0 and denotes $\operatorname{diam} Q \coloneqq ((2n)^2+1)^{1/4} R > (2n)^{1/2} R$,

$$\begin{aligned} A_{k,R}^+ &\coloneqq Q_R \cap \{u > k\}, \\ u^+ &\coloneqq \max\{u, 0\}. \end{aligned} \tag{62}$$

Lemma 19 [16]. *Suppose that F is a bounded nonnegative function in $[r,R]$ and W satisfies the strong Δ_2 -condition in $[0,\infty)$, if there exists $\theta \in [0,1)$ such that for any $r \le \iota < s \le R$,*

$$F(\iota) \le W\left(\frac{1}{s-\iota}\right) + \theta F(s), \tag{63}$$

then we have

$$F(r) \le cW\left(\frac{1}{R-r}\right), \tag{64}$$

where c depends only on θ and $\Delta_2(W)$.

Lemma 20 (Caccioppoli inequality). *Let $\varphi \in \Phi_w(\Omega)$ and $u \in HW_{loc}^{1,\varphi}(\Omega)$ be a local quasiminimizer of (1). Then for all $k \in \mathbb{R}$, there holds*

$$\int_{A_{k,r}^+} \varphi\left(x, \left|\nabla_H (u-k)^+\right|\right)dx \le c\int_{A_{k,R}^+} \varphi\left(x, \frac{u-k}{R-r}\right)dx, \tag{65}$$

where c depends on K in definition 1 and R.

The proof is similar to Lemma 27 in [16].

Lemma 21. *Suppose that $\varphi \in \Phi_w(\Omega)$ satisfies (A1), (A2), (A3), and (A4) and define*

$$\bar{\varphi}(x,\tau) \coloneqq \varphi_{Q_R}^-(\tau)\chi_{[0,1)}(\tau) + \varphi(x,\tau)\chi_{[1,\infty)}(\tau). \tag{66}$$

If $u \in HW_{loc}^{1,\varphi}(\Omega)$ satisfies (65), $R/2 \le \iota < s \le R$, $Q_{3R} \subset \Omega$, then there exists R_0 such that

$$\int_{Q_\iota} \bar{\varphi}(x,(u-k)^+)dx \le c\left(\int_{Q_s} \frac{\bar{\varphi}(x,(u-h)^+)}{\bar{\varphi}(x,k-h)}dx\right)^\alpha \int_{Q_s} \bar{\varphi}\left(x, \frac{(u-h)^+}{s-\iota}\right)dx \tag{67}$$

as $R \in (0,R_0]$, $k-h \ge s-\iota$, where $\alpha \coloneqq \gamma^-/\wp^2\gamma^+ - \wp(\gamma^+ - \gamma^-)$, c, depends only on the parameters of (A1), (A2), (A3), and (A4), $\Delta_2(\varphi)$ and $\wp$. Here, R_0 satisfies $\int_{3Q_{R_0}} \varphi(x, |\nabla_H u|)dx \le 1$.

Though the proof is similar to Lemma 4.6 in [16], we need to replace the results about the Riesz potential and maximal function with our Lemma 13 and (51). For completeness, let us write the detailed proof.

Proof. Let $h < k$, $Q := Q_{\iota+s/2}$ $(\iota < s)$, $\eta \in C_0^{\infty}(Q)$, satisfy

$$0 \le \eta \le 1,\ |\nabla_H \eta| \le \frac{4}{s-\iota} \text{ and } \eta = 1 \text{ in } Q_\iota. \tag{68}$$

Denoting

$$v := (u-k)^+\eta,\ A := \{v > 1\} \cap Q,\ B := \{v \le 1\} \cap Q, \text{and } D := \{u > k\} \cap Q, \tag{69}$$

then we have

$$\int_{Q_\iota} \bar{\varphi}(x, (u-k)^+)dx \le \int_Q \bar{\varphi}(x, (u-k)^+\eta)dx = \int_A \bar{\varphi}(x, v)dx + \int_B \bar{\varphi}(x, v)dx. \tag{70}$$

We first estimate the integral $\int_A \bar{\varphi}(x, v)dx$. If $\tau > 1$, then from (A3), (A4), and Lemma 7,

$$\begin{aligned}\bar{\varphi}\left(x, \tau^{\wp/\wp-1}\right) &\ge c\tau^{\gamma^-/\wp-1}\bar{\varphi}(x,\tau) \ge c\left(\frac{\bar{\varphi}(x,\tau)}{\bar{\varphi}(x,1)}\right)^{\gamma^-/(\wp-1)\gamma^+}\bar{\varphi}(x,\tau)\\ &\sim \bar{\varphi}(x,\tau)^{1+\gamma^-/(\wp-1)\gamma^+} := \bar{\varphi}(x,\tau)^{1+\varepsilon},\end{aligned} \tag{71}$$

so

$$\bar{\varphi}(x,\tau) \le c\left(\bar{\varphi}\left(x,\tau^{\wp/\wp-1}\right)\right)^{1/1+\varepsilon},\ \varepsilon = \gamma - /(\wp-1)\gamma^+. \tag{72}$$

We use it and the Hölder inequality to gain

$$\begin{aligned}\int_A \bar{\varphi}(x,v)dx &\le c\int_A \left(\bar{\varphi}\left(x, v^{\wp/\wp-1}\right)\right)^{1/1+\varepsilon}dx \le c|A|^{\varepsilon/1+\varepsilon}\\ &\quad\cdot\left(\int_A \bar{\varphi}\left(x, v^{\wp/\wp-1}\right)dx\right)^{1/1+\varepsilon}.\end{aligned} \tag{73}$$

Denoting $f_1 := |\nabla_H v|\chi_{\{|\nabla_H v|>1\}}$, then we know that from (51),

$$v \le c_1 I(\nabla_H v) \le c_1 I\left(|\nabla_H v|\chi_{\{|\nabla_H v \le 1|\}}\right) + c_1 I(f_1). \tag{74}$$

If R_0 is small enough, then

$$I\left(|\nabla_H v|\chi_{\{|\nabla_H v \le 1|\}}\right) \le I\left(\chi_Q\right) \le cR_0 \le \frac{1}{2c_1}. \tag{75}$$

Because of $v > 1$ in the set A, it yields from (74) and (75) that

$$v \le \frac{1}{2} + c_1 I(f_1) \le \frac{v}{2} + c_1 I(f_1). \tag{76}$$

Therefore, in the set A, it implies

$$v \le 2c_1 I(f_1). \tag{77}$$

Suppose that R_0 is small enough such that $|3Q_{R_0}| \le 1$, then from Lemma 13 and $\|f_1\|_{L^\varphi(3Q)} \le cMf_1$, we know

$$(I(f_1))^{\wp/\wp-1} \le cMf_1. \tag{78}$$

Noting

$$\|f_1\|_{L^\varphi(\mathbb{H}^n)} = \|f_1\|_{L^\varphi(3Q)} \le \|\nabla_H u\|_{L^\varphi(3Q)} \le 1, \tag{79}$$

the conditions of Lemma 18 are satisfied. Now, we combine (77), (78), the strong Δ_2-condition and Lemma 18 to obtain

$$\begin{aligned}\int_A \bar{\varphi}\left(x, v^{\wp/\wp-1}\right)dx &\le c\int_A \bar{\varphi}\left(x, (I(f_1))^{\wp/\wp-1}\right)dx\\ &\le c\int_A \bar{\varphi}(x, Mf_1)dx \le c\int_A \bar{\varphi}(x, |\nabla_H v|)dx.\end{aligned} \tag{80}$$

Denoting

$$\tilde{\varphi}(\tau) := \varphi_{Q_R}^-(\tau)\chi_{[0,1)}(\tau) + \tau^{\gamma^-}\varphi_{Q_R}^+(1)\chi_{[1,\infty)}(\tau), \tag{81}$$

we see from (A3) that for every x,

$$\begin{aligned}\tilde{\varphi}(\tau) &\le \varphi_{Q_R}^-(\tau)\chi_{[0,1]}(\tau) + \frac{\varphi(x,\tau)}{\varphi(x,1)}\varphi_{Q_R}^+(1)\chi_{(1,\infty)}(\tau)\\ &\le \varphi_{Q_R}^-(\tau)\chi_{[0,1]}(\tau) + \varphi(x,\tau)\chi_{(1,\infty)}(\tau) = \bar{\varphi}(x,\tau).\end{aligned} \tag{82}$$

Let us estimate the measure of A. When $v \in A$, we have $\chi_A(v) = 1$,

$$\varphi^{\sim}(v)^{\wp/\wp-1} = \left(v^{\gamma^-}\varphi_{Q_R}^+(1)\right)^{\wp/\wp-1} \ge (\varphi(x,v))^{\wp/\wp-1} \ge 1, \tag{83}$$

and deduce from Lemmas 16, 17, and (82) that

$$\begin{aligned}|A| &\le \int_Q \varphi^{\sim}(v)^{\wp/\wp-1}dx \le c\left(\int_Q \varphi^{\sim}(|\nabla_H v|)dx\right)^{\wp/\wp-1}\\ &\le c\left(\int_Q \bar{\varphi}(x, |\nabla_H v|)dx\right)^{\wp/\wp-1}.\end{aligned} \tag{84}$$

On the other hand, $|A| \le |D|$, so

$$|A|^{\varepsilon/1+\varepsilon} \le |D|^{\varepsilon/\wp(1+\varepsilon)}|A|^{\varepsilon(\wp-1)/\wp(1+\varepsilon)}. \tag{85}$$

Combining (73), (80), (85), (84), and $\bar{\varphi} \le \varphi$, we know

$$\int_A \bar{\varphi}(x, v)dx \begin{array}{l} \le c|D|^{\varepsilon/\wp(1+\varepsilon)}|A|^{\varepsilon(\wp-1)/\wp(1+\varepsilon)}\left(\int_A \bar{\varphi}(x, |\nabla_H v|)dx\right)^{1/1+\varepsilon} \\ \le c|D|^{\alpha}\int_Q \varphi(x, |\nabla_H v|)dx, \end{array} \quad (86)$$

where $\alpha = \varepsilon/\wp(1+\varepsilon)$.

Next, we estimate the integral $\int_B \bar{\varphi}(x, v)dx$. Observing that $\bar{\varphi} = \tilde{\varphi}$ and they do not depend on x in the set B, the usual Hölder inequality yields

$$\int_B \bar{\varphi}(x, v)dx \le \int_D \tilde{\varphi}(v)dx \le |D|^{1/\wp}\left(\left(\int_Q \varphi^{\sim}(v)^{\wp/\wp-1}dx\right)^{\wp-1/\wp}\right). \quad (87)$$

Because of $|D| \le |Q| \le 1$, $\alpha = \varepsilon/\wp(1+\varepsilon) < 1/\wp$, one has $|D|^{1/\wp} \le |D|^{\alpha}$. Using Lemmas 16, 17, and $c\tilde{\varphi} \le \bar{\varphi} \le \varphi$, it deduces

$$\int_B \bar{\varphi}(x, v)dx \le |D|^{\alpha}\int_Q \varphi(x, |\nabla_H v|)dx. \quad (88)$$

In conclusion, the integrals $\int_A \bar{\varphi}(x, v)dx$ and $\int_B \bar{\varphi}(x, v)dx$ have same upper bound. At present, we estimate the integral $\int_Q \varphi(x, |\nabla_H v|)dx$. Using the

$$|\nabla_H v| \le \left|\nabla_H (u-k)^+\right| + |\nabla_H \eta|(u-k)^+, |\nabla_H \eta| \le \frac{4}{s-\iota}, \quad (89)$$

strong Δ_2-condition and (65), we obtain

$$\int_Q \varphi(x, |\nabla_H v|)dx \le c\left(\int_Q \varphi\left(x, \left|\nabla_H (u-k)^+\right|\right)dx + \int_D \varphi \cdot \left(x, \frac{(u-k)^+}{s-\iota}\right)dx\right) \le c\int_{Q_s} \varphi\left(x, \frac{(u-k)^+}{s-\iota}\right)dx. \quad (90)$$

Note that $u > k$ in D, so $u-h > k-h \ge s-\iota$. Thus,

$$\varphi\left(x, \frac{(u-k)^+}{s-\iota}\right) \le \varphi\left(x, \frac{(u-h)^+}{s-\iota}\right)\chi_D \le \bar{\varphi}\left(x, \frac{(u-h)^+}{s-\iota}\right). \quad (91)$$

Since $\bar{\varphi}$ is increasing, it follows

$$|D| = \int_{Q_s} \chi_D dx \le \int_{Q_s} \frac{\bar{\varphi}\left(x, (u-h)^+\right)}{\bar{\varphi}\left(x, (k-h)^+\right)}dx. \quad (92)$$

Combining (86)–(92), we get

$$\int_Q \bar{\varphi}(x, v)dx \le c\left(\int_{Q_s} \frac{\bar{\varphi}\left(x, (u-h)^+\right)}{\bar{\varphi}\left(x, (k-h)^+\right)}dx\right)^{\alpha}\int_{Q_s} \bar{\varphi}\left(x, \frac{(u-h)^+}{s-\iota}\right)dx. \quad (93)$$

Returning to (70), it reaches (67).

Lemma 22 [16]. *Let $\alpha > 0$ and $\{\Theta_i\}$ be a sequence of real numbers satisfying*

$$\Theta_{i+1} \le D\ell^i \Theta_i^{1+\alpha} \quad (94)$$

for $D > 0$, $\ell > 1$. If

$$\Theta_0 \le D^{-\frac{1}{\alpha}}\ell^{-\frac{1}{\alpha^2}}, \quad (95)$$

then $\Theta_i \to 0$ as $i \to \infty$.

Lemma 23. *Let $\varphi \in \Phi_w(\Omega)$ satisfy (A1), (A2), (A3), and (A4). If $u \in HW^{1,\varphi}_{loc}(\Omega)$ satisfies (65) and $R_0 \in (0, 1)$ as shown in Lemma 21, then for $0 < R \le R_0$ and any $k_0 \in \mathbb{R}$, we have*

$$\operatorname*{esssup}_{Q_{R/2}} u \le k_0 + 1 + cR^{-\frac{\gamma^+}{\alpha\gamma^-}}\left(\int_{Q_R} \varphi\left(x, (u-k_0)^+\right)dx\right)^{1/\gamma^-}, \quad (96)$$

where α as shown in Lemma 21, c depends only on the parameters of (A1), (A2), (A3), and (A4), $\Delta_2(\varphi)$ and $\wp$.

The proof is similar to Theorem 4.11 in [16].

We use (96) for u and $-u$ to immediately obtain the following.

Theorem 24 (local boundedness). *Let $\varphi \in \Phi_w(\Omega)$ satisfy (A1), (A2), (A3), and (A4), then every local quasiminimizer of (1) is locally bounded.*

4. Improvement of the Upper Bound of Bounded Solutions

Lemma 25. *Let $\varphi \in \Phi_w(\Omega)$ satisfy (A1) and (A4) and define*

$$\tilde{\varphi}(x, \tau) := \tau^{\gamma^+}\varphi^-(1)\chi_{[0,1)}(\tau) + \varphi(x, \tau)\chi_{[1,\infty)}(\tau). \quad (97)$$

If $u \in HW^{1,\varphi}_{loc}(\Omega)$ satisfies (65), $R/2 \le \iota < s \le R$, $Q_R \subset \Omega$, then we have

$$\int_{Q_\iota} \tilde{\varphi}^-_{Q_R}\left((u-k)^+\right)dx \le c\left(\int_{Q_s} \frac{\tilde{\varphi}\left(x, (u-h)^+\right)}{\tilde{\varphi}(x, k-h)}dx\right)^{\frac{1}{\wp}}\int_{Q_s} \tilde{\varphi}\left(x, \frac{(u-h)^+}{s-\iota}\right)dx \quad (98)$$

as $k-h \ge s-\iota$, where c depends only on the parameters of (A1) and (A4), $\Delta_2(\varphi)$ and $\wp$.

The proof is similar to the proof of Lemma 31 in [16], and a necessary change is to replace the Sobolev inequality in the Euclidean space to the inequality in Lemma 17.

For $s > 0$, define

$$u_s(x) := \frac{u(sx)}{s} = \frac{u(sx_1, \cdots, sx_{2n}, s^2 t)}{s}, \quad (99)$$

$$\varphi_s(x, f(x)) := \varphi(sx, f(x)). \quad (100)$$

Note that

$$\nabla_H(u_s(x)) = \nabla_H u(sx), \Delta_2(\varphi_s) = \Delta_2(\varphi) \quad (101)$$

and the constants of (A2) and (A2)$'$ will be changed under this scaling (100).

Lemma 26. *Let $\varphi \in \Phi_w(\Omega)$. If $u \in HW^{1,\varphi}_{loc}(\Omega)$ is a local quasiminimizer of (1) in Q_{sR} with $s \in (0, 1]$, $Q_{sR} \subset\subset \Omega$, then u_s is a local quasiminimizer of the functional $\int \varphi_s(x, |\nabla_H u|)dx$ in Q_R.*

Proof. Suppose $v_s \in HW^{1,\varphi}(Q_R)$ and $sptv_s \subset Q_R$, obviously, we have

$$\{v \neq 0\} = s\{v_s \neq 0\} \subset Q_{sR}. \quad (102)$$

Using the transformation of variable and the quasiminimality of u in Q_{sR}, it implies

$$\begin{aligned} \int_{\{v_s \neq 0\}} \varphi_s(x, |\nabla_H u_s(x)|)dx &= \int_{\{v_s \neq 0\}} \varphi_s(x, |\nabla_H u(sx)|)dx \\ &= s^{-\wp} \int_{\{v \neq 0\}} \varphi(z, |\nabla_H u(z)|)dz \leq K s^{-\wp} \int_{\{v \neq 0\}} \varphi \\ &\quad \cdot (z, |\nabla_H (u + v)(z)|)dz \\ &= K \int_{\{v_s \neq 0\}} \varphi_s(x, |\nabla_H (u_s + v_s)(x)|)dx. \end{aligned} \quad (103)$$

Lemma 27. *Let $\varphi \in \Phi_w(\Omega)$ satisfy (A1), (A2)$'$, and (A4). If $u \in HW^{1,\varphi}_{loc}(\Omega)$ is a bounded local quasiminimizer of (1) in Q_{sR}, then for $0 < R < R_0 < 1/3\sqrt{2n\wp}\|u\|_\infty$, $s \in (0, 1]$, $Q_{sR} \subset\subset \Omega$, $k_0 > -2\|u_s\|_{L^\infty(Q_R)}$, we have*

$$\operatorname*{esssup}_{Q_{R/2}} u_s - k_0 \leq c \left(\left(\int_{Q_R} \left((u_s - k_0)^+\right)^{\gamma^+} dx \right)^{1/\gamma^+} + R \right), \quad (104)$$

where R_0 depends only on $\wp$ and $\|u\|_{L^\infty(Q_{sR})}$, c depends only on the parameters of (A1), (A2)$'$, and (A4), $\wp$, R, $\Delta_2(\varphi)$, and $\|u\|_{L^\infty(Q_{sR})}$ does not depend on s.

Proof. The proof follows the line proving proposition 28 in [16]. Writing

$$\tilde{\varphi}(x, \tau) := \tau^{\gamma^+} \varphi^-(1) \chi_{[0,1)}(\tau) + \varphi(x, \tau) \chi_{[1,\infty)}(\tau). \quad (105)$$

Assuming $\tau_0 \in [0, (3\|u\|_\infty / s)]$, we claim

$$(\tilde{\varphi}_s)^+_{Q_R}(\beta\tau_0) \leq (\tilde{\varphi}_s)^-_{Q_R}(\tau_0) \Leftrightarrow \tilde{\varphi}^+_{Q_{sR}}(\beta\tau_0) \leq \tilde{\varphi}^-_{Q_{sR}}(\tau_0). \quad (106)$$

In fact, obviously, it holds

$$\begin{aligned} (\tilde{\varphi}_s)^+_{Q_R}(\tau) &= \sup_{x \in Q_R} \tilde{\varphi}_s(x, \tau) = \sup_{x \in Q_R} \left(\tau^{\gamma^+} \varphi_s^-(1) \chi_{[0,1)}(\tau) + \varphi_s(x, \tau) \chi_{[1,\infty)}(\tau) \right) \\ &= \tau^{\gamma^+} \inf_{x \in Q_R} \varphi_s(x, 1) \chi_{[0,1)}(\tau) + \sup_{x \in Q_R} \left(\varphi_s(x, \tau) \chi_{[1,\infty)}(\tau) \right) \\ &= \frac{1}{s} \left(\tau^{\gamma^+} \inf_{x \in Q_{sR}} \varphi(x, 1) \chi_{[0,1)}(\tau) + \sup_{x \in Q_{sR}} \left(\varphi(x, \tau) \chi_{[1,\infty)}(\tau) \right) \right) \\ &= \frac{1}{s} \tilde{\varphi}^+_{Q_{sR}}(\tau). \end{aligned} \quad (107)$$

Similarly, we have $(\tilde{\varphi}_s)^-_{Q_R}(\tau) = (1/s)\tilde{\varphi}^-_{Q_{sR}}(\tau)$, so two inequalities in (106) are equivalent.

In the sequel, we prove the latter of (106). Let us consider cases $\tau_0 < 1/\beta$ and $\tau_0 \geq 1/\beta$.

(1) If $\tau_0 < 1/\beta$, then

$$\tilde{\varphi}^+_{Q_{sR}}(\beta\tau_0) = (\beta\tau_0)^{\gamma^+} \varphi^-(1). \quad (108)$$

On one hand, $(\beta\tau_0)^{\gamma^+} \varphi^-(1) \leq \tau_0^{\gamma^+} \varphi^-(1)$; on the other hand, (A4) and definition 4 yield

$$(\beta\tau_0)^{\gamma^+} \varphi^-(1) \leq \varphi^-(\beta\tau_0) \leq \varphi^-(\tau_0), \quad (109)$$

so

$$\tilde{\varphi}^+_{Q_{sR}}(\beta\tau_0) \leq \tau_0^{\gamma^+} \varphi^-(1) \chi_{[0,1)}(\tau_0) + \varphi^-_{Q_{sR}}(\tau_0) \chi_{[1,\infty)}(\tau_0) = \tilde{\varphi}^-_{Q_{sR}}(\tau_0). \quad (110)$$

(2) If $\tau_0 \geq 1/\beta$, then

$$\tilde{\varphi}^+_{Q_{sR}}(\beta\tau_0) = \varphi^+_{Q_{sR}}(\beta\tau_0), \ \tilde{\varphi}^-_{Q_{sR}}(\tau_0) = \varphi^-_{Q_{sR}}(\tau_0). \quad (111)$$

By Lemma 10, for $\tau_0 \in [1, (1/\sqrt{\wp}\operatorname{diam} Q_{sR})]$, we have $\varphi^+_{Q_{sR}}(\beta\tau_0) \leq \varphi^-_{Q_{sR}}(\tau_0)$; so when $3\|u\|_\infty / s \leq 1/\sqrt{\wp}\operatorname{diam} Q_{sR} < 1/\sqrt{2n\wp}sR$ (i.e., $R_0 < 1/3\sqrt{2n\wp}\|u\|_\infty$), it holds

$$\varphi^+_{Q_{sR}}(\beta\tau_0) \leq \varphi^-_{Q_{sR}}(\tau_0). \quad (112)$$

Thus, the latter of (106) is proved.

Denote

$$d_0 := \max\left\{\left(\left(\int_{Q_R}((u_s-k_0)^+)^{\gamma^+}dx\right)^{1/\gamma^+}, R\right\}, \ d := Md_0, \quad (113)$$

where $M \geq 1$ is to be determined. Note $d_0 \in [0, (3\|u\|_\infty/s)]$ by using $k_0 > -2\|u_s\|_{L^\infty(Q_R)}$. Denoting

$$\psi := (\tilde{\varphi}_s)^+_{Q_R}, \quad (114)$$

it deduces by the almost increasing of $\varphi(x,\tau)$, (106), and the strong Δ_2-condition that

$$\tilde{\varphi}_s(x,d) \geq LM\tilde{\varphi}_s(x,d_0) \geq LM\psi(\beta d_0) \geq cLM\psi(d_0). \quad (115)$$

For $i \in \mathbb{N}$, we take

$$k_i := k_0 + d\left(1-2^{-i}\right), \ \sigma_i := \frac{R}{2}\left(1+2^{-i}\right), \quad (116)$$

and see $k_{i+1} - k_i \geq \sigma_{i+1} - \sigma_i$. Defining

$$\phi_i := \int_{Q_{\sigma_i}} \tilde{\varphi}_s\left(x,(u_s-k_i)^+\right)dx \quad (117)$$

and applying the strong Δ_2-condition and (106), we get

$$\begin{aligned}\phi_{i+1} &= \int_{Q_{\sigma_{i+1}}} \tilde{\varphi}_s\left(x,(u_s-k_{i+1})^+\right)dx \leq c\int_{Q_{\sigma_{i+1}}} \tilde{\varphi}_s\left(x,\beta(u_s-k_{i+1})^+\right)dx \\ &\leq c\int_{Q_{\sigma_{i+1}}} (\tilde{\varphi}_s)^+_{Q_R}\left(\beta(u_s-k_{i+1})^+\right)dx \leq c\int_{Q_{\sigma_{i+1}}} (\tilde{\varphi}_s)^-_{Q_R}\left((u_s-k_{i+1})^+\right)dx.\end{aligned} \quad (118)$$

By Lemma 26, u_s is a local quasiminimizer, so we know from Lemma 20 that u_s satisfies (65). Hence, using (118) and Lemma 25 yields

$$\begin{aligned}\phi_{i+1} &\leq c\int_{Q_{\sigma_{i+1}}} (\tilde{\varphi}_s)^-_{Q_R}\left((u_s-k_{i+1})^+\right)dx \leq c \\ &\cdot \left(\int_{Q_{\sigma_i}} \frac{\tilde{\varphi}_s\left(x,(u_s-k_i)^+\right)}{\tilde{\varphi}_s(x,k_{i+1}-k_i)}dx\right)^{1/\wp} \int_{Q_{\sigma_i}} \tilde{\varphi}_s\left(x,\frac{(u_s-k_i)^+}{\sigma_i-\sigma_{i+1}}\right)dx \\ &= c\left(\int_{Q_{\sigma_i}} \frac{\tilde{\varphi}_s\left(x,(u_s-k_i)^+\right)}{\tilde{\varphi}_s\left(x,d2^{-i-1}\right)}dx\right)^{1/\wp} \int_{Q_{\sigma_i}} \tilde{\varphi}_s\left(x,\frac{(u_s-k_i)^+}{R2^{-i-2}}\right)dx.\end{aligned} \quad (119)$$

By (A4), definition 4, (106), and the strong Δ_2-condition, we have

$$\begin{aligned}\tilde{\varphi}_s\left(x,Md_0 2^{-i-1}\right) &\geq cM2^{-(i+1)\gamma^+}\tilde{\varphi}_s(x,d_0) \geq cM2^{-(i+1)\gamma^+}\tilde{\varphi}^-_s(d_0) \\ &\geq cM2^{-(i+1)\gamma^+}\tilde{\varphi}^+_s(\beta d_0) \geq cM2^{-(i+1)\gamma^+}\psi(d_0),\end{aligned}$$

$$\tilde{\varphi}_s\left(x,\frac{(u_s-k_i)^+}{R2^{-i-2}}\right) \leq c\left(R2^{-i-2}\right)^{-\gamma^+}\tilde{\varphi}_s\left(x,(u_s-k_i)^+\right), \quad (120)$$

so it deduces from (119) that

$$\begin{aligned}\phi_{i+1} &\leq c\left(\int_{Q_{\sigma_i}} \frac{\tilde{\varphi}_s\left(x,(u_s-k_i)^+\right)}{M2^{-(i+1)\gamma^+}\psi(d_0)}dx\right)^{1/\wp}\int_{Q_{\sigma_i}} \\ &\cdot \left(R2^{-i-2}\right)^{-\gamma^+}\tilde{\varphi}_s\left(x,(u_s-k_i)^+\right)dx \\ &= c\left[2^{-(i+1)\gamma^+}M\psi(d_0)\right]^{-1/\wp}\left(R2^{-i-2}\right)^{-\gamma^+}\phi_i^{1+1/\wp} \\ &= c2^{i\gamma^+(1+1/\wp)}M^{-1/\wp}\psi(d_0)^{-1/\wp}R^{-\gamma^+}\phi_i^{1+1/\wp}.\end{aligned} \quad (121)$$

Select $\alpha := 1/\wp$, $D := cR^{-\gamma^+}M^{-\alpha}\psi(d_0)^{-\alpha}$, $\ell := 2^{(1+\alpha)\gamma^+}$ in Lemma 22. If

$$\phi_0 \leq c2^{-\frac{1+\alpha}{\alpha^2}\gamma^+}R^{\frac{\gamma^+}{\alpha}}M\psi(d_0), \quad (122)$$

then we have by Lemma 22 that

$$u_s \leq k_\infty = k_0 + Md_0 \quad (123)$$

almost everywhere in $Q_{\sigma_\infty} = Q_{R/2}$.

The remaining job is to prove (122). We will point out that $\phi_0/\psi(d_0)$ has a bound not depending on s and then choose M such that (4.9) holds. Since $\tau^{-\gamma^+}\tilde{\varphi}_s(x,\tau)$ is almost decreasing, it follows that $\tau^{-\gamma^+}\psi(\tau)$ is also almost decreasing, and $\tau^{-1}\psi^{-1}(\tau)^{\gamma^+}$ is almost increasing by Lemma 8. Then, we know from ([15], Lemma 5) that $(\psi^{-1})^{\gamma^+}$ is equivalent to a convex function ς. Because of $\tilde{\varphi}_s \leq \psi$, it yields by the Jensen inequality that

$$\begin{aligned}\psi^{-1}(\phi_0) &= \psi^{-1}\left(\int_{Q_R}\tilde{\varphi}_s\left(x,(u_s-k_0)^+\right)dx\right) \\ &\leq \psi^{-1}\left(\int_{Q_R}\psi\left((u_s-k_0)^+\right)dx\right) \\ &\sim \left(\varsigma\left(\int_{Q_R}\psi\left((u_s-k_0)^+\right)dx\right)\right)^{1/\gamma^+} \\ &\leq \left(\int_{Q_R}\varsigma\left(\psi\left((u_s-k_0)^+\right)\right)dx\right)^{1/\gamma^+} \\ &\sim \left(\int_{Q_R}\left((u_s-k_0)^+\right)^{\gamma^+}dx\right)^{1/\gamma^+} \leq d_0.\end{aligned} \quad (124)$$

Since ψ satisfies the strong Δ_2-condition, we have

$$\phi_0 \leq \psi(cd_0) \leq c\psi(d_0), \quad (125)$$

and (122) is proved.

Theorem 28. *Let $\varphi \in \Phi_w(\Omega)$ satisfy (A1), (A2)′ and (A4), and $u \in HW^{1,\varphi}_{loc}(\Omega)$ be a bounded local quasiminimizer of (1). When $0 < r < R_0$, $k \in \mathbb{R}$, we have*

$$\operatorname*{esssup}_{Q_{r/2}} u - k \le c\left(\left(\frac{1}{|Q_r|}\int_{Q_r}\left((u-k)^+\right)^{\gamma^+}dx\right)^{\frac{1}{\gamma^+}} + r\right), \quad (126)$$

where c depends only on the parameters of (A1), (A2)′, and (A4), $\Delta_2(\varphi)$, $\wp$, R_0 and $\|u\|_{L^\infty(Q_r)}$.

The proof is similar to Theorem 5.7 in [16] by using Lemma 27 here. We omit it and describe two interesting corollaries.

Corollary 29. *Let $\varphi \in \Phi_w(\Omega)$ satisfy (A1), (A2)′, and (A4) and $u \in HW^{1,\varphi}_{loc}(\Omega)$ be a bounded local quasiminimizer of (1). When $0 < r < R_0$, $k \in \mathbb{R}$, we have for any $\varepsilon \in (0, 1)$,*

$$\operatorname*{esssup}_{Q_{\varepsilon R}} u - k \le c\left(\left(\frac{1}{(1-\varepsilon)^{\wp}|Q_R|}\int_{Q_R}\left((u-k)^+\right)^{\gamma^+}dx\right)^{\frac{1}{\gamma^+}} + R\right), \quad (127)$$

where c does not depend on R and ε.

The proof is similar to Corollary 5.8 in [16].

Corollary 30. *Let $\varphi \in \Phi_w(\Omega)$ satisfy (A1), (A2)′, and (A4) and $u \in HW^{1,\varphi}_{loc}(\Omega)$ be a bounded local quasiminimizer of (1). When $0 < r < R_0$, $k \in \mathbb{R}$, we have for any $q \in (0,\infty)$,*

$$\operatorname*{esssup}_{Q_{R/2}} u - k \le c\left(\left(\frac{1}{|Q_R|}\int_{Q_R}\left((u-k)^+\right)^{q}dx\right)^{\frac{1}{q}} + R\right), \quad (128)$$

where c does not depend on R and depends only on the parameters of (A1), (A2)′, and (A4), $\Delta_2(\varphi)$, $\wp$, R_0, and $\|u\|_{L^\infty(Q_r)}$.

The proof is similar to Corollary 5.9 in [16].

5. Weak Harnack Inequality and the Proof of the Main Result

Denote

$$D_\theta \coloneqq \{u < \theta\} \cap Q_R. \quad (129)$$

Lemma 31. *If $u \ge 0$, $-u$ satisfies (128) with $q = 1$, $c = c_1$, $k = -\theta$, and for some $\theta > 0$,*

$$|D_\theta| \le \frac{1}{2c_1}|Q_R|, \quad (130)$$

then

$$\operatorname*{ess\,inf}_{Q_{\frac{R}{2}}} u + c_1 R \ge \frac{\theta}{2}. \quad (131)$$

The proof is similar to Lemma 6.1 in [16].

Lemma 32. *Let $\varphi \in \Phi_w(\Omega)$ satisfy (A1), (A2), (A2)′, (A4), and (A3), and $u \in HW^{1,\varphi}_{loc}(\Omega)$ be a bounded local quasiminimizer of (1), R_0 as shown in Lemma 27. If $u \ge 0$, and for all $r \in (0, R_0)$, $\theta > 0$, and $\kappa \in (0, 1)$, there exists $\mu > 0$ such that*

$$|D_\theta| \le \kappa|Q_R|; \quad (132)$$

then, we have

$$\operatorname*{ess\,inf}_{Q_{\frac{R}{2}}} u + cR \ge \mu\theta. \quad (133)$$

The proof is similar to the proof of Lemma 6.2 in [16]. It only needs to replace theorem 3.16 in [19] with Lemma 15 in this paper.

Lemma 33 (covering Lemma). *Suppose that $Q_R \subset \mathbb{H}^n$, E and G ($E \subset G \subset Q_R$) are measurable sets, if there exists $0 < \delta < 1$ such that*

$$|E| \le \delta|Q_R|; \quad (134)$$

(1) For any cube Q with $|Q \cap E| \ge \delta|Q|$, it holds $Q' \subset G$, where Q' is a subset of Q

Then,

$$|E| \le \delta|G|. \quad (135)$$

Proof. From the Calderón-Zygmund decomposition Theorem ([20], p17), we know that for every $f \in L^1(Q_R)$ and $0 < \delta < 1$, there is a sequence of disjoint cubes $\{Q_j\}$ such that for almost all $x \in Q_R \setminus \cup_j Q_j$, we have

$$|f(x)| \le \delta \text{ and } \delta \le \frac{1}{|Q_j|}\int_{Q_j}|f|dx < 2^{\wp}\delta. \quad (136)$$

If we take $f = \chi_E$, then

$$E \subset \left(\cup_j Q_j\right) \cup N,\ |N| = 0 \text{ and } \delta \le \frac{|E \cap Q_j|}{|Q_j|} < 2^{\wp}\delta. \quad (137)$$

Let us decompose Q_j until the cube $\tilde{Q}_j$ satisfies $|E \cap \tilde{Q}_j|/|\tilde{Q}_j| < \delta$; so,

$$E \setminus N \subset \cup_j \tilde{Q}_j \subset G. \quad (138)$$

Hence, it follows

$$|E| \le \sum_j |E \cap \tilde{Q}_j| \le \delta\sum_j |\tilde{Q}_j| \le \delta|G|. \quad (139)$$

Lemma 34. *Suppose $u \ge 0$ and for all $R \in (0, R_0)$ and every $\kappa > 0$, there exists $\mu > 0$ such that if*

$$|D_\theta| \le \kappa|Q_R|, \quad (140)$$

then

$$\underset{Q_{\frac{R}{2}}}{ess\ \inf} u + R \geq \mu\theta ; \tag{141}$$

then, there exists $q > 0$ such that for every $R \in (0, R_0)$, we have

$$\left(\frac{1}{|Q_R|} \int_{Q_R} u^q dx \right)^{\frac{1}{q}} \leq c \left(\underset{Q_{\frac{R}{2}}}{ess\ \inf} u + R \right). \tag{142}$$

Proof. The proof is similar to the proof of Lemma 6.3 in [16], and the key difference is that we replace the covering lemma there with Lemma 5.3. Fix $0 < \delta < 1$, $\kappa := 1 - \delta$ and write

$$E\left(s\mu^i, z, R\right) := \left\{ y \in Q_R(z) \colon u(y) + R > s\mu^i \right\}. \tag{143}$$

Take

$$E = E_s^i := \left\{ y \in Q_R : u(y) + R > s\mu^i \right\},\ G = E_s^{i+1} \tag{144}$$

in Lemma 5.3, where μ is the constant in (141) with respect to κ. Suppose that for some $r < R$ and $z \in Q_R$, we have

$$\left| Q_r(z) \cap E_s^i \right| \geq \delta |Q_r(z)|. \tag{145}$$

Therefore, it follows under (145) that

$$\begin{aligned} \delta |Q_r| &\leq \left| Q_r(z) \cap E_s^i \right| = \left| Q_r(z) \cap Q_R \cap \left\{ u(y) + R > s\mu^i \right\} \right| \\ &= \left| Q_R \cap Q_r(z) \cap \left\{ u(y) + R > s\mu^i \right\} \right| = \left| Q_R \cap E\left(s\mu^i, z, r\right) \right| \\ &\leq \left| E\left(s\mu^i, z, r\right) \right|, \end{aligned}$$

$$\begin{aligned} \left| D\left(s\mu^i, z, r\right) \right| &= \left| \left\{ u(y) < s\mu^i \right\} \cap Q_r(z) \right| \leq \left| \left\{ u(y) + r < s\mu^i \right\} \cap Q_r(z) \right| \\ &\leq |Q_r| - \left| E\left(s\mu^i, z, r\right) \right| \leq (1 - \delta) |Q_r| = \kappa |Q_r|, \end{aligned} \tag{146}$$

so (140) holds. Thus,

$$\underset{Q\left(z, \frac{r}{2}\right)}{\text{ess} \inf} u + r \geq s\mu^{i+1}. \tag{147}$$

From the above discussion, we can infer (147) from the hypothesis (145), so $Q_{r/2}(z) \cap Q_R \subset E_s^{i+1}$. It deduces that (2) in Lemma 33 holds. Hence, we know that from Lemma 33 that if (1) in Lemma 33 does not holds, then $E_s^{i+1} = Q_R$, which implies

$$u + R \geq \underset{Q_{R/2}}{\text{ess} \inf} u + R \geq s\mu^{i+1}. \tag{148}$$

If (1) in Lemma 33 holds, then $|E_s^{i+1}| \geq \delta^{-1} |E_s^i|$, which gives

$$\left| E_s^j \right| \geq \delta^{-1} \left| E_s^{j-1} \right| \geq \delta^{-2} \left| E_s^{j-2} \right| \geq \cdots \geq \delta^{-j} \left| E_s^0 \right|. \tag{149}$$

If $|E_s^0| > 0$, we choose j to be the smallest integral satisfying $j \geq (1/\log \delta) \log (|E_s^0| / |Q_R|)$, and then

$$\delta^j |Q_R| \leq \delta^{\frac{1}{\log \delta} \log \frac{|E_s^0|}{|Q_R|}} |Q_R| = \delta^{\frac{\log |E_s^0| - \log |Q_R|}{\log \delta}} |Q_R| = \left| E_s^0 \right| \leq \delta^j \left| E_s^j \right|, \tag{150}$$

so $|E_s^j| \geq |Q_R|$, i.e., $E_s^j = Q_R$, Thus, as (1), we have

$$\underset{Q_{R/2}}{\text{ess} \inf} u + R \geq s\mu^j. \tag{151}$$

Combining (1) and (2), we obtain

$$\begin{aligned} \underset{Q_{R/2}}{\text{ess} \inf} u + R &\geq s\mu^{j+1} = cs\mu^{\frac{1}{\log \delta} \log \frac{|E_s^0|}{|Q_R|}} = cs\mu^{\log \left(\frac{|E_s^0|}{|Q_R|} \right)^{\frac{1}{\log \delta}}} \\ &= cs \left(\frac{|E_s^0|}{|Q_R|} \right)^{\frac{\log \mu}{\log \delta}}. \end{aligned} \tag{152}$$

If we write $\xi = \underset{Q_{R/2}}{\text{ess} \inf} u + R$, $a = \log \delta / \log \mu$, then

$$\left| E_s^0 \right| \leq c |Q_R| \xi^a s^{-a}. \tag{153}$$

Taking $0 < q < a$, it gets

$$\begin{aligned} \int_{Q_R} (u + R)^q dx &= q \int_0^\infty s^{q-1} \left| E_s^0 \right| ds = q \int_0^\xi s^{q-1} \left| E_s^0 \right| ds + q \int_\xi^\infty s^{q-1} \left| E_s^0 \right| ds \\ &= q \int_0^\xi s^{q-1} |Q_R| ds + q \int_\xi^\infty s^{q-1} \left| E_s^0 \right| ds \\ &\leq |Q_R| \xi^q + c |Q_R| \xi^a \int_\xi^\infty s^{q-a-1} ds = c |Q_R| \xi^q, \end{aligned} \tag{154}$$

so

$$\left(\frac{1}{|Q_R|} \int_{Q_R} u^q dx \right)^{1/q} \leq c \left(\underset{Q_{R/2}}{ess\ \inf} u + R \right). \tag{155}$$

Thus, (142) is proved.

Lemma 35 (weak Harnack inequality). *Let $\varphi \in \Phi_w(\Omega)$ satisfy (A1), (A2), (A2)$'$, (A4), and (A3). If $u \in HW_{loc}^{1,\varphi}(\Omega)$ is a nonnegative local quasiminimizer of (1), R_0 as shown in Lemma 27, then for all $R \in (0, R_0)$ and $k \in \mathbb{R}$, we have*

$$\left(\frac{1}{|Q_R|} \int_{Q_R} u^q dx \right)^{\frac{1}{q}} \leq c \left(\underset{Q_{\frac{R}{2}}}{ess\ \inf} + R \right), \tag{156}$$

where c depends only on the parameters of (A1), (A2), (A2)$'$, (A4), and (A3), $\wp$, R_0, and $\|u\|_{L^\infty(Q_R)}$.

Proof. According to Lemma 32, we see that the conditions of Lemma 34 are satisfied, so (156) is true by using Lemma 34.

Proof of Theorem 2. Combining Corollary 30 and Lemma 35, we prove immediately Theorem 2.

Proof of Theorem 3. The Harnack inequality in Theorem 2 implies the Hölder continuity.

Data Availability

No data used.

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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