Hindawi
Journal of Function Spaces
Volume 2020, Article ID 8879366, 14 pages
https://doi.org/10.1155/2020/8879366

Research Article

Polynomial Decay Rate for a Coupled Lamé System with Viscoelastic Damping and Distributed Delay Terms

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Received 16 August 2020; Revised 28 September 2020; Accepted 22 October 2020; Published 16 November 2020

Academic Editor: Mohamed Boussairi Jleli

In this paper, we prove a general energy decay results of a coupled Lamé system with distributed time delay. By assuming a more general of relaxation functions and using some properties of convex functions, we establish the general energy decay results to the system by using an appropriate Lyapunov functional.

1. Introduction

In this work, we shall be concerned with studying the general decay rate of the following Lamé system in $\Omega \times \mathbb{R}^+$:

$$\begin{cases} u_{tt} - \Delta_e u + \int_0^t g_1(t-s)\Delta u(s)ds - k_1 \Delta u_t - \int_{\tau_1}^{\tau_2} \mu_1(\rho)\Delta u_t(x, t-\rho)d\rho = f_1(u, v), \\ v_{tt} - \Delta_e v + \int_0^t g_2(t-s)\Delta v(s)ds - k_2 \Delta v_t - \int_{\tau_1}^{\tau_2} \mu_2(\rho)\Delta v_t(x, t-\rho)d\rho = f_2(u, v). \end{cases} \quad (1)$$

Equations (1) are associated with the following boundary and initial conditions

$$\begin{cases} u(x,t) = v(x,t) = 0, \; on \; \partial\Omega \times \mathbb{R}^+, \\ u(x,0) = u_0(x), v(x,0) = v_0(x), u_t(x,0) = u_1(x), v_t(x,0) = v_1(x), x \in \Omega, \\ (u_t(x,-t), v_t(x,-t)) = (f_0(x,t), g_0(x,t)), \text{ in } \Omega \times (0, \tau_2), \end{cases} \quad (2)$$

where Ω is a bounded domain in $\mathbb{R}^n (n = 1, 2, 3)$, with smooth boundary $\partial\Omega$. The elasticity differential operator Δ_e is given by

$$\Delta_e u = \mu \Delta u + (\mu + \lambda)\nabla(divu), \quad (3)$$

and the Lamé constants μ and λ are satisfying the following conditions

$$\mu > 0, \mu + \lambda > 0. \quad (4)$$

The parameters k_1, k_2, τ_1, and τ_2 are positive constants, with $\tau_1 < \tau_2$. The functions $\mu_1, \mu_2 : [\tau_1, \tau_2] \to \mathbb{R}$ are bounded. The functions $f_1(u, v)$ and $f_2(u, v)$ which represent the source terms will be specified later.

After several authors have studied the problems of coupled systems and hyperbolic systems, their stability is associated with velocities and is proven under conditions imposed on the subgroup [1]. The researchers also studied behavior of the energy in a limited field with nonlinear damping and external force and a varying delay of time to find solutions to the Lame system [1–9].

Recently, problems that contain viscoelasticity have been addressed, and many results have been found regarding the global existence and stability of solutions (see [2, 9]), under conditions on the relaxation function, whether exponential or polynomial decay. In addition, in [10], Boulaaras obtained the stability result of the global solution to the Lamé system with the flexible viscous term by adding logarithmic nonlinearity, even though the kernel is not necessarily decreasing in contrast to what he studied [2].

Introducing a distributed delay term makes our problem different from those considered so far in the literature.

The importance of this term appears in many works, and this is due to the fact that many phenomena depends on their past. Also, it is influence on the asymptotic behavior of the solution for the different types of problems such that Timoshenko system [3, 11–13], transmission problem [14], wave equation [15], and thermoelastic system [16, 17].

In the present work, we extend the general decay result obtained by Feng in [18] to the case of distributed term delay, namely, we will make sure that the result is achieved if the distributed delay term exists.

This paper is organized as follows. In the second section, we give some preliminaries related to problem (1). In Section 3, we prove our main result.

2. Preliminaries

In this section, we provide some materials and necessary assumptions which we need in the prove of our results. We use the standard Lebesgue and Sobolev spaces with their scaler products and norms. For simplicity, we would write $\|.\|$ instead of $\|.\|_2$. Throughout this work, we used a generic positive constant c.

For the relaxation functions g_1 and g_2, we assume, for $i = 1, 2$,

(A1) $g_i(t)$: $\mathbb{R}^+ \to \mathbb{R}^+$ are nonincreasing C^1 functions satisfying

$$g_i(0) > 0 \text{ and } \mu - \int_0^\infty g_i(s)ds = l_i > 0. \tag{5}$$

We assume further that for $i = 1, 2$:

(A2) There exist two C^1 functions $G_i : \mathbb{R}^+ \to \mathbb{R}^+$, with $G_i(0) = G_i'(0) = 0$, which are linear or are strictly increasing and strictly convex functions of class $C^2(\mathbb{R}^+)$ on $(0, r]$, $r \le g_i(0)$, such that

$$g_i'(t) \le -\xi_i(t)G_i(g_i(t)),\ \forall t \ge 0, \tag{6}$$

where $\xi_i(t)$ are C^1 functions satisfying

$$\xi_i(t) > 0,\ \xi_i'(t) \le 0,\ \forall t \ge 0. \tag{7}$$

(A3) For the source terms f_1 and f_2, we take

$$\begin{aligned} f_1(u, v) &= \alpha|u + v|^{p-1}(u + v) + \beta|u|^{(p-3)/2}u|v|^{(p+1)/2}, \quad \forall(u, v) \in \mathbb{R}^n]^2, \\ f_2(u, v) &= \alpha|u + v|^{p-1}(u + v) + \beta|v|^{(p-3)/2}v|u|^{(p+1)/2}, \quad \forall(u, v) \in \mathbb{R}^n]^2, \end{aligned} \tag{8}$$

with $\alpha, \beta > 0$. Clearly,

$$uf_1(u, v) + vf_2(u, v) = (p + 1)F(u, v),\ \forall(u, v) \in \mathbb{R}^n]^2, \tag{9}$$

where

$$\begin{aligned} F(u, v) &= \frac{1}{(p + 1)}\left[\alpha|u + v|^{p+1} + 2\beta|uv|^{(p+1)/2}\right], \quad \forall(u, v) \in \mathbb{R}^n]^2, \\ f_1(u, v) &= \frac{\partial F}{\partial u},\ f_2(u, v) = \frac{\partial F}{\partial v}. \end{aligned} \tag{10}$$

Further, we assume that there is $C > 0$, such that

$$\left|\frac{\partial f_i}{\partial u}(u, v)\right| + \left|\frac{\partial f_i}{\partial v}(u, v)\right| \le C\left(|u|^{p-1} + |v|^{p-1}\right),\ i = 1, 2 \text{ where } 1 \le p \le 6. \tag{11}$$

(A4)

$$\text{if } n = 1, 2\,; p \ge 3,\ \text{if } n = 3\,; p = 3. \tag{12}$$

So, we have the embedding

$$\begin{gathered} H_0^1(\Omega)^\circ L^q(\Omega) \text{ for } 2 \le q \le \frac{2n}{n - 2} \text{ if } n \ge 3 \text{ or } q \ge 2 \text{ if } n = 1, 2, \\ L^{r\circ}L^q \text{ for } q < r. \end{gathered} \tag{13}$$

Let c_s the same embedding constant, so we have

$$\|v\|_q \le c_s\|\nabla v\|_2,\ \|v\|_q \le c_s\|v\|_r \text{ for } v \in H_0^1(\Omega). \tag{14}$$

Remark 1. There exist two constants $\Lambda_1 > 0$ and $\Lambda_2 > 0$ such that

$$\int_\Omega |f_i(u, v)|^2dx \le \Lambda_i\left(l_1\|\nabla u\|^2 + l_2\|\nabla v\|^2\right)^p,\ i = 1, 2. \tag{15}$$

As in many papers, we introduce the following new variables

$$\begin{cases} z(x, \rho, \rho, t) = u_t(x, t - \rho\rho), \\ y(x, \rho, \rho, t) = v_t(x, t - \rho\rho), \end{cases} \tag{16}$$

then, we obtain

$$\begin{cases} \rho z_t(x, \rho, \rho, t) + z_\rho(x, \rho, \rho, t) = 0, \\ z(x, 0, \rho, t) = u_t(x, t), \end{cases} \tag{17}$$

$$\begin{cases} \rho y_t(x, \rho, \rho, t) + y_\rho(x, \rho, \rho, t) = 0, \\ y(x, 0, \rho, t) = v_t(x, t). \end{cases} \tag{18}$$

Consequently, the problem (1) is equivalent to

$$\begin{cases} u_{tt}-\Delta_e u+\int_0^t g_1(t-s)\Delta u(s)ds-k_1\Delta u_t-\int_{\tau_1}^{\tau_2}\mu_1(\rho)\Delta z(x,1,\rho,t)d\rho=f_1(u,v), \\ v_{tt}-\Delta_e v+\int_0^t g_2(t-s)\Delta v(s)ds-k_2\Delta v_t-\int_{\tau_1}^{\tau_2}\mu_2(\rho)\Delta y(x,1,\rho,t)d\rho=f_2(u,v), \\ \rho z_t(x,\rho,\rho,t)+z_\rho(x,\rho,\rho,t)=0, \\ \rho y_t(x,\rho,\rho,t)+y_\rho(x,\rho,\rho,t)=0, \end{cases} \tag{19}$$

with the initial data and boundary conditions

$$\begin{cases} (u(x,0),v(x,0))=(u_0(x),v_0(x)), \text{ in } \Omega, \\ (u_t(x,0),v_t(x,0))=(u_1(x),v_1(x)), \text{ in } \Omega, \\ (u_t(x,-t),v_t(x,-t))=(f_0(x,t),g_0(x,t)), \text{ in } \Omega\times(0,\tau_2), \\ u(x,t)=v(x,t)=0, \text{ in } \partial\Omega\times(0,\infty), \\ (z(x,\rho,\rho,0),y(x,\rho,\rho,0))=(f_0(x,\rho\rho),g_0(x,\rho\rho)), \text{ in } \Omega\times(0,1)\times(0,\tau_2), \end{cases} \tag{20}$$

where

$$(x,\rho,\rho,t)\in\Omega\times(0,1)\times(\tau_1,\tau_2)\times(0,\infty). \tag{21}$$

We recall the following notations

$$\begin{aligned} (h*\varphi)&=\int_0^t h(t-s)\varphi(s)dsdx, \\ (h\circ\varphi)(t)&=\int_0^t h(t-s)|\varphi(t)-\varphi(s)|^2ds. \end{aligned} \tag{22}$$

Thus, we have the following important property

$$\begin{aligned} \int_\Omega (h*\varphi)\varphi_t dx=&-\frac{1}{2}h(t)\|\varphi(t)\|^2+\frac{1}{2}\left(h'\circ\varphi\right)(t)-\frac{1}{2}\frac{d}{dt} \\ &\cdot\left[(h\circ\varphi)(t)-\left(\int_0^t h(s)ds\right)\|\varphi(t)\|^2\right]. \end{aligned} \tag{23}$$

The energy modified associated to the problem (19) is defined by

$$\begin{aligned} E(t)=&\frac{1}{2}\left[\|u_t\|^2+\left(\mu-\int_0^t g_1(s)ds\right)\|\nabla u\|^2+(\lambda+\mu)\|divu\|^2\right] \\ &+\frac{1}{2}\left[(g_1\circ\nabla u)(t)+\eta\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2}\rho|\mu_1(\rho)|\|\nabla z(x,\rho,\rho,t)\|^2d\rho d\rho dx\right] \\ &+\frac{1}{2}\left[\|v_t\|^2+\left(\mu-\int_0^t g_2(s)ds\right)\|\nabla v\|^2+(\lambda+\mu)\|divv\|^2\right] \\ &+\frac{1}{2}\left[(g_2\circ\nabla v)(t)+\eta\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2}\rho|\mu_2(\rho)|\|\nabla y(x,\rho,\rho,t)\|^2d\rho d\rho dx\right] \\ &-\int_\Omega F(u,v)dx. \end{aligned} \tag{24}$$

First, we prove in the following theorem the result of energy identity.

Lemma 2. *Assume that*

$$\int_{\tau_1}^{\tau_2}|\mu_i(\rho)|d\rho<k_i,\ i=1,2. \tag{25}$$

Then, the energy modified defined by (24) satisfies, along the solution (u,v,z,y) of (19), the estimate

$$\begin{aligned} \frac{d}{dt}E(t)\le&-\frac{1}{2}g_1(t)\|\nabla u\|^2+\frac{1}{2}\left(g_1'\circ\nabla u\right)(t)-\frac{1}{2}g_2(t)\|\nabla v\|^2 \\ &+\frac{1}{2}\left(g_2'\circ\nabla v\right)(t)-\left[k_1-\left(\frac{\eta+1}{2}\right)\left(\int_{\tau_1}^{\tau_2}|\mu_1(\rho)|d\rho\right)\right] \\ &\cdot\|\nabla u_t\|^2-\left(\frac{\eta-1}{2}\right)\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_1(\rho)|\|\nabla z(x,1,\rho,t)|^2d\rho dx \\ &-\left[k_2-\left(\frac{\eta+1}{2}\right)\left(\int_{\tau_1}^{\tau_2}|\mu_2(\rho)|d\rho\right)\right]\|\nabla v_t\|^2 \\ &-\left(\frac{\eta-1}{2}\right)\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(\rho)|\|\nabla y(x,1,\rho,t)|^2d\rho dx\le 0, \end{aligned} \tag{26}$$

for

$$1<\eta<\min\left(\frac{2k_1}{\left(\int_{\tau_1}^{\tau_2}|\mu_1(\rho)|d\rho\right)},\frac{2k_2}{\left(\int_{\tau_1}^{\tau_2}|\mu_2(\rho)|d\rho\right)}\right)-1. \tag{27}$$

Proof. First multiplying the equation $(0.14)_1$ by u_t and integrating by parts over Ω, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left[\|u_t\|^2+\mu\|\nabla u\|^2+(\lambda+\mu)\|divu\|^2\right] \\ &\quad-\int_\Omega\nabla u_t\int_0^t g_1(t-s)\nabla u(s)ds+k_1\|\nabla u_t\|^2 \\ &\quad+\int_\Omega\nabla u_t\int_{\tau_1}^{\tau_2}|\mu_1(\rho)\nabla z(x,1,\rho,t)d\rho dx \\ &\quad=\int_\Omega f_1(u,v).udx, \end{aligned} \tag{28}$$

by using (23), we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left[\|u_t\|^2+\left(\mu-\int_0^t g_1(s)ds\right)\|\nabla u\|^2+(\lambda+\mu)\|divu\|^2+(g_1\circ\nabla u)(t)\right] \\ &\quad=-\frac{1}{2}g_1\|\nabla u\|^2+\frac{1}{2}\left(g_1'\circ\nabla u\right)(t)+\int_\Omega u_tf_1(u,v)dx-k_1\|\nabla u_t\|^2 \\ &\qquad+\int_\Omega\nabla u_t\int_{\tau_1}^{\tau_2}\mu_1(\rho)\nabla z(x,1,\rho,t)d\rho dx. \end{aligned} \tag{29}$$

Similarly, multiplying the equation (19) by v_t and integrating over Ω, we obtain

$$\frac{1}{2}\frac{d}{dt}\left[\|v_t\|^2+\left(\mu-\int_0^t g_2(s)ds\right)\|\nabla v\|^2+(\lambda+\mu)\|divv\|^2+(g_2\circ\nabla v)(t)\right]$$
$$=-\frac{1}{2}g_2\|\nabla v\|^2+\frac{1}{2}\left(g_2'\circ\nabla v\right)(t)+\int_\Omega v_tf_2(u,v)dx-k_2\|\nabla v_t\|^2$$
$$+\int_\Omega \nabla v_t\int_{\tau_1}^{\tau_2}\mu_2(\rho)\nabla y(x,1,\rho,t)d\rho dx. \tag{30}$$

Multiplying the equation (19) by $-\eta|\mu_1(\rho)|\Delta z(x,\rho,\rho,t)$ and integrating by parts over $\Omega\times(0,1)\times(\tau_1,\tau_2)$, we obtain

$$\eta\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2}\rho|\mu_1(\rho)|\nabla z(x,\rho,\rho,t)\nabla z_t(x,\rho,\rho,t)d\rho d\rho dx$$
$$=-\eta\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2}|\mu_1(\rho)|\nabla z(x,\rho,\rho,t)\nabla z_\rho(x,\rho,\rho,t)d\rho d\rho dx, \tag{31}$$

therefore

$$\frac{d}{dt}\frac{\eta}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2}\rho|\mu_1(\rho)||\nabla z(x,\rho,\rho,t)|^2d\rho d\rho dx$$
$$=-\frac{\eta}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2}|\mu_1(\rho)|\frac{d}{d\rho}|\nabla z(x,\rho,\rho,t)|^2d\rho d\rho dx$$
$$=-\frac{\eta}{2}\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_1(\rho)||\nabla z(x,1,\rho,t)|^2d\rho dx$$
$$+\frac{\eta}{2}\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_1(\rho)||\nabla u_t(x,t)|^2d\rho dx. \tag{32}$$

Multiplying the fourth equation of (19) by $-\eta|\mu_2(\rho)|\Delta y(x,\rho,\rho,t)$ and integrating over $\Omega\times(0,1)\times(\tau_1,\tau_2)$, we obtain

$$\frac{d}{dt}\left(\frac{\eta}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2}\rho|\mu_2(\rho)||\nabla y(x,\rho,\rho,t)|^2d\rho d\rho dx\right)$$
$$=-\frac{\eta}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(\rho)|\frac{d}{d\rho}|\nabla y(x,\rho,\rho,t)|^2d\rho d\rho dx$$
$$=-\frac{\eta}{2}\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(\rho)||\nabla y(x,1,\rho,t)|^2d\rho dx$$
$$+\frac{\eta}{2}\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(\rho)||\nabla v_t(x,t)|^2d\rho dx. \tag{33}$$

For the source term, we have

$$\int_\Omega u_tf_1(u,v)dx+\int_\Omega v_tf_2(u,v)dx$$
$$=\int_\Omega u_t\left(\alpha|u+v|^{p-1)}(u+v)+\beta|u|^{(p-3)/2}u|v|^{(p+1)/2}\right)$$
$$+\int_\Omega v_t\left(\alpha|u+v|^{p-1}(u+v)+\beta|v|^{(p-3)/2}v|u|^{(p+1)/2}\right)$$
$$=\int_\Omega\left(\alpha|u+v|^{p-1}(u+v)(u_t+v_t)+\beta\left(|u|^{(p-3)/2}u.u_t\right)\right.$$
$$\left.\times|v|^{(p+1)/2}+\beta\left(|v|^{(p-3)/2}v.v_t\right)|u|^{(p+1)/2}\right)dx$$
$$=\frac{d}{dt}\int_\Omega\left(\frac{\alpha}{p+1}|u+v|^{p+1}+\frac{2\beta}{p+1}|uv|^{(p+1)/2}\right)dx$$
$$=\frac{d}{dt}\int_\Omega F(u,v)dx. \tag{34}$$

By collecting the previous equations (29)–(34), we get

$$\frac{d}{dt}E(t)=-\frac{1}{2}g_1\|\nabla u\|^2+\frac{1}{2}\left(g_1'\circ\nabla u\right)(t)-k_1\|\nabla u_t\|$$
$$+\int_\Omega\nabla u_t\int_{\tau_1}^{\tau_2}\mu_1(\rho)\nabla z(x,1,\rho,t)d\rho dx-\frac{1}{2}g_2\|\nabla v\|^2$$
$$+\frac{1}{2}\left(g_2'\circ\nabla v\right)(t)-k_2\|\nabla v_t\|$$
$$+\int_\Omega\nabla v_t\int_{\tau_1}^{\tau_2}\mu_2(\rho)\nabla y(x,1,\rho,t)d\rho dx$$
$$-\frac{\eta}{2}\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_1(\rho)||\nabla z(x,1,\rho,t)|^2d\rho dx$$
$$+\frac{\eta}{2}\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_1(\rho)||\nabla u_t(x,t)|^2d\rho dx$$
$$-\frac{\eta}{2}\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(\rho)||\nabla y(x,1,\rho,t)|^2d\rho dx$$
$$+\frac{\eta}{2}\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(\rho)||\nabla v_t(x,t)|^2d\rho dx. \tag{35}$$

Using Young's inequality, we obtain

$$\int_\Omega\nabla u_t\int_{\tau_1}^{\tau_2}\mu_1(\rho)\nabla z(x,1,\rho,t)d\rho dx$$
$$\le\int_\Omega\int_{\tau_1}^{\tau_2}\left(\nabla u_t\sqrt{|\mu_1(\rho)|}\right)\left(\sqrt{|\mu_1(\rho)|}\nabla z(x,1,\rho,t)\right)d\rho dx$$
$$\le\frac{1}{2}\left(\int_{\tau_1}^{\tau_2}|\mu_1(\rho)|d\rho\right)\|\nabla u_t\|$$
$$+\frac{1}{2}\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_1(\rho)||\nabla z(x,1,\rho,t)|^2d\rho dx, \tag{36}$$

similarly

$$\int_\Omega \nabla v_t \int_{\tau_1}^{\tau_2} \mu_2(\rho)\nabla y(x,1,\rho,t)d\rho dx \leq \frac{1}{2}\left(\int_{\tau_1}^{\tau_2} |\mu_2(\rho)|d\rho\right)\|\nabla v_t\| + \frac{1}{2}\int_\Omega\int_{\tau_1}^{\tau_2} |\mu_2(\rho)||\nabla y(x,1,\rho,t)|^2 d\rho dx. \tag{37}$$

This completes the proof.

3. General Decay

In this section we will prove that the solution of problems (19)–(20) decay generally to trivial solution. Using the energy method and suitable Lyapunov functional.

In the following, we will present our main stability result:

Theorem 3 (Decay rates of energy). *Assume that (A1)–(A3) hold. Then, for every $t_0 > 0$, there exist two positive constants α_1 and α_2 such that the energy defined by (24) satisfies the following decay*

$$E(t) \leq \alpha_2 G_4^{-1}\left(\alpha_1 \int_{g^{-1}(r)}^t \xi(s)ds\right), \forall t \geq g^{-1}(r), \tag{38}$$

where

$$G_4(t) = \int_t^r \frac{1}{sG_0(s)}ds, G_0(t) = \min\left\{G_1'(t), G_2'(t)\right\}, \tag{39}$$

and $\xi(t) = \min\{\xi_1(t), \xi_2(t)\}, g(t) = \max\{g_1(t), g_2(t)\}$.

This theorem will be proved later after providing some remarks.

Remark 4.

(1) In case $\int_0^\infty \xi_i(t)dt = \infty$, Theorem 3 ensures $\lim_{t\to\infty} E(t) = 0$.

(2) From (A2), we infer that $\lim_{t\to\infty} g_i(t) = 0$. Then, there exists some $t_1 \geq 0$ large enough such that

(a)

$$g_i(t_1) = r \Rightarrow g_i(t) \leq r, \forall t \geq t_1. \tag{40}$$

As G_i is positive continuous functions, and g_i and ξ_i are positive nonincreasing continuous functions, then, for all $0 \leq t \leq t_1$,

$$0 < g_i(t_1) \leq g_i(t) \leq g_i(0) \text{ and } 0 < \xi_i(t_1) \leq \xi_i(t) \leq \xi_i(0), \tag{41}$$

which implies for some positive constants a_i and b_i,

$$a_i \leq \xi_i(t)G_i(g_i(t)) \leq b_i. \tag{42}$$

Consequently,

$$g_i'(t) \leq -\xi_i(t)G_i(g_i(t)) \leq -\frac{a_i}{g_i(0)}g_i(0) \leq -\frac{a_i}{g_i(0)}g_i(t), \text{fort} \in [0, t_1]. \tag{43}$$

(3) We also mention Johnson's inequality, which is very important for proving our result. If G is a convex function on $[a, b]$, $g : \Omega \to a, b]$, we have

$$G\left[\frac{1}{k}\int_\Omega g(x)h(x)dx\right] \leq \frac{1}{k}\int_\Omega G[g(x)]h(x)dx, \tag{44}$$

where h is a function that satisfies

$$h(x) \geq 0 \text{ and } \int_\Omega h(x)dx = k > 0. \tag{45}$$

To prove the desired result, we create a Lyapunov functional equivalent to E. For this, we define some functions that allow us to construct this Lyapunov function.

As in Baowei [18] and Mustafa ([19, 20]), we define

$$C_{\zeta,i} = \int_0^\infty \frac{g_i^2(s)}{\sqrt{\zeta g_i(s) - g_i'(s)}} \text{ and } h_i(t) = \zeta g_i(t) - g_i'(t), i = 1, 2, \tag{46}$$

for any $0 < \zeta < 1$.

Lemma 5. *Let (u, v, z, y) be a solution of the problem (19). Then, the functional*

$$\varphi(t) = \int_\Omega u(t)u_t(t)dx + \int_\Omega v(t)v_t(t)dx, \tag{47}$$

satisfies the estimate

$$\begin{aligned}\varphi'(t) \leq & -\frac{l_1}{2}\|\nabla u(t)\|^2 - \frac{l_2}{2}\|\nabla v(t)\|^2 + \|u_t(t)\|^2 + \|v_t(t)\|^2 \\ & - (\lambda+\mu)\|divu(t)\|^2 - (\lambda+\mu)\|divv(t)\|^2 \\ & + \frac{3C_{\zeta,1}}{2l_1}(h_1\circ\nabla u)(t) + \frac{3k_1^2}{2l_1}\|\nabla u_t\|^2 + \frac{3k_2^2}{2l_2}\|\nabla v_t\|^2 \\ & + \frac{3C_{\zeta,2}}{2l_2}(h_2\circ\nabla v)(t) + (p+1)\int_\Omega F(u(t), v(t))dx \\ & + \frac{3k_1}{2l_1}\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_1(\rho)||\nabla z(x,1,\rho,t)|^2 d\rho dx \\ & + \frac{3k_2}{2l_2}\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(\rho)||\nabla y(x,1,\rho,t)|^2 d\rho dx.\end{aligned} \tag{48}$$

Proof. Taking the derivative of (47), we obtain

$$\begin{aligned}\varphi'(t) = & \int_\Omega |u_t(t)|^2 dx + \int_\Omega u(t)u_{tt}(t)dx + \int_\Omega |v_t(t)|^2 dx \\ & + \int_\Omega v(t)v_{tt}(t)dx.\end{aligned} \tag{49}$$

From problem (19) and using integration by parts, we get

$$\begin{aligned}
\varphi'(t) &= \|u_t(t)\|^2 + \|v_t(t)^2\| + \int_\Omega u(t)\left(\Delta_e u - \int_0^t g_1(t-s)\Delta u(s)ds\right.\\
&\quad \left. + k_1\Delta u_t + \int_{\tau_1}^{\tau_2}\mu_1(\varrho)\Delta z(x,1,\varrho,t)d\varrho + f_1(u,v)\right)dx\\
&\quad + \int_\Omega v(t)\left(\Delta_e v - \int_0^t g_2(t-s)\Delta v(s)ds + k_2\Delta v_t\right.\\
&\quad \left. + \int_{\tau_1}^{\tau_2}\mu_2(\varrho)\Delta y(x,1,\varrho,t)d\varrho + f_2(u,v)\right)dx\\
&= \|u_t(t)\|^2 + \|v_t(t)\|^2 - k_1\int_\Omega \nabla u\nabla u_t dx - k_2\int_\Omega \nabla v\nabla v_t dx\\
&\quad - u\|\nabla u(t)\|^2 - (\lambda+\mu)\|divu(t)\|^2\\
&\quad + \int_\Omega \nabla u(t)\int_0^t g_1(t-s)\nabla u(s)ds - \int_\Omega \nabla u(t)\int_{\tau_1}^{\tau_2}\mu_1(\varrho)\nabla z(x,1,\varrho,t)d\varrho\\
&\quad + \int_\Omega u(t)f_1(u,v)dx - \mu\|\nabla v(t)\|^2 - (\lambda+\mu)\|divv(t)\|^2\\
&\quad + \int_\Omega \nabla v(t)\int_0^t g_2(t-s)\nabla v(s)ds - \int_\Omega \nabla v(t)\int_{\tau_1}^{\tau_2}\mu_2(\varrho)\nabla y(x,1,\varrho,t)d\varrho\\
&\quad + \int_\Omega v(t)f_2(u,v)dx = \|u_t(t)\|^2 + \|v_t(t)\|^2 - k_1\int_\Omega \nabla u\nabla u_t dx\\
&\quad - k_2\int_\Omega \nabla v\nabla v_t dx - \left(\mu - \int_0^t g_1(s)ds\right)\|\nabla u(t)\|^2 - (\lambda+\mu)\|divu(t)\|^2\\
&\quad + \int_\Omega \nabla u(t)\int_0^t g_1(t-s)(\nabla u(s)-\nabla u(t))dsdx\\
&\quad - \int_\Omega \nabla u(t)\int_{\tau_1}^{\tau_2}\mu_1(\varrho)\nabla z(x,1,\varrho,t)d\varrho dx + \int_\Omega u(t)f_1(u,v)dx\\
&\quad - \left(\mu - \int_0^t g_2(s)ds\right)\|\nabla v(t)\|^2 - (\lambda+\mu)\|divv(t)\|^2\\
&\quad + \int_\Omega \nabla v(t)\int_0^t g_2(t-s)(\nabla v(s)-\nabla v(t))dsdx\\
&\quad - \int_\Omega \nabla u(t)\int_{\tau_1}^{\tau_2}\mu_2(\varrho)\nabla y(x,1,\varrho,t)d\varrho dx + \int_\Omega v(t)f_2(u,v)dx.
\end{aligned} \tag{50}$$

By using Hölder and Young's inequalities, we have

$$\begin{aligned}
&\int_\Omega \nabla u(t)\int_0^t g_1(t-s)(\nabla u(s)-\nabla u(t))dsdx\\
&\le \frac{l_1}{6}\|\nabla u(t)\|^2 + \frac{3}{2l_1}\int_\Omega\int_0^t (g_1(t-s)(\nabla u(s)-\nabla u(t))ds)^2 dx\\
&\le \frac{l_1}{6}\|\nabla u(t)\|^2 + \frac{3}{2l_1}\int_\Omega\int_0^t\\
&\quad \times\left(\frac{g_1(t-s)}{\sqrt{\zeta g_1(t-s)-g_1'(t-s)}}\sqrt{\zeta g_1(t-s)-g_1'(t-s)}(\nabla u(s)-\nabla u(t))ds\right)^2 dx\\
&\le \frac{l_1}{6}\|\nabla u(t)\|^2 + \frac{3}{2l_1}\left(\int_0^t \frac{g_1(s)}{\sqrt{\zeta g_1(s)-g_1'(s)}}ds\right)\int_\Omega\int_0^t\\
&\quad \times\left(\zeta g_1(t-s)-g_1'(t-s)\right)|\nabla u(s)-\nabla u(t)|^2 ds\\
&\le \frac{l_1}{6}\|\nabla u(t)\|^2 + \frac{3C_{\zeta,1}}{2l_1}(h_1\circ\nabla u)(t).
\end{aligned} \tag{51}$$

Similarly, we obtain

$$\int_\Omega \nabla v(t)\int_0^t g_2(t-s)(\nabla v(s)-\nabla v(t))dsdx \le \frac{l_2}{6}\|\nabla v(t)\|^2 + \frac{3C_{\zeta,2}}{2l_2}(h_2\circ\nabla v)(t),$$

$$\begin{aligned}
&\int_\Omega \nabla u(t)\int_{\tau_1}^{\tau_2}\mu_1(\rho)\nabla z(x,1,\rho,t)d\rho dx\\
&\le \frac{l_1}{6}\|\nabla u(t)\|^2 + \frac{3}{2l_1}\int_\Omega\left(\int_{\tau_1}^{\tau_2}|\mu_1(\rho)|\nabla z(x,1,\rho,t)d\rho\right)^2 dx\\
&\le \frac{l_1}{6}\|\nabla u(t)\|^2 + \frac{3}{2l_1}\int_\Omega\left(\int_{\tau_1}^{\tau_2}\sqrt{|\mu_1(\rho)|}\sqrt{|\mu_1(\rho)|}\nabla z(x,1,\rho,t)d\rho\right)^2\\
&\le \frac{l_1}{6}\|\nabla u(t)\|^2 + \frac{3}{2l_1}\left(\int_{\tau_1}^{\tau_2}|\mu_1(\rho)|d\rho\right)\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_1(\rho)||\nabla z(x,1,\rho,t)|^2 d\rho\\
&\le \frac{l_1}{6}\|\nabla u(t)\|^2 + \frac{3k_1}{2l_1}\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_1(\rho)||\nabla z(x,1,\rho,t)|^2 d\rho,
\end{aligned}$$

$$\begin{aligned}
&\int_\Omega \nabla v(t)\int_{\tau_1}^{\tau_2}\mu_2(\rho)\nabla y(x,1,\rho,t)d\rho dx\\
&\le \frac{l_2}{6}\|\nabla v(t)\|^2 + \frac{3}{2l_2}\left(\int_{\tau_1}^{\tau_2}|\mu_2(\rho)|d\rho\right)\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(\rho)||\nabla y(x,1,\rho,t)|^2 d\rho\\
&\le \frac{l_2}{6}\|\nabla v(t)\|^2 + \frac{3k_2}{2l_2}\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(\rho)||\nabla y(x,1,\rho,t)|^2 d\rho.
\end{aligned} \tag{52}$$

The Young's inequality gives

$$\begin{aligned}
k_1\int_\Omega \nabla u(t)\nabla u_t(t)dx &\le \frac{l_1}{6}\|\nabla u(t)\|^2 + \frac{3k_1^2}{2l_1}\|\nabla u_t(t)\|^2,\\
k_2\int_\Omega \nabla v(t)\nabla v_t(t)dx &\le \frac{l_2}{6}\|\nabla v(t)\|^2 + \frac{3k_2^2}{2l_2}\|\nabla v_t(t)\|^2.
\end{aligned} \tag{53}$$

For the source term, we have

$$\int_\Omega u(t)f_1(u,v)dx + \int_\Omega v(t)f_2(u,v)dx = (p+1)\int_\Omega F(u,v)dx. \tag{54}$$

Combining the equations (51)–(54), thus, our proof is completed.

Lemma 6. *Let (u, v, z, y) be a solution of the problem (19). Then, the functional*

$$\begin{aligned}
\psi(t) &= \int_\Omega u_t(t)\int_0^t g_1(s)(u(s)-u(t))dsdx\\
&\quad + \int_\Omega v_t(t)\int_0^t g_2(s)(v(s)-v(t))dsdx = \psi_1(t)+\psi_2(t),
\end{aligned} \tag{55}$$

satisfies for any $\delta > 0$ the estimate

$$\begin{aligned}\psi'(t) \le{}& (\delta + \delta\Lambda_3 l_1)\|\nabla u(t)\|^2 + \delta\|divu\|^2 + \delta\Lambda_3 l_2\|\nabla v(t)\|^2 \\ &+ \left(\delta - \int_0^t g_1(s)ds\right)\|u_t(t)\|^2 + \frac{c[C_{\zeta,1}+1]}{\delta}(h_1\circ\nabla u)(t) \\ &+ \delta k_1\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_1(\rho)|\|\nabla z(x,1,\rho,t)|^2 d\rho dx + \delta k_1\|\nabla u_t\|^2 \\ &+ (\delta + \delta\Lambda_4 l_2)\|\nabla v(t)\|^2 + \delta\|divv\|^2 + \delta\Lambda_4 l_1\|\nabla u(t)\|^2 \\ &+ \left(\delta - \int_0^t g_2(s)ds\right)\|v_t(t)\|^2 + \frac{c[C_{\zeta,2}+1]}{\delta}(h_2\circ\nabla v)(t) \\ &+ \delta k_2\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(\rho)|\|\nabla y(x,1,\rho,t)|^2 d\rho dx + \delta k_1\|\nabla v_t\|^2,\end{aligned} \tag{56}$$

where Λ_3 and Λ_4 are two positive constants.

Proof. First, we begin to estimate $\psi_1'(t)$

$$\begin{aligned}\psi_1'(t) ={}& \int_\Omega u_{tt}(t)\int_0^t g_1(t-s)(u(s)-u(t))dsdx \\ &+ \int_\Omega u_t(t)\int_0^t g_1'(t-s)(u(s)-u(t))dsdx - \left(\int_0^t g_1(s)ds\right)\|u_t(t)\|^2 \\ ={}& \int_\Omega\left(\Delta_e u - \int_0^t g_1(t-s)\Delta u(s)ds\right. \\ &\left.+ k_1\Delta u + \int_{\tau_1}^{\tau_2}\mu_1(\rho)\Delta z(x,1,\rho,t)d\rho + f_1(u,v)\right) \\ &\times\left(\int_0^t g_1(t-s)(u(s)-u(t))dsdx\right) \\ &+ \int_\Omega u_t(t)\int_0^t g_1'(t-s)(u(s)-u(t))dsdx \\ &- \left(\int_0^t g_1(s)ds\right)\|u_t(t)\|^2 \\ ={}& \int_\Omega u_t(t)\int_0^t g_1'(t-s)(u(s)-u(t))dsdx \\ &- \left(\int_0^t g_1(s)ds\right)\|u_t(t)\|^2 - \mu\int_\Omega\nabla u\int_0^t g_1(t-s)(\nabla u(s)-\nabla u(t))dsdx \\ &- (\lambda+\mu)\int_\Omega divu(t)\int_0^t g_1(t-s)(divu(s) \\ &- divu(t))dsdx + \left(\int_0^t g_1(t-s)\nabla u(s)ds\right) \\ &\cdot\left(\int_0^t g_1(t-s)(\nabla u(s)-\nabla u(t))dsdx\right) \\ &+ \int_\Omega\left(\int_{\tau_1}^{\tau_2}\mu_1(\rho)\nabla z(x,1,\rho,t)d\rho\int_0^t g_1(t-s)(\nabla u(s)-\nabla u(t))ds\right)dx \\ &+ \int_\Omega f_1(u,v)\int_0^t g_1(t-s)(u(s)-u(t))dsdx \\ &- k_1\int_\Omega\nabla u_t\int_0^t g_1(t-s)(\nabla u(s)-\nabla u(t))dsdx.\end{aligned} \tag{57}$$

Then, we have

$$\begin{aligned}\psi_1'(t) ={}& \int_\Omega u_t(t)\int_0^t g_1'(t-s)(u(s)-u(t))dsdx \\ &- \left(\int_0^t g_1(s)ds\right)\|u_t(t)\|^2 \\ &- \left(\mu - \int_0^t g_1(s)ds\right)\int_\Omega\nabla u\int_0^t g_1(t-s)(\nabla u(s)-\nabla u(t))dsdx \\ &- (\lambda+\mu)\int_\Omega divu(t)\int_0^t g_1(t-s)(divu(s)-divu(t))dsdx \\ &+ \left(\int_\Omega\int_0^t g_1(t-s)(\nabla u(s)-\nabla u(t))dsdx\right)^2 \\ &- k_1\int_\Omega\nabla u_t\int_0^t g_1(t-s)(\nabla u(s)-\nabla u(t))dsdx \\ &+ \int_\Omega\left(\int_{\tau_1}^{\tau_2}\mu_1(\rho)\nabla z(x,1,\rho,t)d\rho.\int_0^t g_1(t-s)(\nabla u(s)-\nabla u(t))ds\right)dx \\ &+ \int_\Omega f_1(u,v)\int_0^t g_1(t-s)(u(s)-u(t))dsdx.\end{aligned} \tag{58}$$

As in previous proof and by using Young's inequality, we conclude that for any $\delta > 0$,

$$\begin{aligned}&\left(\mu - \int_0^t g_1(s)ds\right)\int_\Omega\nabla u\int_0^t g_1(t-s)(\nabla u(s)-\nabla u(t))dsdx \\ &\quad\le \delta\|\nabla u\|^2 + \frac{c.C_{\zeta,1}}{\delta}(h_1\circ\nabla u)(t).\end{aligned} \tag{59}$$

Similarly and by using the fact $\|divu\|^2 \le c\|\nabla u\|^2$, we have

$$\begin{aligned}&(\lambda+\mu)\int_\Omega divu(t)\int_0^t g_1(t-s)(divu(s)-divu(t))dsdx \\ &\quad\le \delta\|divu\|^2 + \frac{c.C_{\zeta,1}}{\delta}(h_1\circ\nabla u)(t).\end{aligned} \tag{60}$$

The same argument for

$$\begin{aligned}&\int_\Omega f_1(u,v)\int_0^t g_1(t-s)(u(s)-u(t))dsdx \\ &\quad\le \delta\int_\Omega|f_1(u,v)|^2dx + \frac{1}{4\delta}\int_\Omega\left(\int_0^t g_1(t-s)(u(s)-u(t))ds\right)^2 dx \\ &\quad\le \delta\Lambda_1\left(l_1\|\nabla u\|^2 + l_2\|\nabla v\|^2\right)^p + \frac{c.C_{\zeta,1}}{\delta}(h_1\circ\nabla u)(t) \\ &\quad\le \delta\Lambda_1\left[\frac{2(p+1)}{p-1}E(0)\right]^{p-1}\left(l_1\|\nabla u\|^2 + l_2\|\nabla v\|^2\right) + \frac{c.C_{\zeta,1}}{\delta}(h_1\circ\nabla u)(t) \\ &\quad\le \delta\Lambda_3\left(l_1\|\nabla u\|^2 + l_2\|\nabla v\|^2\right) + \frac{c.C_{\zeta,1}}{\delta}(h_1\circ\nabla u)(t),\end{aligned} \tag{61}$$

where $\Lambda_3 = [((2(p+1))/(p-1))E(0)]^{p-1}$.

From (46), we have

$$\begin{aligned}
&\int_\Omega u_t(t)\int_0^t g_1'(t-s)(u(s)-u(t))dsdx \\
&= \int_\Omega u_t(t)\int_0^t h_1(t-s)(u(s)-u(t))dsdx \\
&\quad - \int_\Omega u_t(t)\int_0^t \zeta g_1(t-s)(u(s)-u(t))dsdx \le \delta\|u_t(t)\|^2 \\
&\quad + \frac{c}{\delta}\int_\Omega\left(\int_0^t \sqrt{h_1(t-s)}\sqrt{h_1(t-s)}(u(s)-u(t))ds\right)^2 dx \\
&\quad + \frac{c\zeta^2}{\delta}\int_\Omega\left(\int_0^t g_1(t-s)(u(s)-u(t))ds\right)^2 dx \le \delta\|u_t(t)\|^2 \\
&\quad + \frac{c}{\delta}\left(\int_0^t h_1(s)ds\right)(h_1\circ u)(t) + \frac{c\zeta^2 C_{\zeta,1}}{\delta}(h_1\circ u)(t) \le \delta\|u_t(t)\|^2 \\
&\quad + \frac{c}{\delta}(h_1\circ\nabla u)(t) + \frac{c\zeta^2 C_{\zeta,1}}{\delta}(h_1\circ\nabla u)(t) \le \delta\|u_t(t)\|^2 \\
&\quad + \frac{c\left[C_{\zeta,1}+1\right]}{\delta}(h_1\circ\nabla u)(t),
\end{aligned}$$

$$\begin{aligned}
&\int_\Omega\left(\int_{\tau_1}^{\tau_2}\mu_1(\rho)\nabla z(x,1,\rho,t)d\rho . \int_0^t g_1(t-s)(\nabla u(s)-\nabla u(t))ds\right)dx \\
&\le \delta\int_\Omega\left(\int_{\tau_1}^{\tau_2}|\mu_1(\rho)|\nabla z(x,1,\rho,t)d\rho\right)^2 dx \\
&\quad + \frac{1}{4\delta}\int_\Omega\left(\int_0^t g_1(t-s)(\nabla u(s)-\nabla u(t))ds\right)^2 dx \\
&\le \delta\left(\int_{\tau_1}^{\tau_2}|\mu_1(\rho)|d\rho\right)\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_1(\rho)||\nabla z(x,1,\rho,t)|^2 d\rho dx \\
&\quad + \frac{cC_{\zeta,1}}{\delta}(h_1\circ\nabla u)(t) \le \delta k_1\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_1(\rho)||\nabla z(x,1,\rho,t)|^2 d\rho dx \\
&\quad + \frac{cC_{\zeta,1}}{\delta}(h_1\circ\nabla u)(t).
\end{aligned} \tag{62}$$

Finally, Young's inequality gives

$$\begin{aligned}
&k_1\int_\Omega \nabla u_t\int_0^t g_1(t-s)(u(s)-u(t))dsdx \\
&\le \delta k_1\|\nabla u_t\|^2 + \frac{k_1}{4\delta}\int_\Omega\left(\int_0^t g_1(t-s)(u(s)-u(t))ds\right)^2 dx \\
&\le \delta k_1\|\nabla u_t\| + \frac{cC_{\zeta,1}}{\delta}(h_1\circ\nabla u)(t).
\end{aligned} \tag{63}$$

Then

$$\begin{aligned}
\psi_1'(t) \le{}& \left(\delta - \left(\int_0^t g_1(s)ds\right)\right)\|u_t(t)\|^2 + (\delta+\delta\Lambda_3 l_1)\|\nabla u\|^2 \\
&+ \delta\|divu\|^2 + \delta\Lambda_3 l_2\|\nabla v\|^2 + \frac{c\left[C_{\zeta,1}+1\right]}{\delta}(h_1\circ\nabla u)(t) \\
&+ \delta k_1\|\nabla u_t\|^2 + \delta k_1\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_1(\rho)||\nabla z(x,1,\rho,t)|^2 d\rho dx.
\end{aligned} \tag{64}$$

The same steps can be taken to get the next estimate for $\psi_2'(t)$,

$$\begin{aligned}
\psi_2'(t) \le{}& \left(\delta - \left(\int_0^t g_2(s)ds\right)\right)\|v_t(t)\|^2 + (\delta+\delta\Lambda_4 l_2)\|\nabla v\|^2 \\
&+ \delta\|divv\|^2 + \delta\Lambda_4 l_1\|\nabla u\|^2 + \frac{c\left[C_{\zeta,2}+1\right]}{\delta}(h_2\circ\nabla v)(t) \\
&+ \delta k_2\|\nabla v_t\|^2 + \delta k_2\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(\rho)||\nabla y(x,1,\rho,t)|^2 d\rho dx,
\end{aligned} \tag{65}$$

where $\Lambda_4 = \Lambda_2[((2(p+1))/(p-1))E(0)]^{(p-1)}$.

Lemma 7. *Let (u, v, z, y) be a solution of the problem (19). Then, the functional*

$$\begin{aligned}
I(t) = \int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} \rho e^{-\rho\rho}\left[|\mu_1(\rho)||\nabla z(x,\rho,\rho,t)|^2\right. \\
\left. + |\mu_2(\rho)||\nabla y(x,\rho,\rho,t)|^2\right] dxd\rho d\rho,
\end{aligned} \tag{66}$$

satisfies the estimate

$$\begin{aligned}
I'(t) \le{}& -e^{-\tau_2}\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_1(\rho)||\nabla z(x,1,\rho,t)|^2 dxd\rho \\
&- e^{-\tau_2}\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(\rho)||\nabla y(x,1,\rho,t)|^2 dxd\rho \\
&+ k_1\|\nabla u_t(t)\|^2 + k_2\|\nabla v_t(t)\|^2 - I(t).
\end{aligned} \tag{67}$$

Proof. Differentiating (66) with respect to t, we get

$$\begin{aligned}
\frac{d}{dt}I(t) = 2\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} \rho e^{-\rho\rho}[|\mu_1(\rho)|\nabla z(x,\rho,\rho,t)\nabla z_t(x,\rho,\rho,t) \\
+ |\mu_2(\rho)|\nabla y(x,\rho,\rho,t)\nabla y_t(x,\rho,\rho,t)]dxd\rho d\rho.
\end{aligned} \tag{68}$$

By using (17) and (18), we have

$$\begin{aligned}
\frac{d}{dt}I(t) ={}& -\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} e^{-\rho\rho}\left[|\mu_1(\rho)|\frac{\partial}{\partial\rho}|\nabla z(x,\rho,\rho,t)|^2\right. \\
&\left. + |\mu_2(\rho)|\frac{\partial}{\partial\rho}|\nabla y(x,\rho,\rho,t)|^2\right] dxd\rho d\rho \\
={}& -\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2}\left[|\mu_1(\rho)|\frac{\partial}{\partial\rho}\left(e^{-\rho\rho}\nabla z(x,\rho,\rho,t)^2\right)\right. \\
&\left. + |\mu_2(\rho)|\frac{\partial}{\partial\rho}\left(e^{-\rho\rho}|\nabla y(x,\rho,\rho,t)|^2\right)\right] dxd\rho d\rho \\
&- \int_\Omega\int_0^1\int_{\tau_1}^{\tau_2}\rho e^{-\rho\rho}\left[|\mu_1(\rho)||\nabla z(x,\rho,\rho,t)|^2\right. \\
&\left. + |\mu_2(\rho)||\nabla y(x,\rho,\rho,t)|^2\right] dxd\rho d\rho.
\end{aligned} \tag{69}$$

Thus,

$$\begin{aligned}\frac{d}{dt}I(t) = &-\int_\Omega \int_{\tau_1}^{\tau_2} e^{-\rho}|\mu_1(\rho)|\|\nabla z(x,1,\rho,t)|^2 dx d\rho \\ &+ \left(\int_{\tau_1}^{\tau_2} |\mu_1(\rho)| d\rho\right) \|\nabla u_t(x,\rho,t)\|^2 \\ &- \int_\Omega \int_{\tau_1}^{\tau_2} e^{-\rho}|\mu_2(\rho)|\|\nabla y(x,1,\rho,t)|^2 dx d\rho \\ &+ \left(\int_{\tau_1}^{\tau_2} |\mu_2(\rho)| d\rho\right) \|\nabla v_t(x,\rho,t)\|^2 - \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} \rho e^{-\rho\rho} \\ &\cdot \left[|\mu_1(\rho)|\|\nabla z(x,\rho,\rho,t)|^2 + |\mu_2(\rho)|\|\nabla y(x,\rho,\rho,t)|^2\right] dx d\rho d\rho. \end{aligned} \tag{70}$$

Since $e^{-\rho}$ is deceasing function over (τ_1, τ_2), the desired estimate (67) follows immediately from (25).

The following lemmas are needed to prove the general decay when the functions $G_i(t) (i = 1, 2)$ are nonlinear. The proof can be found in Mustafa [19].

Lemma 8. *The functional*

$$\theta_1(t) = \int_\Omega \int_0^t \sigma_1(t-s)|\nabla u(s)|^2 ds dx, \tag{71}$$

where $\sigma_1(t) = \int_t^\infty g_1(s)ds$ *satisfies*

$$\theta_1'(t) \le -\frac{1}{2}(g_1 \circ \nabla u)(t) + 3(\mu - l_1)\|\nabla u\|^2. \tag{72}$$

Lemma 9. *The functional*

$$\theta_2(t) = \int_\Omega \int_0^t \sigma_2(t-s)|\nabla v(s)|^2 ds dx, \tag{73}$$

where $\sigma_2(t) = \int_t^\infty g_2(s)ds$ *satisfies*

$$\theta_2'(t) \le -\frac{1}{2}(g_2 \circ \nabla v)(t) + 3(\mu - l_2)\|\nabla v\|^2. \tag{74}$$

Now, we define the following functional

$$\mathscr{F}(t) = NE(t) + N_1\phi(t) + N_2\psi(t) + I(t), \tag{75}$$

where N, N_1, and N_2 are positive constants. It is easy to prove $F(t)$ and $E(t)$ are equivalent, namely, there exist two positive constants κ_1 and κ_2 such that

$$\kappa_1 E(t) \le \mathscr{F}(t) \le \kappa_2 E(t). \tag{76}$$

By Young's inequality, we get

$$\begin{aligned}\mathscr{F}(t) \le &\left(\frac{N}{2} + \frac{N_1}{2} + \frac{N_2}{2}\right)\left[\|u_t\|^2 + \|v_t\|^2\right] \\ &+ \left(\frac{N}{2}\left(\mu - \int_0^t g_1(s)ds\right) + c\frac{N_1}{2} + c\frac{N_2}{2}\right)\|\nabla u\|^2 \\ &+ \left(\frac{N}{2}\left(\mu - \int_0^t g_2(s)ds\right) + c\frac{N_1}{2} + c\frac{N_2}{2}\right)\|\nabla v\|^2 \\ &+ \left(\frac{N}{2} + c\frac{N_2}{2}\left(\int_0^t g_1(s)ds\right)\right)(g_1 \circ \nabla u)(t) \\ &+ \left(\frac{N}{2} + c\frac{N_2}{2}\left(\int_0^t g_2(s)ds\right)\right)(g_2 \circ \nabla u)(t) \\ &+ \left(\frac{N}{2}\eta + C\right)\int_\Omega \int_0^t \int_{\tau_1}^{\tau_2} \rho|\mu_1(\rho)|\|\nabla z(x,\rho,\rho,t)|^2 d\rho d\rho dx \\ &+ \left(\frac{N}{2}\eta + C\right)\int_\Omega \int_0^t \int_{\tau_1}^{\tau_2} \rho|\mu_2(\rho)|\|\nabla y(x,\rho,\rho,t)|^2 d\rho d\rho dx \\ &+ \frac{N}{2}(\lambda + \mu)\left[\|divu\|^2 + \|divv\|^2\right] - N\int_\Omega F(u,v)dx. \end{aligned} \tag{77}$$

Then, for any N, there exists $\kappa_1 > 0$ such that

$$\mathscr{F} \le \kappa_1 E(t). \tag{78}$$

On the other hand, we can find

$$\begin{aligned}\mathscr{F}(t) \ge &\left(\frac{N}{2} - \frac{N_1}{2} - \frac{N_2}{2}\right)\left[\|u_t\|^2 + \|v_t\|^2\right] \\ &+ \left(\frac{N}{2}\left(\mu - \int_0^t g_1(s)ds\right) - c\frac{N_1}{2} - c\frac{N_2}{2}\right)\|\nabla u\|^2 \\ &+ \left(\frac{N}{2}\left(\mu - \int_0^t g_2(s)ds\right) - c\frac{N_1}{2} - c\frac{N_2}{2}\right)\|\nabla v\|^2 \\ &+ \left(\frac{N}{2} - c\frac{N_2}{2}\left(\int_0^t g_1(s)ds\right)\right)(g_1 \circ \nabla u)(t) \\ &+ \left(\frac{N}{2} - c\frac{N_2}{2}\left(\int_0^t g_2(s)ds\right)\right)(g_2 \circ \nabla u)(t) \\ &+ \left(\frac{N}{2}\eta + c\right)\int_\Omega \int_0^t \int_{\tau_1}^{\tau_2} \rho|\mu_1(\rho)|\|\nabla z(x,\rho,\rho,t)|^2 d\rho d\rho dx \\ &+ \left(\frac{N}{2}\eta + c\right)\int_\Omega \int_0^t \int_{\tau_1}^{\tau_2} \rho|\mu_2(\rho)|\|\nabla y(x,\rho,\rho,t)|^2 d\rho d\rho dx \\ &+ \frac{N}{2}(\lambda + \mu)\left[\|divu\|^2 + \|divv\|^2\right] - N\int_\Omega F(u,v)dx. \end{aligned} \tag{79}$$

We choose N large enough so that

$$\begin{aligned}&\frac{N}{2} - \frac{N_1}{2} - \frac{N_2}{2} > 0,\ \frac{N}{2}\left(\mu - \int_0^t g_i(s)ds\right) - c\frac{N_1}{2} - c\frac{N_2}{2} > 0, \\ &\frac{N}{2} - c\frac{N_2}{2}\left(\int_0^t g_i(s)ds\right) > 0,\ i = 1, 2. \end{aligned} \tag{80}$$

Then, there exist $\kappa_2 > 0$ such that

$$\mathscr{F}(t) \geq \kappa_2 E(t). \quad (81)$$

Lemma 10. *The functional $\mathscr{F}(t)$ satisfies for any $t \geq t_1$,*

$$\begin{aligned}\mathscr{F}'(t) \leq & -4(\mu - l_1)\|\nabla u(t)\|^2 - 4(\mu - l_2)\|\nabla v(t)\|^2 - \|u_t(t)\|^2 \\ & - \|v_t(t)\|^2 - \|divu(t)\|^2 - \|divv(t)\|^2 + \frac{1}{4}(g_1 \circ \nabla u)(t) \\ & + \frac{1}{4}(g_2 \circ \nabla v)(t) + c\int_\Omega F(u(t), v(t))dx \\ & - \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} \rho|\mu_1(\rho)|\|\nabla z(x, \rho, \rho, t)\|^2 d\rho d\rho dx \\ & - \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} \rho|\mu_2(\rho)|\|\nabla y(x, \rho, \rho, t)\|^2 d\rho d\rho dx.\end{aligned} \quad (82)$$

Proof. Let

$$g_0 = \min\left\{\int_0^{t_1} g_1(s)ds, \int_0^{t_1} g_2(s)ds\right\}. \quad (83)$$

From Lemmas 5, 6, and 7, noting that $g_i' = \zeta g_i - h_i$ we have for any $t \geq t_1$,

$$\begin{aligned}\mathscr{F}'(t) \leq & -\left(\frac{l_1}{2}N_1 - N_2\delta(1 + \Lambda_3 l_1) - N_2\delta\Lambda_4 l_1\right)\|\nabla u(t)\|^2 \\ & - \left(\frac{l_2}{2}N_1 - N_2\delta(1 + \Lambda_4 l_2) - N_2\delta\Lambda_3 l_2\right)\|\nabla v(t)\|^2 \\ & - (g_0 N_2 - \delta N_2 - N_1)\left[\|u_t(t)\|^2 + \|v_t(t)\|^2\right] \\ & + \frac{\zeta N}{2}[(g_1 \circ \nabla u)(t) + (g_2 \circ \nabla v)(t)] \\ & + N_1(p+1)\int_\Omega F(u(t), v(t))dx \\ & - \left[\frac{N}{2} - N_2\frac{c[C_{\zeta,1} + 1]}{\delta} - \frac{3N_1 C_{\zeta,1}}{2l_1}\right](h_1 \circ \nabla u)(t) \\ & - \left[\frac{N}{2} - N_2\frac{c[C_{\zeta,2} + 1]}{\delta} - \frac{3N_1 C_{\zeta,2}}{2l_2}\right](h_2 \circ \nabla v)(t) \\ & - [(\lambda + \mu)N_1 - \delta N_2]\left[\|divu(t)\|^2 + \|divv(t)\|^2\right] \\ & - \left[N\sigma + e^{-\tau_2} - N_1\frac{3}{2l_1}k_1 - \delta N_2 k_1\right]\int_\Omega \int_{\tau_1}^{\tau_2} \\ & \cdot |\mu_1(\rho)|\|\nabla z(x, 1, \rho, t)\|^2 d\rho dx \\ & - \left[N\sigma + e^{-\tau_2} - N_1\frac{3}{2l_2}k_2 - \delta N_2 k_2\right]\int_\Omega \int_{\tau_1}^{\tau_2} \\ & \cdot |\mu_2(\rho)|\|\nabla y(x, 1, \rho, t)\|^2 d\rho dx \\ & - \left[N\sigma_1 - N_1\frac{3k_1^2}{2l_1} - \delta N_2 k_1 - k_1\right]\|\nabla u_t(t)\|^2 \\ & - \left[N\sigma_2 - N_1\frac{3k_2^2}{2l_2} - \delta N_2 k_2 - k_2\right]\|\nabla v_t(t)\|^2 - I(t),\end{aligned} \quad (84)$$

where

$$\sigma_1 = \left[k_1 - \left(\frac{\eta + 1}{2}\right)\left(\int_{\tau_1}^{\tau_2} |\mu_1(\rho)|d\rho\right)\right], \sigma_2 = \left[k_2 - \left(\frac{\eta + 1}{2}\right)\left(\int_{\tau_1}^{\tau_2} |\mu_2(\rho)|d\rho\right)\right],$$

$$\sigma = \left(\frac{\eta - 1}{2}\right). \quad (85)$$

Taking $\delta = 1/2N_2$, we can get

$$\begin{aligned}\mathscr{F}'(t) \leq & -\left(\frac{l_1}{2}N_1 - \frac{1}{2}(1 + \Lambda_3 l_1) - \frac{1}{2}\Lambda_4 l_1\right)\|\nabla u(t)\|^2 \\ & - \left(\frac{l_2}{2}N_1 - \frac{1}{2}(1 + \Lambda_4 l_2) - \frac{1}{2}\Lambda_3 l_2\right)\|\nabla v(t)\|^2 \\ & - \left(g_0 N_2 - \frac{1}{2} - N_1\right)\left[\|u_t(t)\|^2 + \|v_t(t)\|^2\right] \\ & + \frac{\zeta N}{2}[(g_1 \circ \nabla u)(t) + (g_2 \circ \nabla v)(t)] \\ & + N_1(p+1)\int_\Omega F(u(t), v(t))dx \\ & - \left[\frac{N}{2} - 2cN_2^2 - C_{\zeta,1}\left(2cN_2^2 + \frac{3N_1}{2l_1}\right)\right](h_1 \circ \nabla u)(t) \\ & - \left[\frac{N}{2} - 2cN_2^2 - C_{\zeta,2}\left(2cN_2^2 + \frac{3N_1}{2l_2}\right)\right](h_2 \circ \nabla v)(t) \\ & - \left[(\lambda + \mu)N_1 - \frac{1}{2}\right]\left[\|divu(t)\|^2 + \|divv(t)\|^2\right] \\ & - \left[N\sigma + e^{-\tau_2} - N_1\frac{3}{2l_1}k_1 - \frac{k_1}{2}\right]\int_\Omega \int_{\tau_1}^{\tau_2} \\ & \cdot |\mu_1(\rho)|\|\nabla z(x, 1, \rho, t)\|^2 d\rho dx \\ & - \left[N\sigma + e^{-\tau_2} - N_1\frac{3}{2l_2}k_2 - \frac{k_2}{2}\right]\int_\Omega \int_{\tau_1}^{\tau_2} \\ & \cdot |\mu_2(\rho)|\|\nabla y(x, 1, \rho, t)\|^2 d\rho dx \\ & - \left[N\sigma_1 - N_1\frac{3k_1^2}{2l_1} - \frac{3k_1}{2}\right]\|\nabla u_t(t)\|^2 \\ & - \left[N\sigma_2 - N_1\frac{3k_2^2}{2l_2} - \frac{3k_2}{2}\right]\|\nabla v_t(t)\|^2 - I(t).\end{aligned} \quad (86)$$

First, we take $N_1 > 0$ large such that

$$\begin{aligned}(\lambda + \mu)N_1 - \frac{1}{2} > 0, \left(\frac{l_1}{2}N_1 - \frac{1}{2}(1 + \Lambda_3 l_1) - \frac{1}{2}\Lambda_4 l_1\right) > 4(\mu - l_1), \\ \left(\frac{l_2}{2}N_1 - \frac{1}{2}(1 + \Lambda_3 l_2) - \frac{1}{2}\Lambda_4 l_2\right) > 4(\mu - l_2).\end{aligned} \quad (87)$$

We choose $N_2 > 0$ large enough so that

$$g_0 N_2 - \frac{1}{2} - N_1 > 1. \tag{88}$$

Note that

$$0 < \frac{\zeta g_i^2(s)}{\zeta g_i(s) - g_i'(s)} < \frac{\zeta g_i^2(s)}{-g_i'(s)},\ i = 1, 2. \tag{89}$$

Then, for any $s \in [0,\infty)$, we get

$$\lim_{\zeta\to 0} \frac{\zeta g_i^2(s)}{\zeta g_i(s) - g_i{}'(s)} = 0,\ i = 1, 2. \tag{90}$$

By using the fact $\zeta g_i^2(s)/(\zeta g_i(s) - g_i'(s)) < g_i(s), i = 1, 2$ and using Lebesgue-dominated convergence theorem, we can get

$$\lim_{\zeta\to 0} \zeta C_{\zeta,i} = \lim_{\zeta\to 0} \int_0^\infty \frac{\zeta g_i^2(s)}{\zeta g_i(s) - g_i'(s)} = 0,\ i = 1, 2. \tag{91}$$

Thus, there exist some $\zeta_0 (0 < \zeta_0 < 1)$ such that if $\zeta < \zeta_0$, then

$$\zeta C_{\zeta,2} < \frac{1}{8\left[(N_1/2l_1) + 2cN_2^2\right]} \text{ and } \zeta C_{\zeta,2} < \frac{1}{8\left[(N_1/2l_2) + 2cN_2^2\right]}. \tag{92}$$

At last, we choose N large enough and choose ζ satisfying

$$\frac{1}{4}N - 2cN_2^2 > 0 \text{ and } \zeta = \frac{1}{2N} > \zeta_0, \tag{93}$$

so, we arrive at

$$\begin{aligned} &\left[\frac{N}{2} - 2cN_2^2 - C_{\zeta,1}\left(2cN_2^2 + \frac{3N_1}{2l_1}\right)\right] > 0 \text{ and} \\ &\left[\frac{N}{2} - 2cN_2^2 - C_{\zeta,1}\left(2cN_2^2 + \frac{3N_1}{2l_1}\right)\right] > 0. \end{aligned} \tag{94}$$

Therefore, we choose N even larger (if needed) so that

$$\begin{aligned} &\left[Nn_1 + e^{-\tau_2} - N_1 \frac{3}{2l_1} k_1 - \frac{k_1}{2}\right] > 0, \left[Nn_1 + e^{-\tau_2} - N_1 \frac{3}{2l_2} k_2 - \frac{k_2}{2}\right] > 0, \\ &\left[Nm - N_1 \frac{3k_1^2}{2l_1} - \frac{3k_1}{2}\right] > 0 \text{ and } \left[Nm - N_1 \frac{3k_2^2}{2l_2} - \frac{3k_2}{2}\right] > 0. \end{aligned} \tag{95}$$

Thus, (82) is established.

Proof of Theorem 11. Taking into account (43) and (26), we obtain that for any $t \ge t_1$,

$$\begin{aligned} &\int_0^{t_1} g_1(s) \int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 dxds \\ &\quad \le -\frac{g_1(0)}{a_1} \int_0^{t_1} g_1'(s) \int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 dxds \le -cE'(t), \\ &\int_0^{t_1} g_2(s) \int_\Omega |\nabla v(t) - \nabla v(t-s)|^2 dxds \\ &\quad \le -\frac{g_2(0)}{a_2} \int_0^{t_1} g_2'(s) \int_\Omega |\nabla v(t) - \nabla v(t-s)|^2 dxds \le -cE'(t). \end{aligned} \tag{96}$$

Noting (82), we shall see that there exists a constant $m > 0$ such that for all $t \ge t_1$,

$$\begin{aligned} \mathscr{F}'(t) \le{}& -mE(t) - cE' + c\int_{t_1}^t g_1(s) \int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 dxds \\ &+ c\int_{t_1}^t g_2(s) \int_\Omega |\nabla v(t) - \nabla v(t-s)|^2 dxds. \end{aligned} \tag{97}$$

Denote $\mathscr{L}(t) = \mathscr{F}(t) + cE(t)$. It is obvious that $\mathscr{L}(t)$ is equivalent to $E(t)$. It follows from (97) that

$$\begin{aligned} \mathscr{L}'(t) \le{}& -mE(t) + c\int_{t_1}^t g_1(s) \int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 dxds \\ &+ c\int_{t_1}^t g_2(s) \int_\Omega |\nabla v(t) - \nabla v(t-s)|^2 dxds. \end{aligned} \tag{98}$$

We consider two cases.

Case 11. $G(t)$ is linear: By multiplying (98) by $\xi(t)$ and using (A2) and (26), we obtain

$$\begin{aligned} \xi(t)\mathscr{L}'(t) \le{}& -m\xi(t)E(t) + c\xi(t)\int_{t_1}^t g_1(s) \int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 dxds \\ &+ c\xi(t)\int_{t_1}^t g_2(s) \int_\Omega |\nabla v(t) - \nabla v(t-s)|^2 dxds \\ \le{}& -m\xi(t)E(t) + c\int_{t_1}^t \xi_1(t) g_1(s) \int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 dxds \\ &+ c\int_{t_1}^t \xi_2(t) g_2(s) \int_\Omega |\nabla v(t) - \nabla v(t-s)|^2 dxds \\ \le{}& -m\xi(t)E(t) - c\int_{t_1}^t g_1'(s) \int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 dxds \\ &- c\int_{t_1}^t g_2'(s) \int_\Omega |\nabla v(t) - \nabla v(t-s)|^2 dxds \\ \le{}& -m\xi(t)E(t) - cE'(t), \end{aligned} \tag{99}$$

which gives as $\xi(t)$ is nonincreasing,

$$[\xi(t)\mathscr{L}(t) + cE(t)]' \leq \xi(t)\mathscr{L}'(t) + cE'(t) \leq -m\xi(t)E(t)\ \forall t \geq t_1. \tag{100}$$

Denote $K(t) = \xi(t)\mathscr{L}(t) + cE(t)$. We get

$$K(t)' \leq -m\xi(t)E(t). \tag{101}$$

Hence, using the fact that $K(t) \sim E(t)$, we easily obtain

$$E(t) \leq c_1 \exp\left(-c_2\int_{t_1}^{t} \xi(s)ds\right). \tag{102}$$

Case 12. $G(t)$ is nonlinear: First, we use Lemmas 8 and 9 to deduce that

$$J(t) = \mathscr{F}(t) + \theta_1(t) + \theta_2(t), \tag{103}$$

is nonnegative, and it satisfies for some positive constant k and for any $t \geq t_1$,

$$\begin{aligned} J'(t) \leq& -(\mu - l_1)\|\nabla u\|^2 - (\mu - l_2)\|\nabla v\|^2 - \|u_t\|^2 - \|v_t\|^2 \\ &- \|divu\|^2 - \|divv\|^2 - \frac{1}{4}(g_1\circ\nabla u) - \frac{1}{4}(g_2\circ\nabla v) \\ &+ c\int_\Omega F(u, v)dx - \int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} \rho|\mu_1(\rho)||\nabla z(x, \rho, \rho, t)|^2 d\rho d\rho dx \\ &- \int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} \rho|\mu_2(\rho)||\nabla y(x, \rho, \rho, t)|^2 d\rho d\rho dx \leq -kE(t) \leq 0. \end{aligned} \tag{104}$$

Therefore,

$$k\int_{t_1}^{t} E(s)ds \leq J(t_1) - J(t) \leq J(t_1), \tag{105}$$

this implies that

$$\int_0^\infty E(s)ds < \infty. \tag{106}$$

Now, we define $I_i(t)$ by

$$\begin{aligned} I_1(t) &:= q\int_{t_1}^{t}\int_\Omega |\nabla u(t) - \nabla u(t - s)|^2 dxds, \\ I_2(t) &:= q\int_{t_1}^{t}\int_\Omega |\nabla v(t) - \nabla v(t - s)|^2 dxds, \end{aligned} \tag{107}$$

where (106) allows for a constant $0 < q < 1$ chosen so that for all $t \geq t_1$

$$I_i(t) < 1,\ i = 1, 2. \tag{108}$$

We also assume without loss of generality that $I_i(t) > 0$ for all $t \geq t_1$; otherwise, (98) yields an exponential decay. Also, we define $\lambda_1(t)$ and $\lambda_2(t)$ by

$$\begin{aligned} \lambda_1(t) &:= -\int_{t_1}^{t} g_1'(s)\int_\Omega |\nabla u(t) - \nabla u(t - s)|^2 dxds, \\ \lambda_2(t) &:= -\int_{t_1}^{t} g_2'(s)\int_\Omega |\nabla v(t) - \nabla v(t - s)|^2 dxds. \end{aligned} \tag{109}$$

We observe that $\lambda_i(t) \leq -cE'(t)\ i = 1, 2$. Noting that $G_i(t)$ is strictly convex on $(0, r]$ and $G_i(0) = 0$, then

$$G_i(\nu x) \leq \nu G_i(x),\ i = 1, 2, \tag{110}$$

provided $0 \leq \nu \leq 1$ and $x \in (0, r]$. By using (A2), (108), and Jensen's inequality, we can obtain

$$\begin{aligned} \lambda_1(t) &= \frac{1}{qI_1(t)}\int_{t_1}^{t} I_1(t)\left(-g_1'(s)\right)\int_\Omega q|\nabla u(t) - \nabla u(t - s)|^2 dxds \\ &\geq \frac{1}{qI_1(t)}\int_{t_1}^{t} I_1(t)\xi_1(s)G_1(g_1(s))\int_\Omega q|\nabla u(t) - \nabla u(t - s)|^2 dxds, \\ &\geq \frac{\xi_1(t)}{qI_1(t)}\int_{t_1}^{t} G_1(I_1(t)g_1(s))\int_\Omega q|\nabla u(t) - \nabla u(t - s)|^2 dxds, \\ &\geq \frac{\xi_1(t)}{q}G_1\left(\frac{1}{I_1(t)}\int_{t_1}^{t} I_1(t)g_1(s)\int_\Omega q|\nabla u(t) - \nabla u(t - s)|^2 dxds\right) \\ &= \frac{\xi_1(t)}{q}G_1\left(q\int_{t_1}^{t} g_1(s)\int_\Omega |\nabla u(t) - \nabla u(t - s)|^2 dxds\right) \\ &= \frac{\xi_1(t)}{q}\bar{G}_1\left(q\int_{t_1}^{t} g_1(s)\int_\Omega |\nabla u(t) - \nabla u(t - s)|^2 dxds\right), \end{aligned} \tag{111}$$

where $\bar{G}_1$ is an extension of G_1 such that $\bar{G}_1$ is strictly increasing and strictly convex C^2 function on $(0, +\infty)$, see [19]. We have from (111)

$$\int_{t_1}^{t} g_1(s)\int_\Omega |\nabla u(t) - \nabla u(t - s)|^2 dxds \leq \frac{1}{q}\bar{G}_1^{-1}\left(\frac{q\lambda_1(t)}{\xi_1(t)}\right). \tag{112}$$

Similarly, we have

$$\int_{t_1}^{t} g_2(s)\int_\Omega |\nabla v(t) - \nabla v(t - s)|^2 dxds \leq \frac{1}{q}\bar{G}_2^{-1}\left(\frac{q\lambda_2(t)}{\xi_2(t)}\right). \tag{113}$$

We infer from (98), (112), and (113) that for any $t \geq t_1$

$$\mathscr{L}'(t) \leq -mE(t) + c\bar{G}_1^{-1}\left(\frac{q\lambda_1(t)}{\xi_1(t)}\right) + c\bar{G}_2^{-1}\left(\frac{q\lambda_2(t)}{\xi_2(t)}\right). \tag{114}$$

Let us denote

$$G_0(t) = \min\left\{\bar{G}_1^{\prime -1}, \bar{G}_2^{\prime -1}\right\}. \tag{115}$$

For $\varepsilon_0 < r$, the functional $\mathcal{K}_1(t)$ defined by:

$$\mathcal{K}_1(t) = G_0\left(\varepsilon_0 \frac{E(t)}{E(0)}\right)\mathcal{L}(t) + E(t), \tag{116}$$

is equivalent to E, and by using the fact that $E' \le 0$, $\bar{G}_i{}' > 0$, and $\bar{G}_i{}'' > 0$, we infer from (114) that

$$\begin{aligned}\mathcal{K}_1'(t) &= \varepsilon_0 \frac{E'(t)}{E(0)} G_0'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right)\mathcal{L}(t) + G_0\left(\varepsilon_0 \frac{E(t)}{E(0)}\right)\mathcal{L}'(t) \\ &\quad + E'(t) \le -mE(t)G_0\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) + cG_0\left(\varepsilon_0 \frac{E(t)}{E(0)}\right)\bar{G}_1{}^{-1} \\ &\quad \cdot \left(\frac{q\lambda_1(t)}{\xi_1(t)}\right) + cG_0\left(\varepsilon_0 \frac{E(t)}{E(0)}\right)\bar{G}_2{}^{-1}\left(\frac{q\lambda_2(t)}{\xi_2(t)}\right).\end{aligned} \tag{117}$$

Let $\bar{G}^*$ be the convex conjugate of $\bar{G}$ in the sense of Young (see Arnold [21]), then

$$\bar{G}_i{}^*(s) = s\left(\bar{G}_i{}'\right)^{-1}(s) - \bar{G}_i\left[\left(\bar{G}_i{}'\right)^{-1}(s)\right],\ i = 1, 2, \tag{118}$$

and $\bar{G}^*$ satisfies the following Young's inequality

$$MD_i \le \bar{G}_i{}^*(M) + \bar{G}_i(D_i),\ i = 1, 2. \tag{119}$$

With $M = G_0(\varepsilon_0(E(t)/E(0)))$ and $D_i = \bar{G}_i{}^{-1}((q\lambda_i(t)/\xi_i(t)))$ and noting $\bar{G}_i{}^*(t) \le t(\bar{G}_i{}')^{-1}(t)$ and (117), we conclude

$$\begin{aligned}\mathcal{K}_1'(t) &\le -mE(t)G_0\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) + c\bar{G}_1{}^*\left(G_0\left(\varepsilon_0 \frac{E(t)}{E(0)}\right)\right) \\ &\quad + c\left(\frac{q\lambda_1(t)}{\xi_1(t)}\right) + c\bar{G}_2{}^*\left(G_0\left(\varepsilon_0 \frac{E(t)}{E(0)}\right)\right) + c\left(\frac{q\lambda_2(t)}{\xi_2(t)}\right) \\ &\le -mE(t)G_0\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) + cG_0\left(\varepsilon_0 \frac{E(t)}{E(0)}\right)\left(\bar{G}_1{}'\right)^{-1} \\ &\quad \cdot \left(G_0\left(\varepsilon_0 \frac{E(t)}{E(0)}\right)\right) + c\left(\frac{q\lambda_1(t)}{\xi_1(t)}\right) + cG_0\left(\varepsilon_0 \frac{E(t)}{E(0)}\right)\left(\bar{G}_2{}'\right)^{-1} \\ &\quad \cdot \left(G_0\left(\varepsilon_0 \frac{E(t)}{E(0)}\right)\right) + c\left(\frac{q\lambda_2(t)}{\xi_2(t)}\right) \\ &\le -mE(t)G_0\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) + cG_0\left(\varepsilon_0 \frac{E(t)}{E(0)}\right)\left(\bar{G}_1{}'\right)^{-1} \\ &\quad \cdot \left(\bar{G}_1{}'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right)\right) + c\left(\frac{q\lambda_1(t)}{\xi_1(t)}\right) + cG_0\left(\varepsilon_0 \frac{E(t)}{E(0)}\right)\left(\bar{G}_2{}'\right)^{-1} \\ &\quad \cdot \left(\bar{G}_2{}'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right)\right) + c\left(\frac{q\lambda_2(t)}{\xi_2(t)}\right) \\ &\le -(mE(0) - c\varepsilon_0)\frac{E(t)}{E(0)}G_0\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) + cq\left(\frac{\lambda_1(t)}{\xi_1(t)} + \frac{\lambda_2(t)}{\xi_2(t)}\right).\end{aligned} \tag{120}$$

Multiplying by $\xi(t)$, we get

$$\begin{aligned}\xi(t)\mathcal{K}_1'(t) &\le -(mE(0) - c\varepsilon_0)\xi(t)\frac{E(t)}{E(0)}G_0\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) + cq(\lambda_1(t) + \lambda_2(t)) \\ &\le -(mE(0) - c\varepsilon_0)\xi(t)\frac{E(t)}{E(0)}G_0\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) - cE'(t).\end{aligned} \tag{121}$$

Consequently, with $\mathcal{K}_2(t) = \xi(t)\mathcal{K}_1(t) + cE(t)$, which satisfies for some $\beta_1, \beta_2 > 0$,

$$\beta_1\mathcal{K}_2(t) \le E(t) \le \beta_2\mathcal{K}_2(t). \tag{122}$$

Choosing a suitable ε_0, we can get from (121) that there exists a constant $k > 0$,

$$\mathcal{K}_2'(t) \le -k\xi(t)\frac{E(t)}{E(0)}G_0\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) \coloneqq -k\xi(t)G_3\left(\varepsilon_0 \frac{E(t)}{E(0)}\right), \tag{123}$$

where $G_3(t) = tG_0(\varepsilon_0 t)$.

From $0 \le \varepsilon_0(E(t)/E(0)) < r$, we conclude that for any $t > 0$,

$$\begin{aligned}G_0\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) &= \min\left\{\bar{G}_1{}'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right), \bar{G}_2{}'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right)\right\} \\ &= \min\left\{G_1{}'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right), G_2{}'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right)\right\}.\end{aligned} \tag{124}$$

Denote $R(t) = (\beta_1\mathcal{K}_2(t)/E(0))$. Using (122), then

$$R(t) \sim E(t). \tag{125}$$

Since $G_3'(t) = G_0(\varepsilon_0 t) + \varepsilon_0 t G_0'(\varepsilon_0 t)$, then, using the strict convexity of G_0 on $(0, r]$, we arrive at $G_3'(t), G_3(t) > 0$ on $(0, 1]$. We infer from (123) that there exists a constant $b_1 > 0$ such that for all $t \le t_1$,

$$R'(t) \le -b_1\xi(t)G_3(R(t)). \tag{126}$$

Then, by integration over (t_1, t), we have

$$\begin{aligned}\int_{t_1}^{t} \frac{-R'(s)}{G_3(R(s))}ds &\ge b_1\int_{t_1}^{t}\xi(s)ds\int_{\varepsilon_0 R(t)}^{\varepsilon_0 R(t_1)}\frac{1}{sG_0(s)}ds \\ &\ge b_1\int_{t_1}^{t}\xi(s)dsR(t) \le \frac{1}{\varepsilon_0}G_4^{-1}\left(b_1\int_{t_1}^{t}\xi(s)ds\right),\end{aligned} \tag{127}$$

where $G_4(t) = \int_t^r (1/sG_0(s))ds$ is strictly decreasing on $(0, r]$ and $\lim_{t\to 0} G_4(t) = +\infty$. A combination of (125) and (127), estimate (38), is established.

4. Conclusion

In this work, we have proved a general energy decay of a coupled Lamé system with distributed time delay. This result is an natural extension of Feng's work in [18]. In order to complete this work, the study of the global existence and the blow-up of the solutions of (1) and (2) will be the subject of forthcoming works.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there is no conflict of interests regarding the publication of this manuscript.

Authors' Contributions

The authors contributed equally in this article. They have all read and approved the final manuscript.

Acknowledgments

The second author would like to thank all the professors of the mathematics department at the University of Annaba in Algeria, especially his Professors/Scientists Pr. Mohamed Haiour, Pr. Ahmed-Salah Chibi, and Pr. Azzedine Benchettah for the important content of masters and PhD courses in pure and applied mathematics which he received during his studies. Moreover, he thanks them for the additional help they provided to him during office hours in their office about the few concepts/difficulties he had encountered, and he appreciates their talent and dedication for their postgraduate students currently and previously. In addition, the idea and research project in this paper was presented by the second author and was carried out by all authors. The authors are grateful to the anonymous referees for the careful reading and their important observations/suggestions for the sake of improving this paper.

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