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Research Article

On a Nonlocal Multipoint and Integral Boundary Value Problem of Nonlinear Fractional Integrodifferential Equations

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The aim of this paper is to give the existence as well as the uniqueness results for a multipoint nonlocal integral boundary value problem of nonlinear sequential fractional integrodifferential equations. First of all, we give some preliminaries and notations that are necessary for the understanding of the manuscript; second of all, we show the existence and uniqueness of the solution by means of the fixed point theory, namely, Banach's contraction principle and Krasnoselskii's fixed point theorem. Last, but not least, we give two examples to illustrate the results.

1. Introduction

In the last few years, fractional differential equations have gained much attention among mathematicians because of the rapid growth and for their applicability in several fields, such as physics, biology, economics, control theory, and engineering; for more details about the theory of fractional differential equations and their applications, we recommend the following articles [1–15] and the references therein.

Furthermore, fractional differential equations with multipoint boundary conditions have provoked a great deal of attention by many authors; a lot of works have been published on this topic; for more details, we give the following references [16–22].

In a recent paper [23], the existence of solutions for a four-point nonlocal boundary value problem of nonlinear integrodifferential equations of fractional order was proven. In [24], the authors discussed the existence of solutions for fractional differential equations with multipoint boundary conditions. The existence and uniqueness of solutions for multiterm nonlinear fractional integrodifferential equations have been studied in [25]. The existence results for sequential fractional integrodifferential equations with nonlocal multipoint and strip conditions were established in [26], and finally, in [27], the authors studied the existence of solutions for nonlinear fractional integrodifferential equations.

Motivated by all these works, and by the fact that there are no papers dealing with nonlinear fractional integrodifferential equations with multipoint and integral boundary value conditions, in this work, we consider the existence and uniqueness of solutions for the following problem:

$$\begin{cases} {}^{c}D^{\beta}({}^{c}D^{\alpha})x(t) = f(t, x(t), \phi x(t), \varphi x(t)), \quad t \in [0,1], \\ x(0) = \sum_{i=1}^{n} a_i x(t_i), x'(0) = 0, \mu_1 x(1) + \mu_2 x'(1) = \sum_{j=1}^{m} b_j \int_{c_j}^{d_j} x(s)ds, \end{cases} \tag{1}$$

where $0 < \alpha < 1$, $1 < \beta \le 2$, with $\alpha + \beta - 2 \ge 0$, $\mu_1, \mu_2, a_i, b_j \in \mathbb{R}$; $0 < t_i < c_j < d_j < 1$, $i = 1, 2, \cdots, n$, $j = 1, 2, \cdots, m$, ${}^{c}D^{\alpha}$, ${}^{c}D^{\beta}$ are the Caputo fractional derivatives, and $f : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ is a continuous function and $\phi x(t) = \int_0^t \lambda(t,s)x(s)ds$, $\varphi x(t) = \int_0^t \delta$

$(t,s)x(s)ds$, where λ, $\delta : [0,1]\times[0,1] \to [0,+\infty)$, with $\lambda_0 = \sup_{t\in[0,1]} |\int_0^t \lambda(t,s)ds| < \infty$, $\delta_0 = \sup_{t\in[0,1]} |\int_0^t \delta(t,s)ds| < \infty$.

This paper is organized as follows: in the second section, we give some preliminaries and notations that will be useful throughout the work; after that, in the third section, we establish the main results by using the fixed point theory; and in the last section, we give some examples to illustrate the results.

2. Preliminaries and Notations

Throughout this section, we present some notations, definitions, and lemmas which will be used for the rest of the paper.

Definition 1 (see [5]). The fractional integral of order $\alpha > 0$ with the lower limit zero for a function f can be defined as

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds. \tag{2}$$

Definition 2 (see [5]). The Caputo derivative of order $\alpha > 0$ with the lower limit zero for a function f can be defined as

$$^cD^\alpha f = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s)ds. \tag{3}$$

where $n \in \mathbb{N}$, $0 \le n-1 < \alpha < n$, $t > 0$.

Theorem 3 (see [28]). *Let M be a bounded, closed, convex, and nonempty subset of a Banach space X. Let A and B be two operators such that*

(i) $Ax + By \in M$ whenever $x, y \in M$

(ii) A is compact and continuous

(iii) B is a contraction mapping

Then, there exists $z \in M$ such that $z = Az + Bz$.

Lemma 4 (see [5]). *Let $\alpha, \beta \ge 0$, then the following relation holds:*

$$I^\alpha t^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} t^{\alpha+\beta}. \tag{4}$$

Lemma 5 (see [5]). *Let $n \in \mathbb{N}$ and $n-1 < \alpha < n$. If f is a continuous function, then we have*

$$I^{\alpha c}D^\alpha f(t) = f(t) + a_0 + a_1 t + a_2 t^2 + \cdots + a_{n-1} t^{n-1}. \tag{5}$$

Lemma 6. *Let $h \in C([0,1],\mathbb{R})$. Then, the unique solution of the boundary value problem*

$$\begin{cases} ^cD^\beta(^cD^\alpha)x(t) = h(t), \quad t \in [0,1], \\ x(0) = \sum_{i=1}^n a_i x(t_i), x'(0) = 0, \mu_1 x(1) + \mu_2 x'(1) = \sum_{j=1}^m b_j \int_{c_j}^{d_j} x(s)ds, \end{cases} \tag{6}$$

is given by

$$\begin{aligned} x(t) = {} & \frac{1}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1} h(s)ds \\ & - \frac{B_1(t)}{\Gamma(\alpha+\beta)} \sum_{i=1}^n a_i \int_0^{t_i} (t_i-s)^{\alpha+\beta-1} \times h(s)ds \\ & + B_2(t) \left[\sum_{j=1}^m \frac{b_j}{\Gamma(\alpha+\beta)} \int_{c_j}^{d_j} \left(\int_0^s (s-u)^{\alpha+\beta-1} h(u)du \right) ds \right. \\ & - \frac{\mu_1}{\Gamma(\alpha+\beta)} \int_0^1 (1-s)^{\alpha+\beta-1} h(s)ds - \frac{\mu_2}{\Gamma(\alpha+\beta-1)} \\ & \left. \times \int_0^t (1-s)^{\alpha+\beta-2} h(s)ds \right], \end{aligned} \tag{7}$$

where

$$\begin{aligned} & \Delta = L_1K_2 - L_2K_1 \ne 0, L_1 = -\frac{\sum_{i=1}^n a_i t_i^{\alpha+1}}{\Gamma(\alpha+2)}, L_2 = 1 - \sum_{j=1}^n a_i, \\ & K_1 = \frac{\mu_1}{\Gamma(\alpha+2)} + \frac{\mu_2}{\Gamma(\alpha+1)} - \frac{\sum_{j=1}^m b_j \left(d_j^{\alpha+2} - c_j^{\alpha+2} \right)}{\Gamma(\alpha+3)}, \\ & K_2 = \mu_1 - \sum_{j=1}^m b_j (d_j - c_j), B_1(t) = \frac{1}{\Delta} \left(K_1 - \frac{K_2 t^{\alpha+1}}{\Gamma(\alpha+2)} \right), \\ & B_2(t) = \frac{1}{\Delta} \left(L_1 - \frac{L_2 t^{\alpha+1}}{\Gamma(\alpha+2)} \right). \end{aligned} \tag{8}$$

Proof. By applying Lemma 5, we obtain

$$\begin{aligned} & ^cD^\alpha x(t) = I^\beta h(t) + \theta_0 + \theta_1 t, \\ & x(t) = I^{\alpha+\beta} h(t) + I^\alpha \theta_0 + I^\alpha \theta_1 t + \theta_2, \end{aligned} \tag{9}$$

where $\theta_0, \theta_1, \theta_2 \in \mathbb{R}$.

This means that

$$x(t)=\frac{1}{\Gamma(\alpha+\beta)}\int_0^t(t-s)^{\alpha+\beta-1}h(s)ds+\frac{t^{\alpha}}{\Gamma(\alpha+1)}\theta_0+\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\theta_1+\theta_2,$$

$$x'(t)=\frac{1}{\Gamma(\alpha+\beta-1)}\int_0^t(t-s)^{\alpha+\beta-2}h(s)ds+\frac{t^{\alpha-1}}{\Gamma(\alpha)}\theta_0+\frac{t^{\alpha}}{\Gamma(\alpha+1)}\theta_1, \tag{10}$$

and by using the condition $x'(0)=0$, we get $\theta_0=0$.

As a result of the condition $x(0)=\sum_{i=1}^n a_i x(t_i)$, we find that $L_2\theta_2+L_1\theta_1=H_1$, where

$$H_1=\sum_{i=1}^n\frac{a_i}{\Gamma(\alpha+\beta)}\int_0^{t_i}(t_i-s)^{\alpha+\beta-1}h(s)ds. \tag{11}$$

Now, we use the condition $\mu_1 x(1)+\mu_2 x'(1)=\sum_{j=1}^m b_j\int_{c_j}^{d_j}x(s)ds$, to obtain $K_2\theta_2+K_1\theta_1=H_2$, where

$$H_2=\frac{-\mu_1}{\Gamma(\alpha+\beta)}\int_0^1(1-s)^{\alpha+\beta-1}h(s)ds-\frac{\mu_2}{\Gamma(\alpha+\beta-1)}\int_0^1(1-s)^{\alpha+\beta-2}\times h(s)ds+\sum_{j=1}^m\frac{b_j}{\Gamma(\alpha+\beta)}\int_{c_j}^{d_i}\left(\int_0^s(s-u)^{\alpha+\beta-1}h(u)du\right)ds. \tag{12}$$

Finally, we have

$$\theta_1=\frac{K_2H_1-L_2H_2}{\Delta},\theta_2=\frac{L_1H_2-K_1H_1}{\Delta}. \tag{13}$$

By substituting the value of θ_0, θ_1, and θ_2, we get the following:

$$\begin{aligned}x(t)&=\frac{1}{\Gamma(\alpha+\beta)}\int_0^t(t-s)^{\alpha+\beta-1}h(s)ds-\frac{B_1(t)}{\Gamma(\alpha+\beta)}\sum_{i=1}^n a_i\int_0^{t_i}(t_i-s)^{\alpha+\beta-1}\times h(s)ds\\&+B_2(t)\left[\sum_{j=1}^m\frac{b_j}{\Gamma(\alpha+\beta)}\int_{c_j}^{d_j}\left(\int_0^s(s-u)^{\alpha+\beta-1}h(u)du\right)ds-\frac{\mu_1}{\Gamma(\alpha+\beta)}\int_0^1(1-s)^{\alpha+\beta-1}h(s)ds-\frac{\mu_2}{\Gamma(\alpha+\beta-1)}\times\int_0^1(1-s)^{\alpha+\beta-2}h(s)ds\right].\end{aligned} \tag{14}$$

Conversely, by direct computations, we obtain the desired result.

3. Main Results

Let X be the Banach space of all continuous functions from $[0,1]\to\mathbb{R}$ endowed with norm $\|x\|=\sup\{|x(t)|:t\in[0,1]\}$.

Theorem 7. *Suppose that $f:[0,1]\times\mathbb{R}^3\to\mathbb{R}$ is a continuous function satisfying*

(H_1) for all $t\in[0,1]$ and $x_1,x_2,x_3,y_1,y_2,y_3\in\mathbb{R}$, we have $|f(t,x_1,x_2,x_3)-f(t,y_1,y_2,y_3)|\le\sigma(t)(|x_1-y_1|+|x_2-y_2|+|x_3-y_3|)$ with $\sigma(t)\in L^1([0,1];[0,\infty))$.

Then, there exists a unique solution for problem (1) under the following condition: $r_1<1$, where

$$r_1=(1+\lambda_0+\delta_0)\sigma^*\cdot\left(\frac{1+B_1\sum_{i=1}^n|a_i|+B_2\sum_{j=1}^m|b_j|(d_j-c_j)+B_2|\mu_1|}{\Gamma(\alpha+\beta)}+\frac{B_2|\mu_2|}{\Gamma(\alpha+\beta-1)}\right), \tag{15}$$

with $B_1=1/|\Delta|(|K_1|+|K_2|/\Gamma(\alpha+\beta))$, $B_2=1/|\Delta|(|L_1|+|L_2|/\Gamma(\alpha+2))$, and $\sigma^*=\int_0^1\sigma(t)dt$.

Proof. Define $P:X\to X$ by

$$\begin{aligned}Px(t)&=\frac{1}{\Gamma(\alpha+\beta)}\int_0^t(t-s)^{\alpha+\beta-1}f(s,x(s),\phi x(s),\varphi x(s))ds-\frac{B_1(t)}{\Gamma(\alpha+\beta)}\times\sum_{i=1}^n a_i\int_0^{t_i}(t_i-s)^{\alpha+\beta-1}f(s,x(s),\phi x(s),\varphi x(s))ds\\&+B_2(t)\times\left[\sum_{j=1}^m\frac{b_j}{\Gamma(\alpha+\beta)}\int_{c_j}^{d_j}\left(\int_0^s(s-u)^{\alpha+\beta-1}f\cdot(u,x(u),\phi x(u),\varphi x(u))\times du\right)ds-\frac{\mu_1}{\Gamma(\alpha+\beta)}\int_0^1(1-s)^{\alpha+\beta-1}f(s,x(s),\phi x(s),\varphi x(s))ds-\frac{\mu_2}{\Gamma(\alpha+\beta-1)}\int_0^1(1-s)^{\alpha+\beta-2}f(s,x(s),\phi x(s),\varphi x(s))ds\right].\end{aligned} \tag{16}$$

Setting $\sup_{0\le t\le1}|f(t,0,0,0)|=F$.

We consider the following set $B_r=\{x\in X:\|x\|\le r\}$, where $r\ge r_2/(1-r_1)$, with

$$r_2=\frac{F}{\Gamma(\alpha+\beta)}+\frac{B_1}{\Gamma(\alpha+\beta)}\sum_{i=1}^n|a_i|F+B_2\sum_{j=1}^m\frac{|b_j|F(d_j-c_j)}{\Gamma(\alpha+\beta)}+\frac{B_2|\mu_1|F}{\Gamma(\alpha+\beta)}+\frac{B_2|\mu_2|F}{\Gamma(\alpha+\beta-1)}. \tag{17}$$

For each $t \in [0,1]$ and $x \in B_r$, we have

$$\begin{aligned}
|Px(t)| \le{}& \frac{1}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1}|f(s,x(s),\phi x(s),\varphi x(s))|ds \\
&+ \frac{|B_1(t)|}{\Gamma(\alpha+\beta)} \times \sum_{i=1}^n |a_i| \int_0^{t_i} (t_i - s)^{\alpha+\beta-1} \\
&\cdot |f(s,x(s),\phi x(s),\varphi x(s))|ds + |B_2(t)| \\
&\cdot \Bigg[\sum_{j=1}^m \frac{|b_j|}{\Gamma(\alpha+\beta)}\int_{c_j}^{d_j}\Bigg(\int_0^s (s-u)^{\alpha+\beta-1} \\
&\times |f(u,x(u),\phi x(u),\varphi x(u))|du\Bigg)ds + \frac{|\mu_1|}{\Gamma(\alpha+\beta)}\int_0^1 \\
&\cdot (1-s)^{\alpha+\beta-1} \times |f(s,x(s),\phi x(s),\varphi x(s))|ds + \frac{|\mu_2|}{\Gamma(\alpha+\beta-1)} \\
&\cdot \int_0^1 (1-s)^{\alpha+\beta-2} \times |f(s,x(s),\phi x(s),\varphi x(s))|ds\Bigg] \\
\le{}& \frac{1}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1}(|f(s,x(s),\phi x(s),\varphi x(s)) \\
&- f(s,0,0,0)| + |f(s,0,0,0)|)ds + \frac{|B_1(t)|}{\Gamma(\alpha+\beta)}\sum_{i=1}^n |a_i|\int_0^{t_i} \\
&\cdot (t_i - s)^{\alpha+\beta-1} \times (|f(s,x(s),\phi x(s),\varphi x(s)) - f(s,0,0,0)| \\
&+ |f(s,0,0,0)|)ds + |B_2(t)|\Bigg[\sum_{j=1}^m \frac{|b_j|}{\Gamma(\alpha+\beta)}\int_{c_j}^{d_j} \\
&\cdot \Bigg(\int_0^s (s-u)^{\alpha+\beta-1} \times (|f(u,x(u),\phi x(u),\varphi x(u)) \\
&- f(u,0,0,0)| + |f(u,0,0,0)|)du\Bigg)ds + \frac{|\mu_1|}{\Gamma(\alpha+\beta)}\int_0^1 \\
&\cdot (1-s)^{\alpha+\beta-1}(|f(s,x(s),\phi x(s),\varphi x(s)) - f(s,0,0,0)| \\
&+ |f(s,0,0,0)|)ds + \frac{|\mu_2|}{\Gamma(\alpha+\beta-1)}\int_0^1 (1-s)^{\alpha+\beta-2} \\
&\times (|f(s,x(s),\phi x(s),\varphi x(s)) - f(s,0,0,0)| + |f(s,0,0,0)|)ds\Bigg] \\
\le{}& \frac{1}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1}(\sigma(s)(|x(s)| + |\phi x(s)| + |\varphi x(s)|) + F)ds \\
&+ \frac{B_1}{\Gamma(\alpha+\beta)}\sum_{i=1}^n |a_i|\int_0^{t_i}(t_i - s)^{\alpha+\beta-1}(\sigma(s)(|x(s)| + |\phi x(s)| \\
&+ |\varphi x(s)|) + F)ds + B_2\Bigg[\sum_{j=1}^m \frac{|b_j|}{\Gamma(\alpha+\beta)}\int_{c_j}^{d_j} \times \int_0^s (s-u)^{\alpha+\beta-1} \\
&\cdot (\sigma(u)(|x(u)| + |\phi x(u)| + |\varphi x(u)|) + F)duds + \frac{|\mu_1|}{\Gamma(\alpha+\beta)}\int_0^1 \\
&\cdot (1-s)^{\alpha+\beta-1}(\sigma(s)(|x(s)| + |\phi x(s)| + |\varphi x(s)|) + F)ds \\
&+ \frac{|\mu_2|}{\Gamma(\alpha+\beta-1)}\int_0^1 (1-s)^{\alpha+\beta-2}(\sigma(s)(|x(s)| + |\phi x(s)| \\
&+ |\varphi x(s)|) + F)ds\Bigg] \le \frac{(1+\lambda_0+\delta_0)\|x\|}{\Gamma(\alpha+\beta)}\int_0^1 \sigma(s)ds + \frac{F}{\Gamma(\alpha+\beta)}\int_0^t \\
&\cdot (t-s)^{\alpha+\beta-1}ds + \frac{B_1}{\Gamma(\alpha+\beta)}\sum_{i=1}^n |a_i|(1+\lambda_0+\delta_0)\|x\|\int_0^1 \sigma(s)ds \\
&+ \frac{B_1}{\Gamma(\alpha+\beta)}\sum_{i=1}^n |a_i| F \times \int_0^{t_i}(t_i - s)^{\alpha+\beta-1}ds + B_2\sum_{j=1}^m \frac{|b_j|}{\Gamma(\alpha+\beta)}\int_{c_j}^{d_j} \\
&\cdot \left((1+\lambda_0+\delta_0)\|x\| \times \int_0^1 \sigma(u)du\right) + B_2\sum_{j=1}^m \frac{|b_j| F (d_j - c_j)}{\Gamma(\alpha+\beta)} \\
&+ \frac{B_2|\mu_1|}{\Gamma(\alpha+\beta)}(1+\lambda_0+\delta_0)\|x\| \times \int_0^1 \sigma(s)ds + \frac{B_2|\mu_1| F}{\Gamma(\alpha+\beta)}\int_0^1 \\
&\times (1-s)^{\alpha+\beta-1}ds + \frac{B_2|\mu_2|}{\Gamma(\alpha+\beta-1)}(1+\lambda_0+\delta_0)\|x\| \\
&\times \int_0^1 \sigma(s)ds + \frac{B_2|\mu_2| F}{\Gamma(\alpha+\beta-1)}\int_0^1 (1-s)^{\alpha+\beta-2}ds \\
\le{}& \frac{(1+\lambda_0+\delta_0)\|x\|}{\Gamma(\alpha+\beta)}\sigma^* + \frac{F}{\Gamma(\alpha+\beta)} + \frac{B_1}{\Gamma(\alpha+\beta)}\sum_{i=1}^n |a_i| \\
&\times (1+\lambda_0+\delta_0)\|x\|\sigma^* + \frac{B_1}{\Gamma(\alpha+\beta)}\sum_{i=1}^n |a_i| F \\
&+ B_2\sum_{j=1}^m \frac{|b_j|(d_j - c_j)}{\Gamma(\alpha+\beta)}(1+\lambda_0+\delta_0)\|x\|\sigma^* \\
&+ B_2\sum_{j=1}^m \frac{|b_j| F (d_j - c_j)}{\Gamma(\alpha+\beta)} + \frac{B_2|\mu_1|}{\Gamma(\alpha+\beta)}(1+\lambda_0+\delta_0)\|x\|\sigma^* \\
&+ \frac{B_2|\mu_1| F}{\Gamma(\alpha+\beta)} + \frac{B_2|\mu_2|}{\Gamma(\alpha+\beta-1)}(1+\lambda_0+\delta_0)\sigma^*\|x\| + \frac{B_2|\mu_2| F}{\Gamma(\alpha+\beta-1)} \\
\le{}& \Bigg[\frac{(1+\lambda_0+\delta_0)\sigma^*}{\Gamma(\alpha+\beta)} + \frac{B_1}{\Gamma(\alpha+\beta)}\sum_{i=1}^n |a_i|(1+\lambda_0+\delta_0)\sigma^* + B_2 \\
&\times \sum_{j=1}^m \frac{|b_j|(d_j - c_j)}{\Gamma(\alpha+\beta)}(1+\lambda_0+\delta_0)\sigma^* + \frac{B_2|\mu_1|}{\Gamma(\alpha+\beta)} \\
&\times (1+\lambda_0+\delta_0)\sigma^* + \frac{B_2|\mu_2|}{\Gamma(\alpha+\beta-1)}(1+\lambda_0+\delta_0)\sigma^*\Bigg]\|x\| \\
&+ \frac{F}{\Gamma(\alpha+\beta)} + \frac{B_1}{\Gamma(\alpha+\beta)} \times \sum_{i=1}^n |a_i| F + B_2\sum_{j=1}^m \frac{|b_j| F (d_j - c_j)}{\Gamma(\alpha+\beta)} \\
&+ \frac{B_2|\mu_1| F}{\Gamma(\alpha+\beta)} + \frac{B_2|\mu_2| F}{\Gamma(\alpha+\beta-1)}.
\end{aligned} \tag{18}$$

This means that $\|Px\| \le r$.
Therefore, $PB_r \subseteq B_r$.
Next, we prove that P is a contraction mapping.
For $x, y \in B_r$, we have

$$\begin{aligned}
&|Px(t) - Py(t)| \\
&\quad\le \frac{1}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1}(|f(s,x(s),\phi x(s),\varphi x(s)) \\
&\qquad - f(s,y(s),\phi y(s),\varphi y(s))|)ds + \frac{B_1\sum_{i=1}^n |a_i|}{\Gamma(\alpha+\beta)} \\
&\qquad \times \int_0^{t_i}(t_i - s)^{\alpha+\beta-1}(|f(s,x(s),\phi x(s),\varphi x(s)) \\
&\qquad - f(s,y(s),\phi y(s),\varphi y(s))|)ds \\
&\qquad + B_2\Bigg[\sum_{j=1}^m \frac{|b_j|}{\Gamma(\alpha+\beta)} \times \int_{c_j}^{d_j} \\
&\qquad \times \Bigg(\int_0^s (s-u)^{\alpha+\beta-1}(|f(s,x(u),\phi x(u),\varphi x(u)) \\
&\qquad - f(s,y(u),\phi y(u),\varphi y(u))|)du\Bigg)ds
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{|\mu_1|}{\Gamma(\alpha+\beta)} \times \int_0^1 (1-s)^{\alpha+\beta-1} (|f(s,x(s),\phi x(s),\varphi x(s)) \\
&- f(s,y(s),\phi y(s),\varphi y(s))|) ds + \frac{|\mu_2|}{\Gamma(\alpha+\beta-1)} \\
&\times \int_0^1 (1-s)^{\alpha+\beta-2} (|f(s,x(s),\phi x(s),\varphi x(s)) - f(s,y(s),\phi y(s),\varphi y(s))|) ds \Bigg] \\
&\le \frac{1}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1} \sigma(s) (|x(s)-y(s)| + |\phi x(s) \\
&- \phi y(s)| + |\varphi x(s) - \varphi y(s)|) ds + \frac{B_1}{\Gamma(\alpha+\beta)} \sum_{i=1}^n |a_i| \int_0^{t_i} \\
&\times (t_i - s)^{\alpha+\beta-1} \sigma(s) \times (|x(s)-y(s)| + |\phi x(s) \\
&- \phi y(s)| + |\varphi x(s) - \varphi y(s)|) ds + B_2 \Bigg[\sum_{j=1}^m \frac{|b_j|}{\Gamma(\alpha+\beta)} \int_{c_j}^{d_j} \\
&\times \left(\int_0^s (s-u)^{\alpha+\beta-1} \sigma(u) \times (|x(u)-y(u)| + |\phi x(u) \right. \\
&\left. - \phi y(u)| + |\varphi x(u) - \varphi y(u)|) du \right) ds \\
&+ \frac{|\mu_1|}{\Gamma(\alpha+\beta)} \int_0^1 (1-s)^{\alpha+\beta-1} \sigma(s) (|x(s)-y(s)| + |\phi x(s) \\
&- \phi y(s)| + |\varphi x(s) - \varphi y(s)|) ds + \frac{|\mu_2|}{\Gamma(\alpha+\beta-1)} \\
&\times \int_0^1 (1-s)^{\alpha+\beta-2} \sigma(s) (|x(s)-y(s)| + |\phi x(s) \\
&\times - \phi y(s)| + |\varphi x(s) - \varphi y(s)|) ds \Bigg] \\
&\le \frac{(1+\lambda_0+\delta_0)\|x-y\|}{\Gamma(\alpha+\beta)} \int_0^1 \sigma(s) ds \\
&+ \frac{B_1}{\Gamma(\alpha+\beta)} \sum_{i=1}^n |a_i| (1+\lambda_0+\delta_0)\|x-y\| \int_0^1 \sigma(s) ds \\
&+ B_2 \sum_{j=1}^m \frac{|b_j|}{\Gamma(\alpha+\beta)} \int_{c_j}^{d_j} \left((1+\lambda_0+\delta_0)\|x-y\| \int_0^1 \sigma(u) du \right) ds \\
&+ \frac{B_2|\mu_1|}{\Gamma(\alpha+\beta)} (1+\lambda_0+\delta_0)\|x-y\| \int_0^1 \sigma(s) ds \\
&+ \frac{B_2|\mu_2|}{\Gamma(\alpha+\beta-1)} (1+\lambda_0+\delta_0)\|x-y\| \int_0^1 \sigma(s) ds \\
&\le \frac{(1+\lambda_0+\delta_0)\|x-y\|}{\Gamma(\alpha+\beta)} \sigma^* + \frac{B_1}{\Gamma(\alpha+\beta)} \sum_{i=1}^n |a_i| (1+\lambda_0+\delta_0)\|x-y\| \sigma^* \\
&+ B_2 \sum_{j=1}^m \frac{|b_j|(d_j-c_j)}{\Gamma(\alpha+\beta)} (1+\lambda_0+\delta_0)\sigma^* \|x-y\| \\
&+ \frac{B_2|\mu_1|}{\Gamma(\alpha+\beta)} (1+\lambda_0+\delta_0)\sigma^* \|x-y\| \\
&+ \frac{B_2|\mu_2|}{\Gamma(\alpha+\beta-1)} (1+\lambda_0+\delta_0)\sigma^* \|x-y\| \le (1+\lambda_0+\delta_0)\sigma^* \\
&\times \left(\frac{1 + B_1\sum_{i=1}^n |a_i| + B_2 \sum_{j=1}^m |b_j|(d_j-c_j) + B_2|\mu_1|}{\Gamma(\alpha+\beta)} + \frac{B_2|\mu_2|}{\Gamma(\alpha+\beta-1)} \right) \|x-y\|.
\end{aligned} \tag{19}
$$

Since $r_1 < 1$, then P is a contraction. Therefore, system (1) has a unique solution.

Theorem 8. *Assume that (H_1) holds and $f : [0;1] \times \mathbb{R}^3 \to \mathbb{R}$ is a continuous function. Furthermore, we suppose*

$$
(H_3) \quad |f(t,x,y,z)| \le \theta(t), \quad \forall (t,x,y,z) \in [0,1] \times \mathbb{R}^3 \text{ with } \theta \in L^1([0,1];\mathbb{R}^+). \tag{20}
$$

Then, problem (1) has at least one solution on [0,1] if R < 1, where

$$
R = (1+\lambda_0+\delta_0)\sigma^* \cdot \left(\frac{B_1\sum_{i=1}^n |a_i| + B_2\sum_{j=1}^m |b_j|(d_j-c_j) + B_2|\mu_1|}{\Gamma(\alpha+\beta)} + \frac{B_2|\mu_2|}{\Gamma(\alpha+\beta-1)} \right). \tag{21}
$$

Proof. We now consider the closed ball $B_{r'} = \{x \in X : \|x\| \le r'\}$ with fixed radius r':

$$
\begin{aligned}
r' \ge{}& \frac{\|\theta\|_{L^1}}{\Gamma(\alpha+\beta+1)} + \frac{B_1}{\Gamma(\alpha+\beta+1)} \sum_{i=1}^n |a_i| t_i^{\alpha+\beta} \|\theta\|_{L^1} \\
&+ B_2 \left[\sum_{j=1}^m \frac{|b_j| \|\theta\|_{L^1} (d_j-c_j)}{\Gamma(\alpha+\beta+1)} + \frac{|\mu_1| \|\theta\|_{L^1}}{\Gamma(\alpha+\beta+1)} + \frac{|\mu_2| \|\theta\|_{L^1}}{\Gamma(\alpha+\beta)} \right].
\end{aligned} \tag{22}
$$

We define the operators P_1 and P_2 on $B_{r'}$ as

$$
\begin{aligned}
P_1x(t) ={}& \frac{1}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1} f(s,x(s),\phi x(s),\varphi x(s)) ds, \\
P_2x(t) ={}& -\frac{B_1(t)\sum_{i=1}^n a_i}{\Gamma(\alpha+\beta)} \int_0^{t_i} (t_i-s)^{\alpha+\beta-1} f(s,x(s),\phi x(s),\varphi x(s)) ds \\
&+ B_2(t) \times \Bigg[\sum_{j=1}^m \frac{b_j}{\Gamma(\alpha+\beta)} \int_{c_j}^{d_j} \left(\int_0^s (s-u)^{\alpha+\beta-1} f \right. \\
&\left. \cdot (u,x(u),\phi x(u),\varphi x(u)) \times du \right) ds - \frac{\mu_1}{\Gamma(\alpha+\beta)} \int_0^1 \\
&\cdot (1-s)^{\alpha+\beta-1} f(s,x(s),\phi x(s),\varphi x(s)) ds - \frac{\mu_2}{\Gamma(\alpha+\beta-1)} \int_0^1 \\
&\cdot (1-s)^{\alpha+\beta-2} f(s,x(s),\phi x(s),\varphi x(s)) ds \Bigg].
\end{aligned} \tag{23}
$$

For $x, y \in B_{r'}$, we have

$$
|P_1x(t)| \le \frac{\|\theta\|_{L^1}}{\Gamma(\alpha+\beta+1)},
$$

$$
\begin{aligned}
|P_2y(t)| \le{}& \frac{B_1}{\Gamma(\alpha+\beta+1)} \sum_{i=1}^n |a_i| t_i^{\alpha+\beta} \|\theta\|_{L^1} \\
&+ B_2 \Bigg[\sum_{j=1}^m \frac{|b_j| \|\theta\|_{L^1} (d_j-c_j)}{\Gamma(\alpha+\beta+1)} + \frac{|\mu_1| \|\theta\|_{L^1}}{\Gamma(\alpha+\beta+1)} \\
&+ \frac{|\mu_2| \|\theta\|_{L^1}}{\Gamma(\alpha+\beta+1)} \Bigg].
\end{aligned} \tag{24}
$$

Consequently,

$$\begin{aligned}\|P_1x+P_2y\| &\le \frac{\|\theta\|_{L^1}}{\Gamma(\alpha+\beta+1)}+\frac{B_1}{\Gamma(\alpha+\beta+1)}\sum_{i=1}^{n}|a_i|t_i^{\alpha+\beta}\|\theta\|_{L^1}+B_2\\ &\quad\times\left[\sum_{j=1}^{m}\frac{|b_j|\|\theta\|_{L^1}(d_j-c_j)}{\Gamma(\alpha+\beta+1)}+\frac{|\mu_1|\|\theta\|_{L^1}}{\Gamma(\alpha+\beta+1)}+\frac{|\mu_2|\|\theta\|_{L^1}}{\Gamma(\alpha+\beta)}\right].\end{aligned} \tag{25}$$

Then,

$$P_1x+P_2y\in B_{r'}. \tag{26}$$

Next, we show that P_2 is a contraction. For $x,y\in B_{r'}$, we have

$$\begin{aligned}&\|P_2x(t)-P_2y(t)\|\\ &\le \frac{|B_1(t)|}{\Gamma(\alpha+\beta)}\sum_{i=1}^{n}|a_i|\int_{i=1}^{t_i}(t_i-s)^{\alpha+\beta-1}\times|f(s,x(s),\phi x(s),\varphi x(s))\\ &\quad-f(s,y(s),\phi y(s),\varphi y(s))|ds+|B_2(t)|\\ &\quad\times\left[\sum_{j=1}^{m}\frac{|b_j|}{\Gamma(\alpha+\beta)}\int_{c_j}^{d_j}\int_0^s(s-u)^{\alpha+\beta-1}\times|f(s,x(u),\phi x(u),\varphi x(u))\right.\\ &\quad-f(s,y(u),\phi y(u),\varphi y(u))|duds+\frac{|\mu_1|}{\Gamma(\alpha+\beta)}\int_0^1(1-s)^{\alpha+\beta-1}\\ &\quad\times|f(s,x(s),\phi x(s),\varphi x(s))-f(s,y(s),\phi y(s),\varphi y(s))|ds\\ &\quad+\frac{|\mu_2|}{\Gamma(\alpha+\beta-1)}\int_0^1(1-s)^{\alpha+\beta-2}|f(s,x(s),\phi x(s),\varphi x(s))\\ &\quad\left.-f(s,y(s),\phi y(s),\varphi y(s))|ds\right]\\ &\le \frac{B_1}{\Gamma(\alpha+\beta)}\sum_{i=1}^{n}|a_i|\int_0^{t_i}(t-s)^{\alpha+\beta-1}\sigma(s)(|x(s)-y(s)|+|\phi x(s)\\ &\quad-\phi y(s)|+|\varphi x(s)-\varphi y(s)|)ds+B_2\left[\sum_{j=1}^{m}\frac{|b_j|}{\Gamma(\alpha+\beta)}\right.\\ &\quad\times\int_{c_j}^{d_j}\int_0^s(s-u)^{\alpha+\beta-1}\sigma(u)(|x(u)-y(u)|+|\phi x(u)\\ &\quad-\phi y(u)|+|\varphi x(u)-\varphi y(u)|)duds+\frac{|\mu_1|}{\Gamma(\alpha+\beta)}\int_0^1\\ &\quad\times(1-s)^{\alpha+\beta-1}\sigma(s)(|x(s)-y(s)|+|\phi x(s)-\phi y(s)|+|\varphi x(s)\\ &\quad-\varphi y(s)|)ds+\frac{|\mu_2|}{\Gamma(\alpha+\beta-1)}\int_0^1(1-s)^{\alpha+\beta-2}\sigma(s)\\ &\quad\left.\times(|x(s)-y(s)|+|\phi x(s)-\phi y(s)|+|\varphi x(s)-\varphi y(s)|)ds\right]\\ &\le(1+\lambda_0+\delta_0)\sigma^*\left(\frac{B_1\sum_{i=1}^{n}|a_i|+B_2\sum_{j=1}^{m}|b_j|(d_j-c_j)+B_2|\mu_1|}{\Gamma(\alpha+\beta)}\right.\\ &\quad\left.+\frac{B_2|\mu_2|}{\Gamma(\alpha+\beta-1)}\right)\|x-y\|.\end{aligned} \tag{27}$$

since $R<1$, we conclude that P_2 is a contraction. Now, we show that P_1 is compact and continuous.

Continuity of f implies that the operator P_1 is continuous. Also, P_1 is uniformly bounded on $B_{r'}$ as

$$\|P_1x\|\le\frac{\|\theta\|_{L^1}}{\Gamma(\alpha+\beta+1)}. \tag{28}$$

Suppose that $0\le t_1<t_2\le 1$. We have

$$\begin{aligned}&|P_1x(t_2)-P_1x(t_1)|\\ &\le\frac{1}{\Gamma(\alpha+\beta)}\left|\int_0^{t_1}\left((t_2-s)^{\alpha+\beta-1}-(t_1-s)^{\alpha+\beta-1}\right)\right.\\ &\quad\times f(s,x(s),\phi x(s),\varphi x(s))ds+\int_{t_1}^{t_2}(t_2-s)^{\alpha+\beta-1}f\\ &\quad\left.\times(s,x(s),\phi x(s),\varphi x(s))ds\right|\\ &\le\frac{\|\theta\|_{L^1}}{\Gamma(\alpha+\beta+1)}\left[2(t_2-t_1)^{\alpha+\beta}+\left|t_1^{\alpha+\beta}-t_2^{\alpha+\beta}\right|\right].\end{aligned} \tag{29}$$

Hence, $|P_1x(t_2)-P_1x(t_1)|\to 0$, as $t_1\to t_2$ independently from $x\in B_{r'}$.

This shows that the operator P_1 is relatively compact on $B_{r'}$. Hence, by the Arzela-Ascoli theorem, P_1 is compact on $B_{r'}$.

Then, by Krasnoselskii's fixed point theorem, problem (1) has at least one solution on $B_{r'}$.

4. Example

In this section, we give two examples to prove the applicability of our main results.

Example 9. Let us consider the following system:

$$\begin{cases}{}^cD^{11/7}\left({}^cD^{5/7}\right)x(t)=\dfrac{t^2}{200}\left(\dfrac{1}{1+|x(t)|}+\dfrac{1}{100}\displaystyle\int_0^t t^4s^3x(s)ds\right), & t\in[0,1],\\ x(0)=\dfrac{1}{500}\left(x\left(\dfrac{1}{19}\right)+x\left(\dfrac{4}{19}\right)+x\left(\dfrac{8}{19}\right)+x\left(\dfrac{11}{19}\right)\right), & x'(0)=0,\\ \dfrac{1}{300}x(1)+\dfrac{1}{300}x'(1)=\dfrac{1}{400}\left(\displaystyle\int_{2/19}^{3/19}x(s)ds+\int_{5/19}^{7/19}x(s)ds+\int_{9/19}^{10/19}x(s)ds\right).\end{cases} \tag{30}$$

Here, $m=3$, $n=4$, $\beta=11/7$, $\alpha=5/7$, $a_1=a_2=a_3=a_4=1/500$, $\mu_1=\mu_2=1/300$, $b_1=b_2=b_3=1/400$, $c_1=2/19$, $c_2=5/19$, $c_3=9/19$, $d_1=3/19$, $d_2=7/19$, $d_3=10/19$, $f(t,x,y,z)=(t^2/200)(1/(1+|x(t)|)+y(t)+z(t))$, $\lambda(t,s)=\delta(t,s)=t^4s^3/200$, and $\sigma(t)=t^2/200$.

It follows that

$$\lambda_0=\delta_0=\frac{1}{800},\ \sigma^*=\frac{1}{600},\ B_1\approx 1,3367,\ B_2\approx 117,95142,\ r_1\approx 0,0027. \tag{31}$$

By Theorem 7, we obtain that problem (30) has a unique solution.

Example 10. Consider the following problem:

$$\begin{cases} {}^cD^{18/11}\left({}^cD^{6/11}\right)x(t)=\dfrac{t^3}{400}\left(\dfrac{|x(t)|e^{-t}}{1+|x(t)|}+\displaystyle\int_0^t\dfrac{(t+s)^3|x(s)|(\cos(s)+\sin(s))}{400(1+|x(s)|)}\right)t\in[0,1],\\ x(0)=\dfrac{1}{800}\left(x\left(\dfrac{12}{41}\right)+x\left(\dfrac{15}{41}\right)+x\left(\dfrac{18}{41}\right)+x\left(\dfrac{21}{41}\right)\right),\quad x'(0)=0,\\ \dfrac{1}{600}x(1)+\dfrac{1}{600}x'(1)=\dfrac{1}{700}\left(\displaystyle\int_{13/41}^{14/41}x(s)ds+\int_{16/41}^{17/41}x(s)ds+\int_{19/41}^{20/41}x(s)ds\right). \end{cases} \tag{32}$$

Here, $m=3$, $n=4$, $\beta=18/11$, $\alpha=6/11$, $a_1=a_2=a_3=a_4=1/800$, $\mu_1=\mu_2=1/600$, $b_1=b_2=b_3=1/700$, $c_1=13/41$, $c_2=16/41$, $c_3=19/41$, $d_1=14/41$, $d_2=17/41$, $d_3=20/41$, $f(t,x,y,z)=(t^3/400)(|x(t)|e^{-t}/(1+|x(t)|)+|y(t)|\cos(t)/(1+|y(t)|)+|z(t)|\sin(t)/(1+|z(t)|))$, $\lambda(t,s)=\delta(t,s)=(t+s)^3/400$, $\sigma(t)=t^3/400$, and $\theta(t)=3t^3/400$.

It is clear that

$$\lambda_0=\delta_0=\frac{15}{1600},\sigma^*=\frac{1}{1600},B_1\approx 1,3641,B_2\approx 229,3039,R\approx 0,000566. \tag{33}$$

Then, problem (32) has at least one solution.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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