Research Article

On Norm-Attainable Operators in Banach Spaces

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Norm-attainable operators have been studied over a period of time with nice results obtained particularly in Hilbert spaces. In this work, we consider the Banach space setting by characterizing nonpower operators on H and elementary operators. Lastly, we characterize a new notion of norm-attainability for power operators in general Banach spaces.

1. Introduction

Studies on Hilbert space operators have elicited a lot of interest from mathematicians for decades. Characterizations of properties of Hilbert space operators have been done by many mathematicians over a long period of time with interesting results being obtained. Such properties include norms, Green function, numerical ranges, positivity, spectrum, and invertibility among others (see [1–5], [6] and the references therein). Norm-attainability is also a property which has been given keen attention. This property still remains very important as it has a lot of open questions which are unanswered particularly when a super class of Hilbert space operators called supraposinormal operators [3] is considered. In [7], the authors characterized the norm property for elementary operators and gave conditions under which a general elementary operator is norm attainable. In-depth characterization of norm-attainable operators has also been done in detail with considerations given to other properties like orthogonality (see [1–5] and the references therein). Regarding derivations, authors in [5] showed that if V_Γ and W_Γ are Γ -Banach algebras and δ an α -inner derivation, then δ is norm attainable if and only if the adjoint, δ^* , of δ is norm attainable. Moreover, as a consequence, they proved that if δ_N^1 and δ_N^2 are norm attainable then δ_N is norm attainable if either δ_N^1 and δ_N^2 or both are zero derivations and δ_N^1 and δ_N^2 are α -inner derivation and α' -inner derivation, respectively. On elementary operators, necessary and sufficient conditions for norm-attainability for Hilbert space operators were given in [2]

where it was proved that if $S \in B(H)$, $\beta \in W_0(S)$, and $\alpha > 0$, then there exists an operator $Z \in B(H)$ such that $\|S\| = \|Z\|$, with $\|S - Z\| < \alpha$. Furthermore, there exists a unit vector $\eta \in H$ such that $\|Z\eta\| = \|Z\|$ with $\langle Z\eta, \eta \rangle = \beta$, where $W_0(S)$ denotes the maximal numerical range of the operator S . Moreover, norm-attainability conditions for elementary operators and generalized derivations have been given. For orthogonality of elementary operators in norm-attainable classes, a detailed exposition has been given in [4, 8]. A super-class of Hilbert space operators has also been considered in norm-attainable classes. In [3], a good characterization has been done on α -supraposinormality of operators in dense norm-attainable classes. In this paper, we continue in the spirit of characterization of operators in normed spaces. We characterize norm-attainability for functionals in Banach spaces. Moreover, we give a new notion of norm-attainability for power operators and also for elementary operators which generalizes the results of [2].

2. Preliminaries

In this section, we outline preliminary concepts which are useful in the sequel.

Definition 1 [9]. Let V be a linear vector space. A nonnegative real valued function $\|\cdot\| : V \rightarrow \mathbb{R}$ is called a norm on V if it satisfies the following conditions:

- (i) $\|x\| \geq 0$ and $\|x\| = 0$, if and only if $x = 0$, for all $x \in V$

- (ii) $\|\alpha x\| = |\alpha| \|x\|$, for all $x \in V$ and $\alpha \in \mathbb{K}$
- (iii) $\|x + y\| \leq \|x\| + \|y\|$, for all $x, y \in V$

The ordered pair $(V, \|\cdot\|)$ is called a normed linear space or simply a normed space.

Remark 2. Examples of normed spaces are Banach space, Hilbert space, Hardy space, Orlicz space, Sobolev-Folland-Stein space, and $B(H)$ among others.

Definition 3 [9]. An operator $A \in B(H)$ is called a scalar operator of order m if it possesses a spectral distribution of order m , i.e., if there is a continuous unital morphism $\phi : C_0^m(\mathbb{C}) \rightarrow B(H)$ such that $\phi(z) = A$, where z stands for the identity function on $\mathbb{C}$ and $C_0^m(\mathbb{C})$ for the space of compactly supported functions on $\mathbb{C}$ continuously differentiable of order m , $0 \leq m \leq \infty$. An operator $A_0 \in B(H)$ is called subscalar if it is similar to the restriction of a scalar operator to an invariant subspace.

Definition 4 [1]. An operator $A \in B(H)$ is said to be normal if $AA^* = A^*A$ and p -normal if $A^p A^* = A^* A^p$; self-adjoint if $A = A^*$; positive if $A = A^*$ and $\langle Ax, x \rangle \geq 0$, for all $x \in H$; and projection if $A^2 = A = A^*$.

Definition 5 ([2], Definition 1.1). An operator $A \in B(H)$ is said to be norm attainable if there exists a unit vector $x_0 \in H$ such that $\|Ax_0\| = \|A\|$. The set of all norm-attainable operators on a Hilbert space H is denoted by $NA(H)$.

Definition 6 ([2], Definition 1.2). For an operator $A \in B(H)$, we define a numerical range by $W(A) = \{\langle Ax, x \rangle : x \in H, \|x\| = 1\}$ and the maximal numerical range by $W_0(A) = \{\beta \in \mathbb{C} : \langle Ax_n, x_n \rangle \rightarrow \beta, \text{ where } \|x_n\| = 1, \|Ax_n\| \rightarrow \|A\|\}$.

Definition 7 [10]. Let Ω be a Banach lattice then Ω is an abstract M space, i.e., $\Omega \in AM$ if $x \wedge y = 0$ implies $\|x + y\| = \max\{\|x\|, \|y\|\}$. Also Ω is abstract L space, i.e., $\Omega \in AL$ if $x \vee y = 0$ implies $\|x + y\| = \|x\| + \|y\|$.

Definition 8 [9]. Let Ω be a Banach lattice. Then, $\Omega \in AM$ implies $\Omega^* \in AL$ and $\Omega \in AL$ implies $\Omega^* \in AM$. $\Omega \in AM$ if and only if for any $x, y \in \Omega, x, y \geq 0$ implies $\|x \vee y\| = \max\{\|x\|, \|y\|\}$. $\Omega \in AL$ if and only if for any $x, y \in \Omega, x, y \geq 0$ implies $\|x + y\| = \|x\| + \|y\|$.

Definition 9 [10]. Let Ω be a Banach lattice. Then, Ω is said to be σ complete, if for every order bounded sequence $\{x_n\} \in \Omega, \bigvee_{n \geq 1} x_n$ exists in Ω . Also, Ω is said to be bounded σ complete, provided that the any norm bounded and order monotone sequence in Ω is order convergent.

Definition 10 [10]. Let Ω be a Banach space. An element $x \in D(\Omega)$ is called an extreme point of $B(\Omega)$ if $x = \lambda y + (1 - \lambda)z$, $y, z \in B(\Omega)$, and $\lambda \in (0, 1)$, imply $y = z$. In this case, we write $x \in \text{ext}B(\Omega)$.

Definition 11 [1]. Let $\mathcal{A}$ be a Banach algebra and consider $T : \mathcal{A} \rightarrow \mathcal{A}$. The operator T is called an elementary operator if it has the representation $T(X) = \sum_{i=1}^n A_i X B_i$, $\forall X \in \mathcal{A}$, where A_i, B_i are fixed in $\mathcal{A}$ or $\mathcal{M}(\mathcal{A})$, where $\mathcal{M}(\mathcal{A})$ is the multiplier algebra of $\mathcal{A}$.

Example 12. Let $\mathcal{A} = B(H)$. For $A, B \in B(H)$, we define the particular elementary operators:

- (i) The left multiplication operator $L_A : B(H) \rightarrow B(H)$ by $L_A(X) = AX$, for all $X \in B(H)$
- (ii) The right multiplication operator $R_B : B(H) \rightarrow B(H)$ by $R_B(X) = XB$, for all $X \in B(H)$
- (iii) The generalized derivation (implemented by A, B) by $\delta_{A,B} = L_A - R_B$, for all $X \in B(H)$
- (iv) The basic elementary operator (implemented by A, B) by $M_{A,B}(X) = AXB$, for all $X \in B(H)$
- (v) The Jordan elementary operator (implemented by A, B) by $\mathcal{U}_{A,B}(X) = AXB + BXA$, for all $X \in B(H)$

3. Norm-Attainability for Functionals

In this section, we characterize norm-attainability of functionals in Banach spaces. We regard H^* the dual space of a Hilbert space H to be nonzero throughout this section unless otherwise stated. Let $\varphi \in H^*$. Then, φ is said to be norm attainable at $\varphi/\|\varphi\|$ if there exists $T \in B(H)$ such that $\langle \varphi, T \rangle = \|\varphi\| \cdot \|T\| > 0$. $\varphi^*/\|\varphi^*\|$ is called a support for φ . The following proposition shows that any functional is norm attainable in nonzero dual spaces.

Proposition 13. Let $B(W)$ be the set of all bounded linear maps on an Orlicz space W then every $\varphi \in H_+^*$ is norm attainable on $B(W)$.

Proof. Let W be a bounded linear Orlicz space and $b_n \in W$ such that b_n is monotone decreasing to 0. Given $J_n \in B(W)$, we have $\varphi(J_n) > \|\varphi\| - b_n$ since $\pi(J_n) \leq 1 < \infty$ and $\varphi(W_0) = \{0\}$. Suppose that $\pi(J_n) \leq 2^{-n}$ and let $J(t) = \sup |J_n(t)|$. Then, $\pi(J) \leq \sum_{n=1}^{\infty} \pi(J_n) \leq 1$, that is, $J \in B(W)$ and $\varphi(J) \geq \sup_n (\|J_n\|) = \|\varphi\|$, because $\varphi \in H_+^*$.

Theorem 14. Let $B(W)$ be the set of all bounded linear maps on an Orlicz space W then $\varphi \in H^*$ is norm attainable on $B(W)$ if and only there exists L in a subspace $\mathcal{E}$ of $B(W)$ such that $\varphi^+ = \varphi|_L$ and $\varphi^- = \varphi|_{GL}$.

Proof. It is known that we have $P, Q \in B(W)$ such that $\varphi|_L(P) = \varphi^+/\|\varphi^+\|$ and $\varphi|_{GL}(Q) = \varphi^-/\|\varphi^-\|$ by Proposition 13. Assume that $\pi(P|_L) \leq 1/2$ and $\pi(Q|_{GL}) \leq 1/2$. Then, $\pi(P|_L - Q|_{GL}) \leq \pi(P) + \pi(Q) \leq 1$. Indeed, $P|_L - Q|_{GL} \in L(W)$ and $\varphi(P|_L - Q|_{GL}) = \|\varphi^+\| + \|\varphi^-\| = \|\varphi\|$. Suppose that $P \in B(W)$ satisfies $\varphi(P) = \|\varphi\|$. Let $L = \{l \in G : P(l) \geq 0\}$ then we show that $\varphi|_L - \varphi|_{GL} \in H_+^*$. Now, if $\varphi|_L \in H_+^*$, then there exists $Q \in W^+$. If $\varphi|_{L \notin H'}$, then there exists $Q \in W^+$ such that $\varphi|_L$

$(Q) < 0$. Now φ being singular, assume $\pi(Q) \leq 1/2$ and $\pi(P) \leq 1/2$. Then, $J = P|_{G \setminus L} - Q|_L \in B(W)$, and so, $\|\varphi^-\| \geq \varphi^-(-J) = -\varphi^+(J) + \varphi(J) \geq \varphi(J) = \varphi(P|_{G \setminus L} - \varphi|_L(Q)) > \varphi(P|_{GL})$. This is contrary to our earlier assumption. Lastly, $\|\varphi\| = \|\varphi^+\| + \|\varphi^-\| > \varphi(P|_L) + \varphi(P|_{G \setminus L}) = \varphi(P) = \|\varphi\|$. $(\varphi|_{G \setminus L}) \in H_1^*$ follows analogously.

Corollary 15. Let $\varphi \in H^*$ be norm attainable at $P \in B(W)$, then $\varphi(P_A) \|\varphi|_A\|$ for all $A \in \mathcal{E}$.

Proof. $\|\varphi\| = \|\varphi|_A\| + \|\varphi|_{G \setminus A}\| \geq \varphi|_A(P) + \varphi|_{G \setminus A}(P) = \varphi(P) = \|\varphi\|$ is enough.

Theorem 16. Let $\varphi \in H^*$ be singular then the set of all such φ is dense in H^* .

Proof. Let $\varphi \in H^*$ be singular and $\varepsilon > 0$ be given. Then, we have $L \in \mathcal{E}$ such that $\|\varphi^+|_{G \setminus L}\| < \varepsilon$ and $\|\varphi^-|_L\| \leq \varepsilon$. Suppose that $\psi = \varphi^+|_L - \varphi^-|_{G \setminus L}$. Then, by Theorem 14, ψ is norm attainable. Also, $\|\varphi - \psi\| \leq \|\varphi^+ - \psi|_L\| + \|\varphi - \psi|_{G \setminus L}\| = \|\varphi^+|_{G \setminus L}\| + \|\varphi^-|_L\| < 2\varepsilon$.

Theorem 17. Let $B(W)$ be the set of all bounded linear maps on an Orlicz space W . Then, $\phi = \chi + \varphi$ ($0 \neq \chi \in H_0^*$, $\varphi \in H^*$) is norm attainable at $P \in B(W)$ if and only if $\pi(P) = 1$, $\varphi(P) = \|\varphi\|$, and $\int_G k\chi(t)P(t)dt = \pi(x) + \omega(k\chi)$, where $k \in K_N(\chi) = \{k : k^{-1}[1 + \omega(k\chi)] = \|\chi\|_N^0\}$.

Proof. From the statement of the theorem, we have $\|\phi\|^0 = f(P) = k^{-1}\langle k\chi, \chi \rangle + \varphi(P) \leq k^{-1}[\pi(P) + \omega(k\chi)] + \varphi(P) \leq k^{-1}[1 + \omega(k\chi)] + \|\varphi\| = \|\chi\|_N^0 + \|\varphi\| = \|\phi\|^0$. The converse follows from the fact that ϕ is singular from Theorem 16 and an assertion from Proposition 13.

At this point, we consider norm-attainable functionals in Banach lattices. We denote an abstract L space and abstract M space by AL and AM , respectively. For details on AL and AM , see [10]. We state the following lemma.

Lemma 18. Let $\Omega \in AL$ and $\varphi \in B(\Omega^*)$. Then, the following are equivalent.

- (i) φ is norm attainable
- (ii) Both φ^+ and φ^- are norm attainable
- (iii) φ^+ or φ^- is norm one

Proof. (i) $\Rightarrow$ (ii). Choose $x \in B(\Omega)$ such that $\varphi(x) = \|\varphi\| = 1$. From

$$\begin{aligned} 1 = \|\varphi\| &= \varphi(x) = \varphi^+(x^+) + \varphi^-(x^-) - \varphi^+(x^-) - \varphi^-(x^+) \\ &= \|\varphi^+\| \|x^+\| + \|\varphi^-\| \|x^-\| - \varphi^+(x^-) - \varphi^-(x^+) \\ &= \|\varphi\| (\|x^+\| + \|x^-\|) = \|\varphi\| \|x\| = \|\varphi\| = 1, \end{aligned} \quad (1)$$

we obtain $\varphi^+(x^+) = \|\varphi^+\| \|x^+\|$; $\varphi^-(x^-) = \|\varphi^-\| \|x^-\|$ and

$\varphi^+(x^-) = \varphi^-(x^+) = 0$ since $\varphi^\pm(x^\pm) \leq \|\varphi^\pm\| \|x^\pm\|$ and $\varphi^\pm, (x^\pm)$ are nonnegative.

(ii) $\Rightarrow$ (iii). This follows obviously.

(iii) $\Rightarrow$ (i). We suppose that φ^+ is norm one and norm attainable. Choose $x \in B(\Omega^+)$ such that $\varphi^+(x) = \|\varphi\| = 1$. We have $\varphi^-(x) = 0$. Indeed, $1 = \|\varphi\| \geq \varphi(|x|) \geq \varphi^+(x) + \varphi^-(|x|) = 1 + \varphi^-(|x|) \geq 1$, which implies that $\varphi^-(x^+) = \varphi^-(x^-) = 0$. Hence, $\varphi(x) = \varphi^+(x) = \|\varphi\| = 1$.

Theorem 19. Let $\Omega \in AL$ and $\theta \leq \varphi \in B(\Omega^*)$. Then, the following are equivalent.

- (i) φ is norm attainable
- (ii) There exists $\theta \leq x \neq \theta$ such that $\varphi(y) = \|y\|$ for all $\bar{E}_\Omega$, the norm closure of E_Ω where $E_\Omega = \{y \in \Omega : \theta \leq y \leq nx, \text{ for some } n > 0\}$
- (iii) There exists $\theta \neq x \in \Omega^+$ such that among $B(\Omega^*) = \{\psi \in \Omega^* : \|\psi\| = 1\}$, φ is maximal on $\bar{E}_\Omega$

Proof. (i) $\Rightarrow$ (ii). Choose $x \in B(\Omega)$ satisfying $\varphi(x) = \|x\| = 1$. We have $x \in \theta$ for $1 = \varphi(x) = \varphi(x^+) = \varphi(x^+) - \varphi^-(x^+) \leq \|x^+\| - \|x^-\| \leq 1$ by Lemma 18 which implies $\|x^-\| = 0$. Now, if we consider $y \in \bar{E}_\Omega$, we need to prove that $\varphi(y) = \|y\|$. Since φ is continuous, let $y \in \bar{E}_\Omega$, i.e., $\theta \leq y \leq nx$ for some $n \geq 0$. But since $n = \varphi(nx) = \varphi(nx - y) + \varphi(y) \leq \|nx - y\| + \|y\| = \|nx\| = n$, we have $\varphi(y) = \|y\|$.

(ii) $\Rightarrow$ (iii). Follows trivially from (ii) $\Rightarrow$ (iii) in Lemma 18 and abstractness of AL .

(iii) $\Rightarrow$ (i). Let φ be maximal in $B(\Omega^*)$ on $\bar{E}_\Omega$ for some $\theta \neq x \in \Omega^+$. Choose $\psi \in B(\Omega_*)$ such that $\psi(x) = \|x\|$, then by $\varphi(x) \geq \psi(x) = \|x\|$, it is clear that φ is norm attainable at $x/\|x\|$.

Proposition 20. Let $\Omega \in AM$ be σ -complete and $\varphi \in \Omega^*$. Then, for any $\varepsilon > 0$, there exists a subspace E of $\Omega = E + E^\perp$ and $\|\varphi^+|_{E^\perp}\| < \varepsilon$, $\|\varphi^-|_E\| < \varepsilon$.

Proof. Choose $x \in D(\Omega)$ with the property that $\varphi^+(x) > \|\varphi\| - \varepsilon$, and $E = (x^\perp)^\perp$. Then, $x^+ \in E$, $x^- \in E^\perp$ and $\Omega = E + E^\perp$. Furthermore, for $x \in D(\Omega)$, we have that $\|\varphi^+|_E\| + \|\varphi^+|_{E^\perp}\| + \|\varphi^-|_E\| + \|\varphi^-|_{E^\perp}\| = \|\varphi^+\| + \|\varphi^-\| = \|\varphi\| < \varphi(x) + \varepsilon = \varphi^+|_E(x) + \varphi^+|_{E^\perp}(x) - \varphi^-|_E(x) - \varphi^-|_{E^\perp}(x) + \varepsilon$ because $\varphi^+|_{E^\perp}(x) \leq 0$ and $\varphi^-|_E(x) \geq 0$. So, we conclude that $\|\varphi^+|_{E^\perp}\| + \|\varphi^-|_E\| = \|\varphi^+\| - \|\varphi^+|_E\| + \|\varphi^-|_E\| - \|\varphi^-|_{E^\perp}\| \leq \|\varphi^+\| - \varphi^+|_E(x) + \varphi^-|_E(x) - \varphi^-|_{E^\perp}(x) < \varphi^+|_{E^\perp}(x) - \varphi^-|_E(x) + \varepsilon \leq \varepsilon$. This completes the proof as required.

Theorem 21. Let a Banach lattice Ω be bounded σ -complete and $B(\Omega)$ order-closed, then every positive bounded linear $\varphi \in \Omega^*$ is norm attainable.

Proof. Consider $x_n (\geq 0) \in D(\Omega)$ such that $\varphi(x_n) \rightarrow \|\varphi\|$. Since Ω is bounded σ -complete and $B(\Omega)$ is closed under order, $y = \vee_n (x_n)$ exists in Ω and $\|y\| = 1$. Hence, $y \geq x_n \geq 0$ and $\varphi \geq 0$ implies $\|\varphi\| \geq \varphi(y) \geq \varphi(x_n) \rightarrow \|\varphi\|$. So, $x \in D(\Omega)$

exists which satisfies the norm-attainability condition, $\varphi(x) = \|\varphi\|$, for functionals and this completes the proof.

Theorem 22. *Let $\Omega \in AM$ be bounded σ -complete and $B(\Omega)$ order-closed, then $\varphi \in \Omega^*$ is norm attainable if and only if there exists a subspace E of Ω such that $\varphi^+ = \varphi|_E$, $\varphi^- = -\varphi|_{E^\perp}$.*

Proof. Necessity. Let $x \in B(\Omega)$ be having the property that $\varphi(x) = \|\varphi\|$, and define $E = (x^-)^\perp$. Then, $\Omega = E + E^\perp$ and $x^+ \in E$, $x^- \in E^\perp$. Now, $\|\varphi\| = \|\varphi|_E\| + \|\varphi|_{E^\perp}\|$; we need to show that $\varphi^+ = \varphi|_E$ and $\varphi^- = -\varphi|_{E^\perp}$, it is enough that $\varphi|_E \geq 0$ and $-\varphi|_{E^\perp} \geq 0$. Indeed, if $\varphi_E(y) < 0$ for some $y(\geq 0) \in D(\Omega)$ then we let $y \in E$. So, $z = -x^- - y$ satisfies $\|z\| = \max\{\|x\|, \|y\|\} = 1$, and hence, $\|\varphi^-\| = \varphi^-(z) = \varphi(z) - \varphi^+(z) \geq \varphi(z) = \varphi|_{E^\perp}(-x^-) - \varphi|_E(y) > \varphi|_{E^\perp}(-x^-) = -\varphi|_{E^\perp}(x^-)$. Now since $\|\varphi^+\| \geq \varphi(x_E) = \varphi|_E(x)$, this is contrary to $\|\varphi\| = \|\varphi^+\| + \|\varphi^-\| > \varphi|_E(x) - \varphi|_{E^\perp}(x) = \varphi(x) = \|\varphi\|$. Also, $-\varphi|_{E^\perp} \geq 0$ follows analogously.

Sufficiency. We know that there exists $x, y(\geq 0) \in D(\Omega)$ such that $\varphi^+(x) = \|\varphi^+\|$ and $\varphi^-(y) = \|\varphi^-\|$ from Theorem 21. Since $\varphi^+ = \varphi|_E$ and $\varphi^- = -\varphi|_{E^\perp}$, we let $x \in E$ and $y \in E^\perp$ and $u \in x - y$. Then, $u = \|x - y\| = \max\{\|x\|, \|y\|\} = 1$ and so we have $\|\varphi\| = \|\varphi^+\| + \|\varphi^-\| = \varphi^+(x) + \varphi^-(y) = \varphi|_E(x) + \varphi|_{E^\perp}(-y) = \varphi(u)$.

Corollary 23. *Let $\Omega \in AM$ be a σ -complete and $\varphi \in D(\Omega^*)$. Then, $\varphi \in \text{ext}B(\Omega^*)$ if and only if $\varphi(x)\varphi(y) = 0$ for all $x, y \in \Omega$ satisfying $x \wedge y = 0$.*

Proof. Necessity. If there exists $x, y \in \Omega$ satisfying $x \wedge y = 0$ but $\varphi(x) > 0$ and $\varphi(y) > 0$, then we set $E = y^\perp$, and $\Omega = E + E^\perp$. Let $\psi = \varphi|_E$ and $\tau = \varphi|_{E^\perp}$. Then, $\|\psi\| > 0$, $\|\tau\| > 0$ since $x \in E$, $y \in E^\perp$. Therefore, $\varphi = \|\psi\|(\psi/\|\psi\|) + \|\tau\|(\tau/\|\tau\|)$ and $\|\psi\| + \|\tau\| = \|\varphi\| = 1$. Hence, $\varphi \in \text{ext}B(\Omega)$.

Sufficiency. First, we show $\|\varphi^+\| \|\varphi^-\| = 0$. In fact, for any $\varepsilon > 0$, by Theorem 21, there exists two orthogonal subspaces $E, F \in \Omega$ such that $\Omega = E + F$ and $\|\varphi^-|_E\| < \varepsilon$, $\|\varphi^+|_F\| < \varepsilon$. Choose $x \in B(\Omega)$ such that $\varphi(x) > \|\varphi\| - \varepsilon$, and let $x = u + v$, where $u \in E$ and $v \in F$. Then, $\varphi(u)\varphi(v) = 0$ since $u \wedge v = 0$. If $\varphi(v) = 0$ then $\|\varphi\| - \varepsilon < \varphi(x) = \varphi^+|_E(u) - \varphi^-|_E(u) \leq \|\varphi^+|_E\| + \|\varphi^-|_E\| < \|\varphi^+\| + \varepsilon$. Let $\varepsilon \rightarrow 0$, we find $\|\varphi^-\| = \|\varphi\| - \|\varphi^+\| = 0$. Similarly, if $\varphi(u) = 0$, then $\|\varphi^+\| = 0$. Hence, without loss of generality, we assume $\varphi = \varphi^+$. Let $\psi, \tau \in D(\Omega^*)$ satisfy $2\varphi = \psi + \tau$. Then, $2\varphi = (\psi^+ + \tau^+) - (\psi^- + \tau^-)$, and hence, $\|2\varphi\| = \|\psi^+\| + \|\tau^+\| + \|\psi^-\| + \|\tau^-\| = \|\psi\| + \|\tau\| = 2 = \|2\varphi\|$. Thus, $\psi^+ + \tau^+ = 2\varphi$ and $\psi^- = \tau^- = 0$. Now, we show $\psi = \tau = \varphi$, that is, $\varphi \in \text{ext}B(\Omega^*)$. This follows if we prove that $\psi(y) = \tau(y) = 0$ whenever $\varphi(y) = 0$; this means $\varphi = a\psi = b\tau$, but $\varphi, \psi, \tau \in D(\Omega^*)$ and $2\varphi = \psi + \tau$, so $a = b = 1$, this means assume $y \geq 0$; then from $\psi(y) \geq 0, \tau(y) \geq 0$ and $\psi(y) + \tau(y) = 2\varphi(y) = 0$. We have $\psi(y) = \tau(y) = 0$. For the general case, since $\varphi(y) = 0$ and by the condition given in theorem $\varphi(y^+)\varphi(y^-) = 0$, we have $\varphi(y^+) = \varphi(y^-) = 0$; hence, the condition $\psi(y) = \tau(y) = 0$ follows from the first case.

4. Norm-Attainability for Nonpower Operators

This section deals with norm-attainability for usual Hilbert space operators. To avoid ambiguity and without loss of generality, we use the term “nonpower” to differentiate these operators from those in the next section. The first result is a proof of norm-attainability condition for compact self adjoint operators in a dense separable infinite dimensional complex Hilbert space.

Theorem 24. *Let H be a dense separable infinite dimensional complex Hilbert space. Let $T : H \rightarrow H$ be a compact and self-adjoint operator on Hilbert space H . Let D be unit sphere in H . Then, there exists a vector $x_0 \in D$ such that $\|Tx_0\| = \|T\|$.*

Proof. From an analogy of the definition of the usual operator norm, we know that $\|T\| = \sup_{x \in D} \|Tx\|$, so there exists a sequence of elements $x_1, x_2, \dots \in D$ such that $\lim_{n \rightarrow \infty} \|Tx_n\| = \|T\|$. Since T let $y_0 = \lim_{n \rightarrow \infty} Tx_n$ exists in H . Let $Y = \text{Span}\{x_1, x_2, \dots\}$. Then, it is a closed subspace of H , hence a Hilbert space. So, Y is reflexive and separable. Thus, there exists a subsequence of (x_n) , say $(x_{j_1}, x_{j_2}, x_{j_3}, \dots)$ with $1 \leq j_1 < j_2 < \dots$, such that x_{j_k} converge weakly in Y to some point $x_0 \in Y$. Now, for each $z \in Y$, $\langle Tx_0, z \rangle = \langle x_0, Tx \rangle = \lim_{k \rightarrow \infty} \langle x_{j_k}, Tz \rangle = \lim_{k \rightarrow \infty} \langle Tx_{j_k}, z \rangle = \langle y_0, z \rangle$. Now, the second equality holds since x_{j_k} converge weakly to x_0 . Indeed, $x \mapsto \langle x, Tz \rangle$ is a bounded linear functional on Y , and the last equality holds because we know from the fact that $y_0 = \lim_{n \rightarrow \infty} Tx_n$ exists in H . So, $\lim_{k \rightarrow \infty} Tx_{j_k} = y_0$ exists and $\langle \cdot, \cdot \rangle$ is continuous. Since $\langle Tx_0, z \rangle = \langle x_0, Tx \rangle = \lim_{k \rightarrow \infty} \langle x_{j_k}, Tz \rangle = \lim_{k \rightarrow \infty} \langle Tx_{j_k}, z \rangle = \langle y_0, z \rangle$ holds for $z \in Y$, we conclude that $Tx_0 = y_0$. Hence, $\|Tx_0\| = \|y_0\| = \|\lim_{k \rightarrow \infty} Tx_{j_k}\| = \lim_{k \rightarrow \infty} \|Tx_{j_k}\| = \|T\|$. Now, since x_{j_k} converge weakly to x_0 , we have $\langle x_0, x_0 \rangle = \lim_{k \rightarrow \infty} \langle x_{j_k}, x_0 \rangle$ and $|\langle x_{j_k}, x_0 \rangle| \leq \|x_{j_k}\| \cdot \|x_0\| = 1$ for all k , hence $\|x_0\| \leq 1$. We cannot have $\|x_0\| < 1$ since then $\|Tx_0\| \leq \|T\| \cdot \|x_0\| < \|T\|$ which is a contradiction. Hence, $\|x_0\| = 1$, i.e., $x_0 \in D$. Hence, the existence of x_0 is proved as required.

Corollary 25. *Let H be a dense separable infinite dimensional complex Hilbert space. Let $T : H \rightarrow H$ be a compact and self-adjoint operator on Hilbert space H and let $Y \subset H$ be a subspace such that $T(Y) \subset Y$. Let $T(Y^\perp) \subset Y^\perp$ be a subspace such that $T(Y) \subset Y$. Then, $T(Y^\perp) \subset Y^\perp$, and $T|_{Y^\perp} : Y^\perp \rightarrow Y^\perp$ is a bounded self-adjoint linear operator on Hilbert space $Y^\perp$, with the norm $\|T|_{Y^\perp}\| \leq \|T\|$.*

Proof. Choose $z \in Y^\perp$. Now, for each $y \in Y$, $\langle Tz, y \rangle = \langle z, Ty \rangle = 0$ since $Ty \in T(Y) \subset Y$ and $z \in Y^\perp$. Clearly, $Tz \in Y^\perp$. Hence, $T(Y^\perp) \subset Y^\perp$.

The next proposition in this section is a very important refined result for a characterization of operators in $B(H)$ whose analogies can be found in [2] Theorem 2.1 or [4] Proposition 2.1.

Lemma 26. Let H be a reflexive, dense, separable, infinite dimensional complex Hilbert space. Let $S \in B(H)$, $\beta \in W_0(S)$, and $\alpha > 0$ be given. Then, there exists a self-adjoint compact operator $Z \in B(H)$ such that $\|S\| = \|Z\|$, with $\|S - Z\| < \alpha$. Furthermore, there exists a unit vector $\eta \in H$ such that $\|Z\eta\| = \|Z\|$ with $\langle Z\eta, \eta \rangle = \beta$.

Corollary 27. Let $S, T \in B(H)$. If both S and T are norm attainable, then the basic elementary operator $M_{S,T}$ is also norm attainable.

Proof. For any pair (S, T) , it is known that $\|M_{S,T}\| = \|S\| \|T\|$. We can assume that $\|S\| = \|T\| = 1$. If both S and T are norm attainable, then there exists unit vectors ζ and η with $\|S\zeta\| = \|T\eta\| = 1$. We can therefore define an operator X by $X = \langle \cdot, T\eta \rangle \zeta$. Clearly, $\|X\| = 1$. Therefore, we have $\|SXT\| \geq \|SXT\eta\| = \|\langle T\eta, T\eta \rangle S\zeta\| = 1$. Hence, $\|M_{S,T}(X)\| = \|SXT\| = 1$, that is, $M_{S,T}$ is also norm attainable.

Proposition 28. If the elementary operator T_{A_i, B_i} is norm attainable, then there exists an isometry or a coisometry Q_0 such that $\|T_{A_i, B_i}\| = \|\sum_{i=1}^n A_i Q_0 B_i\|$.

Proof. From [7], if the elementary operator T_{A_i, B_i} is norm attainable, then there exists a contractive operator $X_0 \in B(H)$ such that $\|T_{A_i, B_i}\| = \|\sum_{i=1}^n A_i X_0 B_i\|$. Lastly, it is sufficient enough to show that $X_0 = (1/2)(\Gamma_1 + \Gamma_2)$ for two isometries or coisometries $\Gamma_1, \Gamma_2 \in B(H)$. This follows immediately from Lemma 3.1 in [7].

5. New Notion of Norm-Attainability for Power Operators

In this section, we give new notions of norm-attainability for power operators. First, we begin with some auxiliary results.

Proposition 29. Let $T \in NA(H)$. Then, T is p -norm attainable if it is p -normal.

Proof. Let $T \in NA(H)$ be p -normal, i.e., $T^p T^* = T^* T^p$. By replacing T^* with its power raised to p , it is easy to see that $T^p (T^*)^p = (T^*)^p T^p$. Hence, T^p is normal. Now, if we consider T^p to be normal, then $T^p T = T T^p$ since $T^* T^p = T^p T^*$ by Fuglede property. Therefore, T is p -normal. But $T \in NA(H)$ and T^p is normal so it follows that there exists a unit vector $x \in H$ such that $\|T^p x\| = \|T^p\|$, for any $p \in \mathbb{N}$.

Remark 30. Every norm-attainable operator and every bounded normal operator is p -norm attainable and p -normal, respectively, for any $p \in \mathbb{N}$. However, the converses are not true in general.

Theorem 31. Let $NA_p(H)$ be the set of all p -norm-attainable operators on H . Then, $NA_p(H)$ is a closed subset of $NA(H)$ which is closed under scalar multiplication if and only if for any $T \in NA(H)$, T is p -normal.

Proof. Consider $\pi \in \mathbb{K}$ and let T be p -normal. By carrying out a simple calculation, we find that $(\pi T)^p (\pi T)^* = (\pi T)^* (\pi T)^p$ and so πT is p -normal. If $T \in NA(H)$, then the converse follows immediately by taking limits over a sequence of vectors in H . Therefore, T is p -normal.

Corollary 32. Let $T \in NA(H)$ be p -norm attainable. Then, the following conditions hold:

- (i) UTU^* is p -norm attainable for a unitary operator U
- (ii) T^* is p -norm attainable
- (iii) T^{-1} is p -norm attainable if it exists
- (iv) If there exists a unitary equivalence between $T_0 \in NA(H)$ and T , then T_0 is p -norm attainable
- (v) If G is a unitarily invariant subspace of H such that G reduces T , then $T_0 = T|_G$ is a p -norm-attainable operator

Proof. (i) Since U is unitary then $UU^* = U^*U = I$, where I is the identity operator then by Definition 5 and by Theorem 31 we obtain the result. (ii) Since T is p -norm attainable from Proposition 29, T^p is p -norm attainable and so $(T^*)^p$ is p -norm attainable. Consequently, T^* is p -norm attainable. (i) (ii) Suppose that T^{-1} exists. Since T is p -norm attainable, T^p is p -norm attainable. Since $(T^p)^{-1} = (T^{-1})^p$ is p -norm attainable, T^{-1} is a p -norm-attainable operator. (iv) Follows immediately from (i). (v) Follows from the fact that G is invariant under T . The rest is clear.

Theorem 33. Let $A, B \in NA_p(H)$ be commuting p -norm-attainable operators, then AB is a p -norm-attainable operator.

Proof. Suppose that $A, B \in NA_p(H)$. Since A, B are commuting p -norm-attainable operators, then A^p, B^p are commuting normal operator. From [3], $A^p B^p$ is a p -norm-attainable operator and hence norm attainable. Lastly, $A^p B^p = (AB)^p = (BA)^p$ is normal and norm attainable. Hence, AB is a p -norm-attainable operator.

Remark 34. Not all p -norm-attainable operators are p -normal. In fact, the following example shows that the sum of two commuting p -normal operators need not be p -normal.

Example 35. Consider

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad (2)$$

$$B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Clearly, A and B are commuting 2-normal operators. Now,

$$\begin{aligned} A + B &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \\ (A + B)^2 &= \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \end{aligned} \quad (3)$$

are not normal. So $A + B$ is not 2-normal. Recall that B is self-adjoint.

6. Open Questions

In conclusion, we have characterized norm-attainability for functionals in Banach spaces. Moreover, we have given a new notion of norm-attainability for power operators and also for elementary operators in normed spaces. The following two open questions arise naturally:

- (1) Let H be a reflexive, dense, separable, infinite dimensional complex Hilbert space. Does there exist a unit vector $x_0 \in H$ for a bounded self-adjoint operator $A : H \rightarrow H$ such that $\|Ax_0\| < \|A\|$?
- (2) When does p -norm-attainability and p -normality coincide in general Banach spaces?

Data Availability

No data used in the manuscript.

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Conflicts of Interest

The author declares that there is no conflict of interest.

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