Research Article

Generalized (p, q) -Gamma-type operators

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In the present paper, the generalized (p, q) -gamma-type operators based on (p, q) -calculus are introduced. The moments and central moments are obtained, and some local approximation properties of these operators are investigated by means of modulus of continuity and Peetre $\mathcal{K}$ -functional. Also, the rate of convergence, weighted approximation, and pointwise estimates of these operators are studied. Finally, a Voronovskaja-type theorem is presented.

1. Introduction

In [1], Mazhar introduced gamma operators preserving linear functions as follows:

$$\begin{aligned}\bar{G}_n(f; x) &= \int_0^\infty g_n(x; u) du \int_0^\infty g_{n-1}(u; t) f(t) dt \\ &= \frac{(2n)!x^{n-1}}{n!(n-1)!} \int_0^\infty \frac{t^{n-1}}{(x+t)^{2n+1}} f(t) dt,\end{aligned}\quad (1)$$

where $g_n(x; u) = (x^{n+1}/n!)e^{-xu}u^n$, $n > 1$, $x > 0$. In [2], Karsli considered new gamma operators preserving x^2 as follows:

$$\begin{aligned}\tilde{G}_n(f; x) &= \int_0^\infty g_{n+2}(x; u) du \int_0^\infty g_n(u; t) f(t) dt \\ &= \frac{(2n+3)!x^{n+3}}{n!(n+2)!} \int_0^\infty \frac{t^n}{(x+t)^{2n+4}} f(t) dt,\end{aligned}\quad (2)$$

where $x > 0$. In [3], Mao defined generalized gamma operators as follows:

$$\begin{aligned}G_{n,k}(f; x) &= \int_0^\infty g_n(x; u) du \int_0^\infty g_{n-k}(u; t) f(t) dt \\ &= \frac{(2n-k+1)!x^{n+1}}{n!(n-k)!} \int_0^\infty \frac{t^{n-k}}{(x+t)^{2n-k+2}} f(t) dt,\end{aligned}\quad (3)$$

where $x > 0$, $n \geq k$. Obviously, $G_{n,1}(f; x) = \bar{G}_n(f; x)$ and $G_{n,2}(f; x) = \tilde{G}_n(f; x)$. In [4, 5], some approximation properties of operators ((1)–(3)) were discussed.

In Bernstein polynomials, some of their modifications and corresponding operators have been studied in many papers (see [6–10]). The q -analogue of well-known positive probability operators were widely studied and discussed (see books [11–13]) since Bernstein polynomials were proposed by Lupas [14] and Phillips [15]. In [16], the q -analogue of the operators (1) was defined and discussed. In [17], Cai and Zeng constructed and studied a q -analogue of the operators (2). Meantime, modifications and generaliza-

tions of the operators (2) were introduced and researched in [18–20]. In [21], Karsli constructed a q -analogue of the operators (3) and extended the works of [16, 17, 20]. Recently, many operators are constructed with two-parameter (p, q) -integers based on postquantum calculus ((p, q) -calculus) which have been used widely in many areas of sciences such as Lie group, different equations, hypergeometric series, and physical sciences. First, we recall some useful concepts and notations from (p, q) -calculus, which can be found in [11–13]. The (p, q) -integers $[m]_{p,q}$ are defined by

$$[m]_{p,q} = p^{m-1} + p^{m-2}q + p^{m-3}q^2 + \cdots + pq^{m-2} + q^{m-1} \\ = \begin{cases} \frac{p^m - q^m}{p - q}, & p \neq q \neq 1, \\ mp^{m-1}, & p = q \neq 1, \\ [m]_q, & p = 1, \\ m, & p = q = 1. \end{cases} \quad (4)$$

By some simple calculation, for any $i, j \in \mathbb{N}$, we have the following relation:

$$q^i[j - i]_{p,q} = [j]_{p,q} - p^{j-i}[i]_{p,q}. \quad (5)$$

The (p, q) -factorial is defined by

$$[m]_{p,q}! = \begin{cases} [1]_{p,q}[2]_{p,q} \cdots [m]_{p,q}, & m \geq 1, \\ 1, & m = 0. \end{cases} \quad (6)$$

The (p, q) -power basis is defined by

$$(x \oplus t)_{p,q}^m = (x + t)(px + qt)(p^2x + q^2t) \cdots (p^{m-1}x + q^{m-1}t), \\ (x!t)_{p,q}^m = (x - t)(px - qt)(p^2x - q^2t) \cdots (p^{m-1}x - q^{m-1}t). \quad (7)$$

Let f be an arbitrary function. The improper (p, q) -integral of f on $[0, \infty)$ is defined as (see [22])

$$\int_0^\infty f(x) d_{p,q}x = (p - q) \sum_{m=-\infty}^\infty \frac{q^m}{p^{m+1}} f\left(\frac{q^m}{p^{m+1}}\right), \quad 0 < \frac{q}{p} < 1. \quad (8)$$

Let m be a nonnegative integer. The (p, q) -gamma function is defined as

$$\Gamma_{p,q}(m + 1) = \frac{(p!q)_{p,q}^m}{(p - q)^m} = [m]_{p,q}!, \quad 0 < q < p \leq 1. \quad (9)$$

Aral and Gupta [23] proposed a (p, q) -beta function of the second kind for $i, j \in \mathbb{N}_+ := \{1, 2, \dots\}$, as

$$B_{p,q}(i, j) = \int_0^\infty \frac{x^{i-1}}{(1 \oplus px)_{p,q}^{i+j}} d_{p,q}x, \quad (10)$$

and gave the relation of the (p, q) -analogues of beta and gamma functions:

$$B_{p,q}(i, j) = \frac{q\Gamma_{p,q}(i)\Gamma_{p,q}(j)}{(p^{i+1}q^{j-1})^{i/2}\Gamma_{p,q}(i+j)}. \quad (11)$$

As a special case, if $p = q = 1$, $B(i, j) = \Gamma(i)\Gamma(j)/\Gamma(i+j)$. It is obvious that the order is important for the (p, q) -setting, which is the reason why the (p, q) -variant of beta function does not satisfy commutativity property, i.e., $B_{p,q}(i, j) \neq B_{p,q}(j, i)$.

Since Mursaleen et al. firstly introduced (p, q) -calculus in approximation theory and constructed the (p, q) -analogue of Bernstein operators [24] and (p, q) -Bernstein-Stancu operators [25], generalizations of many well-known approximation operators based on (p, q) -calculus were widely introduced and discussed by several authors (see (p, q) -Szász-Mirakjan operators [26], (p, q) -Baskakov-Durrmeyer-Stancu operators [27], (p, q) -Bernstein-Stancu-Schurer-Kantorovich operators [28], (p, q) -Baskakov-beta operators [29], (p, q) -Lorentz polynomials [30], (p, q) -Szász-Mirakjan Kantorovich operators [31], (p, q) -Bleimann-Butzer-Hahn operators [32], (p, q) -Bernstein operators [33, 34], and so on). In [35], Cheng and Zhang constructed a (p, q) -analogue of the operators (1) using the (p, q) -beta function of the second kind and studied their approximation properties. Later, Cheng et al. defined the (p, q) -analogue of the operators (2) and researched their approximation properties in [36]. All these achievements motivate us to construct the (p, q) -analogue of the gamma operator (3) and generalize the works of [35, 36]. Now, we construct generalized (p, q) -gamma-type operators as follows:

Definition 1. Let $k \in \mathbb{N}_+$, $n = k, k + 1, \dots$, $x \in (0, \infty)$, and $0 < q < p \leq 1$. For $f \in C(0, \infty)$, then the (p, q) -analogue of the gamma operator (3) can be defined by

$$G_{n,k}^{p,q}(f; x) = \frac{x^{n+1}p^{n^2 - ((k-1)n/2) + ((k-1)/2)}q^{n^2 + ((3-k)n/2) - ((k-1)/2)}}{B_{p,q}(n - k + 1, n + 1)} \\ \cdot \int_0^\infty \frac{t^{n-k}}{\left((pq)^{n + ((1-k)/2)}x \oplus t\right)_{p,q}^{2n-k+2}} f(t) d_{p,q}t. \quad (12)$$

In the case $p = q = 1, k = 1$, we obtain the operators (1); in the case $p = q = 1, k = 2$, we obtain the operators (2); in the case $p = q = 1$, we obtain the operators (3); in the case $p = 1, k = 1$, we obtain the operators [16]; in the case $p = 1, k = 2$, we obtain the operators [17]; in the case $p = 1$, we obtain the operators [21]; in the case $k = 1$, we obtain the operators [35]; and in the case $k = 2$, we obtain the operators [36].

The paper is organized as follows: In Section 1, we introduce the history of gamma-type operators and recall some basic notations about (p, q) -calculus; then, we construct the generalized (p, q) -gamma operators with the (p, q) -beta

function. In Section 2, we obtain the auxiliary lemmas and corollaries about the moment computation formulas. And the second- and fourth-order central moment computation formula and limit equalities are also obtained. In Section 3, we discuss the local approximation about the operators by means of modulus of continuity and Peetre $\mathcal{K}$ -functional. In Sections 4 and 5, the rate of convergence and weighted approximation for these operators are researched. In Section 6, two pointwise estimates are given by using the Lipschitz-type maximal function. In Section 7, the Voronovskaja-type asymptotic formula is presented.

2. Moment Estimates

In order to obtain the approximation properties of the operators $G_{n,k}^{p,q}(f; x)$, we need the following lemmas and corollaries.

Lemma 2. For $x \in (0, \infty)$, $0 < q < p \leq 1$, $k \in \mathbb{N}_+$, $n = k, k+1, \dots$, and $m = 0, 1, 2, \dots, n$, we have

$$G_{n,k}^{p,q}(t^m, x) = (pq)^{\frac{m(k-m)}{2}} \frac{[n-k+m]_{p,q}! [n-m]_{p,q}!}{[n-k]_{p,q}! [n]_{p,q}!} x^m. \quad (13)$$

Proof. Set $C = n^2 - ((k-1)n/2) + (k-1)/2$, $D = n^2 + ((3-k)n/2) - ((k-1)/2)$, and $A = B = n + ((1-k)/2)$, we have

$$\begin{aligned} G_{n,k}^{p,q}(t^m; x) &= \frac{x^{n+1} p^C q^D}{B_{p,q}(n-k+1, n+1)} \\ &\quad \cdot \int_0^\infty \frac{t^{n-k}}{(p^A q^B x \oplus t)_{p,q}^{2n-k+2}} t^m d_{p,q} t \\ &= \frac{x^{n+1} p^C q^D}{B_{p,q}(n-k+1, n+1)} \frac{1}{(p^A q^B x)^{2n-k+2}} \\ &\quad \cdot \int_0^\infty \frac{t^{n-k+m}}{(1 \oplus (pt/p^{1+A} q^B x))_{p,q}^{2n-k+2}} d_{p,q} t \\ &= \frac{x^m p^C q^D (p^{1+A} q^B)^{n-k+m+1}}{B_{p,q}(n-k+1, n+1) (p^A q^B)^{2n-k+2}} \\ &\quad \cdot \int_0^\infty \frac{(t/p^{1+A} q^B x)^{n-k+m}}{(1 \oplus (pt/p^{1+A} q^B x))_{p,q}^{2n-k+2}} d_{p,q} \left(\frac{t}{p^{1+A} q^B x} \right) \\ &= \frac{x^m p^C q^D (p^{1+A} q^B)^{n-k+m+1}}{(p^A q^B)^{2n-k+2}} \frac{B_{p,q}(n-k+m+1, n-m+1)}{B_{p,q}(n-k+1, n+1)} \\ &= \frac{x^m p^C q^D (p^{1+A} q^B)^{n-k+m+1}}{(p^A q^B)^{2n-k+2}} \frac{(p^{n-k+2} q^{n-k})^{n-k+1/2}}{(p^{n-k+m+2} q^{n-k+m})^{n-k+m+1/2}} \\ &\quad \cdot \frac{[n-k+m]_{p,q}! [n-m]_{p,q}!}{[n-k]_{p,q}! [n]_{p,q}!} \\ &= (pq)^{m(k-m)/2} \frac{[n-k+m]_{p,q}! [n-m]_{p,q}!}{[n-k]_{p,q}! [n]_{p,q}!} x^m. \end{aligned} \quad (14)$$

Then, the following corollary can be obtained immediately.

Corollary 3. For $x \in (0, \infty)$, $0 < q < p \leq 1$, and $n \geq \max\{k, 4\}$, $k \in \mathbb{N}$, the following equalities hold:

$$\begin{aligned} G_{n,k}^{p,q}(1; x) &= 1, \quad G_{n,k}^{p,q}(t; x) = (pq)^{(k-1)/2} \frac{[n-k+1]_{p,q}}{[n]_{p,q}} x, \\ G_{n,k}^{p,q}(t^2; x) &= (pq)^{k-2} \frac{[n-k+1]_{p,q} [n-k+2]_{p,q}}{[n]_{p,q} [n-1]_{p,q}} x^2, \\ G_{n,k}^{p,q}(t^3; x) &= (pq)^{3(k-3)/2} \frac{[n-k+1]_{p,q} [n-k+2]_{p,q} [n-k+3]_{p,q}}{[n]_{p,q} [n-1]_{p,q} [n-2]_{p,q}} x^3, \\ G_{n,k}^{p,q}(t^4; x) &= (pq)^{2(k-4)} \frac{[n-k+1]_{p,q} [n-k+2]_{p,q} [n-k+3]_{p,q} [n-k+4]_{p,q}}{[n]_{p,q} [n-1]_{p,q} [n-2]_{p,q} [n-3]_{p,q}} x^4. \end{aligned} \quad (15)$$

Corollary 4. For $x \in (0, \infty)$, $0 < q < p \leq 1$, and $n \geq \max\{k, 4\}$, using Corollary 3, we can easily obtain the following explicit formulas for the first and second central moments:

$$\begin{aligned} A_{n,k}^{p,q}(x) &:= G_{n,k}^{p,q}(t-x; x) = \left((pq)^{(k-1)/2} \frac{[n-k+1]_{p,q}}{[n]_{p,q}} - 1 \right) x, \\ B_{n,k}^{p,q}(x) &:= G_{n,k}^{p,q}((t-x)^2; x) \\ &= \left((pq)^{k-2} \frac{[n-k+1]_{p,q} [n-k+2]_{p,q}}{[n]_{p,q} [n-1]_{p,q}} \right. \\ &\quad \left. - 2(pq)^{(k-1)/2} \frac{[n-k+1]_{p,q}}{[n]_{p,q}} + 1 \right) x^2. \end{aligned} \quad (16)$$

Corollary 5. The sequences $(p_n), (q_n)$ satisfy $0 < q_n < p_n \leq 1$ such that $p_n \rightarrow 1, q_n \rightarrow 1$ and $p_n^n \rightarrow \alpha \in [0, 1], q_n^n \rightarrow \beta \in [0, 1], [n]_{p_n, q_n} \rightarrow \infty$ as $n \rightarrow \infty$; then, for any $x \in (0, \infty)$, we have

$$\lim_{n \rightarrow \infty} [n]_{p_n, q_n} A_{n,k}^{p_n, q_n}(x) = -\frac{k-1}{2} (\alpha + \beta) x, \quad (17)$$

$$\lim_{n \rightarrow \infty} [n]_{p_n, q_n} B_{n,k}^{p_n, q_n}(x) = (\alpha + \beta) x^2, \quad (18)$$

$$\lim_{n \rightarrow \infty} [n]_{p_n, q_n} G_{n,k}^{p_n, q_n}((t-x)^4; x) = 0. \quad (19)$$

Proof. Using (5), we have $[n-k+1]_{p_n, q_n} = q_n^{1-k} [n]_{p_n, q_n} - p_n^n (p_n q_n)^{1-k} [k-1]_{p_n, q_n}$. Hence,

$$\begin{aligned} \lim_{n \rightarrow \infty} [n]_{p_n, q_n} A_{n,k}^{p_n, q_n}(x) &= \lim_{n \rightarrow \infty} [n]_{p_n, q_n} \left((p_n q_n)^{(k-1)/2} \frac{[n-k+1]_{p_n, q_n}}{[n]_{p_n, q_n}} - 1 \right) \\ &\quad \cdot x = \lim_{n \rightarrow \infty} [n]_{p_n, q_n} \left(\left(\frac{p_n}{q_n} \right)^{(k-1)/2} - 1 - \frac{p_n^n (p_n q_n)^{(1-k)/2} [k-1]_{p_n, q_n}}{[n]_{p_n, q_n}} \right) \\ &\quad \cdot x = \lim_{n \rightarrow \infty} [n]_{p_n, q_n} \left(p_n^{(k-1)/2} - q_n^{(k-1)/2} \right) x - (k-1) \alpha x \\ &= \lim_{n \rightarrow \infty} \frac{p_n^n - q_n^n}{p_n - q_n} \frac{p_n^{k-1} - q_n^{k-1}}{p_n^{(k-1)/2} x + q_n^{(k-1)/2}} - (k-1) \\ &\quad \cdot \alpha x = \frac{1}{2} \lim_{n \rightarrow \infty} (p_n^n - q_n^n) [k-1]_{p_n, q_n} - (k-1) \alpha x = -\frac{k-1}{2} (\alpha + \beta) x. \end{aligned} \quad (20)$$

Using $[n-k+1]_{p_n, q_n} / [n]_{p_n, q_n} = q_n^{1-k} - (p_n^n (p_n q_n)^{1-k})$
 $[k-1]_{p_n, q_n} / [n]_{p_n, q_n}$ and $[n-k+2]_{p_n, q_n} / [n-1]_{p_n, q_n} = q_n^{3-k} - ($
 $p_n^{n-1} (p_n q_n)^{3-k} [k-3]_{p_n, q_n} / [n-1]_{p_n, q_n})$, we have

$$\begin{aligned} & \lim_{n \rightarrow \infty} [n]_{p_n, q_n} G_{n,k}^{p_n, q_n}(t(t-x); x) \\ &= \lim_{n \rightarrow \infty} [n]_{p_n, q_n} \left((p_n q_n)^{k-2} \left(q_n^{1-k} - \frac{p_n^n (p_n q_n)^{1-k} [k-1]_{p_n, q_n}}{[n]_{p_n, q_n}} \right) \right. \\ & \quad \times \left(q_n^{3-k} - \frac{p_n^{n-1} (p_n q_n)^{3-k} [k-3]_{p_n, q_n}}{[n-1]_{p_n, q_n}} \right) \\ & \quad \left. - \left(\left(\frac{p_n}{q_n} \right)^{(k-1)/2} - \frac{p_n^n (p_n q_n)^{(1-k)/2} [k-1]_{p_n, q_n}}{[n]_{p_n, q_n}} \right) \right) \\ & \quad \times x^2 = \lim_{n \rightarrow \infty} [n]_{p_n, q_n} \left(\left(\frac{p_n}{q_n} \right)^{k-2} - \left(\frac{p_n}{q_n} \right)^{(k-1)/2} \right) \\ & \quad \times x^2 - (k-3) \alpha x^2 = \lim_{n \rightarrow \infty} [n]_{p_n, q_n} \left(p_n^{(k-3)/2} - q_n^{(k-3)/2} \right) \\ & \quad - (k-3) \alpha x^2 = \lim_{n \rightarrow \infty} \left[\frac{k-3}{2} \right]_{p_n, q_n} (p_n^n - q_n^n) x^2 - (k-3) \alpha x^2 \\ &= -\frac{k-3}{2} (\alpha + \beta) x^2. \end{aligned} \quad (21)$$

Thus,

$$\begin{aligned} & \lim_{n \rightarrow \infty} [n]_{p_n, q_n} B_{n,k}^{p_n, q_n}(x) = \lim_{n \rightarrow \infty} [n]_{p_n, q_n} G_{n,k}^{p_n, q_n}((t-x)t; x) \\ & \quad - x \lim_{n \rightarrow \infty} [n]_{p_n, q_n} G_{n,k}^{p_n, q_n}((t-x); x) = (\alpha + \beta) x^2. \end{aligned} \quad (22)$$

Now, we prove the limit equality (19) while $k \geq 4$, $k = 1, 2, 3$ is similar. Set $r_n = p_n/q_n$ and $i \leq k$, by (5); we can obtain

$$\begin{aligned} & (p_n q_n)^{(k-i)/2} \frac{[n-k+i]_{p_n, q_n}}{[n]_{p_n, q_n}} \\ &= r_n^{(k-i)/2} \frac{q_n^{k-i} [n-(k-i)]_{p_n, q_n}}{[n]_{p_n, q_n}} \\ &= r_n^{(k-i)/2} \frac{[n]_{p_n, q_n} - p_n^{n-(k-i)} [k-i]_{p_n, q_n}}{[n]_{p_n, q_n}} \\ &= r_n^{(k-i)/2} \left(1 - \frac{p_n^{n-(k-i)} [k-i]_{p_n, q_n}}{[n]_{p_n, q_n}} \right) \\ &= r_n^{(k-i)/2} - (p_n q_n)^{-((k-i)/2)} p_n^n \frac{[k-i]_{p_n, q_n}}{[n]_{p_n, q_n}}. \end{aligned} \quad (23)$$

Hence, we can rewrite

$$\begin{aligned} G_{n,k}^{p_n, q_n}(t; x) &= \left(r_n^{(k-i)/2} - (p_n q_n)^{-((k-i)/2)} p_n^n \frac{[k-1]_{p_n, q_n}}{[n]_{p_n, q_n}} \right) \\ & \quad \cdot x = \left(r_n^{(k-i)/2} - \frac{(k-1)\alpha}{[n]_{p_n, q_n}} + o\left(\frac{1}{[n]_{p_n, q_n}}\right) \right) x, \\ G_{n,k}^{p_n, q_n}(t^2; x) &= \left(\prod_{l=0}^1 \left(r_n^{(k-2)/2} - (p_n q_n)^{-((k-2)/2)} p_n^{n-l} \frac{[k-2]_{p_n, q_n}}{[n-l]_{p_n, q_n}} \right) \right) \\ & \quad \cdot x^2 = \left(r_n^{k-2} - \frac{2(k-2)\alpha}{[n]_{p_n, q_n}} + o\left(\frac{1}{[n]_{p_n, q_n}}\right) \right) x^2, \\ G_{n,k}^{p_n, q_n}(t^3; x) &= \left(\prod_{l=0}^2 \left(r_n^{(k-3)/2} - (p_n q_n)^{-((k-3)/2)} p_n^{n-l} \frac{[k-3]_{p_n, q_n}}{[n-l]_{p_n, q_n}} \right) \right) \\ & \quad \cdot x^3 = \left(r_n^{3(k-3)/2} - \frac{3(k-3)\alpha}{[n]_{p_n, q_n}} + o\left(\frac{1}{[n]_{p_n, q_n}}\right) \right) x^3, \\ G_{n,k}^{p_n, q_n}(t^4; x) &= \left(\prod_{l=0}^3 \left(r_n^{(k-4)/2} - (p_n q_n)^{-((k-4)/2)} p_n^{n-l} \frac{[k-4]_{p_n, q_n}}{[n-l]_{p_n, q_n}} \right) \right) \\ & \quad \cdot x^4 = \left(r_n^{2(k-4)} - \frac{4(k-4)\alpha}{[n]_{p_n, q_n}} + o\left(\frac{1}{[n]_{p_n, q_n}}\right) \right) x^4. \end{aligned} \quad (24)$$

Further, we can easily get

$$\begin{aligned} G_{n,k}^{p_n, q_n}((t-x)^4; x) &= \left(r_n^{2(k-4)} - 4r_n^{3(k-3)/2} + 6r_n^{k-2} \right. \\ & \quad \left. - 4r_n^{(k-1)/2} + 1 + o\left(\frac{1}{[n]_{p_n, q_n}}\right) \right) x^4. \end{aligned} \quad (25)$$

For any $l \in \mathbb{N}_+$, $[n]_{p_n, q_n} (r_n^l - 1) \sim [n]_{p_n, q_n} (p_n^l - q_n^l) \sim (\alpha - \beta)l$. We can obtain

$$\begin{aligned} & \lim_{n \rightarrow \infty} [n]_{p_n, q_n} G_{n,k}^{p_n, q_n}((t-x)^4; x) \\ &= \lim_{n \rightarrow \infty} [n]_{p_n, q_n} \left(r_n^{2(k-4)} - 4r_n^{3(k-3)/2} + 6r_n^{k-2} \right. \\ & \quad \left. - 4r_n^{(k-1)/2} + 1 + o\left(\frac{1}{[n]_{p_n, q_n}}\right) \right) x^4 \\ &= \lim_{n \rightarrow \infty} [n]_{p_n, q_n} \left(\left(r_n^{2(k-4)} - 1 \right) - 4 \left(r_n^{3(k-3)/2} - 1 \right) \right. \\ & \quad \left. + 6 \left(r_n^{k-2} - 1 \right) - 4 \left(r_n^{(k-1)/2} - 1 \right) \right) x^4 \\ &= (\alpha - \beta) [2(k-4) - 6(k-3) + 6(k-2) - 2(k-1)] x^4 = 0, \end{aligned} \quad (26)$$

we obtain the required result.

Corollary 6. Let us denote the norm $\|f\| = \sup_{x \in (0, \infty)} |f(x)|$ on $C_B(0, \infty)$ (the class of real valued continuous bounded functions on $(0, \infty)$). For any $f \in C_B(0, \infty)$, we have

$$\|G_{n,k}^{p,q}(f; x)\| \leq \|f\|. \quad (27)$$

Proof. In view of (12) and Corollary 3, the proof of this corollary can be obtained easily.

3. Local Approximation

For any $f \in C_B(0, \infty)$, let us consider the following $\mathcal{K}$ -functional:

$$\mathcal{K}(f; \delta) = \inf_{g \in S^2} \left\{ \|f - g\| + \delta \|g''\| \right\}, \quad (28)$$

where $\delta \in (0, \infty)$ and $S^2 = \{g \in C_B(0, \infty) : g', g'' \in C_B(0, \infty)\}$. The usual modulus of continuity and the second-order modulus of smoothness of f can be defined by

$$\begin{aligned} \omega(f; \delta) &= \sup_{0 < |t| \leq \delta} \sup_{x \in (0, \infty)} |f(x+t) - f(x)|, \\ \omega_2(f; \delta) &= \sup_{0 < |t| \leq \delta} \sup_{x \in (0, \infty)} |f(x+2t) - 2f(x+t) + f(x)|. \end{aligned} \quad (29)$$

By [37] (p.177, Theorem 2.4), there exists an absolute constant $C > 0$ such that

$$\mathcal{K}(f; \delta^2) \leq C\omega_2(f; \delta), \delta > 0. \quad (30)$$

Theorem 7. Let $(p_n), (q_n)$ be the sequences defined in Corollary 5 and $f \in C_B(0, \infty)$. Then, for all $k = 1, 2, 3, \dots, n \geq \max\{k, 2\}$, there exists an absolute positive $C_1 = 4C$ such that

$$\begin{aligned} |G_{n,k}^{p_n, q_n}(f; x) - f(x)| &\leq C_1 \omega_2 \left(f; \sqrt{(A_{n,k}^{p_n, q_n}(x))^2 + B_{n,k}^{p_n, q_n}(x)} \right) \\ &\quad + \omega \left(f; |A_{n,k}^{p_n, q_n}(x)| \right). \end{aligned} \quad (31)$$

Proof. Define the following new operators:

$$G_{n,k}^{p_n, q_n}(f; x) = G_{n,k}^{p_n, q_n}(f; x) - f(A_{n,k}^{p_n, q_n}(x) + x) + f(x), \quad x \in (0, \infty). \quad (32)$$

Let $x, t \in (0, \infty)$ and $g \in S^2$. By Taylor's expansion formula, we get

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t g''(u)(t-u)du. \quad (33)$$

Applying $G_{n,k}^{p_n, q_n}$ to the above equality and $G_{n,k}^{p_n, q_n}(t-x; x) = 0$, we can obtain

$$\begin{aligned} G_{n,k}^{p_n, q_n}(g; x) - g(x) &= G_{n,k}^{p_n, q_n} \left(\int_x^t g''(u)(t-u)du; x \right) \\ &\leq G_{n,k}^{p_n, q_n} \left(\left| \int_x^t |t-u| g''(u)du \right|; x \right) \\ &\quad + \left| \int_x^{A_{n,k}^{p_n, q_n}(x)+x} g''(u) \left(A_{n,k}^{p_n, q_n}(x) + x - u \right) du \right| \\ &\leq G_{n,k}^{p_n, q_n}((t-x)^2; x) \|g''\| + (A_{n,k}^{p_n, q_n}(x))^2 \|g''\| \\ &= (A_{n,k}^{p_n, q_n}(x) + B_{n,k}^{p_n, q_n}(x))^2 \|g''\|. \end{aligned} \quad (34)$$

By Corollary 6 and (32), we easily know $|G_{n,k}^{p_n, q_n}(f; x)| \leq 3\|f\|$. Hence,

$$\begin{aligned} |G_{n,k}^{p_n, q_n}(f; x) - f(x)| &= |G_{n,k}^{p_n, q_n}(f; x) + f(A_{n,k}^{p_n, q_n}(x) + x) \\ &\quad - 2f(x)| \leq |G_{n,k}^{p_n, q_n}(f - g; x) - (f - g)(x)| + |G_{n,k}^{p_n, q_n}(g; x) \\ &\quad - g(x)| + |f(A_{n,k}^{p_n, q_n}(x) + x) - f(x)| \leq 4\|f - g\| \\ &\quad + \left((A_{n,k}^{p_n, q_n}(x) + B_{n,k}^{p_n, q_n}(x))^2 + \right) \|g''\| + \omega(f; |A_{n,k}^{p_n, q_n}(x)|). \end{aligned} \quad (35)$$

Taking the infimum on the right-hand side over all $g \in S^2$ and using (30), we obtain the desired assertion.

Corollary 8. Let $(p_n), (q_n)$ be the sequences defined in Corollary 5 and $f \in C_B(0, \infty)$. Then, for any finite interval $I \subset (0, \infty)$, the sequence $\{G_{n,k}^{p_n, q_n}(f; x)\}$ converges to f uniformly on I .

4. Rate of Convergence

Let

$$\begin{aligned} B_w(0, \infty) &= \{f : |f(x)| \leq M_f w(x)\}, \\ C_w(0, \infty) &= \{f : f \in B_w(0, \infty) \cap C(0, \infty)\}, \\ C_w^0(0, \infty) &= \left\{ f : f \in C_w(0, \infty), \lim_{x \rightarrow \infty} \frac{|f(x)|}{w(x)} < \infty \right\}, \end{aligned} \quad (36)$$

where $w(x)$ is the weighted function given by $w(x) = 1 + x^2$ and M_f is an absolute constant depending only on f . $C_w^0(0, \infty)$ is equipped with the norm $\|f\|_w = \sup_{x \in (0, \infty)} |f(x)|/w(x)$.

As is known, if $f \in C(0, \infty)$ is not uniform, we cannot obtain $\lim_{\delta \rightarrow 0^+} \omega(f; \delta) = 0$. In [38], Ispir defined the following weighted modulus of continuity:

$$\Omega(f; \delta) = \sup_{0 < t \leq \delta, x \in (0, \infty)} \frac{|f(x+t) - f(x)|}{w(x)w(t)}, \quad (37)$$

and proved the properties of monotone increasing about Ω

$(f; \delta)$ as $\delta > 0$, $\lim_{\delta \rightarrow 0^+} \Omega(f; \delta) = 0$, and the inequality

$$\Omega(f; \lambda\delta) \leq 2(1 + \lambda)(1 + \delta^2)\Omega(f; \delta), \quad (38)$$

while $\lambda > 0$ and $f \in C_w^0(0, \infty)$. Meantime, we recall the modulus of continuity of f on the interval $(0, \kappa] \subset (0, \infty)$ by

$$\omega_\kappa(f; \delta) = \sup_{x, t \in (0, \kappa], |x-t| \leq \delta} |f(t) - f(x)|, \quad \delta > 0. \quad (39)$$

Theorem 9. Let $f \in C_w(0, \infty)$, $0 < q < p \leq 1$, and $\kappa > 0$, we have

$$\begin{aligned} \|G_{n,k}^{p,q}(f; x) - f(x)\|_{C(0, \kappa)} &\leq 4M_f(3 + 2\kappa^2)B_{n,k}^{p,q}(\kappa) \\ &+ 2\omega_{\kappa+1}\left(f; \sqrt{B_{n,k}^{p,q}(\kappa)}\right). \end{aligned} \quad (40)$$

Proof. For any $x \in (0, \kappa)$ and $t > \kappa + 1$, we easily have $1 \leq (t - \kappa)^2 \leq (t - x)^2$; thus,

$$\begin{aligned} |f(t) - f(x)| &\leq |f(t)| + |f(x)| \leq M_f(2 + t^2 + x^2) \\ &= M_f(2 + x^2 + (t - x + x)^2) \leq M_f(2 + 2x^2 + (t - x)^2) \\ &\leq M_f(3 + 2x^2)(t - x)^2 \leq M_f(3 + 2\kappa^2)(t - x)^2, \end{aligned} \quad (41)$$

and for any $x \in (0, \kappa)$, $t \in (0, \kappa + 1)$, and $\delta > 0$, we have

$$|f(t) - f(x)| \leq \omega_{\kappa+1}(|t - x|; x) \leq \left(1 + \frac{|t - x|}{\delta}\right) \omega_{\kappa+1}(f; \delta). \quad (42)$$

For (41) and (42), we can get

$$|f(t) - f(x)| \leq M_f(3 + 2\kappa^2)(t - x)^2 + \left(1 + \frac{|t - x|}{\delta}\right) \omega_{\kappa+1}(f; \delta). \quad (43)$$

By Schwarz's inequality, for any $x \in (0, \kappa)$, we can get

$$\begin{aligned} |G_{n,k}^{p,q}(f; x) - f(x)| &\leq G_{n,k}^{p,q}(|f(t) - f(x)|; x) \\ &\leq M_f(3 + 2\kappa^2)G_{n,k}^{p,q}((t - x)^2; x) \\ &\quad + G_{n,k}^{p,q}\left(\left(1 + \frac{|t - x|}{\delta}\right); x\right) \omega_{\kappa+1}(f; \delta) \\ &\leq M_f(3 + 2\kappa^2)G_{n,k}^{p,q}((t - x)^2; x) \\ &\quad + \omega_{\kappa+1}(f; \delta) \left(1 + \frac{1}{\delta} \sqrt{G_{n,k}^{p,q}((t - x)^2; x)}\right) \\ &\leq M_f(3 + 2\kappa^2)B_{n,k}^{p,q} + \omega_{\kappa+1}(f; \delta) \left(1 + \frac{1}{\delta} \sqrt{B_{n,k}^{p,q}(x)}\right) \\ &\leq M_f(3 + 2\kappa^2)B_{n,k}^{p,q}(\kappa) + \omega_{\kappa+1}(f; \delta) \left(1 + \frac{1}{\delta} \sqrt{B_{n,k}^{p,q}(\kappa)}\right). \end{aligned} \quad (44)$$

By taking $\delta = \sqrt{B_{n,k}^{p,q}(\kappa)}$ and the supremum over all $x \in (0, \kappa)$, we accomplish the proof of Theorem 9.

5. Weighted Approximation

In this section, we will discuss the following three theorems about weighted approximation for the operators $G_{n,k}^{p_n,q_n}(f; x)$:

Theorem 10. Let $f \in C_w^0(0, \infty)$ and the sequences $(p_n), (q_n)$ satisfy $0 < q_n < p_n \leq 1$ such that $p_n \rightarrow 1, q_n \rightarrow 1, p_n^n \rightarrow \alpha \in [0, 1], q_n^n \rightarrow \beta \in [0, 1], [n]_{p_n,q_n} \rightarrow \infty$ as $n \rightarrow \infty$; then, there exists $N \in \mathbb{N}_+$ such that for all $n > N$ and $\rho > 0$, the inequality

$$\sup_{x \in (0, \infty)} \frac{|G_{n,k}^{p_n,q_n}(f; x) - f(x)|}{(1 + x^2)^{(5/2)+\rho}} \leq 8(2 + \sqrt{2})\Omega\left(f; \frac{1}{\sqrt{[n]_{p_n,q_n}}}\right) \quad (45)$$

holds.

Proof. Using (37) and (38), we can write

$$\begin{aligned} |f(t) - f(x)| &\leq (1 + (t - x)^2)(1 + x^2)\Omega(f; |t - x|) \\ &\leq 2\left(1 + \frac{|t - x|}{\delta}\right)(1 + \delta^2)\Omega(f; \delta)(1 + (t - x)^2)(1 + x^2) \\ &\leq \begin{cases} 4(1 + \delta^2)^2(1 + x^2)\Omega(f; \delta), & |t - x| \leq \delta, \\ 4(1 + \delta^2)(1 + x^2)\Omega(f; \delta)\frac{|t - x| + |t - x|^3}{\delta}, & |t - x| > \delta. \end{cases} \end{aligned} \quad (46)$$

For any $\delta \in (0, 1)$ and $x, t \in (0, \infty)$, (46) can be rewritten:

$$|f(t) - f(x)| \leq 8(1 + x^2)\Omega(f; \delta) \left(2 + \frac{|t - x| + |t - x|^3}{\delta}\right). \quad (47)$$

Using (18) and (19), there exists $N \in \mathbb{N}_+$ such that for any $n > N$,

$$\begin{aligned} G_{n,k}^{p_n,q_n}((t - x)^2; x) &\leq \frac{2}{[n]_{p_n,q_n}}x^2, \\ G_{n,k}^{p_n,q_n}((t - x)^4; x) &\leq x^4. \end{aligned} \quad (48)$$

By Schwarz's inequality, we can obtain

$$G_{n,k}^{p_n,q_n}(|t - x|; x) \leq \sqrt{G_{n,k}^{p_n,q_n}((t - x)^2; x)} \leq \sqrt{2} \frac{x}{\sqrt{[n]_{p_n,q_n}}}, \quad (49)$$

$$\begin{aligned}
G_{n,k}^{p_n,q_n}(|t-x|^3; x) &\leq \sqrt{G_{n,k}^{p_n,q_n}((t-x)^2; x)} \\
&\quad \times \sqrt{G_{n,k}^{p_n,q_n}((t-x)^4; x)} \\
&\leq \sqrt{2} \frac{x^3}{\sqrt{[n]_{p_n,q_n}}}.
\end{aligned} \quad (50)$$

Since $G_{n,k}^{p_n,q_n}$ is linear and positive, using (47), (49), and (50), we can obtain

$$\begin{aligned}
|G_{n,k}^{p_n,q_n}(f; x) - f(x)| &\leq 8(1+x^2)\Omega(f; \delta) \\
&\quad \cdot \left(2 + \frac{G_{n,k}^{p_n,q_n}(|t-x| + |t-x|^3; x)}{\delta}\right) \\
&\leq 8(1+x^2) \left(2 + \frac{\sqrt{2}(x+x^3)}{\delta \sqrt{[n]_{p_n,q_n}}}\right) \Omega(f; \delta).
\end{aligned} \quad (51)$$

By choosing $\delta = 1/\sqrt{[n]_{p_n,q_n}}$, the conclusion holds.

Theorem 11. Let $(p_n), (q_n)$ be the sequences defined in Theorem 10. Then, for any $f \in C_w^0(0, \infty)$, we have

$$\lim_{n \rightarrow \infty} \|G_{n,k}^{p_n,q_n}(f; x) - f\|_w = 0. \quad (52)$$

Proof. By the weighted Korovkin theorem in [39], we see that it is sufficient to verify the following three conditions:

$$\lim_{n \rightarrow \infty} \|G_{n,k}^{p_n,q_n}(t^k; x) - x^k\|_w = 0, \quad k = 0, 1, 2. \quad (53)$$

Since $G_{n,k}^{p_n,q_n}(1; x) = 1$, then (53) holds true for $k = 0$. By Corollary 3, we can obtain

$$\begin{aligned}
\|G_{n,k}^{p_n,q_n}(t; x) - x\|_w &= \sup_{x \in (0, \infty)} \frac{|G_{n,k}^{p_n,q_n}(t; x) - x|}{1+x^2} \\
&\leq \left| (p_n q_n)^{(k-1)/2} \frac{[n-k+1]_{p_n,q_n} - 1}{[n]_{p_n,q_n}} \right| \sup_{x \in (0, \infty)} \frac{x}{1+x^2} \\
&= \frac{1}{2} \left| (p_n q_n)^{(k-1)/2} \frac{[n-k+1]_{p_n,q_n} - 1}{[n]_{p_n,q_n}} \right| \rightarrow 0, \quad n \rightarrow \infty, \\
\|G_{n,k}^{p_n,q_n}(t^2; x) - x^2\|_w &= \sup_{x \in (0, \infty)} \frac{|G_{n,k}^{p_n,q_n}(t^2; x) - x^2|}{1+x^2} \\
&\leq \sup_{x \in (0, \infty)} \frac{x^2}{1+x^2} \left| (p_n q_n)^{k-2} \frac{[n-k+1]_{p_n,q_n} [n-k+2]_{p_n,q_n} - 1}{[n]_{p_n,q_n} [n-1]_{p_n,q_n}} \right| \\
&= \left| (p_n q_n)^{k-2} \frac{[n-k+1]_{p_n,q_n} [n-k+2]_{p_n,q_n} - 1}{[n]_{p_n,q_n} [n-1]_{p_n,q_n}} \right| \rightarrow 0, \quad n \rightarrow \infty.
\end{aligned} \quad (54)$$

Thus, the proof of Theorem 11 is completed.

Theorem 12. Let $(p_n), (q_n)$ be the sequences defined in Theorem 10. Then, for any $f \in C_w^0(0, \infty)$ and $\lambda > 0$, we have

$$\lim_{n \rightarrow \infty} \sup_{x \in \mathbb{R}_+} \frac{|G_{n,k}^{p_n,q_n}(f; x) - f(x)|}{(1+x^2)^{1+\lambda}} = 0. \quad (55)$$

Proof. Let $x_0 \in (0, \infty)$ be arbitrary but fixed. Then,

$$\begin{aligned}
&\sup_{x \in (0, \infty)} \frac{|G_{n,k}^{p_n,q_n}(f; x) - f(x)|}{(1+x^2)^{1+\lambda}} \\
&\leq \sup_{x \in (0, x_0]} \frac{|G_{n,k}^{p_n,q_n}(f; x) - f(x)|}{(1+x^2)^{1+\lambda}} + \sup_{x \in (x_0, \infty)} \frac{|G_{n,k}^{p_n,q_n}(f; x) - f(x)|}{(1+x^2)^{1+\lambda}} \\
&\leq \|G_{n,k}^{p_n,q_n}(f; x) - f\|_{C(0, x_0]} + \|f\|_w \sup_{x \in (x_0, \infty)} \frac{|G_{n,k}^{p_n,q_n}(1+t^2; x)|}{(1+x^2)^{1+\lambda}} \\
&\quad + \sup_{x \in (x_0, \infty)} \frac{|f(x)|}{(1+x^2)^{1+\lambda}}.
\end{aligned} \quad (56)$$

Since $|f(x)| \leq M_f(1+x^2)$, we have $\sup_{x \in (x_0, \infty)} (|f(x)|/(1+x^2)^{1+\lambda}) \leq (M_f \|f\|_w / (1+x_0^2)^\lambda)$. Let $\varepsilon > 0$ be arbitrary; we can choose x_0 to be so large such that

$$\frac{M_f \|f\|_w}{(1+x_0^2)^\lambda} < \frac{\varepsilon}{3}. \quad (57)$$

In view of Corollary 3, while $x \in (x_0, \infty)$, we can obtain

$$\|f\|_w \lim_{x \rightarrow \infty} \lim_{n \rightarrow \infty} \frac{|G_{n,k}^{p_n,q_n}(1+t^2; x)|}{(1+x^2)^{1+\lambda}} = \|f\|_w \lim_{x \rightarrow \infty} \frac{1}{(1+x^2)^\lambda} = 0. \quad (58)$$

Hence, we can choose N and x_0 to be so large such that for any $n > N$, the inequality

$$\sup_{x \in (x_0, \infty)} \|f\|_w \frac{|G_{n,k}^{p_n,q_n}(1+t^2; x)|}{(1+x^2)^{1+\lambda}} < \frac{\varepsilon}{3} \quad (59)$$

holds. Also, the first term of the above inequality tends to zero by Theorem 9, that is,

$$\|G_{n,k}^{p_n,q_n}(f; x) - f\|_{C(0, x_0]} < \frac{\varepsilon}{3}. \quad (60)$$

Thus, combining (57), (59), and (60), we obtain the desired result.

6. Pointwise Estimates

In this section, we establish two pointwise estimates of the operators (12). First, we give the relation between the local

smoothness of f and local approximation. We denote that $f \in C(0, \infty)$ is in $\text{Lip}_M(\gamma, E)$, $\gamma \in (0, 1]$, $E \subset (0, \infty)$ if it satisfies the following condition:

$$|f(t) - f(x)| \leq M|t - x|^\gamma, \quad t \in (0, \infty), x \in E, \quad (61)$$

where M is a constant depending only on γ and f .

Theorem 13. Let $0 < q < p \leq 1$, $\gamma \in (0, 1]$, and E be any bounded subset on $(0, \infty)$. If $f \in C_B(0, \infty) \cap \text{Lip}_M(\gamma, E)$, then, for all $x \in (0, \infty)$, we have

$$|G_{n,k}^{p,q}(f; x) - f(x)| \leq M \left(B_{n,k}^{p,q}(x) \right)^{\gamma/2} + 2d^\gamma(x; E), \quad (62)$$

where $d(x; E)$ denotes the distance between x and E defined by

$$d(x; E) = \inf \{ |t - x| : t \in E \}. \quad (63)$$

Proof. Let $\bar{E}$ be the closure of E . Using the properties of the infimum, there is at least a point $t_0 \in \bar{E}$ such that $d(x; E) = |x - t_0|$. By the triangle inequality

$$|f(t) - f(x)| \leq |f(t) - f(t_0)| + |f(x) - f(t_0)|, \quad (64)$$

we can obtain

$$\begin{aligned} |G_{n,k}^{p,q}(f; x) - f(x)| &\leq G_{n,k}^{p,q}(|f(t) - f(t_0)|; x) \\ &+ G_{n,k}^{p,q}(|f(x) - f(t_0)|; x) \leq M \{ G_{n,k}^{p,q}(|t - t_0|^\gamma; x) \\ &+ |x - t_0|^\gamma \} \leq M \{ G_{n,k}^{p,q}(|t - x|^\gamma + |x - t_0|^\gamma; x) + |x - t_0|^\gamma \} \\ &= M \{ G_{n,k}^{p,q}(|t - x|^\gamma; x) + 2|x - t_0|^\gamma \}. \end{aligned} \quad (65)$$

Choosing $p_1 = 2/\gamma$ and $p_2 = 2/(2 - \gamma)$ and using the well-known Hölder inequality, we have

$$\begin{aligned} |G_{n,k}^{p,q}(f; x) - f(x)| &\leq M \left\{ \left(G_{n,k}^{p,q}(|t - x|^{p_1 \gamma}; x) \right)^{1/p_1} \right. \\ &\quad \times \left. \left(G_{n,k}^{p,q}(1^{p_2}; x) \right)^{1/p_2} + 2d^\gamma(x; E) \right\} \\ &\leq M \left\{ \left(G_{n,k}^{p,q}((t - x)^2; x) \right)^{\gamma/2} + 2d^\gamma(x; E) \right\} \\ &\leq M \left(\left(B_{n,k}^{p,q}(x) \right)^{\gamma/2} + 2d^\gamma(x; E) \right). \end{aligned} \quad (66)$$

Next, we obtain the local direct estimate of the operators $G_{n,k}^{p,q}$, using the Lipschitz-type maximal function of the order γ introduced by Lenze [40] as

$$\tilde{\omega}_\gamma(f; x) = \sup_{x, t \in (0, \infty), x \neq t} \frac{|f(t) - f(x)|}{|t - x|^\gamma}, \quad \gamma \in (0, 1]. \quad (67)$$

Theorem 14. Let $f \in C_B(0, \infty)$ and $\gamma \in (0, 1]$. Then, for all $x \in (0, \infty)$, we have

$$|G_{n,k}^{p,q}(f; x) - f(x)| \leq \tilde{\omega}_\gamma(f; x) \left(B_{n,k}^{p,q}(x) \right)^{\gamma/2}. \quad (68)$$

Proof. From equation (67), we have

$$|G_{n,k}^{p,q}(f; x) - f(x)| \leq \tilde{\omega}_\gamma(f; x) G_{n,k}^{p,q}(|t - x|^\gamma; x). \quad (69)$$

Applying the well-known Hölder inequality, we can get

$$\begin{aligned} |G_{n,k}^{p,q}(f; x) - f(x)| &\leq \tilde{\omega}_\gamma(f; x) \left(G_{n,k}^{p,q}((t - x)^2; x) \right)^{\gamma/2} \\ &\leq \tilde{\omega}_\gamma(f; x) \left(B_{n,k}^{p,q}(x) \right)^{\gamma/2}. \end{aligned} \quad (70)$$

7. Voronovskaja-Type Theorem

In this section, we give a Voronovskaja-type asymptotic formula for the operators (12) by means of the second and fourth central moments.

Theorem 15. Let $(p_n), (q_n)$ be the sequences defined in Corollary 5 and $f \in C_B(0, \infty)$. Suppose that $f''(x)$ exists at a point $x \in (0, \infty)$, then we can obtain

$$\begin{aligned} \lim_{n \rightarrow \infty} [n]_{p_n, q_n} \left(G_{n,k}^{p_n, q_n}(f; x) - f(x) \right) \\ = \frac{\alpha + \beta}{2} \left(x^2 f''(x) - (k - 1) x f'(x) \right). \end{aligned} \quad (71)$$

Proof. Using Taylor's expansion formula, we can obtain

$$f(t) = f(x) + f'(x)(t - x) + \frac{1}{2} f''(x)(t - x)^2 + R(t, x)(t - x)^2, \quad (72)$$

where $R(t, x)$ is the Peano form of the remainder and $\lim_{t \rightarrow x} R(t, x) = 0$. Applying $G_{n,k}^{p_n, q_n}$ to the both sides of (72), we have

$$\begin{aligned} [n]_{p_n, q_n} \left(G_{n,k}^{p_n, q_n}(\zeta; x) - \zeta(x) \right) &= [n]_{p_n, q_n} f'(x) G_{n,k}^{p_n, q_n}(t - x; x) \\ &+ [n]_{p_n, q_n} \frac{f''(x)}{2} G_{n,k}^{p_n, q_n}((t - x)^2; x) \\ &+ [n]_{p_n, q_n} G_{n,k}^{p_n, q_n}(R(t, x)(t - x)^2; x). \end{aligned} \quad (73)$$

By Schwarz's inequality, we have

$$\begin{aligned} G_{n,k}^{p_n, q_n}(R(t, x)(t - x)^2; x) \\ \leq \sqrt{G_{n,k}^{p_n, q_n}(R^2(t, x); x)} \sqrt{G_{n,k}^{p_n, q_n}((t - x)^4; x)}. \end{aligned} \quad (74)$$

We observe that $R^2(x, x) = 0$ and $R^2(\cdot, x) \in C_B(0, \infty)$. Then, it follows from Corollary 8 that

$$\lim_{n \rightarrow \infty} G_{n,k}^{p_n, q_n}(R^2(t, x); x) = R^2(x, x) = 0. \quad (75)$$

Hence, from (19), (74), and (75), we can obtain

$$\lim_{n \rightarrow \infty} [n]_{p_n, q_n} G_{n,k}^{p_n, q_n}(R(t, x)(t - x)^2; x) = 0. \quad (76)$$

Combining (17), (18), and (76), we obtain the required result.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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