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Research Article

Pre-Quasi Simple Banach Operator Ideal Generated by s – Numbers

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Let E be a weighted Nakano sequence space or generalized Cesáro sequence space defined by weighted mean and by using $s-$numbers of operators from a Banach space X into a Banach space Y. We give the sufficient (not necessary) conditions on E such that the components $S_E(X,Y) := \{T \in L(X,Y)\colon\ (s_n(T))_{n=0}^{\infty} \in E\}$ of the class S_E form pre-quasi operator ideal, the class of all finite rank operators are dense in the Banach pre-quasi ideal S_E, the pre-quasi operator ideal formed by the sequence of approximation numbers is strictly contained for different weights and powers, the pre-quasi Banach Operator ideal formed by the sequence of approximation numbers is small, and finally, the pre-quasi Banach operator ideal constructed by $s-$numbers is simple Banach space.

1. Introduction

All through the paper,

$$L(X,Y) = \{T\colon X \longrightarrow Y; T \text{ is a bounded linear operator; } X \text{ and } Y \text{ are Banach spaces}\}, \tag{1}$$

and if $X = Y$, we write $L(X)$; by ω, we denote the space of all real sequences and θ is the zero vector of E. Due to the immense applications in geometry of Banach spaces, spectral theory, geometry of Banach spaces, theory of eigenvalue distributions etc., the theory of operator ideals goals possesses an uncommon essentialness in useful examination. Some of operator ideals in the class of Banach spaces or Hilbert spaces are defined by different scalar sequence spaces. For example, the ideal of compact operators is defined by the space c_0 of null sequences and Kolmogorov numbers. Pietsch [1] examined the quasi-ideals formed by the approximation numbers and classical sequence space $\ell^p\,(0 < p < \infty)$. He showed that the ideals of nuclear operators and of Hilbert Schmidt operators between Hilbert spaces are defined by ℓ^1 and ℓ^2, respectively. Also, he proved that the class of all finite rank operators is dense in the Banach quasi ideal and the algebra $L(\ell^p)$, where $(1 \le p < \infty)$ contains one and only one nontrivial closed ideal. Pietsch [2] showed that the quasi Banach operator ideal formed by the sequence of approximation numbers is small. Makarov and Faried [3] proved that the quasi-operator ideal formed by the sequence of approximation numbers is strictly contained for different powers, i.e., for any infinite dimensional Banach spaces X and Y and for any $q > p > 0$, it is true that $S^{app}_{\ell^p}(X,Y) \subsetneqq S^{app}_{\ell^q}(X,Y) \subsetneqq L(X,Y)$. In [4], Faried and Bakery studied the operator ideals constructed by approximation numbers and generalized Cesáro and Orlicz sequence spaces ℓ_M. In [5], Faried and Bakery introduced the concept of pre-quasi operator ideal which is more general than the usual classes of operator ideal; they studied the operator ideals constructed by $s-$ numbers, generalized Cesáro and Orlicz sequence spaces ℓ_M, and showed that the operator ideal formed by the previous sequence spaces and approximation numbers is small under certain conditions. The idea of this paper is to study a generalized class S_E by using the sequence of s-numbers and E (weighted Nakano sequence space or generalized Cesáro sequence space); we give sufficient (not necessary) conditions on E such that S_E constructs a pre-quasi operator ideal, which gives a negative answer of Rhoades [6] open problem about the linearity of $E-$type spaces S_E. The components of S_E as a pre-quasi Banach operator ideal containing finite dimensional operators as a dense subset and its completeness are proved. The pre-quasi operator ideal formed by the sequence of approximation numbers is strictly contained for different weights and powers are determined. Finally, we show that the pre-quasi Banach operator ideal formed by E and approximation numbers is small under certain conditions. Furthermore, the sufficient conditions for which the pre-quasi Banach operator ideal constructed by $s-$numbers is a simple Banach space.

2. Definitions and Preliminaries

Definition 1 (see[7]). An s-number function is a map defined on $L(X,Y)$ which associates to each operator $T \in L(X,Y)$, a nonnegative scaler sequence $(s_n(T))_{n=0}^{\infty}$, assuming that the the following states are verified:

(a) $\|T\| = s_0(T) \geq s_1(T) \geq s_2(T) \geq \cdots \geq 0$, for $T \in L(X, Y)$

(b) $s_{m+n-1}(T_1+T_2) \leq s_m(T_1) + s_n(T_2)$ for all $T_1, T_2 \in L(X,Y)$, m, $n \in \mathbb{N}$

(c) Ideal property: $s_n(RVT) \leq \|R\| s_n(V) \|T\|$ for all $T \in L(X_0, X)$, $V \in L(X,Y)$ and $R \in L(Y, Y_0)$, where X_0 and Y_0 are arbitrary Banach spaces

(d) If $G \in L(X,Y)$ and $\lambda \in \mathbb{R}$, we obtain $s_n(\lambda G) = |\lambda| s_n(G)$

(e) Rank property: if $rank(T) \leq n$ then $s_n(T) = 0$ for each $T \in L(X,Y)$

(f) Norming property: $s_{r\geq n}(I_n) = 0$ or $s_{r<n}(I_n) = 1$, where I_n represents the unit operator on the n-dimensional Hilbert space ℓ_2^n

There are several examples of s-numbers, we mention the following:

(1) The n-th approximation number, denoted by $\alpha_n(T)$, is defined by

$$\alpha_n(T) = \inf\{\|T-B\|: B \in L(X,Y) \text{ and } \operatorname{rank}(B) \leq n\}. \tag{2}$$

(2) The n-th Gel'fand number, denoted by $c_n(T)$, is defined by $c_n(T) = \alpha_n(J_Y T)$, where J_Y is a metric injection from the normed space Y to a higher space $l_\infty(\Lambda)$ for an adequate index set Λ. This number is independent of the choice of the higher space $l_\infty(\Lambda)$.

(3) The n-th Kolmogorov number, denoted by $d_n(T)$, is defined by

$$d_n(T) = \inf_{\dim Y \leq n} \sup_{\|x\|\leq 1} \inf_{y \in Y} \|Tx - y\|. \tag{3}$$

(4) The n-th Weyl number, denoted by $x_n(T)$, is defined by

$$x_n(T) = \inf\left\{\alpha_n(TB): \left\|B: \ell_2 \longrightarrow X\right\| \leq 1\right\}. \tag{4}$$

(5) The n-th Chang number, denoted by $y_n(T)$, is defined by

$$y_n(T) = \inf\left\{\alpha_n(BT): \left\|B: Y \longrightarrow \ell_2\right\| \leq 1\right\}. \tag{5}$$

(6) The n-th Hilbert number, denoted by $h_n(T)$, is defined by

$$\begin{aligned} h_n(T) = \sup\big\{\alpha_n(BTA): &\left\|B: Y \longrightarrow \ell_2\right\| \\ &\leq 1 \text{ and } \left\|A: \ell_2 \longrightarrow X\right\| \leq 1\big\}. \end{aligned} \tag{6}$$

Remark (see[7]). Among all the s-number sequences defined above, it is easy to verify that the approximation number, $\alpha_n(T)$, is the largest and the Hilbert number, $h_n(T)$, is the smallest s-number sequence, i.e., $h_n(T) \leq s_n(T) \leq \alpha_n(T)$ for any bounded linear operator T. If T is compact and defined on a Hilbert space, then all the s-numbers coincide with the eigenvalues of $|T|$, where $|T| = (T * T)^{1/2}$.

Theorem 1 (see [7], p.115). If $T \in L(X,Y)$, then

$$\begin{aligned} h_n(T) &\leq x_n(T) \leq c_n(T) \leq \alpha_n(T), \\ h_n(T) &\leq y_n(T) \leq d_n(T) \leq \alpha_n(T). \end{aligned} \tag{7}$$

Definition 2 (see [1]). A finite rank operator is a bounded linear operator whose dimension of the range space is finite. The space of all finite rank operators on E is denoted by $F(E)$.

Definition 3 (see[1]). A bounded linear operator $A: E \longrightarrow E$ (where E is a Banach space) is called approximable if there are $S_n \in F(E)$, for all $n \in \mathbb{N}$ such that $\lim_{n\longrightarrow\infty} \|A - S_n\| = 0$. The space of all approximable operators on E is denoted by $\Lambda(E)$.

Lemma 1 (see [1]). *Let $T \in L(X,Y)$. If T is not approximable, then there are operators $G \in L(X,X)$ and $B \in L(Y,Y)$, such that $BTGe_k = e_k$ for all $k \in \mathbb{N}$.*

Definition 4 (see [1]). A Banach space X is called simple if the algebra $L(X)$ contains one and only one nontrivial closed ideal.

Definition 5 (see [1]). A bounded linear operator $A: E \longrightarrow E$ (where E is a Banach space) is called compact if $A(B_1)$ has compact closure, where B_1 denotes the closed unit ball of E. The space of all compact operators on E is denoted by $L_c(E)$.

Theorem 2 (see [1]). *If E is infinite dimensional Banach space, we have*

$$F(E) \subsetneqq \Lambda(E) \subsetneqq L_c(E) \subsetneqq L(E). \quad (8)$$

Definition 6 (see [1]). Let L be the class of all bounded linear operators between any arbitrary Banach spaces. A subclass U of L is called an operator ideal if each element $U(X,Y) = U \cap L(X,Y)$ fulfills the following conditions:

(i) $I_{\mathsf{F}} \in U$ where F represents Banach space of one dimension

(ii) The space $U(X,Y)$ is linear over $\mathbb{R}$

(iii) If $T \in L(X_0, X)$, $V \in U(X,Y)$ and $R \in L(Y,Y_0)$; then, $RVT \in U(X_0, Y_0)$ (see [8, 9])

The concept of pre-quasi operator ideal is more general than the usual classes of operator ideal.

Definition 7 (see [5]). A function $g: \Omega \longrightarrow [0,\infty)$ is said to be a pre-quasi norm on the ideal Ω if the following conditions holds:

(1) For all $T \in \Omega(X,Y)$, $g(T) \geq 0$ and $g(T) = 0$, if and only if $T = 0$

(2) There exists a constant $M \geq 1$ such that $g(\lambda T) \leq M|\lambda| g(T)$, for all $T \in \Omega(X,Y)$ and $\lambda \in \mathbb{R}$

(3) There exists a constant $K \geq 1$ such that $g(T_1 + T_2) \leq K[g(T_1) + g(T_2)]$, for all $T_1, T_2 \in \Omega(X,Y)$

(4) There exists a constant $C \geq 1$ such that if $T \in L(X_0, X)$, $P \in \Omega(X,Y)$ and $R \in L(Y,Y_0)$; then $g(RPT) \leq C\|R\| g(P)\|T\|$, where X_0 and Y_0 are normed spaces.

Theorem 3 (see [5]). *Every quasi norm on the ideal Ω is a pre-quasi norm on the ideal Ω.*

Let $p = (p_n)$ be a positive real and $(\beta_n)_{n\in\mathbb{N}}$ be a sequence of positive real; the weighted Nakano sequence space is defined by

$$\ell_\beta^{(p_n)} = \{x = (x_k) \in \omega: \rho(\lambda x) < \infty \text{ for some } \lambda > 0\} \text{ where } \rho(x) = \sum_{k=0}^{\infty} \beta_k (|x_k|)^{p_k}. \quad (9)$$

And $(\ell_\beta^{(p_n)}, \|\cdot\|)$ is a Banach space, where

$$\|x\| = \inf\left\{\eta > 0: \rho\left(\frac{x}{\eta}\right) \leq 1\right\}. \quad (10)$$

When (p_n) is bounded, we mark

$$\ell_\beta^{(p_n)} = \left\{x = (x_k) \in \omega: \sum_{k=0}^{\infty} \beta_k |x_k|^{p_k} < \infty\right\}. \quad (11)$$

If $\beta_n = 1$ for all $n \in \mathbb{N}$, then $\ell_\beta^{(p_n)}$ will reduce to $\ell^{(p_n)}$ studied in [10, 11].

In [12], Şengönül defined the sequence space as

$$\text{ces}((a_n),(p_n)) = \{x = (x_k) \in \omega: \exists \lambda > 0 \text{ with } \rho(\lambda x) < \infty\},$$
$$\rho(x) = \sum_{n=0}^{\infty} \left(a_n \sum_{k=0}^{n} |x_k|\right)^{p_k}, \quad (12)$$

where (a_n) and (p_n) are the sequences of positive real and $p_n \geq 1$ for all $n \in \mathbb{N}$. With the norm,

$$\|x\| = \inf\left\{\eta > 0: \rho\left(\frac{x}{\eta}\right) \leq 1\right\}. \quad (13)$$

Note

(1) Taking $a_n = (1/(n+1))$ for all $n \in \mathbb{N}$, then $\text{ces}((a_n),(p_n))$ is reduced to $\text{ces}(p_n)$ studied by Sanhan and Suantai [13]

(2) Taking $a_n = (1/(n+1))$ and $p_n = p$ for all $n \in \mathbb{N}$, then *ces*$((a_n),(p_n))$ is reduced to ces_p studied by many authors (see [14–16])

Definition 8 (see [5]). Let E be a linear space of sequences, then E is called a (sss) if

(1) For $n \in \mathbb{N}$, $e_n \in E$

(2) E is solid; i.e., assuming $x = (x_n) \in w$, $y = (y_n) \in E$ and $|x_n| \leq |y_n|$ for all $n \in \mathbb{N}$, then $x \in E$

(3) $(x_{[n/2]})_{n=0}^{\infty} \in E$, where $[n/2]$ indicates the integral part of $[n/2]$, whenever $(x_n)_{n=0}^{\infty} \in E$

Definition 9 (see [5]). A subclass of the (sss) is called a premodular (sss) assuming that we have a map $\rho: E \longrightarrow [0,\infty[$ with the following:

(i) For $x \in E$, $x = \theta \Longleftrightarrow \rho(x) = 0$ with $\rho(x) \geq 0$

(ii) For each $x \in E$ and scalar λ, we get a real number $L \geq 1$ for which $\rho(\lambda x) \leq |\lambda| L \rho(x)$

(iii) $\rho(x+y) \leq K(\rho(x) + \rho(y))$ for each $x, y \in E$, holds for a few numbers $K \geq 1$

(iv) For $n \in \mathbb{N}$, $|x_n| \leq |y_n|$, we obtain $\rho(x_n) \leq \rho(y_n)$

(v) The inequality, $\rho(x_n) \leq \rho(x_{[n/2]}) \leq K_0 \rho(x_n)$ holds, for some numbers $K_0 \geq 1$

(vi) $\overline{F} = E_\rho$, where F is the space of finite sequences

(vii) There is a steady $\xi > 0$ such that $\rho(\lambda, 0, 0, 0, \ldots) \geq \xi |\lambda| \rho(1, 0, 0, 0, \ldots)$ for any $\lambda \in \mathbb{R}$

Condition (ii) gives the continuity of $\rho(x)$ at θ. The linear space E enriched with the metric topology formed by the premodular ρ will be indicated by E_ρ. Moreover, condition (16) in Definition 8 and condition (vi) in Definition 9 explain that $(e_n)_{n\in\mathbb{N}}$ is a Schauder basis of E_ρ.

Notations. The sets S_E, $S_E(X,Y)$, S_E^{app}, and $S_E^{\mathrm{app}}(X,Y)$ (cf. [5]) as follows:

$$\begin{gathered} S_E := \{S_E(X,Y);\ X \text{ and } Y \text{ are Banach spaces}\}, \text{ where} \\ S_E(X,Y) := \{T \in L(X,Y)\colon\ ((s_i(T))_{i=0}^{\infty} \in E\}. \text{ Also,} \\ S_E^{\mathrm{app}} := \{S_E^{\mathrm{app}}(X,Y);\ X \text{ and } Y \text{ are Banach spaces}\}, \text{ where} \\ S_E^{\mathrm{app}}(X,Y) := \{T \in L(X,Y)\colon\ ((\alpha_i(T))_{i=0}^{\infty} \in E\}. \end{gathered} \tag{14}$$

Theorem 4 (see [5]). *If E is a (sss), then S_E is an operator ideal.*

Theorem 5 (see [3]). *If X and Y are infinite dimensional Banach spaces and (μ_i) is a monotonic decreasing sequence to zero, then there exists a bounded linear operator T such that*

$$\frac{1}{16}\mu_{3i} \le \alpha_i(T) \le 8\mu_{i+1}. \tag{15}$$

Now and after, define $e_n = \{0,0,\ldots,1,0,0,\ldots\}$ where 1 appears at the n^{th} place for all $n \in \mathbb{N}$ and the given inequality will be used in the sequel:

$$|a_n + b_n|^{p_n} \le H\left(|a_n|^{p_n} + |b_n|^{p_n}\right), \tag{16}$$

where $H = \max\{1, 2^{h-1}\}$, $h = \sup_n p_n$, and $p_n \ge 1$ for all $n \in \mathbb{N}$ (see [17]).

3. Main Results

3.1. Linear Problem. We examine here the operator ideals created by $s-$numbers and also weighted Nakano sequence space or generalized Cesáro sequence space defined by weighted mean such that those classes of all bounded linear operators T between arbitrary Banach spaces with $(\alpha_n(T))$ in these sequence spaces type an ideal operator.

Theorem 6. *$\ell_\beta^{(p_n)}$ is a (sss), if the following conditions are satisfied:*

(a1) The sequence (p_n) is increasing and bounded from above with $p_n > 0$ for all $n \in \mathbb{N}$

(a2) Either (β_n) is monotonic decreasing or monotonic increasing such that there exists a constant $C \ge 1$, for which $\beta_{2n+1} \le C\beta_n$

Proof (1) Let $x, y \in \ell_\beta^{(p_n)}$. Since (p_n) is bounded, we get

$$\sum_{n=0}^{\infty} \beta_n |x_n + y_n|^{p_n} \le H\left(\sum_{n=0}^{\infty} \beta_n |x_n|^{p_n} + \sum_{n=0}^{\infty} \beta_n |y_n|^{p_n}\right) < \infty, \tag{17}$$

then $x + y \in \ell_\beta^{(p_n)}$

(2) Let $\lambda \in \mathbb{R}$ and $x \in \ell_\beta^{(p_n)}$. Since (p_n) is bounded, we have

$$\sum_{n=0}^{\infty} \beta_n |\lambda x_n|^{p_n} \le \sup_n |\lambda|^{p_n} \sum_{n=0}^{\infty} \beta_n |x_n|^{p_n} < \infty \tag{18}$$

Then, $\lambda x \in \ell_\beta^{(p_n)}$. Therefore, by using Parts (1) and (2), we have that the space $\ell_\beta^{(p_n)}$ is linear. Also, $e_n \in \ell_\beta^{(p_n)}$ for all $n \in \mathbb{N}$, since

$$\sum_{i=0}^{\infty} \beta_i |e_n(i)|^{p_i} = \beta_n. \tag{19}$$

(2) Let $|x_n| \le |y_n|$ for all $n \in \mathbb{N}$ and $y \in \ell_\beta^{(p_n)}$. Since $\beta_n > 0$ for all $n \in \mathbb{N}$, then

$$\sum_{n=0}^{\infty} \beta_n |x_n|^{p_n} \le \sum_{n=0}^{\infty} \beta_n |y_n|^{p_n} < \infty, \tag{20}$$

and we get $x \in \ell_\beta^{(p_n)}$.

(3) Let $(x_n) \in \ell_\beta^{(p_n)}$, (β_n) be an increasing sequence. There exists $C > 0$ such that $\beta_{2n+1} \le C\beta_n$ and (p_n) be increasing; then, we have

$$\begin{aligned} \sum_{n=0}^{\infty} \beta_n |x_{[n/2]}|^{p_n} &= \sum_{n=0}^{\infty} \beta_{2n} |x_n|^{p_{2n}} + \sum_{n=0}^{\infty} \beta_{2n+1} |x_n|^{p_{2n+1}} \\ &\le \sum_{n=0}^{\infty} \beta_{2n} |x_n|^{p_n} + \sum_{n=0}^{\infty} \beta_{2n+1} |x_n|^{p_n} \le 2C \sum_{n=0}^{\infty} \beta_n |x_n|^{p_n}, \end{aligned} \tag{21}$$

and then $(x_{[n/2]}) \in \ell_\beta^{(p_n)}$. □

Theorem 7. *$ces((a_n),(p_n))$ is a (sss), if the following conditions are satisfied:*

(b1) The sequence (p_n) is increasing and bounded with $p_0 > 1$

(b2) $\sum_{n=0}^{\infty} (a_n)^{p_n} < \infty$

Proof (1) Given that $x, y \in ces((a_n),(p_n))$ and $\lambda_1, \lambda_2 \in \mathbb{R}$. Since (p_n) is bounded, we have

$$\begin{aligned} \sum_{n=0}^{\infty} \left(a_n \sum_{k=0}^{n} |\lambda_1 x_k + \lambda_2 y_k|\right)^{p_n} \le H\Bigg(&\sup_n |\lambda_1|^{p_n} \sum_{n=0}^{\infty} \left(a_n \sum_{k=0}^{n} |x_k|\right)^{p_n} \\ &+ \sup_n |\lambda_2|^{p_n} \sum_{n=0}^{\infty} \left(a_n \sum_{k=0}^{n} |y_k|\right)^{p_n}\Bigg) < \infty. \end{aligned} \tag{22}$$

Hence, $\lambda_1 x + \lambda_2 y \in \mathrm{ces}((a_n),(p_n))$; then, the space $\mathrm{ces}((a_n),(p_n))$ is linear. Also prove that $e_m \in \mathrm{ces}((a_n),(p_n))$ for all $m \in \mathbb{N}$, since $\sum_{n=0}^{\infty} (a_n)^{p_n} < \infty$. So we get

$$\sum_{n=0}^{\infty} \left(a_n \sum_{k=0}^{n} |e_m(k)|\right)^{p_n} = \sum_{n=m}^{\infty} (a_n)^{p_n} \le \sum_{n=0}^{\infty} (a_n)^{p_n} < \infty. \tag{23}$$

Hence, $e_m \in \mathrm{ces}((a_n),(p_n))$.

(2) Let $|x_n| \le |y_n|$ for all $n \in \mathbb{N}$ and $y \in \text{ces}((a_n),(p_n))$. Since $a_n > 0$ for all $n \in \mathbb{N}$, then

$$\sum_{n=0}^{\infty}\left(a_n\sum_{k=0}^{n}|x_k|\right)^{p_n} \le \sum_{n=0}^{\infty}\left(a_n\sum_{k=0}^{n}|y_k|\right)^{p_n} < \infty, \tag{24}$$

and we get $x \in \text{ces}((a_n),(p_n))$.

(3) Let $(x_n) \in \text{ces}((a_n),(p_n))$. Since (p_n) is increasing and the sequence (a_n) with $\sum_{n=0}^{\infty}(a_n)^{p_n} < \infty$ is decreasing, then we have

$$\begin{aligned}
\sum_{n=0}^{\infty}\left(a_n\sum_{k=0}^{n}|x_{[k/2]}|\right)^{p_n} &= \sum_{n=0}^{\infty}\left(a_{2n}\sum_{k=0}^{2n}|x_{[k/2]}|\right)^{p_{2n}} + \sum_{n=0}^{\infty}\left(a_{2n+1}\sum_{k=0}^{2n+1}|x_{[k/2]}|\right)^{p_{2n+1}}\\
&\le \sum_{n=0}^{\infty}\left(a_{2n}\left(\sum_{k=0}^{2n}2|x_k|+|x_n|\right)\right)^{p_n} + \sum_{n=0}^{\infty}\left(a_{2n+1}\sum_{k=0}^{2n+1}2|x_k|\right)^{p_n}\\
&\le 2^{h-1}\left(\sum_{n=0}^{\infty}\left(2a_n\sum_{k=0}^{n}|x_k|\right)^{p_n} + \sum_{n=0}^{\infty}\left(a_n\sum_{k=0}^{n}|x_k|\right)^{p_n}\right) + \sum_{n=0}^{\infty}\left(2a_n\sum_{k=0}^{n}|x_{[k/2]}|\right)^{p_n}\\
&\le 2^{h-1}\left(2^h+1\right)\sum_{n=0}^{\infty}\left(a_n\sum_{k=0}^{n}|x_k|\right)^{p_n} + 2^h\sum_{n=0}^{\infty}\left(a_n\sum_{k=0}^{n}|x_k|\right)^{p_n}\\
&\le \left(2^{2h-1}+2^{h-1}+2^h\right)\sum_{n=0}^{\infty}\left(a_n\sum_{k=0}^{n}|x_k|\right)^{p_n} < \infty.
\end{aligned} \tag{25}$$

Hence, $(x_{[n/2]}) \in \text{ces}((a_n),(p_n))$.

By using Theorem 4, we can get the following corollaries:

Corollary 1. *Let conditions (a1) and (a2) be satisfied; then, $S^{app}_{\ell_\beta^{(p_n)}}$ be an operator ideal.*

Corollary 2. *$S^{app}_{\ell^{(p_n)}}$ is an operator ideal, if the sequence (p_n) is increasing and bounded from above with $p_n > 0$ for all $n \in \mathbb{N}$.*

Corollary 3. *If $p \in (0,\infty)$, then $S^{app}_{\ell^p}$ is an operator ideal.*

Corollary 4. *Conditions (b1) and (b2) are satisfied; hence, $S^{app}_{ces((a_n),(p_n))}$ is an operator ideal.*

Corollary 5. *Assume (p_n) is increasing with $p_0 > 1$ and bounded, so $S^{app}_{ces(p_n)}$ is an operator ideal.*

Corollary 6. *If $p \in (1,\infty)$, then $S^{app}_{ces_p}$ is an operator ideal.*

4. Topological Problem

The following question arises naturally: which sufficient conditions (not necessary) on the sequence space E (weighted Nakano sequence space and generalized Cesáro sequence space defined by weighted mean) are the ideal of the finite rank operators in the class of Banach spaces dense in S_E? This gives a negative answer of Rhoades [6] open problem about the linearity of E−type spaces (S_E).

Theorem 8. $\overline{F(X,Y)} = S_{\ell_\beta^{(p_n)}}(X,Y)$, *whenever conditions (a1) and (a2) are satisfied.*

Proof. First, we substantiate that each finite operator $T \in F(X,Y)$ belongs to $S_{\ell_\beta^{(p_n)}}(X,Y)$. Given that $\{e_n\}_{n\in\mathbb{N}} \subset \ell_\beta^{(p_n)}$ and the space $\ell_\beta^{(p_n)}$ is linear, then for all finite operators $T \in F(X,Y)$, i.e., the sequence $(s_n(T))_{n\in\mathbb{N}}$ contains only finitely many numbers different from zero. Currently, we substantiate that $S_{\ell_\beta^{(p_n)}}(X,Y) \subseteq \overline{F(X,Y)}$. On taking $T \in S_{\ell_\beta^{(p_n)}}(X,Y)$,we obtain $(s_n(T))_{n=0}^{\infty} \in \ell_\beta^{(p_n)}$; hence, $\rho((s_n(T))_{n=0}^{\infty}) < \infty$; let $\varepsilon \in (0,1)$; at that point, there exists a $m \in \mathbb{N}-\{0\}$ such that $\rho((s_n(T))_{n=m}^{\infty}) < (\varepsilon/4C^2)$ for some $C \ge 1$. While $(s_n(T))_{n\in\mathbb{N}}$ is decreasing, we get

$$\begin{aligned}
\sum_{n=m+1}^{2m}\beta_n(s_{2m}(T))^{p_n} &\le \sum_{n=m+1}^{2m}\beta_n(s_n(T))^{p_n}\\
&\le \sum_{n=m}^{\infty}\beta_n(s_n(T))^{p_n} < \frac{\varepsilon}{4C^2}.
\end{aligned} \tag{26}$$

Hence, there exists $A \in F_{2m}(X,Y)$; rank $A \le 2m$ and

$$\sum_{n=2m+1}^{3m}\beta_n(\|T-A\|)^{p_n} \le \sum_{n=m+1}^{2m}\beta_n(T-A)^{p_n} < \frac{\varepsilon}{4C^2}. \tag{27}$$

Since (p_n) is a bounded, consider

$$\sum_{n=0}^{m}\beta_n(\|T-A\|)^{p_n} < \frac{\varepsilon}{4C^2}. \tag{28}$$

Let (β_n) be monotonic increasing such that there exists a constant $C \ge 1$ for which $\beta_{2n+1} \le C\beta_n$. Then, we have for $n \ge m$ that

$$\beta_{2m+n} \le \beta_{2m+2n+1} \le C\beta_{m+n} \le C\beta_{2n} \le C\beta_{2n+1} \le C^2\beta_n. \tag{29}$$

Since (p_n) is increasing, inequalities (26)–(29) give

$$
\begin{aligned}
d(T,A) &= \rho\left(s_n(T-A)\right)_{n=0}^{\infty} \\
&= \sum_{n=0}^{3m-1} \beta_n \left(s_n(T-A)\right)^{p_n} + \sum_{n=3m}^{\infty} \beta_n \left(s_n(T-A)\right)^{p_n} \\
&\le \sum_{n=0}^{3m} \beta_n \left(\|T-A\|\right)^{p_n} + \sum_{n=m}^{\infty} \beta_{n+2m} \left(s_{n+2m}(T-A)\right)^{p_{n+2m}} \\
&\le 3\sum_{n=0}^{m} \beta_n \left(\|T-A\|\right)^{p_n} + C^2 \sum_{n=m}^{\infty} \beta_n \left(s_n(T)\right)^{p_n} < \varepsilon.
\end{aligned} \tag{30}
$$

Since $I_3 \in S_{\varrho^{(n)}_{(1)}}$, condition (a1) is not satisfied which gives a counter example of the converse statement. This finishes the proof.

From Theorem 8, we can say that if (a1) and (a2) are satisfied, then every compact operator would be approximated by finite rank operators and the converse is not always true.

Theorem 9. *$\overline{F(X,Y)} = S_{ces((a_n),(p_n))}(X,Y)$; assume that states (b1) and (b2) are fulfilled and the converse is not always true.*

Proof. Primary since $e_n \in \text{ces}((a_n),(p_n))$, for each $n \in \mathbb{N}$ and the space $\text{ces}((a_n),(p_n))$ is linear, then for every finite mapping $T \in F(X,Y)$, i.e., the sequence $(s_n(T))_{n\in\mathbb{N}}$ contains only fnitely many numbers different from zero. Hence, $\overline{F(X,Y)} \subseteq S_{\text{ces}((a_n),(p_n))}(X,Y)$. By letting $T \in S_{\text{ces}((a_n),(p_n))}(X,Y)$, we obtain $(s_n(T))_{n\in\mathbb{N}} \in \text{ces}((a_n),(p_n))$, while $\rho((\alpha_n(T))_{n\in\mathbb{N}}) < \infty$. Let $\varepsilon \in (0,1)$. Then, there exists a number $m \in \mathbb{N} - \{0\}$ such that $\rho((s_n(T))_{n=m}^{\infty}) < (\varepsilon/2^{h+2}\delta C)$ for some $C \ge 1$, where $\delta = \max\left\{1, \sum_{n=m}^{\infty} a_n^{p_n}\right\}$. As $s_n(T)$ is decreasing for every $n \in \mathbb{N}$, we obtain

$$
\begin{aligned}
\sum_{n=m+1}^{2m}\left(a_n \sum_{k=0}^{n} s_{2m}(T)\right)^{p_n} &\le \sum_{n=m+1}^{2m}\left(a_n \sum_{k=0}^{n} s_k(T)\right)^{p_n} \\
&\le \sum_{n=m}^{\infty}\left(a_n \sum_{k=0}^{n} s_k(T)\right)^{p_n} < \frac{\varepsilon}{2^{h+2}\delta C}.
\end{aligned} \tag{31}
$$

Then, there exists $A \in F_{2m}(X,Y)$ and rank $A \le 2m$ and

$$
\sum_{n=2m+1}^{3m}\left(a_n \sum_{k=0}^{n} \|T-A\|\right)^{p_n} \le \sum_{n=m+1}^{2m}\left(a_n \sum_{k=0}^{n} T-A\right)^{p_n} < \frac{\varepsilon}{2^{h+2}\delta C}, \tag{32}
$$

and since (p_n) is bounded, consider

$$
\sup_{n=m}^{\infty}\left(\sum_{k=0}^{m} \|T-A\|\right)^{p_n} < \frac{\varepsilon}{2^h \delta}. \tag{33}
$$

Hence, set

$$
\sum_{n=0}^{m}\left(a_n \sum_{k=0}^{n} \|T-A\|\right)^{p_n} < \frac{\varepsilon}{2^{h+3}\delta C}. \tag{34}
$$

However, (p_n) is increasing and (a_n) is decreasing for each $n \in \mathbb{N}$; by using inequalities (31)–(34), we have

$$
\begin{aligned}
d(T,A) &= \rho\left(s_n(T-A)\right)_{n=0}^{\infty} \\
&= \sum_{n=0}^{3m-1}\left(a_n \sum_{k=0}^{n} s_k(T-A)\right)^{p_n} + \sum_{n=3m}^{\infty}\left(a_n \sum_{k=0}^{n} s_k(T-A)^{p_n}\right) \\
&\le \sum_{n=0}^{3m}\left(a_n \sum_{k=0}^{n} \|T-A\|\right)^{p_n} + \sum_{n=m}^{\infty}\left(a_{n+2m} \sum_{k=0}^{n+2m} s_k(T-A)\right)^{p_{n+2m}} \\
&\le 3\sum_{n=0}^{m}\left(a_n \sum_{k=0}^{n} \|T-A\|\right)^{p_n} + \sum_{n=m}^{\infty}\left(a_n \sum_{k=0}^{2m-1} s_k(T-A) + a_n \sum_{k=2m}^{n+2m} s_k(T-A)\right)^{p_n} \\
&\le 3\sum_{n=0}^{m}\left(a_n \sum_{k=0}^{n} \|T-A\|\right)^{p_n} + 2^{h-1}\left(\sum_{n=m}^{\infty}\left(a_n \sum_{k=0}^{2m-1} s_k(T-A)\right)^{p_n} + \sum_{n=m}^{\infty}\left(a_n \sum_{k=2m}^{n+2m} s_k(T-A)\right)^{p_n}\right) \\
&\le 3\sum_{n=0}^{m}\left(a_n \sum_{k=0}^{n} \|T-A\|\right)^{p_n} + 2^{h-1}\left(\sum_{n=m}^{\infty}\left(a_n \sum_{k=0}^{m} \|T-A\|\right)^{p_n} + \sum_{n=m}^{\infty}\left(a_n \sum_{k=0}^{n} s_{k+2m}(T-A)\right)^{p_n}\right) \\
&\le 3\sum_{n=0}^{m}\left(a_n \sum_{k=0}^{n} \|T-A\|\right)^{p_n} + 2^{h-1}\sup_{n=m}^{\infty}\left(\sum_{k=0}^{m} \|T-A\|)^{p_n} \sum_{n=m}^{\infty} (a_n)^{p_n} + 2^{h-1}\sum_{n=m}^{\infty}\left(a_n \sum_{k=0}^{n} s_k(T)\right)^{p_n}\right) < \varepsilon.
\end{aligned} \tag{35}
$$

Since $I_3 \in S_{\text{ces}((1),(1))}$, condition (b2) is not satisfied which gives a counter example of the converse statement. This finishes the proof.

From Theorem 9, we can say that if conditions (b1) and (b2) are satisfied, then every compact operators would be approximated by finite rank operators and the converse is not always true. □

Corollary 7. *If (p_n) is an increasing with $p_n > 0$ for all $n \in \mathbb{N}$ and bounded from above, then $S_{l(p)}(X,Y) = \overline{F(X,Y)}$.*

Corollary 8. *If $0 < p < \infty$, then $S_{l^p}(X, Y) = \overline{F(X, Y)}$.*

Corollary 9. $S_{ces(p_n)}(X, Y) = \overline{F(X, Y)}$*, if (p_n) is increasing with $p_0 > 1$ and $lim_{n\longrightarrow\infty}\sup p_n < \infty$.*

Corollary 10. $S_{ces_p}(X, Y) = \overline{F(X, Y)}$*, if $1 < p < \infty$.*

5. Completeness of the Pre-Quasi Ideal Components

For which sequence space E are the components of pre-quasi operator ideal S_E complete?

Theorem 10. $\ell_\beta^{(p_n)}$ *is a premodular (sss), if conditions (a1) and (a2) are satisfied.*

Proof. We define the functional ρ on $\ell_\beta^{(p_n)}$ as $\rho(x) = \sum_{n=0}^{\infty} \beta_n |x_n|^{p_n}$:

(i) Evidently, $\rho(x) \geq 0$ and $\rho(x) = 0 \Longleftrightarrow x = \theta$
(ii) There is a steady $L = \max\{1, \sup_n |\lambda|^{p_n}\} \geq 1$ such that $\rho(\lambda x) \leq L|\lambda|\rho(x)$ for all $x \in \ell_\beta^{(p_n)}$ and $\lambda \in \mathbb{R}$
(iii) We have the inequality $\rho(x + y) \leq H(\rho(x) + \rho(y))$ for all $x, y \in \ell_\beta^{(p_n)}$
(iv) Clearly follows from inequality (20) of Theorem 6
(v) It obtained from (27) Theorem 6 that $K_0 \geq 2$
(vi) It is clear that $\overline{F} = \ell_\beta^{(p_n)}$
(vii) There exists a steady $0 < \xi \leq |\lambda|^{p_0 - 1}$ such that

$$\rho(\lambda, 0, 0, 0, \ldots) \geq \xi|\lambda|\rho(1, 0, 0, 0, \ldots) \text{ for any } \lambda \in \mathbb{R} \tag{36}$$

Theorem 11. $ces((a_n), (p_n))$ *is a premodular (sss), if conditions (b1) and (b2) are satisfied.*

Proof. We define the functional ρ on $\mathrm{ces}((a_n), (p_n))$ as $\rho(x) = \sum_{n=0}^{\infty} (a_n \sum_{k=0}^{n} |x_k|)^{p_n}$:

(i) Clearly, $\rho(x) \geq 0$ and $\rho(x) = 0 \Longleftrightarrow x = \theta$.
(ii) There is a number $L = \max\{1, \sup_n |\lambda|^{p_n}\} \geq 1$ with $\rho(\lambda x) \leq L|\lambda|\rho(x)$ for all $x \in \mathrm{ces}((a_n), (p_n))$ and $\lambda \in \mathbb{R}$.
(iii) We have the inequality $\rho(x + y) \leq H(\rho(x) + \rho(y))$ for all $x, y \in \mathrm{ces}((a_n), (p_n))$.
(iv) It clearly follows from inequality (24) of Theorem 7.
(v) It is clear from (27) Theorem 7, that $K_0 \geq (2^{2h-1} + 2^{h-1} + 2^h) \geq 1$.
(vi) It is clear that $\overline{F} = \mathrm{ces}((a_n), (p_n))$.
(vii) There exists a steady $0 \langle \xi \leq |\lambda|^{p_0 - 1}$ such that $\rho(\lambda, 0, 0, 0, \ldots) \geq \xi|\lambda|\rho(1, 0, 0, 0, \ldots)$ for any $\lambda \in \mathbb{R}$. We state the following theorem without proof, and this can be established using standard technique.

Theorem 12. *The function $g(P) = \varrho(s_i(P))_{i=0}^{\infty}$ is a pre-quasi norm on S_{E_ϱ}, where E_ϱ is a premodular (sss).*

Theorem 13. *If X and Y are Banach spaces and E_ρ is a premodular (sss), then (S_{E_ρ}, g), where $g(T) = \rho((s_n(T))_{n=0}^{\infty})$ is a pre-quasi Banach operator ideal.*

Proof. Since E_ρ is a premodular (sss), then the function $g(T) = \rho((s_n(T))_{n=0}^{\infty})$ is a pre-quasi norm on S_{E_ρ}. Let (T_m) be a Cauchy sequence in $S_{E_\rho}(X, Y)$, then by utilizing Part (vii) of Definition 9 and since $L(X, Y) \supseteq S_{E_\rho}(X, Y)$, we get

$$\begin{aligned} g\left(T_i - T_j\right) &= \rho\left(\left(s_n\left(T_i - T_j\right)\right)_{n=0}^{\infty}\right) \geq \rho\left(s_0\left(T_i - T_j\right), 0, 0, 0, \ldots\right) \\ &= \rho\left(\left\|T_i - T_j\right\|, 0, 0, 0, \ldots\right) \geq \xi\left\|T_m - T_j\right\|\rho(1, 0, 0, 0, \ldots). \end{aligned} \tag{37}$$

Then, $(T_m)_{m\in\mathbb{N}}$ is a Cauchy sequence in $L(X, Y)$. While the space $L(X, Y)$ is a Banach space, so there exists $T \in L(X, Y)$ with $\lim_{m\longrightarrow\infty}\|T_m - T\| = 0$, and while $(s_n(T_m))_{n=0}^{\infty} \in E_\rho$ for each $m \in \mathbb{N}$; hence, using Parts (iii) and (iv) of Definition 9 and as ρ is continuous at θ, we obtain

$$\begin{aligned} g(T) &= \rho\left(\left(s_n(T)\right)_{n=0}^{\infty}\right) = \rho\left(\left(s_n\left(T - T_m + T_m\right)\right)_{n=0}^{\infty}\right) \\ &\leq K\rho\left(\left(s_{[n/2]}\left(T - T_m\right)\right)_{n=0}^{\infty}\right) + K\rho\left(\left(s_{[n/2]}\left(T_m\right)_{n=0}^{\infty}\right)\right) \\ &\leq K\rho\left(\left(\left\|T_m - T\right\|\right)_{n=0}^{\infty}\right) + K\rho\left(\left(s_n\left(T_m\right)_{n=0}^{\infty}\right)\right) < \varepsilon, \end{aligned} \tag{38}$$

and we have $(s_n(T))_{n=0}^{\infty} \in E_\rho$, then $T \in S_{E_\rho}(X, Y)$.

Corollary 11. *If X and Y are Banach spaces and conditions (a1) and (a2) are satisfied, then $S_{\ell_\beta^{(p_n)}}$ is a pre-quasi Banach operator ideal.*

Corollary 12. *If X and Y are Banach spaces, (p_n) is increasing with $p_n > 0$ for all $n \in \mathbb{N}$ and bounded from above, then $S_{\ell^{(p_n)}}$ is a pre-quasi Banach operator ideal.*

Corollary 13. *If X and Y are Banach spaces and $0 < p < \infty$, then S_{ℓ^p} is a pre-quasi Banach operator ideal.*

Corollary 14. *If X and Y are Banach spaces and the conditions (b1) and (b2) are satisfied, then $S_{ces((a_n),(p_n))}$ is a pre-quasi Banach operator ideal.*

Corollary 15. *If X and Y are Banach spaces and (p_n) is increasing with $p_0 > 1$ and $\lim_{n\longrightarrow\infty}\sup p_n < \infty$, then $S_{ces(p_n)}$ is a pre-quasi Banach operator ideal.*

Corollary 16. *If X and Y are Banach spaces and $p \in (1, \infty)$, then $U_{ces_p}^{app}(X, Y)$ is complete.*

6. Smallness of the Pre-Quasi Banach Operator Ideal

We give here the sufficient conditions on the weighted Nakano sequence space such that the pre-quasi operator ideal formed by the sequence of approximation numbers and this sequence space is strictly contained for different weights and powers.

Theorem 14. *For any infinite dimensional Banach spaces X and Y and for any $0 < p_n < q_n$ and $0 < a_n < b_n$ for all $n \in \mathbb{N}$, it is true that*

$$S^{\mathrm{app}}_{\ell^{(p_n)}_{(b_n)}}(X,Y) \subsetneq S^{\mathrm{app}}_{\ell^{(q_n)}_{(a_n)}}(X,Y) \subsetneq L(X,Y). \tag{39}$$

Proof. Let X and Y be infinite dimensional Banach spaces and for any $0 < p_n < q_n$ and $0 < a_n < b_n$ for all $n \in \mathbb{N}$, if $T \in S^{\mathrm{app}}_{\ell^{(p_n)}_{(b_n)}}(X,Y)$, then $(\alpha_n(T)) \in \ell^{(p_n)}_{(b_n)}$. One can see that

$$\sum_{n=0}^{\infty} a_n (\alpha_n(T))^{q_n} < \sum_{n=0}^{\infty} b_n (\alpha_n(T))^{p_n} < \infty. \tag{40}$$

Hence, $T \in S^{\mathrm{app}}_{\ell^{(q_n)}_{(a_n)}}(X,Y)$. Next, if we take (a_n) with $\sup a_n < \infty$, $(b_n^{-1}) \in \ell^{(q_n/p_n)}$and $\mu_n = (1/\sqrt[p_n]{b_n})$. So by using Theorem 5, one can find $T \in L(X,Y)$ with

$$\frac{1}{16\sqrt[p_{3n}]{b_{3n}}} \leq \alpha_n(T) \leq \frac{8}{\sqrt[p_{n+1}]{b_{n+1}}}, \tag{41}$$

such that T does not belong to $S^{\mathrm{app}}_{\ell^{(p_n)}_{(b_n)}}(X,Y)$ and $T \in S^{\mathrm{app}}_{\ell^{(q_n)}_{(a_n)}}(X,Y)$.

It is easy to see that $S^{\mathrm{app}}_{\ell^{(q_n)}_{(a_n)}}(X,Y) \subset L(X,Y)$. Next, if we take $\mu_n = (1/\sqrt[q_n]{a_n})$. So by using Theorem 5, one can find $T \in L(X,Y)$ with

$$\frac{1}{16\sqrt[q_{3n}]{a_{3n}}} \leq \alpha_n(T) \leq \frac{8}{\sqrt[q_{n+1}]{a_{n+1}}}, \tag{42}$$

such that T does not belong to $S^{\mathrm{app}}_{\ell^{(q_n)}_{(a_n)}}(X,Y)$. This finishes the proof.

Corollary 17 (see [3]). *For any infinite dimensional Banach spaces X and Y and for any $q > p > 0$, it is true that $S^{app}_{\ell^p}(X,Y) \subsetneq S^{app}_{\ell^q}(X,Y) \subsetneq L(X,Y)$.*

Lemma 2. *If (a_n) is a sequence of positive real with $a_n \leq 1$ for all $n \in \mathbb{N}$, (p_n) is monotonic increasing bounded sequence with $p_n \geq 1$ for all $n \in \mathbb{N}$ or $a_n > 1$ for all $n \in \mathbb{N}$, and (p_n) is monotonic decreasing bounded sequence with $p_n \geq 1$ for all $n \in \mathbb{N}$, one has the following inequality:*

$$\left(\sum_{k=0}^{n} a_k\right)^{p_n} < 2^{n(\sup p_n - 1)} \sum_{k=0}^{n} a_k^{p_k}. \tag{43}$$

Proof. By using inequality (16) and the sufficient conditions, one has

$$\begin{aligned}
\left(\sum_{k=0}^{n} a_k\right)^{p_n} &= \left(a_0 + \sum_{k=1}^{n} a_k\right)^{p_n} < 2^{\sup p_n - 1}\left(a_0^{p_n} + \left(\sum_{k=1}^{n} a_k\right)^{p_n}\right) \\
&= 2^{\sup p_n - 1}\left(a_0^{p_n} + \left(a_1 + \sum_{k=2}^{n} a_k\right)^{p_n}\right) \\
&< 2^{\sup p_n - 1}\left(a_0^{p_n} + 2^{\sup p_n - 1}\left(a_1^{p_n} + \left(\sum_{k=2}^{n} a_k\right)^{p_n}\right)\right) \\
&< 2^{\sup p_n - 1}\left(2^{\sup p_n - 1} a_0^{p_n} + 2^{\sup p_n - 1}\left(a_1^{p_n} + \left(\sum_{k=2}^{n} a_k\right)^{p_n}\right)\right) \\
&< 2^{2(\sup p_n - 1)}\left(a_0^{p_n} + a_1^{p_n} + \left(\sum_{k=2}^{n} a_k\right)^{p_n}\right) \\
&< 2^{3(\sup p_n - 1)}\left(a_0^{p_n} + a_1^{p_n} + a_2^{p_n} + \left(\sum_{k=3}^{n} a_k\right)^{p_n}\right) \\
&\vdots \\
&< 2^{(n-1s)(\sup p_n - 1)}\left(a_0^{p_n} + a_1^{p_n} + a_2^{p_n} + \cdots + a_{n-2}^{p_n}\right. \\
&\quad \left. + \left(\sum_{k=n-1}^{n} a_k\right)^{p_n}\right) \\
&< 2^{n(\sup p_n - 1)} \sum_{k=0}^{n} a_k^{p_n} < 2^{n(\sup p_n - 1)} \sum_{k=0}^{n} a_k^{p_k}.
\end{aligned} \tag{44}$$

We give here the sufficient conditions on the generalized Cesáro sequence space defined by weighted mean such that the pre-quasi operator ideal formed by the sequence of approximation numbers, and this sequence space is strictly contained for different weights and powers.

Theorem 15. *For any infinite dimensional Banach spaces X and Y and for any $1 < p_n < q_n$ for all $n \in \mathbb{N}$, it is true that $S^{app}_{ces((b_n),(p_n))}(X,Y) \subsetneq S^{app}_{ces((a_n),(q_n))}(X,Y) \subsetneq L(X,Y)$, where (p_n) and (q_n) are the monotonic increasing bounded sequences.*

Proof. Let X and Y be infinite dimensional Banach spaces and for any $0 < p_n < q_n$ and $0 < a_n < b_n$ for all $n \in \mathbb{N}$; if $T \in S^{\mathrm{app}}_{\mathrm{ces}((b_n),(p_n))}(X,Y)$, then $(\alpha_n(T)) \in \mathrm{ces}((b_n),(p_n))$. One has

$$\sum_{n=0}^{\infty}\left(a_n \sum_{k=0}^{n} \alpha_k(T)\right)^{q_n} < \sum_{n=0}^{\infty}\left(b_n \sum_{k=0}^{n} \alpha_k(T)\right)^{p_n} < \infty. \tag{45}$$

Hence, $T \in S^{\mathrm{app}}_{\mathrm{ces}((a_n),(q_n))}(X,Y)$. Next, if we take (a_n) with $\sup 2^{(\sup q_n - 1)n} a_n^{q_n} < \infty$, $(b_n^{-1}) \in \ell^{(q_n/p_n)}$ and $\mu_n = (1/\sqrt[p_n]{b_n})$. So by using Theorem 5 and Lemma 2, one can find $T \in L(X,Y)$ with

$$\frac{1}{16\sqrt[p_{3n}]{b_{3n}}} \leq \alpha_n(T) \leq \frac{8}{\sqrt[p_{n+1}]{b_{n+1}}}, \tag{46}$$

such that T does not belong to $S^{\mathrm{app}}_{\mathrm{ces}((b_n),(p_n))}(X,Y)$ and $T \in S^{\mathrm{app}}_{\mathrm{ces}((a_n),(q_n))}(X,Y)$.

It is easy to see that $S^{app}_{ces((a_n),(q_n))}(X,Y) \subset L(X,Y)$. Next, we take $\mu_n = (1/\sqrt[qn]{a_n})$. So by using Theorem 5, one can find $T \in L(X,Y)$ with

$$\frac{1}{16\sqrt[q3n]{a_{3n}}} \le \alpha_n(T) \le \frac{8}{\sqrt[qn+1]{a_{n+1}}}, \tag{47}$$

such that T does not belong to $S^{app}_{ces((a_n),(q_n))}(X,Y)$. This finishes the proof.

Corollary 18. *For any infinite dimensional Banach spaces X and Y and $1 < p < q < \infty$, then*

$$S^{app}_{ces_p}(X,Y) \subsetneqq S^{app}_{ces_q}(X,Y) \subsetneqq L(X,Y). \tag{48}$$

In this part, we give the conditions for which the pre-quasi Banach Operator ideal $S^{app}_{ces((a_i),q_i)}$ is small.

Theorem 16. *If conditions (b1), (b2), and $(na_n) \notin \ell^{(p_n)}$ are satisfied, then the pre-quasi Banach operator ideal $S^{app}_{ces((a_i),p_i)}$ is small.*

Proof. Since (p_i) is an increasing sequence with $p_0 > 1$ and $(a_i) \in \ell^{(p_i)}$, take $\lambda = (\sum_{i=0}^{\infty} a_i^{p_i})^{1/h}$. Then, $(S^{app}_{ces((a_i),p_i)}, g)$, where $g(T) = (1/\lambda)(\sum_{i=0}^{\infty}(a_i\sum_{j=0}^{i}\alpha_j(T))^{p_i})^{1/h}$ is a pre-quasi Banach operator ideal. Let X and Y be any two Banach spaces. Suppose that $S^{app}_{ces((a_i),p_i)}(X,Y) = L(X,Y)$; then, there exists a constant $C > 0$ such that $g(T) \le C\|T\|$ for all $T \in L(X,Y)$. Assume that X and Y be infinite dimensional Banach spaces. Hence, by Dvoretzky's Theorem [18] for $m \in \mathbb{N}$, we have quotient spaces X/N_m and subspaces M_m of Y which can be mapped onto ℓ_2^m by isomorphisms H_m and A_m such that $H_m\|H_m^{-1}\| \le 2$ and $A_m\|A_m^{-1}\| \le 2$. Let I_m be the identity map on ℓ_2^m, Q_m be the quotient map from X onto X/N_m, and J_m be the natural embedding map from M_m into Y. Let u_n be the Bernstein numbers [19]; then,

$$\begin{aligned} 1 = u_n(I_m) &= u_n\left(A_mA_m^{-1}I_mH_mH_m^{-1}\right) \\ &\le \|A_m\|u_n\left(A_m^{-1}I_mH_m\right)\|H_m^{-1}\| \\ &= \|A_m\|u_n\left(J_mA_m^{-1}I_mH_m\right)\|H_m^{-1}\| \\ &\le \|A_m\|d_n\left(J_mA_m^{-1}I_mH_m\right)\|H_m^{-1}\| \\ &= \|A_m\|d_n\left(J_mA_m^{-1}I_mH_mQ_m\right)\|H_m^{-1}\| \\ &\le \|A_m\|\alpha_n\left(J_mA_m^{-1}I_mH_mQ_m\right)\|H_m^{-1}\|, \end{aligned} \tag{49}$$

for $1 \le i \le m$. Now,

$$\begin{gathered} \sum_{j=0}^{i}(a_i) \le \sum_{j=0}^{i}\|A_m\|\, a_i\alpha_j\left(J_mA_m^{-1}I_mH_mQ_m\right)\|H_m^{-1}\| \Longrightarrow \\ (i+1)a_i \le \|A_m\|\left(a_i\sum_{j=0}^{i}\alpha_j\left(J_mA_m^{-1}I_mH_mQ_m\right)\right)\|H_m^{-1}\| \Longrightarrow \\ ((i+1)a_i)^{p_i} \le \left(\|A_m\|\|H_m^{-1}\|\right)^{p_i}\left(a_i\sum_{j=0}^{i}\alpha_j\left(J_mA_m^{-1}I_mH_mQ_m\right)\right)^{p_i}. \end{gathered} \tag{50}$$

Therefore,

$$\begin{gathered} \left(\sum_{i=0}^{m}((i+1)a_i)^{p_i}\right)^{1/h} \le L\|A_m\|\|H_m^{-1}\|\left[\sum_{i=0}^{m}\left(a_i\sum_{j=0}^{i}\alpha_j\left(J_mA_m^{-1}I_mH_mQ_m\right)\right)^{p_i}\right]^{1/h} \Longrightarrow \\ \frac{1}{\lambda}\left(\sum_{i=0}^{m}((i+1)a_i)^{p_i}\right)^{1/h} \le L\|A_m\|\|H_m^{-1}\|\frac{1}{\lambda}\left[\sum_{i=0}^{m}\left(a_i\sum_{j=0}^{i}\alpha_j\left(J_mA_m^{-1}I_mH_mQ_m\right)\right)^{p_i}\right]^{1/h} \Longrightarrow \\ \frac{1}{\lambda}\left(\sum_{i=0}^{m}((i+1)a_i)^{p_i}\right)^{1/h} \le L\|A_m\|\|H_m^{-1}\|g\left(J_mA_m^{-1}I_mH_mQ_m\right) \Longrightarrow \\ \frac{1}{\lambda}\left(\sum_{i=0}^{m}((i+1)a_i)^{p_i}\right)^{1/h} \le LC\|A_m\|\|H_m^{-1}\|\|J_mA_m^{-1}I_mH_mQ_m\| \Longrightarrow \\ \frac{1}{\lambda}\left(\sum_{i=0}^{m}((i+1)a_i)^{p_i}\right)^{1/h} \le LC\|A_m\|\|H_m^{-1}\|\|J_mA_m^{-1}\|\|I_m\|\|H_mQ_m\| = LC\|A_m\|\|H_m^{-1}\|\|A_m^{-1}\|\|I_m\|\|H_m\| \Longrightarrow \\ \frac{1}{\lambda}\left(\sum_{i=0}^{m}((i+1)a_i)^{p_i}\right)^{1/h} \le 4LC, \end{gathered} \tag{51}$$

for some $L \geq 1$. Thus, we arrive at a contradiction since m is an arbitrary and $(na_n) \notin \ell^{(p_n)}$. Thus, X and Y both cannot be infinite dimensional when $S^{\mathrm{app}}_{\mathrm{ces}((a_i),(p_i))}(X,Y) = L(X,Y)$ and hence, the result. $\square$

Theorem 17. *If (p_i) is an increasing and $p_0 > 1$, then the pre-quasi Banach operator ideal $S^{Kol}_{ces(p_i)}$ is small.*

Corollary 19. *If $1 < p < \infty$, then the quasi Banach operator ideal $S^{app}_{ces_p}$ is small.*

Corollary 20. *If $1 < p < \infty$, then the quasi Banach operator ideal $S^{Kol}_{ces_p}$ is small.*

In this part, we give the conditions for which the pre-quasi Banach operator ideal $S^{app}_{\ell^{(p_n)}_\beta}$ is small.

Theorem 18. *If conditions (a1), (a2), and $(\beta_n) \notin \ell^1$ are satisfied, then the pre-quasi Banach operator ideal $S^{app}_{\ell^{(p_n)}_\beta}$ is small.*

Proof. Since conditions (a1), (a2), and $(\beta_n) \notin \ell^1$ are satisfied, then $(S^{\mathrm{app}}_{\ell^{(p_n)}_\beta}, g)$, where $g(T) = \sum_{i=0}^{\infty} \beta_i (\alpha_i(T))^{p_i}$, is a pre-quasi Banach operator ideal. Let X and Y be any two Banach spaces. Suppose that $S^{\mathrm{app}}_{\ell^{(p_n)}_\beta}(X,Y) = L(X,Y)$. Then, there exists a constant $C > 0$ such that $g(T) \leq C\|T\|$ for all $T \in L(X,Y)$. Assume that X and Y are infinite dimensional Banach spaces. By using inequality (49) and $(\beta_n) \notin \ell^1$, one obtains

$$\begin{aligned}
1 &\leq \left(\|A_m\|\|H_m^{-1}\|\right)^{p_i}\left(\alpha_i\left(J_m A_m^{-1} I_m H_m Q_m\right)\right)^{p_i} \Longrightarrow \\
\beta_i &\leq L\|A_m\|\beta_i\left(\alpha_i\left(J_m A_m^{-1} I_m H_m Q_m\right)\right)^{p_i}\|H_m^{-1}\| \Longrightarrow \\
\sum_{i=0}^{m}\beta_i &\leq L\|A_m\|\|H_m^{-1}\|\sum_{i=0}^{m}\beta_i\left(\alpha_i\left(J_m A_m^{-1} I_m H_m Q_m\right)\right)^{p_i} \Longrightarrow \\
\sum_{i=0}^{m}\beta_i &\leq L\|A_m\|\|H_m^{-1}\| g\left(J_m A_m^{-1} I_m H_m Q_m\right) \Longrightarrow \\
\sum_{i=0}^{m}\beta_i &\leq LC\|A_m\|\|H_m^{-1}\|\|J_m A_m^{-1} I_m H_m Q_m\| \Longrightarrow \\
\sum_{i=0}^{m}\beta_i &\leq LC\|A_m\|\|H_m^{-1}\|\|J_m A_m^{-1}\|\|I_m\|\|H_m Q_m\| \\
&= LC\|A_m\|\|H_m^{-1}\|\|A_m^{-1}\|\|I_m\|\|H_m\| \Longrightarrow \\
\sum_{i=0}^{m}\beta_i &\leq 4LC,
\end{aligned} \tag{52}$$

for some $L \geq 1$. Thus, we arrive at a contradiction since m is an arbitrary. Thus, X and Y both cannot be infinite dimensional when $S^{\mathrm{app}}_{\ell^{(p_n)}_\beta}(X,Y) = L(X,Y)$ and hence, the result. $\square$

Corollary 21 (see [2]). *If $0 < p < \infty$, then the quasi Banach operator ideal $S^{app}_{\ell^p}$ is small.*

Corollary 22. *If $0 < p < \infty$, then the quasi Banach operator ideal $S^{Kol}_{\ell^p}$ is small.*

7. Pre-Quasi Simple Banach Operator Ideal

The following question arises naturally: for which weighted Nakano sequence space or generalized Cesáro sequence space defined by weighted mean is the pre-quasi Banach ideal simple?

Theorem 19. *If (p_n) and (q_n) are bounded sequences with $1 \leq p_n < q_n$ and $0 < a_n < b_n$ for all $n \in \mathbb{N}$, then*

$$L\left(S_{\ell^{(q_n)}_{(a_n)}}, S_{\ell^{(p_n)}_{(b_n)}}\right) = \Lambda\left(S_{\ell^{(q_n)}_{(a_n)}}, S_{\ell^{(p_n)}_{(b_n)}}\right). \tag{53}$$

Proof. Suppose that there exists $T \in L(S_{\ell^{(q_n)}_{(a_n)}}, S_{\ell^{(p_n)}_{(b_n)}})$ which is not approximable. According to Lemma 1, we can find $X \in L(S_{\ell^{(q_n)}_{(a_n)}}, S_{\ell^{(q_n)}_{(a_n)}})$ and $B \in L(S_{\ell^{(p_n)}_{(b_n)}}, S_{\ell^{(p_n)}_{(b_n)}})$ with $BTXI_k = I_k$. Then, it follows for all $k \in \mathbb{N}$ that

$$\begin{aligned}
\|I_k\|_{S_{\ell^{(p_n)}_{(b_n)}}} &= \left(\sum_{n=0}^{\infty} b_n\left(\alpha_n(I_k)\right)^{p_n}\right)^{1/\sup p_n} = \left(\sum_{n=0}^{k-1} b_n\right)^{1/\sup p_n} \\
&\leq \|BTX\|\|I_k\|_{S_{\ell^{(q_n)}_{(a_n)}}} \leq \left(\sum_{n=0}^{\infty} a_n\left(\alpha_n(I_k)\right)^{q_n}\right)^{1/\sup q_n} \\
&= \left(\sum_{n=0}^{k-1} a_n\right)^{1/\sup q_n}.
\end{aligned} \tag{54}$$

But this is impossible. $\square$

Corollary 23. *If (p_n) and (q_n) are bounded sequences with $1 \leq p_n < q_n$ and $0 < a_n < b_n$ for all $n \in \mathbb{N}$, then*

$$L\left(S_{\ell^{(q_n)}_{(a_n)}}, S_{\ell^{(p_n)}_{(b_n)}}\right) = L_C\left(S_{\ell^{(q_n)}_{(a_n)}}, S_{\ell^{(p_n)}_{(b_n)}}\right). \tag{55}$$

Proof. Every approximable operator is compact. $\square$

Theorem 20. *If (p_n) and (q_n) are bounded sequences with $1 < p_n < q_n$ and $0 < a_n < b_n$ for all $n \in \mathbb{N}$, then*

$$\begin{aligned}
&L\left(S_{\mathrm{ces}((a_n),(q_n))}, S_{\mathrm{ces}((b_n),(p_n))}\right) \\
&\quad = \Lambda\left(S_{\mathrm{ces}((a_n),(q_n))}, S_{\mathrm{ces}((b_n),(p_n))}\right).
\end{aligned} \tag{56}$$

Proof. Suppose that there exists $T \in L(S_{\mathrm{ces}((a_n),(q_n))}, S_{\mathrm{ces}((b_n),(p_n))})$ which is not approximable. According to Lemma 1, we can find $X \in L(S_{\mathrm{ces}((a_n),(q_n))}, S_{\mathrm{ces}((a_n),(q_n))})$ and $B \in L(S_{\mathrm{ces}((b_n),(p_n))}, S_{\mathrm{ces}((b_n),(p_n))})$ with $\mathrm{BTX}I_k = I_k$. Then, it follows for all $k \in \mathbb{N}$ that

$$
\begin{aligned}
\|I_k\|_{S_{\mathrm{ces}((b_n),(p_n))}} &= \left(\sum_{n=0}^{\infty}\left(b_n\sum_{i=0}^{n}\alpha_i(I_k)\right)^{p_n}\right)^{1/\sup p_n} \\
&= \left(\sum_{n=0}^{\infty}\left(b_n\sum_{i=0}^{k-1}\alpha_i(I_k)\right)^{p_n}\right)^{1/\sup p_n} \\
&= \left(\sum_{n=0}^{\infty}(kb_n))^{p_n}\right)^{1/\sup p_n} \\
&\le \|\mathrm{BTX}\|\|I_k\|_{S_{\mathrm{ces}((a_n),(q_n))}} \\
&\le \left(\sum_{n=0}^{\infty}\left(a_n\sum_{i=0}^{n}\alpha_i(I_k)\right)^{q_n}\right)^{1/\sup q_n} \\
&= \left(\sum_{n=0}^{\infty}\left(a_n\sum_{i=0}^{k-1}\alpha_i(I_k)\right)^{q_n}\right)^{1/\sup q_n} \\
&= \left(\sum_{n=0}^{\infty}\left(ka_n\right)^{q_n}\right)^{1/\sup q_n}.
\end{aligned}
\tag{57}
$$

But this is impossible. □

Corollary 24. *If (p_n) and (q_n) are bounded sequences with $1 < p_n < q_n$ and $0 < a_n < b_n$ for all $n \in \mathbb{N}$, then*

$$
\begin{aligned}
&L\left(S_{\mathrm{ces}((a_n),(q_n))}, S_{\mathrm{ces}((b_n),(p_n))}\right) \\
&= L_C\left(S_{\mathrm{ces}((a_n),(q_n))}, S_{\mathrm{ces}((b_n),(p_n))}\right).
\end{aligned}
\tag{58}
$$

Theorem 21. *For a bounded sequence (p_n) with $1 \le p_n < \infty$ and $b_n > 0$ for all $n \in \mathbb{N}$, the pre-quasi Banach space $S_{\ell^{(p_n)}_{(b_n)}}$ is simple.*

Proof. Suppose that the closed ideal $L_C(S_{\ell^{(p_n)}_{(b_n)}})$ contains an operator T which is not approximable. According to Lemma 1, we can find $X, B \in L(S_{\ell^{(p_n)}_{(b_n)}})$ with $BTXI_k = I_k$. This means that $I_{S_{\ell^{(p_n)}_{(b_n)}}} \in L_C(S_{\ell^{(p_n)}_{(b_n)}})$. Consequently, $L(S_{\ell^{(p_n)}_{(b_n)}}) = L_C(S_{\ell^{(p_n)}_{(b_n)}})$. Therefore, $\Lambda(S_{\ell^{(p_n)}_{(b_n)}})$ is the only nontrivial closed ideal in $L(S_{\ell^{(p_n)}_{(b_n)}})$. □

Theorem 22. *For a bounded sequence (p_n) with $1 < p_n < \infty$ and $b_n > 0$ for all $n \in \mathbb{N}$, the pre-quasi Banach space $(S_{ces((b_n),(p_n))})$ is simple.*

Proof. Suppose that the closed ideal $L_C(S_{\mathrm{ces}((b_n),(p_n))})$ contains an operator T which is not approximable. According to Lemma 1, we can find $X, B \in L(S_{\mathrm{ces}((b_n),(p_n))})$ with $\mathrm{BTX}I_k = I_k$. This means that $I_{S_{\mathrm{ces}((b_n),(p_n))}} \in L_C(S_{\mathrm{ces}((b_n),(p_n))})$. Consequently, $L(S_{\mathrm{ces}((b_n),(p_n))}) = L_C(S_{\mathrm{ces}((b_n),(p_n))})$. Therefore, $\Lambda(S_{\mathrm{ces}((b_n),(p_n))})$ is the only nontrivial closed ideal in $L(S_{\mathrm{ces}((b_n),(p_n))})$. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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