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Research Article

Three-Order Multipoint Boundary Value Problems for p-Laplacian Operator on Time Scales

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In this paper, the existence of positive solutions for the nonlinear four-point singular BVP for there-order with p-Laplacian operator on time scales will be studied. By using the fixed-point theory, the existence of positive solutions for nonlinear singular boundary value problem with p-Laplacian operator on time scales is obtained.

1. Introduction

In recent years, the nonlinear boundary value problems have been extensively studied. Recently, for the existence of positive solutions of multipoint boundary value problems, some authors have obtained the existence results. The differential equations offer wonderful tools for describing various natural phenomena arising from natural sciences and engineering, many numerical and analytical results, for example [1–20]. However, the multipoint boundary value problems treated in the above mentioned references do not discuss the problems with singularities and the there-order p-Laplacian operator. For the singular case of multipoint boundary value problems for higher-order p-Laplacian operator, with the author's acknowledge, no one has studied the existence of positive solutions in this case.

In this paper, we study the following equation with p-Laplacian on time scale:

$$\left(\phi_p\left(u^{\Delta\Delta}\right)\right)^{\nabla} + g(t)f\left(u(t), u^{\Delta}(t)\right) = 0, \quad 0 < t < T, \tag{1}$$

with the following boundary value conditions:

$$\begin{cases} u(0) = 0, \\ u^{\Delta}(0) - M_0\left(u^{\Delta\Delta}(\xi)\right) = 0, \quad u^{\Delta}(T) + M_1\left(u^{\Delta\Delta}(\eta)\right) = 0, \end{cases} \tag{2}$$

where $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $\phi_q = \phi_p^{-1}$, $(1/p) + (1/q) = 1$. $\xi, \eta \in (0, T)$ is prescribed and $\xi < \eta$, $g : (0, T) \to [0,\infty)$, M_0, M_1 are both nondecreasing continuous odd functions defined on $(-\infty, +\infty)$.

A time scale $\mathbf{T}$ is a nonempty subset and closed subset of $\mathbf{R}$. By an internal $(0, T)$, we always mean the intersection of the real internal $(0, T)$ with the given time scale, that is $(0, T) \bigcap \mathbf{T}$. The operators σ and ρ from $\mathbf{T}$ to $\mathbf{T}$ which defined by [21],

$$\sigma(t) = \inf\{\tau \in \mathbf{T} \mid \tau > t\} \in \mathbf{T}, \rho(t) = \sup\{\tau \in \mathbf{T} \mid \tau < t\} \in \mathbf{T}. \tag{3}$$

are called the forward jump operator and the backward jump operator, respectively.

The point $t \in \mathbf{T}$ is left-dense, left-scattered, right-dense, right-scattered if $\rho(t) = t$, $\rho(t) < t$, $\sigma(t) = t$, $\sigma(t) > t$, respectively. If $\mathbf{T}$ has a right scattered minimum m, define $\mathbf{T}_k = \mathbf{T} - \{m\}$; otherwise set $\mathbf{T}_k = \mathbf{T}$. If $\mathbf{T}$ has a left scattered maximum M, define $\mathbf{T}^k = \mathbf{T} - \{M\}$; otherwise set $\mathbf{T}^k = \mathbf{T}$.

In this paper, by constructing an integral equation which is equivalent to the problem (1), (2), we research the existence of positive solutions for nonlinear singular boundary value problem (1), (2) when g and f satisfy some suitable conditions.

2. Preliminaries and Lemmas

For convenience, in the rest of this article, $\mathbf{T}$ is a closed subset of $\mathbf{R}$ with $0 \in \mathbf{T}_k$, $T \in \mathbf{T}^k$.

Letting

$$B = \{u \in C_{ld}[0, T]: u(0) = 0\}. \tag{4}$$

Then, B is a Banach space with the norm $\|u\| = \max_{t\in[0,T]} | u^{\Delta}(t) |$. Suppose

$$K = \{u \in B: \ u^{\Delta}(t) \geq 0, \ u^{\Delta}(t) \ \text{is concave function on} \ t \in [0, T]\}. \tag{5}$$

Obviously, K is a cone in B and $0 \leq u^{\Delta}(t) \leq \|u\|$ on $[0, T]$. Set $K_r = \{u \in K : \|u\| \leq r\}$.

In the rest of the paper, we make the following assumptions:

(H_1) $f \in C([0,+\infty)^2, [0,+\infty))$;

(H_2) $g \in C_{ld}((0, T), [0,+\infty))$ and there exists $t_0 \in (0, T)$ which satisfy

$$g(t_0) > 0, 0 < \int_0^T g(t)\nabla t < \infty; \tag{6}$$

(H_3) M_0, $M_1 \in ((-\infty,+\infty), R)$ are both increasing, continuous, odd functions, and at least one of them satisfies the condition that there exists one $b > 0$ which satisfy

$$0 < M_i(v) \leq bv, \forall v \geq 0, i = 0 \text{ or } 1. \tag{7}$$

By direct account, From paper [22], we can easy to obtain the following results.

Lemma 1. *Suppose condition (H_2) holds. Then, there exists a constant $L \in (0, (T/2))$ satisfies*

$$0 < \int_L^{T-L} g(t)\nabla t < \infty. \tag{8}$$

Furthermore, the function

$$A(t) = \int_L^t \phi_q\left(\int_s^t g(s_1)\nabla s_1\right)\Delta s + \int_t^{T-L} \phi_q\left(\int_t^s g(s_1)\nabla s_1\right)\Delta s, \quad t \in [L, T-L], \tag{9}$$

is positive ld-continuous function on $[L, T-L]$, therefore, $A(t)$ has minimum on $[L, T-L]$. Hence, we suppose that there exists constant $B' > 0$ which satisfy $A \geq B'$ on $t \in [L, T-L]$.

Lemma 2. *Suppose that conditions (H_1), (H_2), (H_3) hold, $u(t) \in K \cap C^2(0, T)$ is a solution of boundary value problems (1), (2) if and only if $u(t) \in B$ is a solution of the following integral equation*

$$u(t) = \int_0^t w(s)\Delta s, \tag{10}$$

where

$$w(t) = \begin{cases} M_0\phi_q\left(\int_\xi^\delta g(s)f\left(u(s), u^{\Delta}(s)\right)\Delta s\right) + \int_0^t \phi_q\left(\int_s^\delta g(r)f\left(u(r), u^{\Delta}(r)\right)\Delta r\right)\nabla s, 0 \leq t \leq \delta, \\ M_1\phi_q\left(\int_\delta^\eta g(s)f\left(u(s), u^{\Delta}(s)\right)\Delta s\right) + \int_t^T \phi_q\left(\int_\delta^s g(r)f\left(u(r), u^{\Delta}(r)\right)\Delta r\right)\nabla s, \delta \leq t \leq T. \end{cases} \tag{11}$$

Here, δ is a unique solution of the equation $g_1(t) = g_2(t)$, where

$$g_1(t) = M_0\phi_q\left(\int_\xi^\delta g(s)f\left(u(s), u^{\Delta}(s)\right)\Delta s\right) + \int_0^t \phi_q\left(\int_s^\delta g(r)f\left(u(r), u^{\Delta}(r)\right)\Delta r\right)\nabla s,$$

$$g_2(t) = M_1\phi_q\left(\int_\delta^\eta g(s)f\left(u(s), u^{\Delta}(s)\right)\Delta s\right) + \int_t^T \phi_q\left(\int_\delta^s g(r)f\left(u(r), u^{\Delta}(r)\right)\Delta r\right)\nabla s. \tag{12}$$

Equation $g_1(t) = g_2(t)$ has a unique solution in $(0, T)$. Because $g_1(t)$ is strictly monotone increasing on $[0, T)$, and

$g_1(0)=0$, $g_2(t)$ is strictly monotone decreasing on $(0,T]$, and $g_2(T)=0$.

Proof. Necessity. By the equation of the boundary condition and (H_3), we have

$$u^{\Delta\Delta}(\xi)\geq 0, u^{\Delta\Delta}(\eta)\leq 0. \tag{13}$$

Then, there exist a constant $\delta\in[\xi,\eta]\subset(0,T)$ which satisfy $u^{\Delta\Delta}(\delta)=0$. Firstly, by integrating the equation of the problems (1) on (δ,t), we have

$$\phi_p\left(u^{\Delta\Delta}(t)\right)=\phi_p\left(u^{\Delta\Delta}(\delta)\right)-\int_\delta^t g(s)f\left(u(s),u^{\Delta}(s)\right)\Delta s, \tag{14}$$

then

$$u^{\Delta\Delta}(t)=-\phi_q\left(\int_\delta^t g(s)f\left(u(s),u^{\Delta}(s)\right)\Delta s\right), \tag{15}$$

$$u^{\Delta}(t)=u^{\Delta}(\delta)-\int_\delta^t \phi_q\left(\int_\delta^s g(r)f\left(u(r),u^{\Delta}(r)\right)\Delta r\right)\nabla s. \tag{16}$$

By $u^{\Delta\Delta}(\delta)=0$ and condition (2), let $t=\eta$ on (15), we have

$$u^{\Delta\Delta}(1)=-M_1\left(u^{\Delta\Delta}(\eta)\right)=M_1\phi_q\left(\int_\delta^\eta g(s)f\left(u(s),u^{\Delta}(s)\right)\Delta s\right). \tag{17}$$

Then, we have

$$\begin{aligned}u^{\Delta}(\delta)&=M_1\phi_q\left(\int_\delta^\eta g(s)f\left(u(s),u^{\Delta}(s)\right)\Delta s\right)\\&\quad+\int_\delta^T \phi_q\left(\int_\delta^s g(r)f\left(u(r),u^{\Delta}(r)\right)\Delta r\right)\nabla s.\end{aligned} \tag{18}$$

Then

$$\begin{aligned}u^{\Delta\Delta}(t)&=M_1\phi_q\left(\int_\delta^\eta g(s)f\left(u(s),u^{\Delta}(s)\right)\Delta s\right)\\&\quad+\int_t^T \phi_q\left(\int_\delta^s g(r)f\left(u(r),u^{\Delta}(r)\right)\Delta r\right)\nabla s.\end{aligned} \tag{19}$$

Therefore, by integrating the above equation (19) on $(0,t)$, we can east to have

$$\begin{aligned}u(t)&=\int_0^t M_1\phi_q\left(\int_\delta^\eta g(s)f\left(u(s),u^{\Delta}(s)\right)\Delta s\right)\nabla t\\&\quad+\int_0^t\left(\int_{s_1}^T \phi_q\left(\int_\delta^s g(r)f\left(u(r),u^{\Delta}(r)\right)\Delta r\right)\nabla s\right)\nabla s_1.\end{aligned} \tag{20}$$

Similarly, for $t\in(0,\delta)$, by integrating the equation of problems (1) on $(0,\delta)$, we have

$$\begin{aligned}u(t)&=\int_0^t M_0\phi_q\left(\int_\xi^\delta g(s)f\left(u(s),u^{\Delta}(s)\right)\Delta s\right)\nabla t\\&\quad+\int_0^t\left(\int_{s_1}^T \phi_q\left(\int_s^\delta g(r)f\left(u(r),u^{\Delta}(r)\right)\Delta r\right)\nabla s\right)\nabla s_1.\end{aligned} \tag{21}$$

Therefore, for any $t\in[0,T]$, $u(t)$ can be expressed as equation $u(t)=\int_0^t w(s)\Delta s$, where $w(t)$ is expressed as Lemma 3.

Sufficiency. Suppose $u(t)=\int_0^t w(s)\Delta s$. Then we have

$$u^{\Delta\Delta}(t)=\begin{cases}\phi_q\left(\int_t^\delta g(s)f\left(u(s),u^{\Delta}(s)\right)\Delta s\right)\geq 0, & 0\leq t<\delta,\\ -\phi_q\left(\int_\delta^t g(s)f\left(u(s),u^{\Delta}(s)\right)\Delta s\right)\leq 0, & \delta<t\leq T,\end{cases} \tag{22}$$

So, $(\phi_p(u^{\Delta\Delta}))^{\nabla}+g(t)f(u(t),u^{\Delta}(t))=0$, $0<t<T$, $t\neq\delta$. These imply that the equation (1) holds. Furthermore, we can easily obtain the boundary value equations of (2). This completes the proof of Lemma 3.

Now, we define an operator $T:K\to C^2[0,T]$ given by

$$(Tu)(t)=\int_0^t w(s)\Delta s, \tag{23}$$

where $w(t)$ is given by (15). And we can easily obtain the following Lemma.

Lemma 3. *Let $u\in K$ and L in Lemma 1. Then*

$$u^{\Delta}(t)\geq L\|u\|, t\in[L,T-L]. \tag{24}$$

Lemma 4. *Suppose that conditions (H_1), (H_2) hold, then for $L\in(0,T/2)$ in Lemma 1, we have*

$$u(t)\leq\frac{1}{L}u^{\Delta}(t), t\in[L,T-L]. \tag{25}$$

Proof. If $u(t)$ is the solution of problem (1), (2), then $u^{\Delta}(t)$ is a concave function, and $u(t)\geq 0$, $u^{\Delta}(t)\geq 0, t\in[0,T]$.

Thus for $t\in[L,T-L]$, we have $u^{\Delta}(t)\geq L\|u^{\Delta}\|$. Then by $u(t)=\int_0^t u^{\Delta}(s)\Delta s\leq\|u^{\Delta}\|$, we have

$$u(t)\leq\frac{1}{L}u^{\Delta}(t), t\in[L,1-L]. \tag{26}$$

The proof is complete.

Remark. : Obviously, we can obtain the following results,

$$w(0) - M_0 w'(\xi) = 0, \quad w(1) + M_1 w'(\eta) = 0. \tag{27}$$

Furthermore, by Arzela-Ascoli Theorem, it is easy to obtain the following Lemma.

Lemma 5. *$T : K \to K$ is completely continuous.*

For convenience, we set

$$R^* = 2\left(B'\right)^{-1}, R_* = \left[(b+1)\phi_q\left(\int_0^T g(r)\Delta r\right)\right]^{-1}. \tag{28}$$

where B' and L are given as Lemma 1.

3. The Existence of Single and Many Positive Solution

In this section, we present the following five main results.

Theorem 6. *Suppose that condition (H_1), (H_2), (H_3) hold. Assume that f also satisfy.*

(A_1) For $Lr \le u_2 \le r$, $0 \le u_1 \le (1/L)u_2$, we have $f(u_1, u_2) \ge (mr)^{p-1}$;

(A_2) For $0 \le u_2 \le R, 0 \le u_1 \le (1/L)u_2$, we have $f(u_1, u_2) \le (MR)^{p-1}$,where $m \in (R^*, \infty)$, $M \in (0, R_*)$. Then, the boundary value problem (1), (2) has at last one solution u such that $\|u\|$ lies between r and R.

The proof of Theorem 6. From Condition (H_3), for $\forall v \ge 0$, we can suppose that $0 < M_0(v) \le bv$, and $r < R$. By Lemma 3, for any $u \in K$, we can obtain that

$$u^{\Delta}(t) \ge L\|u\|, t \in [L, T-L]. \tag{29}$$

We define the following two open subset Ω_1 and Ω_2 of E:

$$\Omega_1 = \{u \in K : \|u\| < r\}, \Omega_2 = \{u \in K : \|u\| < R\}. \tag{30}$$

For any $u \in \partial\Omega_1$, by (29), we have

$$r = \|u\| \ge u^{\Delta}(t) \ge L\|u\| = Lr, t \in [L, T-L]. \tag{31}$$

For $t \in [L, T-L]$ and $u \in \partial\Omega_1$, we shall discuss it from three perspectives.

(i) If $\delta \in [L, T-L]$, thus for $u \in \partial\Omega_1$, by (A_1) and Lemma 3, we have

$$\begin{aligned} 2(Tu)^{\Delta}(\delta) &\ge \int_0^{\delta} \phi_q\left(\int_s^{\delta} g(r) f\left(u(r), u^{\Delta}(r)\right)\nabla r\right)\Delta s \\ &\quad + \int_{\delta}^{T} \phi_q\left(\int_{\delta}^{s} g(r) f\left(u(r), u^{\Delta}(r)\right)\nabla r\right)\Delta s \\ &\ge \int_L^{\delta} \phi_q\left(\int_s^{\delta} g(r) f\left(u(r), u^{\Delta}(r)\right)\nabla r\right)\Delta s \\ &\quad + \int_{\delta}^{T-L} \phi_q\left(\int_{\delta}^{s} g(r) f\left(u(r), u^{\Delta}(r)\right)\nabla r\right)\Delta s \\ &\ge mrA(\delta) \ge mrB' > 2r = 2\|u\|. \end{aligned} \tag{32}$$

Then, with the case of $\delta \in [L, T-L]$ and $u \in \partial\Omega_1$, we have $\|Tu\| \ge \|u\|$.

(ii) If $\delta \in (T-L, T]$, thus for $u \in \partial\Omega_1$, by (A_1) and Lemma 3, we have

$$\begin{aligned} (Tu)^{\Delta}(\delta) &\ge M_0\phi_q\left(\int_{\xi}^{\delta} g(r) f\left(u(r), u^{\Delta}(r)\right)\nabla r\right) \\ &\quad + \int_0^{\delta} \phi_q\left(\int_s^{\delta} g(r) f\left(u(r), u^{\Delta}(r)\right)\nabla r\right)\Delta s \\ &\ge \int_L^{T-L} \phi_q\left(\int_s^{T-L} g(r) f\left(u(r), u^{\Delta}(r)\right)\nabla r\right)\Delta s \\ &\ge mrA(1-L) \ge mrB' > 2r > r = \|u\|. \end{aligned} \tag{33}$$

Then, with the case of $\delta \in (T-L, T]$ and $u \in \partial\Omega_1$, we have $\|Tu\| \ge \|u\|$.

(iii) If $\delta \in (0, L)$, thus for $u \in \partial\Omega_1$, by (A_1) and Lemma 3, we have

$$\begin{aligned} (Tu)^{\Delta}(\delta) &\ge M_1\phi_q\left(\int_{\delta}^{\eta} g(r) f\left(u(r), u^{\Delta}(r)\right)\nabla r\right) \\ &\quad + \int_{\delta}^{1} \phi_q\left(\int_{\delta}^{s} g(r) f\left(u(r), u^{\Delta}(r)\right)\nabla r\right)\Delta s \\ &\ge \int_L^{T-L} \phi_q\left(\int_L^{s} g(r) f\left(u(r), u^{\Delta}(r)\right)\nabla r\right)\Delta s \\ &\ge mrA(L) \ge mrB' > 2r > r = \|u\|. \end{aligned} \tag{34}$$

Then, with the case of $\delta \in (0, L)$ and $u \in \partial\Omega_1$, we have $\|Tu\| \ge \|u\|$.

Therefore, for any case of $\delta \in (0, T-L]$, we all easy to obtain that

$$\|Tu\| > \|u\|, \forall u \in \partial\Omega_1. \tag{35}$$

Then, by fixed point theorem of cone expansion-compression type in [23, 24], we have

$$i(T, \Omega_1, K) = 0. \tag{36}$$

Secondly, for $u \in \partial\Omega_2$, using $u^{\Delta}(t) \le \|u\| = R$, from (A_2), we can easily know that

$$\begin{aligned}(Tu)^{\Delta}(\delta) &\le M_0\phi_q\left(\int_0^T g(r)f\left(u(r), u^{\Delta}(r)\right)\Delta r\right) \\ &\quad + \int_0^T \phi_q\left(\int_s^{\delta} g(r)f\left(u(r), u^{\Delta}(r)\right)\nabla r\right)\Delta s \\ &\le bMR\phi_q\left(\int_0^T g(r)\Delta r\right) + MR\phi_q\left(\int_0^T g(r)\Delta r\right) \\ &= (b+1)MR\phi_q\left(\int_0^T g(r)\Delta r\right) \le R = \|u\|.\end{aligned} \tag{37}$$

Thus, we have

$$\|Tu\| < \|u\|, \forall u \in \partial\Omega_2. \tag{38}$$

Then, by fixed point theorem of cone expansion-compression type in [23, 24], we have

$$i(T, \Omega_2, K) = 1. \tag{39}$$

Therefore, by (36), (39), $r < R$ we have

$$i\left(T, \Omega_2 \setminus \bar{\Omega}_1, K\right) = 1. \tag{40}$$

Then, operator T has at last one fixed point $u \in (\Omega_2 \setminus \bar{\Omega}_1)$, and $r \le \|u\| \le R$. This completes the proof of Theorem 6.

Theorem 7. *Suppose that condition (H_1), (H_2), (H_3) hold. Assume that f also satisfy.*

(A_3) $f^0 = \lim_{u_2\to 0}\max_{0\le u_1\le(1/L)u_2}(f(u_1,u_2)/(u_2)^{p-1}) = \varphi \in [0, (R_*/4)^{p-1})$;

(A_4) $f_\infty = \lim_{u_2\to\infty}\min_{0\le u_1\le(1/L)u_2}(f(u_1,u_2)/(u_2)^{p-1}) = \psi \in ((2R^*/L)^{p-1}, \infty)$.

Then, the boundary value problem (1), (2) has at last one solution u.

The proof of Theorem 7.

First, by $0 \le \lim_{u_2\to 0}\max_{0\le u_1\le(1/L)u_2}(f(u_1,u_2)/(u_2)^{p-1}) = \varphi < (R_*/4)^{p-1}$, letting $\varepsilon = (R_*/4)^{p-1} - \varphi > 0$, we know that there exists an appropriately small positive number ρ which satisfy as $0 \le u_2 \le \rho$, $u_2 \ne 0$, and we have

$$f(u_1, u_2) \le (\varphi + \varepsilon)(u_2)^{p-1} \le (R_*/4)^{p-1}\rho^{p-1} = ((R_*/4)\rho)^{p-1}. \tag{41}$$

Then, letting $R = \rho$, $0 < M = R_*/4 < R_*$, thus by the above equation, we can have

$$f(u_1, u_2) \le (MR)^{p-1}, 0 \le u_2 \le R, 0 \le u_1 \le \frac{1}{L}u_2. \tag{42}$$

So condition (A_2) in Theorem 6 holds.

Next, by $(2R^*/L)^{p-1} < \lim_{u_2\to\infty}\min_{0\le u_1\le(1/L)u_2}(f(u_1,u_2)/(u_2)^{p-1}) = \psi < \infty$, letting $\varepsilon = \psi - (2R^*/L)^{p-1}$, we know that there exists an adequately big positive number $r \ne R$ which satisfy as $u_2 \ge Lr$, $0 \le u_1 \le (1/L)u_2$, and we have

$$f(u_1, u_2) \ge (\psi - \varepsilon)(u_2)^{p-1} \ge \left(\frac{2R^*}{L}\right)^{p-1}(Lr)^{p-1} = (2R^*r)^{p-1}, \tag{43}$$

Letting $m = 2R^* > R^*$, thus by the above equation, we have that (A_1) in Theorem 6 holds. Therefore, by Theorem 6, we can easily obtain the results of Theorem 7 holds. The proof of Theorem 3.2 is complete.

Corollary 8. *Suppose that condition (H_1), (H_2), (H_3) hold. Assume that f also satisfy.*

(A_5) $f^\infty = \lim_{u_2\to\infty}\max_{0\le u_1\le(1/L)u_2}(f(u_1,u_2)/(u_2)^{p-1}) = \lambda \in [0, (R_*/4)^{p-1})$;

(A_6) $f_0 = \lim_{u_2\to 0}\min_{0\le u_1\le(1/L)u_2}(f(u_1,u_2)/(u_2)^{p-1}) = \varphi \in ((2R^*/L)^{p-1}, \infty)$.

Then, the boundary value problem (1), (2) has at last one solution u.

The proof of Corollary 8. Similar to the proof of Theorem 7, we can obtain Corollary 8.

Theorem 9. *Suppose that conditions (H_1), (H_2), (H_3), and (A_2) in Theorem 6 hold. Assume that f also satisfy.*

(A_7) $f_0 = \lim_{u_2\to 0}\min_{0\le u_1\le(1/L)u_2}(f(u_1,u_2)/(u_2)^{p-1}) = +\infty$;

(A_8) $f_\infty = \lim_{u_2\to\infty}\min_{0\le u_1\le(1/L)u_2}(f(u_1,u_2)/(u_2)^{p-1}) = +\infty$.

Then, the boundary value problem (1), (2) have at least two solutions u_1, u_2 which satisfy

$$0 < \|u_1\| < R < \|u_2\|. \tag{44}$$

The proof of Theorem 9.

Firstly, by $\lim_{u_2\to 0}\min_{0\le u_1\le(1/L)u_2}(f(u_1,u_2)/(u_2)^{p-1}) = +\infty$, for any $M > (2/B'L)$, there exists a constant $\rho_* \in (0, R)$ which satisfy

$$f(u_1, u_2) \ge (Mu_2)^{p-1}, 0 < u_2 \le \rho_*, 0 \le u_1 \le \frac{1}{L}u_2. \tag{45}$$

Set $\Omega_{\rho_*} = \{u \in K : \|u\| < \rho_*\}$, similar to the previous proof of Theorem 6, for any $u \in \partial\Omega_{\rho_*}$, from the above discussion and Lemma 2, we can have from three perspectives

$$\|Tu\| \geq \|u\|, \forall u \in \partial\Omega_{\rho_*}. \quad (46)$$

Then, by fixed point theorem of cone expansion-compression type, we can have

$$i\left(T, \Omega_{\rho_*}, K\right) = 0. \quad (47)$$

Secondly, for any $\bar{M} > (2/B'L)$, by $\lim_{u_2\to\infty} \min_{0\leq u_1\leq (1/L)u_2} (f(u_1, u_2)/(u_2)^{p-1}) = +\infty$, there exists a constant $\rho_0 > 0$ which satisfy

$$f(u_1, u_2) \geq \left(\bar{M}u_2\right)^{p-1}, u_2 > \rho_0, 0 \leq u_1 \leq \frac{1}{L}u_2. \quad (48)$$

Therefore, we choose a constant $\rho^* > \max\{R, \rho_0/L\}$, obviously $\rho_* < R < \rho^*$. Set $\Omega_{\rho^*} = \{u \in K : \|u\| < \rho^*\}$. For any $u \in \partial\Omega_{\rho^*}$, by Lemma 2, we can easily obtain

$$u^{\Delta}(t) \geq L\|u\| = L\rho^* > \rho_0, t \in [L, T-L]. \quad (49)$$

Then, by the above discussion and also similar to the previous proof of Theorem 6, we can also have from three perspectives

$$\|Tu\| \geq \|u\|, \forall u \in \partial\Omega_{\rho^*}. \quad (50)$$

Then, by fixed point theorem of cone expansion-compression type, we have

$$i\left(T, \Omega_{\rho^*}, K\right) = 0. \quad (51)$$

Finally, imitating the latter proof of Theorem 6, for any $u \in \partial\Omega_R$, by (A_2), setting $\Omega_R = \{u \in K : \|u\| < R\}$, we can also easy to have

$$\|Tu\| \leq \|u\|, \forall u \in \partial\Omega_R. \quad (52)$$

Then, by fixed point theorem of cone expansion-compression type, we have

$$i(T, \Omega_R, K) = 1. \quad (53)$$

Therefore, by (47), (51), (53), $\rho_* < R < \rho^*$ we have

$$i\left(T, \Omega_R \setminus \bar{\Omega}_{\rho_*}, K\right) = 1, i\left(T, \Omega_{\rho^*} \setminus \bar{\Omega}_R, K\right) = -1. \quad (54)$$

Then, T have fixed point $u_1 \in \Omega_R \setminus \bar{\Omega}_{\rho_*}$, and fixed point $u_2 \in \Omega_{\rho^*} \setminus \bar{\Omega}_R$.

Obviously, u_1, u_2 are all positive solutions of problem (1), (2) and $0 < \|u_1\| < R < \|u_2\|$. The proof of Theorem 9 is complete.

Theorem 10. *Suppose that conditions (H_1), (H_2), (H_3) and (A_1) in Theorem 6 hold. Assume that f also satisfy.*

(A_9) $f^0 = \lim_{u_2\to 0} \max_{0\leq u_1\leq (1/L)u_2} (f(u_1, u_2)/(u_2)^{p-1}) = 0$;

(A_{10}) $f^\infty = \lim_{u_2\to\infty} \max_{0\leq u_1\leq (1/L)u_2} (f(u_1, u_2)/(u_2)^{p-1}) = 0$.

Then, the boundary value problem (1), (2) have at least two solutions u_1, u_2 which satisfy $0 < \|u_1\| < r < \|u_2\|$.

The proof of Theorem 10.

Firstly, by $\lim_{u_2\to\infty} \max_{0\leq u_1\leq (1/L)u_2} (f(u_1, u_2)/(u_2)^{p-1}) = 0$, for $\eta_1 \in (0, R_*)$, there exists a constant $\rho_* \in (0, r)$ which satisfy

$$f(u_1, u_2) \leq (\eta_1 u_2)^{p-1}, 0 < u_2 \leq \rho_*, 0 \leq u_1 \leq \frac{1}{L}u_2. \quad (55)$$

Set $\Omega_{\rho_*} = \{u \in K : \|u\| < \rho_*\}$, for any $u \in \partial\Omega_{\rho_*}$, by (23), we have

$$\begin{aligned}\|Tu\| = (Tu)^{(n-2)}(\delta) &\leq M_0\phi_q\left(\int_0^T g(r)f\left(u(r), u^{\Delta}(r)\right)\nabla r\right) \\ &\quad + \int_0^T \phi_q\left(\int_s^\delta g(r)f\left(u(r), u^{\Delta}(r)\right)\nabla r\right)\Delta s \\ &\leq M_0\phi_q\left(\int_0^T g(r)f\left(u(r), u^{\Delta}(r)\right)\nabla r\right) \\ &\quad + \phi_q\left(\int_0^T g(r)f\left(u(r), u^{\Delta}(r)\right)\nabla r\right) \\ &\leq (b+1)\eta_1\rho_*\phi_q\left(\int_0^T g(r)dr\right) \leq \rho_* = \|u\|.\end{aligned} \quad (56)$$

Then, by fixed point theorem of cone expansion-compression type, we have

$$i\left(T, \Omega_{\rho_*}, K\right) = 1. \quad (57)$$

Secondly, letting $f^*(x) = \max_{0\leq u_{n-1}\leq x, 0\leq u_1\leq (1/L)u_2} f(u_1, u_2)$, we can easy to know that $f^*(x)$ is monotone increasing with respect to $x \geq 0$.

Therefore by $\lim_{u_2\to\infty} \max_{0\leq u_1\leq (1/L)u_2} (f(u_1, u_2)/(u_2)^{p-1}) = 0$, we can easy to have $\lim_{x\to\infty} (f^*(x)/x^{p-1}) = 0$.

Therefore, for any $\eta_2 \in (0, R_*)$, there exists a constant $\rho^* > r$ which satisfy

$$f^*(x) \leq (\eta_2 x)^{p-1}, x \geq \rho^*. \quad (58)$$

Set $\Omega_{\rho^*} = \{u \in K : \|u\| < \rho^*\}$, for any $u \in \partial\Omega_{\rho^*}$, by (4.8), we have

$$\begin{aligned}\|Tu\| = (Tu)^{\Delta}(\delta) &\le M_0\phi_q\left(\int_0^T g(r)f\left(u(r),u^{\Delta}(r)\right)\nabla r\right)\\ &\quad + \int_0^T \phi_q\left(\int_s^{\delta} g(r)f\left(u(r),u^{\Delta}(r)\right)\nabla r\right)\Delta s\\ &\le M_0\phi_q\left(\int_0^T g(r)f\left(u(r),u^{\Delta}(r)\right)\nabla r\right)\\ &\quad + \phi_q\left(\int_0^T g(r)f\left(u(r),u^{\Delta}(r)\right)\nabla r\right)\\ &\le (b+1)\phi_q\left(\int_0^T g(r)f^*(\rho^*)dr\right)\\ &\le (b+1)\eta_2\rho^*\phi_q\left(\int_0^T g(r)dr\right) \le \rho^* = \|u\|.\end{aligned} \tag{59}$$

Then, by fixed point theorem of cone expansion-compression type, we have

$$i\left(T,\Omega_{\rho^*},K\right) = 1. \tag{60}$$

Finally, imitating the previous proof of Theorem 6, for any $u \in \partial\Omega_r$, by (A_1), setting $\Omega_r = \{u \in K : \|u\| < r\}$, For any $u \in \partial\Omega_r$, we can also easy to have

$$\|Tu\| \ge \|u\|, \forall u \in \partial\Omega_r. \tag{61}$$

Then, by fixed point theorem of cone expansion-compression type, we have

$$i(T,\Omega_r,K) = 0. \tag{62}$$

Therefore, by (57), (60), (62), $\rho_* < r < \rho^*$, we have

$$i\left(T,\Omega_r \setminus \bar{\Omega}_{\rho_*},K\right) = -1, i\left(T,\Omega_{\rho^*} \setminus \bar{\Omega}_r,K\right) = 1. \tag{63}$$

Then, T have fixed point $u_1 \in \Omega_r \setminus \bar{\Omega}_{\rho_*}$, and fixed point $u_2 \in \Omega_{\rho_*} \setminus \bar{\Omega}_r$.

Obviously, u_1, u_2 are all positive solutions of problem (1),(2) and $0 < \|u_1\| < r < \|u_2\|$. The proof of Theorem 10 is complete.

4. Application

Example. Consider the following three-order BVP with p-Laplacian

$$\begin{cases} \left(\phi_p\left(u''\right)\right)' + \dfrac{1}{64\pi^4}t^{-\frac{1}{2}}(1-t)\left[\ln u^5 + e^{(u')^2}\right] = 0, 0 < t < 1,\\ u(0) = 0,\\ u'(0) - u''(0.25) = 0, u'(1) + 5u''(0.3) = 0, \end{cases} \tag{64}$$

where $p = 4, \xi = 0.25, \eta = 0.3,\ B' = 1/4$,

$$g(t) = \frac{1}{64\pi^4}t^{-\frac{1}{2}}(1-t), f(u_1,u_2) = \ln u^5 + e^{(u')^2}. \tag{65}$$

Then obviously, $q = 4/3, \int_0^1 g(t)dt = 1/64\pi^3, f_{\infty} = +\infty$, $f_0 = +\infty$,

$$M_0(v) = v < 2v = bv, M_1(v) = 5v, \forall v \ge 0, \tag{66}$$

so conditions (H_1), (H_2), (H_3), (A_7), (A_8) hold.

Next, $\phi_q(\int_0^1 g(t)dt) = 1/4\pi$, $R_* = 4\pi/3$, we choose $R = 3$, $M = 2$ and for $B' = 1/4$, because of the monotone increasing of $f(u_1,\ u_2)$ on $[0,\infty) \times [0,\infty)$, then

$$f(u_1,\ u_2) \le f(12,\ 3) = 286,\ \ 0 \le u_2 \le 3,\ \ 0 \le u_1 \le 4u_2. \tag{67}$$

Therefore, using $0 < M < R_*$, we have $(MR)^{p-1} = 328$, we know

$$f(u_1,\ u_2) \le (MR)^{p-1},\ \ 0 \le u_2 \le 3,\ \ 0 \le u_1 \le 4u_2, \tag{68}$$

so conditions (A_2) holds. Then, by Theorem 9, the Example has at least two positive solutions v_1, v_2 and $0 < \|v_1\| < 3 < \|v_2\|$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

Authors' Contributions

The study was carried out in collaboration among all authors. All authors read and approved the final manuscript.

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References

[1] X. Liu, M. Jia, and X. Xiang, "On the solvability of a fractional differential equation model involving the p-Laplacian operator," *Computers and Mathematics with Applications*, vol. 64, no. 10, pp. 3267–3275, 2012.

[2] X. Yang, "Green's function and positive solutions for higher-order ODE," *Applied Mathematics and Computation*, vol. 136, no. 2-3, pp. 379–393, 2003.

[3] F. Atici and G. S. Guseinov, “On Green's functions and positive solutions for boundary value problems on time scales,” *Journal of Computational and Applied Mathematics*, vol. 141, no. 1-2, pp. 75–99, 2002.

[4] X. Zhang, Z. Shao, and Q. Zhong, “Positive solutions for semipositone $(k, n-k)$ conjugate boundary value problems with singularities on space variables,” *Applied Mathematics Letters*, vol. 72, pp. 50–57, 2017.

[5] X. Zhang, L. Liu, and Y. Wu, “The entire large solutions for a quasilinear Schrödinger elliptic equation by the dual approach,” *Applied Mathematics Letters*, vol. 55, pp. 1–9, 2016.

[6] R. Liang, J. Peng, and J. Shen, “Double positive solutions for a nonlinear four-point boundary value problem with a p -Laplacian operator,” *Nonlinear Analysis: Theory, Methods & Applications*, vol. 68, no. 7, pp. 1881–1889, 2008.

[7] E. R. Kaufmann, “Positive solutions of a three-point boundary value problem on a time scale,” *Electronic Journal of Differential Equations (EJDE)*, vol. 82, pp. 1–11, 2003.

[8] X. Hao, P. Zhang, and L. Liu, “Iterative solutions of singular $(k, n-k)$ conjugate boundary value problems with dependence on the derivatives,” *Applied Mathematics Letters*, vol. 27, pp. 64–69, 2014.

[9] X. Zhang, L. Liu, Y. Wu, and Y. Cui, “Existence of infinitely solutions for a modified nonlinear Schrödinger equation via dual approach,” *Electronic Journal of Differential Equations*, vol. 2147, pp. 1–15, 2018.

[10] J. He, X. Zhang, L. Liu, and Y. Wu, “Existence and nonexistence of radial solutions of the Dirichlet problem for a class of general k-hessian equations,” *Nonlinear Analysis: Modelling and Control*, vol. 23, no. 4, pp. 475–492, 2018.

[11] X. Zhang, Y. Wu, and Y. Cui, “Existence and nonexistence of blow-up solutions for a Schrödinger equation involving a nonlinear operator,” *Applied Mathematics Letters*, vol. 82, pp. 85–91, 2018.

[12] J. He, X. Zhang, L. Liu, Y. Wu, and Y. Cui, “Existence and asymptotic analysis of positive solutions for a singular fractional differential equation with nonlocal boundary conditions,” *Boundary Value Problems*, vol. 2018, no. 1, 2018.

[13] X. Zhang, L. Liu, Y. Wu, and Y. Cui, “New result on the critical exponent for solution of an ordinary fractional differential problem,” *Journal of Function Spaces*, vol. 2017, Article ID 3976469, 4 pages, 2017.

[14] X. Zhang, L. Liu, Y. Wu, and Y. Cui, “The existence and nonexistence of entire large solutions for a quasilinear Schrödinger elliptic system by dual approach,” *Journal of Mathematical Analysis and Applications*, vol. 464, no. 2, pp. 1089–1106, 2018.

[15] C. Chen, K. Li, Y. Chen, and Y. Huang, “Two-grid finite element methods combined with Crank-Nicolson scheme for nonlinear Sobolev equations,” *Advances in Computational Mathematics*, vol. 45, no. 2, pp. 611–630, 2019.

[16] X. Zhang, J. Xu, J. Jiang, Y. Wu, and Y. Cui, “The convergence analysis and uniqueness of blow-up solutions for a Dirichlet problem of the general k-Hessian equations,” *Applied Mathematics Letters*, vol. 102, article 106124, 2020.

[17] X. Liu, M. Jia, and W. Ge, “The method of lower and upper solutions for mixed fractional four-point boundary value problem with p-Laplacian operator,” *Applied Mathematics Letters*, vol. 65, pp. 56–62, 2017.

[18] H. Zhang, Y. Li, and J. Xu, “Positive solutions for a system of fractional integral boundary value problems involving Hadamard-type fractional derivatives,” *Complexity*, vol. 2019, Article ID 2671539, 11 pages, 2019.

[19] T. Qi, Y. Liu, and Y. Cui, “Existence of solutions for a class of coupled fractional differential systems with nonlocal boundary conditions,” *Journal of Function Spaces*, vol. 2017, Article ID 6703860, 9 pages, 2017.

[20] B. Liu and Y. Liu, “Positive solutions of a two-point boundary value problem for singular fractional differential equations in Banach space,” *Journal of Function Spaces and Applications*, vol. 2013, article 585639, 9 pages, 2013.

[21] S. Hilger, “Analysis on measure chains – a unified approach to continuous and discrete calculus,” *Results in Mathematics*, vol. 18, no. 1-2, pp. 18–56, 1990.

[22] H. R. Sun and W. T. Li, “Positive solutions for nonlinear three-point boundary value problems on time scales,” *Journal of Mathematical Analysis and Applications*, vol. 299, no. 2, pp. 508–524, 2004.

[23] D. Guo and V. Lakshmikantham, *Nonlinear Problems in Abstract Cone*, Academic Press, Sandiego, 1988.

[24] D. Guo, V. Lakshmikantham, and X. Liu, *Nonlinear Integral Equations in Abstract Spaces*, Kluwer Academic Publishers, 1996.