Research Article

Versions of Inequalities Related to p -Schatten Norm

Fugen Gao , Xiaoxiao Liu, and Mengyu Tian

College of Mathematics and Information Science, Henan Normal University, Xinxiang 453007, Henan, China

Correspondence should be addressed to Fugen Gao; gaofugen08@126.com

Received 19 November 2019; Accepted 11 January 2020; Published 24 February 2020

Academic Editor: Kehe Zhu

Copyright © 2020 Fugen Gao et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

In this paper, we investigate some operator inequalities for the p -Schatten norm and obtain some other versions of these inequalities when parameters taking values in different regions. Let $A_1, \dots, A_n, B_1, \dots, B_n \in B_p(H)$ such that $\sum_{i,j=1}^n A_i^* B_j = 0$. Then, for $p \geq 2$, $p \geq \lambda > 0$, and $0 < \mu \leq 2$, $n^{2((1/p)-(1/\lambda))} (\sum_{i,j=1}^n \|A_i \pm B_j\|_p^\lambda)^{1/\lambda} \leq n^{1/2} (\sum_{i=1}^n \|A_i\|_p^2 + \sum_{i=1}^n \|B_i\|_p^2)^{1/2} \leq 2^{(1/2)-(1/\mu)} n^{1-(\mu/4)} (\sum_{i=1}^n \|A_i\|_p^{4/\mu} + \sum_{i=1}^n \|B_i\|_p^{4/\mu})^{\mu/4}$. For $0 < p \leq 2$, $p \leq \lambda$, and $\mu \geq 2$, the inequalities are reversed. Moreover, we get some applications of our results.

1. Introduction

Let $B(H)$ be the C^* -algebra of all bounded linear operators acting on a complex separable Hilbert space H . $|A| = (A^*A)^{1/2}$ denotes the absolute value of an operator $A \in B(H)$. If $A \in B(H)$ is compact, let $\{s_j(A)\}_{j=1}^\infty$ be the sequence of decreasingly ordered singular values of A . For $0 < p < \infty$, let $\|A\|_p = (\text{tr}|A|^p)^{1/p} = (\sum_{j=1}^\infty s_j^p(A))^{1/p}$, where tr is the nuclearity (or trace class) of integral operators. This defines the Schatten p -norm (quasi-norm, resp.) for $1 \leq p < \infty$ ($0 < p < 1$, resp.) on the set

$$B_p(H) = \{X \in B(H): \|X\|_p < \infty\}, \quad (1)$$

which is called the p -Schatten class of $B(H)$ (see [1]). The Schatten p -norms are unitarily invariant and when $p = 1$, $\|A_1\| = \text{tr}|A|$ is called the trace norm of A .

There are some classical Clarkson's inequalities for the Schatten p -norms of operators in $B_p(H)$ (see [2, 3]). If $A, B \in B_p(H)$, then

$$2^{p-1} (\|A\|_p^p + \|B\|_p^p) \leq \|A - B\|_p^p + \|A + B\|_p^p \leq 2 (\|A\|_p^p + \|B\|_p^p), \quad (2)$$

for $0 < p \leq 2$ and

$$2 (\|A\|_p^p + \|B\|_p^p) \leq \|A - B\|_p^p + \|A + B\|_p^p \leq 2^{p-1} (\|A\|_p^p + \|B\|_p^p), \quad (3)$$

for $2 \leq p < \infty$. For $p = 2$, by (2) and (3), we have

$$\|A - B\|_2^2 + \|A + B\|_2^2 = 2 (\|A\|_2^2 + \|B\|_2^2), \quad (4)$$

which is called the parallelogram law.

Bhatia et al. have obtained some generalizations of (2) to n -tuples of operators and many different conclusions by using various methods such as complex interpolation method and concavity and convexity of certain functions (see [3–8]).

Recently, some other versions of some p -Schatten inequalities have been given by Conde and Moslehian in [9] and Gao and Tian in [10].

Theorem 1 (see [9]). Let $A_1, \dots, A_n, B_1, \dots, B_n \in B_p(H)$ such that $\sum_{i,j=1}^n A_i^* B_j = 0$, then for $0 < p \leq 2$, $p \leq \lambda$, and $0 < \mu \leq 2$,

$$2^{(1/2)-(1/\mu)} n^{1-(1/\mu)} \left(\sum_{i=1}^n \|A_i\|_p^\mu + \sum_{i=1}^n \|B_i\|_p^\mu \right)^{1/\mu} \leq n^{1/2} \left(\sum_{i=1}^n \|A_i\|_p^2 + \sum_{i=1}^n \|B_i\|_p^2 \right)^{1/2} \leq n^{2((1/p)-(1/\lambda))} \left(\sum_{i,j=1}^n \|A_i \pm B_j\|_p^\lambda \right)^{1/\lambda}. \quad (5)$$

For $2 \leq p$, $0 < \lambda \leq p$, and $2 \leq \mu$, the inequalities are reversed.

Theorem 2 (see [9]). Let $A_1, \dots, A_n, B_1, \dots, B_n \in B_p(H)$ such that $\sum_{i,j=1}^n A_i^* B_j = 0$, then for $0 < p \leq 2$, $p \leq \lambda$, and $0 < \mu \leq 2$,

$$n \left(\frac{1}{n^2} \sum_{i,j=1}^n \|A_i \pm B_j\|_p^\mu \right)^{1/\mu} \leq n^{(1/2)+(1/p)-(1/\lambda)} \left(\sum_{i=1}^n \left\| (|A_i|^2 + |B_i|^2)^{1/2} \right\|_p^\lambda \right)^{1/\lambda}. \quad (6)$$

For $2 \leq p$, $0 < \lambda \leq p$, and $2 \leq \mu$, the inequality is reversed.

Theorem 3 (see [10]). Let $A_1, \dots, A_n, B_1, \dots, B_n \in B_p(H)$ such that $\sum_{i,j=1}^n A_i^* B_j = 0$. Then, for $0 < p \leq 2$, $p \geq \lambda > 0$, and $\mu \geq 2$,

$$n^{2((1/p)-(1/\lambda))} \left(\sum_{i,j=1}^n \|A_i \pm B_j\|_p^\lambda \right)^{1/\lambda} \leq n^{(2/p)-(1/2)} \left(\sum_{i=1}^n \|A_i\|_p^p + \sum_{i=1}^n \|B_i\|_p^p \right)^{1/p} \leq 2^{(1/p)-(2/p\mu)} n^{(3/p)-(2/p\mu)-(1/2)} \left(\sum_{i=1}^n \|A_i\|_p^{p\mu/2} + \sum_{i=1}^n \|B_i\|_p^{p\mu/2} \right)^{2/p\mu}. \quad (7)$$

For $p \geq 2$, $p \leq \lambda$, and $0 < \mu \leq 2$, the inequalities are reversed.

Theorem 4 (see [10]). Let $A_1, \dots, A_n, B_1, \dots, B_n \in B_p(H)$ such that $\sum_{i,j=1}^n A_i^* B_j = 0$. Then, for $0 < p \leq 2$, $p \geq \lambda > 0$, and $\mu \geq 2$,

$$n \left(\frac{1}{n^2} \sum_{i,j=1}^n \|A_i \pm B_j\|_p^\mu \right)^{1/\mu} \geq n^{2-(2/p)-(1/\lambda)} \cdot \left(\sum_{i=1}^n \left\| (|A_i|^2 + |B_i|^2)^{1/2} \right\|_p^\lambda \right)^{1/\lambda}. \quad (8)$$

For $p \geq 2$, $p \leq \lambda$, and $0 < \mu \leq 2$, the inequality is reversed.

In this paper, motivated by the above conclusions and techniques, we consider some other versions of p -Schatten norm inequalities when p , λ , and μ taking values in different regions.

2. Main Results

In this section, we consider the p -Schatten norm inequalities of (5) and (6) when parameters taking values in different regions. We start our work with the following lemmas that we will use along the paper.

Fact 1. $M_s(\bar{x}) \leq M_{s'}(\bar{x})$ for $0 < s \leq s'$, where $M_s(\bar{x}) = ((1/n) \sum_{i=1}^n x_i^s)^{1/s}$, $\bar{x} = (x_1, \dots, x_n)$ is an n -tuple of non-negative numbers.

Fact 2. $\|T\|_p^2 = \|T\|^2_{p/2}$ for any $T \in B_p(H)$ with $p > 0$.

Lemma 1 (see [9]). Let $A_1, \dots, A_n, B_1, \dots, B_n \in B(H)$ such that $\sum_{i,j=1}^n A_i^* B_j = 0$, then

$$\sum_{i,j=1}^n |A_i \pm B_j|^2 = \sum_{i,j=1}^n |A_i|^2 + |B_j|^2 \pm \sum_{i,j=1}^n A_i^* B_j + B_j^* A_i \quad (9)$$

$$= \sum_{i,j=1}^n |A_i|^2 + |B_j|^2.$$

Lemma 2 (see [2, 11]). If $A_1, \dots, A_n \in B_p(H)$ for some $p > 0$ and $A_1, \dots, A_n$ are positive, then for $0 < p \leq 1$,

$$n^{p-1} \sum_{i=1}^n \|A_i\|_p^p \leq \left(\sum_{i=1}^n \|A_i\|_p \right)^p \leq \left\| \sum_{i=1}^n A_i \right\|_p^p \leq \sum_{i=1}^n \|A_i\|_p^p, \quad (10)$$

and for $1 \leq p < \infty$, the inequalities are reversed.

They are a refinement of Lemma 1 in [6]. A commutative version of the previous lemma for scalars is given in the following:

Let $\bar{x} = (x_1, \dots, x_n)$ be an n -tuple of non-negative numbers, then

$$n^{p-1} \sum_{i=1}^n x_i^p \leq \left(\sum_{i=1}^n x_i \right)^p \leq \sum_{i=1}^n x_i^p, \quad (11)$$

for $0 < p \leq 1$ and

$$\sum_{i=1}^n x_i^p \leq \left(\sum_{i=1}^n x_i \right)^p \leq n^{p-1} \sum_{i=1}^n x_i^p, \quad (12)$$

for $1 \leq p < \infty$.

Theorem 5. Let $A_1, \dots, A_n, B_1, \dots, B_n \in B_p(H)$ such that $\sum_{i,j=1}^n A_i^* B_j = 0$. Then, for $p \geq 2$, $p \geq \lambda > 0$, and $0 < \mu \leq 2$,

$$n^{2((1/p)-(1/\lambda))} \left(\sum_{i,j=1}^n \|A_i \pm B_j\|_p^\lambda \right)^{1/\lambda} \leq n^{1/2} \left\| \sum_{i=1}^n |A_i|^2 + \sum_{i=1}^n |B_i|^2 \right\|_{p/2}^{1/2} \leq 2^{(1/2)-(\mu/4)} n^{1-(\mu/4)} \left(\sum_{i=1}^n \|A_i\|_p^{4/\mu} + \sum_{i=1}^n \|B_i\|_p^{4/\mu} \right)^{\mu/4}. \quad (13)$$

For $0 < p \leq 2$, $p \leq \lambda$, and $\mu \geq 2$, the inequalities are reversed.

Proof. Let $p \geq 2$, $p \geq \lambda > 0$, $0 < \mu \leq 2$. It follows from $M_\lambda(\bar{x}) \leq M_p(\bar{x})$ such that

$$n^{2((1/p)-(1/\lambda))} \left(\sum_{i,j=1}^n \|A_i \pm B_j\|_p^\lambda \right)^{1/\lambda} = n^{2/p} \left(\frac{1}{n^2} \sum_{i,j=1}^n \|A_i \pm B_j\|_p^\lambda \right)^{1/\lambda} \leq \left(\sum_{i,j=1}^n \|A_i \pm B_j\|_p^p \right)^{1/p}. \quad (14)$$

Applying the well-known fact that $\|T\|_p^2 = \| |T|^2 \|_{p/2}$ for any $T \in B_p(H)$ with $p > 0$ and Lemmas 1 and 2, we get

$$\begin{aligned} \left(\sum_{i,j=1}^n \|A_i \pm B_j\|_p^p \right)^{1/p} &= \left(\sum_{i,j=1}^n \| |A_i \pm B_j|^2 \|_{p/2}^{p/2} \right)^{1/p} \\ &\leq \left(\left\| \sum_{i,j=1}^n |A_i \pm B_j|^2 \right\|_{p/2}^{p/2} \right)^{1/p} \\ &= \left\| \sum_{i,j=1}^n |A_i|^2 + |B_j|^2 \right\|_{p/2}^{1/2} \\ &= n^{1/2} \left\| \sum_{i=1}^n |A_i|^2 + \sum_{i=1}^n |B_i|^2 \right\|_{p/2}^{1/2}. \end{aligned} \quad (15)$$

Using Lemma 2 and the convexity of the function $f(x) = x^\alpha$ on $[0, +\infty)$ for $1 \leq \alpha < \infty$, we obtain

$$\begin{aligned} n^{1/2} \left\| \sum_{i=1}^n |A_i|^2 + \sum_{i=1}^n |B_i|^2 \right\|_{p/2}^{1/2} &= n^{1/2} \left(\left\| \sum_{i=1}^n |A_i|^2 + \sum_{i=1}^n |B_i|^2 \right\|_{p/2}^{2/\mu} \right)^{\mu/4} \\ &\leq n^{1/2} \left[\left(\sum_{i=1}^n \| |A_i|^2 \|_{p/2} + \sum_{i=1}^n \| |B_i|^2 \|_{p/2} \right)^{2/\mu} \right]^{\mu/4} \\ &= n^{1/2} \left\{ (2n)^{2/\mu} \left[\frac{1}{2n} \left(\sum_{i=1}^n \| |A_i|^2 \|_{p/2} + \sum_{i=1}^n \| |B_i|^2 \|_{p/2} \right) \right]^{2/\mu} \right\}^{\mu/4} \\ &\leq 2^{1/2} n \left[\frac{1}{2n} \left(\sum_{i=1}^n \| |A_i|^2 \|_{p/2}^{2/\mu} + \sum_{i=1}^n \| |B_i|^2 \|_{p/2}^{2/\mu} \right) \right]^{\mu/4} \\ &= 2^{(1/2)-(\mu/4)} n^{1-(\mu/4)} \left(\sum_{i=1}^n \|A_i\|_p^{4/\mu} + \sum_{i=1}^n \|B_i\|_p^{4/\mu} \right)^{\mu/4}. \end{aligned} \quad (16)$$

For $0 < p \leq 2$, $p \leq \lambda$, and $\mu \geq 2$, we can prove the inequalities by the same reasoning. $\square$

Corollary 1. Let $A_1, \dots, A_n, B_1, \dots, B_n \in B_p(H)$ such that $\sum_{i,j=1}^n A_i^* B_j = 0$. Then,

$$\sum_{i,j=1}^n \|A_i \pm B_j\|_2^2 = n \left(\sum_{i=1}^n \|A_i\|_2^2 + \sum_{i=1}^n \|B_i\|_2^2 \right). \quad (17)$$

Proof. Motivated by Theorem 5, let $\lambda = \mu = p = 2$. $\square$

Corollary 2. Let $A_1, \dots, A_n \in B_p(H)$ such that $\sum_{i=1}^n A_i = 0$. Then,

$$\sum_{i,j=1}^n \|A_i \pm A_j\|_2^2 = 2n \left(\sum_{i=1}^n \|A_i\|_2^2 \right). \quad (18)$$

Proof. $\sum_{i=1}^n A_i = 0$ implies that $\sum_{i,j=1}^n A_i^* A_j = 0$. The statement is a consequence of Corollary 1. $\square$

Theorem 6. Let $A_1, \dots, A_n, B_1, \dots, B_n \in B_p(H)$ such that $\sum_{i,j=1}^n A_i^* B_j = 0$. Then, for $p \geq 2$, $p \geq \lambda > 0$, and $0 < \mu \leq 2$,

$$n \left(\frac{1}{n^2} \sum_{i,j=1}^n \|A_i \pm B_j\|_p^\mu \right)^{1/\mu} \leq n^{2+(1/\lambda)-(4/p)} \left(\sum_{i=1}^n \left\| (|A_i|^2 + |B_i|^2)^{1/2} \right\|_p^\lambda \right)^{1/\lambda}. \quad (19)$$

For $0 < p \leq 2$, $p \leq \lambda$, and $\mu \geq 2$, the inequality is reversed.

Proof. We suppose that $p \geq 2$, $p \geq \lambda > 0$, and $0 < \mu \leq 2$. Then, by Lemma 2 we have

$$\begin{aligned} n \left(\frac{1}{n^2} \sum_{i,j=1}^n \|A_i \pm B_j\|_p^\mu \right)^{1/\mu} &= n \left(\frac{1}{n^2} \sum_{i,j=1}^n \| |A_i \pm B_j| \|_{p/2}^{\mu/2} \right)^{1/\mu} \\ &\leq n \left[\left(\frac{1}{n^2} \sum_{i,j=1}^n \| |A_i \pm B_j| \|_{p/2}^{\mu/2} \right)^{\mu/2} \right]^{1/\mu} \\ &= \left(\sum_{i,j=1}^n \| |A_i \pm B_j| \|_{p/2}^{\mu/2} \right)^{1/2} \\ &\leq \left[(n^2)^{(p/2)-1} \sum_{i,j=1}^n \| |A_i \pm B_j| \|_{p/2}^{p/2} \right]^{1/p} \\ &= n^{1-(2/p)} \left(\sum_{i,j=1}^n \| |A_i \pm B_j| \|_{p/2}^{p/2} \right)^{1/p} \\ &\leq n^{1-(2/p)} \left(\left\| \sum_{i,j=1}^n |A_i \pm B_j|^2 \right\|_{p/2}^{p/2} \right)^{1/p} \\ &= n^{(3/2)-(2/p)} \left(\left\| \sum_{i=1}^n (|A_i|^2 + |B_i|^2) \right\|_{p/2}^{p/2} \right)^{1/p} \\ &\leq n^{(3/2)-(2/p)} \left(n^{(p/2)-1} \sum_{i=1}^n \| |A_i|^2 + |B_i|^2 \|_{p/2}^{p/2} \right)^{1/p} \\ &= n^{2-(3/p)} \left(\sum_{i=1}^n \left\| (|A_i|^2 + |B_i|^2)^{1/2} \right\|_p^p \right)^{1/p} \\ &\leq n^{2-(3/p)} \left[n^{(p/\lambda)-1} \left(\sum_{i=1}^n \left\| (|A_i|^2 + |B_i|^2)^{1/2} \right\|_p^\lambda \right)^{p/\lambda} \right]^{1/p} \\ &= n^{2+(1/\lambda)-(4/p)} \left(\sum_{i=1}^n \left\| (|A_i|^2 + |B_i|^2)^{1/2} \right\|_p^\lambda \right)^{1/\lambda}. \end{aligned} \quad (20)$$

When $0 < p \leq 2$, $p \leq \lambda$, and $\mu \geq 2$, we can prove the inequality by the same reasoning. $\square$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This research was supported by the National Natural Science Foundation of China (11701154) and the Natural Science Foundation of the Department of Education, Henan Province (19A110020 and 20A110020), the graduate education reform and quality improvement project, Henan Province.

References

- [1] I. Gohberg and M. Kreĭn, *Introduction to the Theory of Linear Nonselfadjoint Operators*, Vol. 18, American Mathematical Society, Providence, RI, USA, 1969.
- [2] C. A. McCarthy, " C_p ," *Israel Journal of Mathematics*, vol. 5, no. 4, pp. 249–271, 1967.
- [3] D. J. Kečkić, "Continuous generalization of Clarkson-McCarthy inequalities," *Banach Journal of Mathematical Analysis*, vol. 13, no. 1, pp. 26–46, 2019.
- [4] R. Bhatia and F. Kittaneh, "Clarkson inequalities with several operators," *Bulletin of the London Mathematical Society*, vol. 36, no. 6, pp. 820–832, 2004.
- [5] O. Hirzallah and F. Kittaneh, "Non-commutative Clarkson inequalities for n -tuples of operator," *Integral Equations and Operator Theory*, vol. 60, no. 3, pp. 369–379, 2008.
- [6] O. Hirzallah, F. Kittaneh, and M. S. Moslehian, "Schatten p -norm inequalities related to a characterization of inner product spaces," *Mathematical Inequalities & Applications*, vol. 13, no. 2, pp. 235–241, 2010.
- [7] S. Milošević, "Norm inequalities for elementary operators related to contractions and operators with spectra contained in the unit disk in norm ideals," *Advances in Operator Theory*, vol. 1, no. 2, pp. 147–159, 2016.
- [8] M. S. Moslehian, M. Tominaga, and K.-S. Saito, "Schatten p -norm inequalities related to an extended operator parallelogram law," *Linear Algebra and Its Applications*, vol. 435, no. 4, pp. 823–829, 2011.
- [9] C. Conde and M. S. Moslehian, "Norm inequalities related to p -Schatten class," *Linear Algebra and Its Applications*, vol. 498, pp. 441–449, 2016.
- [10] F. Gao and M. y. Tian, "Some inequalities related to p -Schatten norm," *Journal of Mathematical Inequalities*, vol. 12, no. 2, pp. 339–345, 2018.
- [11] R. Bhatia and F. Kittaneh, "Cartesian decompositions and Schatten norms," *Linear Algebra and Its Applications*, vol. 38, no. 1–3, pp. 109–116, 2000.