Research Article

V-Prox-Regular Functions in Smooth Banach Spaces

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In this paper, we continue the study of the V -prox-regularity that we have started recently for sets. We define an appropriate concept of the V -prox-regularity for functions in reflexive smooth Banach spaces by adapting the one given in Hilbert spaces. Our main goal is to study the relationship between the V -prox-regularity of a given l.s.c. f and the V -prox-regularity of its epigraph.

1. Introduction and Preliminaries

Throughout this work, X will denote a reflexive smooth Banach space unless otherwise specified. We recall from [1] the concept of V -proximal subdifferential.

Definition 1. Let $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous (l.s.c.) function and $x \in X$, where f is finite. We recall that the V -proximal subdifferential of f at x is defined as $x^* \in \partial^\pi f(x)$ if and only if there exists $\sigma > 0$ such that

$$\langle x^*, x' - x \rangle \leq f(x') - f(x) + \sigma V(J(x), x'), \forall x' \text{ near } x. \quad (1)$$

Here, J is the normalised duality mapping on X , and V is the functional defined from $X^* \times X$ to $[0, \infty)$ by

$$V(x^*, x) = \|x^*\|^2 - 2\langle x^*, x \rangle + \|x\|^2, \text{ for any } (x^*, x) \in X^* \times X. \quad (2)$$

The V -proximal normal cone of a nonempty closed subset S in X at $x \in S$ is defined as the V -proximal subdifferential of the indicator function of S , that is, $N^\pi(S; x) = \partial^\pi \psi_S(x)$.

Another proximal subdifferential $\partial_G^\pi f(x)$ is defined (see [5]) geometrically via the V -proximal normal cone of the epi-

graph as follows:

$$\partial_G^\pi f(x) = \{x^* \in X^* : (x^*, -1) \in N^\pi(\text{epi } f; (x, f(x)))\}. \quad (3)$$

It has been proved in [2] that in general, we have the inclusion $\partial^\pi f(x) \subset \partial_G^\pi f(x)$. The generalized projection on a closed nonempty set S is defined as follows: $\bar{x} \in \pi_S(x^*)$ if and only if $V(x^*, \bar{x}) = \inf_{x \in S} V(x^*, x)$ (see [3] for convex sets and see [1] for nonconvex sets). We also recall (see for instance [2]) the definition of the Fréchet subdifferential and Fréchet normal cone as follows: $x^* \in \partial^F f(\bar{x})$ if and only if for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\langle x^*, x - \bar{x} \rangle \leq f(x) - f(\bar{x}) + \varepsilon \|x - \bar{x}\|, \forall x \in \bar{x} + \delta \mathbb{B}. \quad (4)$$

The Fréchet normal cone $N^F(S; x)$ of a nonempty closed subset S in X at $\bar{x} \in S$ is defined as $N^F(S; \bar{x}) = \partial^F \psi_S(\bar{x})$.

2. Generalized V-Prox-Regular Sets

In this section, we recall the concept of the generalized V -prox-regularity of sets introduced and studied in [4] and other results needed in our work.

Definition 2. Let S be a nonempty closed set in a reflexive Banach space X and let $\bar{x} \in S$. We will say that S is generalized V -prox-regular at $\bar{x}$ if and only if there exist $r > 0$ and $\varepsilon > 0$ such that for all $x \in S \cap (\bar{x} + \varepsilon \mathbb{B})$ and for any $x^* \in N^\pi(S; x)$

$\cap \varepsilon \mathbb{B}_*$, the point x is a generalized projection of $Jx + rx^*$ on $S \cap (\bar{x} + \varepsilon \mathbb{B})$, that is, $x \in \pi_{S \cap (\bar{x} + \varepsilon \mathbb{B})}(Jx + rx^*)$.

We quote from [5, 6] the following two results.

Lemma 3. *For any $x \in \text{dom } f$ and any $(x, \alpha) \in \text{epi } f$ (i.e., $f(x) \leq \alpha$), we have*

$$(x^*, 0) \in N^\pi(\text{epi } f; (x, \alpha)) \Rightarrow (x^*, 0) \in N^\pi(\text{epi } f; (x, f(x))). \quad (5)$$

Proposition 4. *Let X be an Asplund space and let $f : X \rightarrow \mathbb{R} \cup \{\infty\}$ be a proper l.s.c. function around $\bar{x} \in \text{dom } f$. Then, for any $x^* \in X^*$ with $(x^*, 0) \in N^F(\text{epi } f; (\bar{x}, f(\bar{x})))$, there exist sequences $x_k \rightarrow \bar{x}$ with $f(x_k) \rightarrow f(\bar{x})$, $\lambda_k \rightarrow 0^+$, and $x_k^* \in \lambda_k \partial^F f(x_k)$ such that $\|x_k^* - x^*\|_* \rightarrow 0$.*

We recall from [2] that X is *V-proximal trustworthy* provided that for any $\varepsilon > 0$, any two functions $f_1, f_2 : X \rightarrow \mathbb{R} \cup \{\infty\}$ and any $u \in X$ such that f_1 is lower semicontinuous and f_2 is Lipschitz around u the following fuzzy sum rule holds:

$$\partial_A^\pi(f_1 + f_2)(u) \subset \bigcup \left\{ \partial_A^\pi f_1(u_1) + \partial_A^\pi f_2(u_2) : u_i \in U_{f_i}(u, \varepsilon), i = 1, 2 \right\} + \varepsilon \mathbb{B}_*. \quad (6)$$

Here, $U_{f_i}(u, \varepsilon) := \{x \in u + \varepsilon \mathbb{B} \text{ such that } |f_i(x) - f_i(u)| < \varepsilon\}$, and $\mathbb{B}_*$ denotes the closed unit ball in X^* . The proof of the following proposition can be found in [2].

Proposition 5. *Let X be a V-proximal trustworthy space. Let S be a closed subset of X with $x \in S$ and let $x^* \in N^F(S; x)$. Then, for any $\varepsilon > 0$, there exists $x_\varepsilon \in (\bar{x} + \varepsilon \mathbb{B}) \cap S$ such that $x^* \in N^\pi(S; x_\varepsilon) + \varepsilon \mathbb{B}_*$.*

3. V-Prox-Regular Functions

In Hilbert spaces setting, the authors in [7] extended and studied the concept of prox-regularity for functions which has been introduced for finite dimensional spaces in [8]. Many papers studied this concept in finite dimensional spaces and its applications to nonsmooth optimization (see for instance [9] and the references therein). Our aim here is to extend the concept of prox-regularity to reflexive smooth Banach spaces by using the concept of the V-proximal normal cone and the functional V . Another way to extend this concept has been proposed in [10].

Definition 6. Assume that X is a reflexive smooth Banach space. A l.s.c. function $f : X \rightarrow \mathbb{R} \cup \{\infty\}$ is said to be a V-prox-regular at $x_0 \in \text{dom } f$ for $x_0^* \in \partial^\pi f(x_0)$ if and only if there exist $\varepsilon > 0$, $r > 0$ such that $\forall (x, x^*) \in \partial_G^\pi f$ with $\|x - x_0\| \leq \varepsilon$, $|f(x) - f(x_0)| < \varepsilon_0$ and $\|x^* - x_0^*\| < \varepsilon_0$ such that

$$f(x') \geq f(x) + \langle x^*, x' - x \rangle - \frac{r}{2} V(Jx, x'), \forall x' \in x_0 + \varepsilon_0 \mathbb{B}. \quad (7)$$

We say that f is V-prox-regular at x_0 if it is V-prox-regular for any $x_0^* \in \partial^\pi f(x_0)$. Obviously, this concept coincides with the one studied in [7] in Hilbert spaces. Using this definition for functions, we can associate to it another definition of the V-prox-regularity for sets via the indicator function. We have the following definition:

Definition 7. Let X be a reflexive smooth Banach space. A closed nonempty set S is said to be V-prox-regular at $x_0 \in S$ for $x_0^* \in N^\pi(S; x_0)$ if and only if the indicator function ψ_S is V-prox-regular at x_0 for $x_0^* \in \partial^\pi \psi_S(x_0)$.

Proposition 8. *Let X be a reflexive smooth Banach space. A closed set S is generalized V-prox-regular at $x_0 \in S$ in the sense of Definition 2, if and only if the indicator function ψ_S is V-prox-regular at x_0 for $x_0^* = 0$ in the sense of Definition 6.*

Proof. Assume that S is generalized V-prox-regular at $x_0 \in S$, then there exist $r > 0$ and $\varepsilon > 0$ such that for all $x \in S \cap (x_0 + \varepsilon \mathbb{B})$ and for any $x^* \in N^\pi(S; x) \cap \varepsilon \mathbb{B}_*$, the point x is a generalized projection of $Jx + rx^*$ on $S \cap (x_0 + \varepsilon \mathbb{B})$, that is, $x \in \pi_{S \cap (x_0 + \varepsilon \mathbb{B})}(Jx + rx^*)$. By definition of generalized projection, we have

$$V(Jx + rx^*; x) \leq V(Jx + rx^*; x'). \forall x' \in S \cap (x_0 + \varepsilon \mathbb{B}). \quad (8)$$

So, for any $x' \in S \cap (x_0 + \varepsilon \mathbb{B})$, we have

$$\|x\|^2 - 2 \langle Jx + rx^*; x \rangle \leq \|x'\|^2 - 2 \langle Jx + rx^*; x' \rangle. \quad (9)$$

Thus,

$$\begin{aligned} 2 \langle rx^*; x' - x \rangle &= 2 \langle Jx + rx^*; x' - x \rangle - 2 \langle Jx; x' - x \rangle \\ &= 2 \langle Jx + rx^*; x' - x \rangle - 2 \langle Jx; x' \rangle + 2 \|x\|^2 \\ &\leq \|x'\|^2 - 2 \langle Jx; x' \rangle + \|x\|^2 = V(Jx, x'), \end{aligned} \quad (10)$$

for any $x' \in S \cap (x_0 + \varepsilon \mathbb{B})$. Hence,

$$\langle x^*; x' - x \rangle \leq \frac{1}{2r} V(Jx, x'), \text{ for all } x' \in S \cap (x_0 + \varepsilon \mathbb{B}) \quad (11)$$

and so

$$\langle x^*; x' - x \rangle \leq \psi_S(x') - \psi_S(x) + \frac{1}{2r} V(Jx, x'), \text{ for all } x' \in x_0 + \varepsilon \mathbb{B}. \quad (12)$$

On the other hand, we have $x^* \in N^\pi(S; x) \cap \varepsilon \mathbb{B}_*$ which implies that $x^* \in \partial_G^\pi \psi_S(x)$ with $\|x^* - x_0^*\| < \varepsilon$ and $x_0^* = 0$. Thus, the inequality (12) ensures that ψ_S is V-prox-regular at x_0 for $x_0^* = 0$. For the opposite direction, we follow the same lines as in the direct one.

In the next theorem, we study the relationship between the V -prox-regularity of a function f and the V -prox-regularity of its epigraph.

Theorem 9. *Let $f : X \rightarrow \mathbb{R} \cup \{\infty\}$ be a function that is l.s.c. at $x_0 \in \text{dom } f$. If f is V -prox-regular at x_0 for $x_0^* \in \partial^\pi f(x_0)$, then the epigraph $\text{epi } f$ is V -prox-regular at $(x_0, f(x_0))$ for $(x_0^*, -1) \in N^\pi(\text{epi } f; (x_0, f(x_0)))$. If, in addition, the space X is 2-uniformly convex, then the previous implication becomes an equivalence, that is, f is V -prox-regular at x_0 for $x_0^* \in \partial^\pi f(x_0)$ if and only if the epigraph $\text{epi } f$ is V -prox-regular at $(x_0, f(x_0))$ for $(x_0^*, -1) \in N^\pi(\text{epi } f; (x_0, f(x_0)))$.*

Proof. Assume that f is V -prox-regular at x_0 for $x_0^* \in \partial^\pi f(x_0)$, fix $\varepsilon_0 \in (0, 1)$ and $r > 0$ such that $\forall(x, x^*) \in \partial_G^\pi f$ with $\|x - x_0\| \leq \varepsilon$, $|f(x) - f(x_0)| < \varepsilon_0$ and $\|x^* - x_0^*\| < \varepsilon_0$ such that

$$f(x') \geq f(x) + \langle x^*, x' - x \rangle - \frac{r}{2} V(Jx, x'), \forall x' \in x_0 + \varepsilon_0 \mathbb{B}. \quad (13)$$

By l.s.c. of f , we choose $\varepsilon \in (0, 1)$ with $(\varepsilon/1 - \varepsilon)(1 + \|x_0^*\|) < \varepsilon_0$ and such that for any $x' \in x_0 + \varepsilon \mathbb{B}$, we have $f(x_0) - \varepsilon < f(x')$. Fix any $(x, \alpha) \in \text{epi } f$, with $\|(x, \alpha) - (x_0, f(x_0))\| < \varepsilon$ and $(x^*, -\lambda) \in N^\pi(\text{epi } f; (x, \alpha))$ with $\|(x^*, -\lambda) - (x_0^*, -1)\| < \varepsilon$. Clearly, $0 < 1 - \varepsilon < \lambda < 1 + \varepsilon$, which ensures that $\alpha = f(x)$ and so $x^*/\lambda \in \partial_G^\pi f(x)$. Take now any $(x', \alpha') \in \text{epi } f$ with $\|(x', \alpha') - (x_0, f(x_0))\| < \varepsilon$. Notice that

$$\begin{aligned} f(x_0) - \varepsilon &< f(x') \leq \alpha' < f(x_0) + \varepsilon_0, \\ \left\| \frac{x^*}{\lambda} - x_0^* \right\| &\leq \frac{1}{\lambda} \|x^* - \lambda x_0^*\| = \frac{1}{\lambda} \|x^* - x_0^* - \lambda x_0^* + x_0^*\| \leq \frac{1}{\lambda} (\|x^* - x_0^*\| + \lambda \|x_0^*\|) \\ &\leq \frac{1}{\lambda} [\varepsilon + (1 - \lambda) \|x_0^*\|] \leq \frac{\varepsilon}{\lambda} \\ &+ \frac{\varepsilon}{1 - \varepsilon} \|x_0^*\| \leq \frac{\varepsilon}{1 - \varepsilon} (1 + \|x_0^*\|) < \varepsilon_0, \end{aligned} \quad (14)$$

which yields by (13)

$$\begin{aligned} \left\langle \frac{x^*}{\lambda}, x' - x \right\rangle - (\alpha' - f(x)) &\leq \left\langle \frac{x^*}{\lambda}, x' - x \right\rangle \\ &- (f(x') - f(x)) \leq \frac{r}{2} V(Jx, x'), \end{aligned} \quad (15)$$

and hence,

$$\langle x^*, x' - x \rangle - \lambda(\alpha' - \alpha) \leq \frac{r\lambda}{2} V(Jx, x'). \quad (16)$$

On the other hand, we have

$$V_{X \times \mathbb{B}}((Jx, \alpha); (x' - \alpha')) = V(Jx, x') + (\alpha' - \alpha)^2. \quad (17)$$

So,

$$\langle (x^*, -\lambda); (x', \alpha') - (x, \alpha) \rangle \leq \frac{r\lambda}{2} V_{X \times \mathbb{B}}((Jx, \alpha); (x' - \alpha')). \quad (18)$$

Thus, for any $(x, \alpha) \in \text{epi } f$, with $\|(x, \alpha) - (x_0, f(x_0))\| < \varepsilon$ and $(x^*, -\lambda) \in N^\pi(\text{epi } f; (x, \alpha))$ with $\|(x^*, -\lambda) - (x_0^*, -1)\| < \varepsilon$, we have

$$\langle (x^*, -\lambda); (x', \alpha') - (x, \alpha) \rangle \leq \frac{r(1 + \varepsilon_0)}{2} V_{X \times \mathbb{B}}((Jx, \alpha); (x' - \alpha')), \quad (19)$$

for any $(x', \alpha') \in \text{epi } f$ with $\|(x', \alpha') - (x_0, f(x_0))\| < \varepsilon$. This means that the epigraph $\text{epi } f$ is V -prox-regular at $(x_0, f(x_0))$ for $(x_0^*, -1) \in N^\pi(\text{epi } f; (x_0, f(x_0)))$.

Assume now that the space X is a 2-uniformly convex space. Assume that the epigraph $\text{epi } f$ is V -prox-regular at $(x_0, f(x_0))$ for $(x_0^*, -1) \in N^\pi(\text{epi } f; (x_0, f(x_0)))$, choose $\varepsilon_0 \in (0, 1)$ and $r > 0$ such that for any $(x, \alpha) \in \text{epi } f$ and any $(x^*, -\lambda) \in N^\pi(\text{epi } f; (x, \alpha))$ with $\|(x, \alpha) - (x_0, f(x_0))\| < \varepsilon_0$ and with $\|(x^*, -\lambda) - (x_0^*, -1)\| < \varepsilon_0$, we have

$$\langle (x^*, -\lambda); (x', \alpha') - (x, \alpha) \rangle \leq \frac{r}{2} \left[V(Jx, x') + (\alpha' - \alpha)^2 \right], \quad (20)$$

for all $(x', \alpha') \in \text{epi } f$ with $\|(x', \alpha') - (x_0, f(x_0))\| < \varepsilon_0$. Using the fact that space X is a 2-uniformly convex space, we can find a positive constant $\bar{\beta} > 0$ depending only on the space X and the constant $M := (\|x_0\| + 4\varepsilon_0)$ such that

$$V(Jx, x') \geq \bar{\beta} \|x - x'\|^2, \forall x', x \in M\mathbb{B}. \quad (21)$$

Choose now a positive number $\varepsilon \in (0, \min \{\varepsilon_0, (1/3r(1 + \|x_0^*\|), 1/(9\beta r)^{1/p})\})$ such that

$$[2\varepsilon(1 + \|x_0^*\|) + \beta\varepsilon^p] \left[1 + \bar{\beta}^{-1} \left(\frac{\varepsilon}{2} + \|x_0^*\| \right)^2 \right] < \min \left\{ \frac{1}{r}, \frac{1}{r^2} \right\} \quad (22)$$

and such that (according to the l.s.c. of f)

$$f(x) > f(x_0) - \frac{1}{2r}, \forall x \in x_0 + \varepsilon \mathbb{B}. \quad (23)$$

Here, β depends only on the space X and M . Then, for any $x \in x_0 + \varepsilon \mathbb{B}$, we have $-\varepsilon < f(x) - f(x_0) < \varepsilon$ and so

$$f(x) - f(x_0) + \frac{1}{r} > f(x_0) - \frac{1}{2r} - f(x_0) + \frac{1}{r} - \varepsilon = \frac{1}{2r} - \varepsilon > 0. \quad (24)$$

Also, we have by (23) one that has

$$r \left[1 + \bar{\beta}^{-1} \left(\frac{\varepsilon}{2} + \|x_0^*\| \right)^2 \right] < \frac{1}{r[2\varepsilon(1+\|x_0^*\|) + \beta\varepsilon^p]}, \quad (25)$$

and hence since $1/r[2\varepsilon(1+\|x_0^*\|) + \beta\varepsilon^p] > 1$ (by (23) again), we may fix some positive number σ such that

$$\max \left\{ 1, r \left[1 + \bar{\beta}^{-1} \left(\frac{\varepsilon}{2} + \|x_0^*\| \right)^2 \right] \right\} < \sigma < \frac{1}{r[2\varepsilon(1+\|x_0^*\|) + \beta\varepsilon^p]}. \quad (26)$$

Take now any $x, x' \in x_0 + (\varepsilon/2)\mathbb{B}$ with $|f(x) - f(x_0)| < \varepsilon/2$ and $|f(x') - f(x_0)| < \varepsilon/2$ and $x^* \in \partial^\pi f(x)$ with $x^* \in x_0^* + (\varepsilon/2)\mathbb{B}_*$. Then, applying (20) with $(x^*, -1)$ instead of $(x^*, -\lambda)$, $f(x)$ instead of α , and $f(x')$ instead of α' , we obtain

$$\begin{aligned} \langle x^*; x' - x \rangle - (f(x') - f(x)) &\leq \frac{r}{2} V(Jx; x') \\ &\quad + \frac{r}{2} (f(x') - f(x))^2, \end{aligned} \quad (27)$$

which is equivalent to (with $\rho := 1/r$)

$$\begin{aligned} 2\rho \langle x^*; x' - x \rangle - V(Jx; x') &\leq (f(x') - f(x))^2 \\ &\quad + 2\rho (f(x') - f(x)) = [f(x') - f(x) + \rho]^2 - \rho^2 \end{aligned} \quad (28)$$

and so

$$\rho^2 + 2\rho \langle x^*; x' - x \rangle - V(Jx; x') \leq [f(x') - f(x) + \rho]^2. \quad (29)$$

Note that

$$\begin{aligned} \rho^2 + 2\rho \langle x^*; x' - x \rangle - V(Jx; x') \\ &\geq \rho^2 - 2\rho \|x^*\| \|x' - x\| - \beta\varepsilon^p \geq \rho^2 - 2\rho\varepsilon(1+\|x_0^*\|) - \beta\varepsilon^p \\ &\geq \rho^2 - 2\rho \frac{\rho}{3} - \frac{\rho^2}{9} = \frac{2\rho^2}{9} > 0. \end{aligned} \quad (30)$$

Thus,

$$f(x') - f(x) + \rho \geq [\rho^2 + 2\rho \langle x^*; x' - x \rangle - V(Jx; x')]^{1/2}. \quad (31)$$

Now, we develop the expression

$$\begin{aligned} E &:= [\rho^2 + 2\rho \langle x^*; x' - x \rangle - V(Jx; x')] \\ &\quad - [\rho + \langle x^*; x' - x \rangle - \sigma V(Jx; x')]^2, \\ E &= -V(Jx; x') + 2\rho\sigma V(Jx; x') \\ &\quad - [\langle x^*; x' - x \rangle - \sigma V(Jx; x')]^2 = (2\rho\sigma - 1)V(Jx; x') \\ &\quad - [\langle x^*; x' - x \rangle]^2 + 2\sigma V(Jx; x') \langle x^*; x' - x \rangle \\ &\quad - \sigma^2 [V(Jx; x')]^2. \end{aligned} \quad (32)$$

Observe that

$$\begin{aligned} \langle x^*; x' - x \rangle &\leq \|x^*\| \|x' - x\| \leq \left(\frac{\varepsilon}{2} + \|x_0^*\| \right) \varepsilon, \\ V(Jx; x') &\leq \beta\varepsilon^p, \\ [\langle x^*; x' - x \rangle]^2 &\leq \|x^*\|^2 \|x' - x\|^2 \leq \left(\frac{\varepsilon}{2} + \|x_0^*\| \right)^2 \bar{\beta}^{-1} V(Jx; x'). \end{aligned} \quad (33)$$

Thus,

$$\begin{aligned} E &\geq (2\rho\sigma - 1)V(Jx; x') - \left[\frac{\varepsilon}{2} + \|x_0^*\| \right]^2 \bar{\beta}^{-1} V(Jx; x') \\ &\quad + 2\sigma V(Jx; x') \langle x^*; x' - x \rangle - \sigma^2 [V(Jx; x')]^2 = V(Jx; x') \\ &\quad \cdot \left[2\rho\sigma - 1 - \left(\frac{\varepsilon}{2} + \|x_0^*\| \right)^2 \bar{\beta}^{-1} - 2 \left(\frac{\varepsilon}{2} + \|x_0^*\| \right) \varepsilon \sigma - \sigma^2 \beta\varepsilon^p \right]. \end{aligned} \quad (34)$$

But as $\sigma > 1$, one has

$$\begin{aligned} 2\rho\sigma - 1 - 2\sigma\varepsilon(1+\|x_0^*\|) - \left(\frac{\varepsilon}{2} + \|x_0^*\| \right)^2 \bar{\beta}^{-1} - \sigma^2 \beta\varepsilon^p \\ &\geq \left[-1 - \left(\frac{\varepsilon}{2} + \|x_0^*\| \right)^2 \bar{\beta}^{-1} + \rho\sigma \right] \\ &\quad + \sigma[\rho - 2\sigma\varepsilon(1+\|x_0^*\|) - \sigma\beta\varepsilon^p]. \end{aligned} \quad (35)$$

Using the choice of σ in the inequality (26), we have

$$\begin{aligned} \sigma &> r \left(1 + \bar{\beta}^{-1} \left(\frac{\varepsilon}{2} + \|x_0^*\| \right)^2 \right), \\ \sigma &< \frac{1}{r[2\varepsilon(1+\|x_0^*\|) + \beta\varepsilon^p]}. \end{aligned} \quad (36)$$

which ensure, respectively,

$$\begin{aligned} -1 - \left(\frac{\varepsilon}{2} + \|x_0^*\|\right)^2 \bar{\beta}^{-1} + \frac{\sigma}{r} &> 0, \\ \frac{1}{r} - 2\sigma\varepsilon[1 + \|x_0^*\|] - \sigma\beta\varepsilon^p &> 0. \end{aligned} \quad (37)$$

Since $\rho = \sigma/r$, we obtain

$$\left[-1 - \left(\frac{\varepsilon}{2} + \|x_0^*\|\right)^2 \bar{\beta}^{-1} + \rho\sigma\right] + \sigma[\rho - 2\sigma\varepsilon[1 + \|x_0^*\|] - \sigma\beta\varepsilon^p] > 0. \quad (38)$$

Therefore, it follows from (35) that $E > 0$ and so

$$\begin{aligned} &[\rho^2 + 2\rho\langle x^*; x' - x \rangle - V(Jx; x')] \\ &> [\rho + \langle x^*, x' - x \rangle - \sigma V(Jx, x')]^2 > 0, \end{aligned} \quad (39)$$

which ensures that

$$\begin{aligned} &[\rho^2 + 2\rho\langle x^*; x' - x \rangle - V(Jx; x')]^{1/2} \\ &> \rho + \langle x^*, x' - x \rangle - \sigma V(Jx, x') > 0. \end{aligned} \quad (40)$$

Thus, by (31) and (35), we obtain

$$f(x') - f(x) + \rho \geq \rho + \langle x^*, x' - x \rangle - \sigma V(Jx, x'). \quad (41)$$

Now, it remains to show the conclusion for x' satisfying $|f(x') - f(x_0)| \geq \varepsilon$. First choose, by the l.s.c. of f , some positive number

$$\eta < \frac{\varepsilon}{4(1 + \|x_0^*\|)}, \quad (42)$$

such that

$$f(x) - f(x_0) > -\varepsilon, \quad \forall x \in x_0 + \eta\mathbb{B}. \quad (43)$$

Now, fix any $x, x' \in x_0 + \eta\mathbb{B}$ with only $f(x) - f(x_0) < \eta$, and fix any $x^* \in x_0^* + \eta\mathbb{B}_*$. Thus, if $|f(x') - f(x_0)| \geq \varepsilon$, then

$$f(x') - f(x_0) \geq \varepsilon \text{ or } f(x) - f(x_0) \leq -\varepsilon. \quad (44)$$

But since $f(x') - f(x_0) > -\varepsilon$ (by (43)), we get only

$$f(x') \geq \varepsilon + f(x_0). \quad (45)$$

Therefore,

$$\begin{aligned} f(x') &= f(x) + [f(x_0) - f(x)] + [f(x') - f(x_0)] \\ &\geq f(x) - \eta + \varepsilon \geq f(x) + \frac{\varepsilon}{2}. \end{aligned} \quad (46)$$

Observe that

$$\langle x^*, x' - x \rangle - \sigma V(Jx, x') \leq \|x^*\| \|x' - x\| \leq (1 + \|x_0^*\|)\eta < \frac{\varepsilon}{2}, \quad (47)$$

and hence, we get

$$f(x') - f(x) + \rho \geq \rho + \langle x^*, x' - x \rangle - \sigma V(Jx, x'). \quad (48)$$

Finally, the relation (41) together with the last inequality conclude that f is V -prox-regular at $\bar{x}$ for x_0^* .

Remark 10. In the previous theorem, we proved the following: if f is V -prox-regular at x_0 for $x_0^* \in \partial^\pi f(x_0)$, then we have the V -prox-regularity of $\text{epi } f$ at $(x_0, f(x_0))$ for $(x_0^*, -1) \in N^\pi(\text{epi } f; (x_0, f(x_0)))$. Unfortunately, we do not obtain the generalized V -prox-regularity of the epigraph in the sense of Definition 2 since $\text{epi } f$ is a generalized V -prox-regularity at $(x_0, f(x_0))$ if and only if it is V -prox-regularity at $(x_0, f(x_0))$ for $(0, 0) \in N^\pi(\text{epi } f; (x_0, f(x_0)))$ which cannot be the case (since $(x_0^*, -1) \neq (0, 0)$) under the V -prox-regularity of f at x_0 . To ensure the generalized V -prox-regularity of the epigraph, we need another kind of regularity called the V -primal lower nice function introduced and studied in [5].

Definition 11. Let $f : X \longrightarrow \mathbb{R} \cup \{\infty\}$ be a l.s.c. function. We will say that f is V -primal lower nice at $\bar{x} \in \text{dom } f$, if there exist $\lambda_1, \lambda_2 > 0, c > 0, T > 0$ such that

$$f(y) \geq f(x) + \langle x^*, y - x \rangle - \frac{t}{2} V(Jx, y), \quad (49)$$

whenever $t \geq T, x \in \mathcal{N}_V(\bar{x}, \lambda_1, \lambda_2), y \in \bar{x} + \lambda_1\mathbb{B}, x^* \in \partial_G^\pi f(x)$, and $\|x^*\| \leq ct$. Here,

$$\mathcal{N}_V(\bar{x}, \lambda_1, \lambda_2) := \{x \in \bar{x} + \lambda_1\mathbb{B} \text{ with } V(J\bar{x}, x) \leq \lambda_2\}. \quad (50)$$

This concept is stronger than the V -prox-regularity. Indeed, as mentioned in [8] for finite dimensional spaces, obviously, any function that is a V -p.l.n. function is V -prox-regular. However, the inverse is not true. For a V -p.l.n. function, the condition (49) must hold for all V -proximal gradients with the *linear growth* condition, whereas for V -prox-regular functions, the condition (7) only has to hold for V -proximal subgradients close to a fixed $x_0^* \in \partial^\pi f(x_0)$ and just in a neighborhood of x_0 with $f(x)$ close to $f(x_0)$. For instance (see [8]), the function f on $\mathbb{R}$ with $f(x) = 1$ for $x > 0$ and $f(x) = 0$ for $x \leq 0$ is easily seen to be V -prox-regular at $x_0 = 0$ for $x^* = 0$ but not V -p.l.n. for this x_0 . We have to notice that this definition of V -p.l.n. functions extends, to reflexive smooth Banach spaces, the definition of primal lower nice functions defined in finite dimension spaces in [11], and in Hilbert spaces in [12]. Our next proposition shows the generalized V -prox-regularity of the epigraph in the sense of Definition 2 of any V -primal lower nice functions.

Proposition 12. *Let X be a V -proximal trustworthy space. If f is V -primal lower nice at $\bar{x} \in \text{dom } f$, then $\text{epi } f$ is generalized V -prox-regular at $(\bar{x}, f(\bar{x}))$.*

Proof. By definition of V -p.l.n. at $\bar{x}$, we have positive numbers $\varepsilon_1 > 0, \varepsilon_2 > 0, r > 0, T > 0$ such that for any $t \geq T$, any $x \in \mathcal{N}(\bar{x}, \varepsilon_1, \varepsilon_2)$, and any $x^* \in \partial_{\mathbb{G}}^\pi f(x) \cap \text{rt}\mathbb{B}_*$, we have

$$\langle x^*, y - x \rangle \leq f(y) - f(x) + \frac{t}{2} V(Jx, y), \forall y \in \bar{x} + \varepsilon_1 \mathbb{B}. \quad (51)$$

Take $(x, \alpha) \in \text{epi } f$ and $(x^*, -\lambda) \in N^\pi(\text{epi } f; (x, \alpha)) \cap \varepsilon_0 \mathbb{B}_{X \times \mathbb{R}}$ and $x \in \bar{x} + \varepsilon_0 \mathbb{B}$ with $|f(\bar{x}) - \alpha| < \varepsilon_0$, where ε_0 is taken in $(0, \varepsilon_1)$ such that

$$V(J\bar{x}, z) < \varepsilon_2, \quad \forall z \in \bar{x} + \varepsilon_0 \mathbb{B}. \quad (52)$$

Fix now any $(x', \alpha') \in \text{epi } f$ with $\|x' - \bar{x}\| < \varepsilon_0$ and $|\alpha' - f(\bar{x})| < \varepsilon_0$. Clearly, $x' \in \mathcal{N}_V(\bar{x}; \varepsilon_1, \varepsilon_2)$.

Case 1. $\lambda > 0$. In this case, we necessarily have $\alpha = f(x)$. Then, $(x^*/\lambda, -1) \in N^\pi(\text{epi } f; (x, f(x)))$, so $x^*/\lambda \in \partial_{\mathbb{G}}^\pi f(x)$ and $\|x^*/\lambda\| \leq \varepsilon_0/\lambda \leq rt$ for every $t \geq \max\{T, (1/r\lambda)\}$. Hence, by (3.18), we obtain

$$\langle \lambda^{-1}x^*, x' - x \rangle \leq f(x') - f(x) + \frac{t}{2} V(Jx, x'), \quad (53)$$

which entails

$$\begin{aligned} \langle x^*, x' - x \rangle &\leq \lambda [f(x') - f(x)] + \frac{t\lambda}{2} V(Jx, x') \\ &\leq \lambda [f(x') - f(x)] + \frac{t\lambda}{2} V(Jx, x') + \frac{t\lambda}{2} [\alpha' - \alpha]^2. \end{aligned} \quad (54)$$

Since $f(x') \leq \alpha'$, we have $\forall t' \geq \max\{(T/2), (1/2r)\}$, and we have $2\lambda^{-1}t' \geq 1/r\lambda$ and $2\lambda^{-1}t' \geq T/\lambda \geq T$ and hence (54) entails with $t = 2\lambda^{-1}t'$

$$\begin{aligned} \langle x^*, x' - x \rangle &\leq \lambda [f(x') - f(x)] + (2\lambda^{-1}t') \frac{\lambda}{2} V(Jx, x') \\ &\quad + (2\lambda^{-1}t') \frac{\lambda}{2} [\alpha' - \alpha]^2 \leq \lambda [f(x') - f(x)] \\ &\quad + t' V(Jx, x') + t' [\alpha' - \alpha]^2 \leq t' \|x\|^2 + t' \|x'\|^2 \\ &\quad - 2t' \langle Jx, x' - x \rangle + \lambda [\alpha' - \alpha] + t' [\alpha' - \alpha]^2. \end{aligned} \quad (55)$$

Hence,

$$\langle x^* + 2t' Jx, x' - x \rangle \leq t' \|x'\|^2 - t' \|x\|^2 + \lambda [\alpha' - \alpha] + t' [\alpha' - \alpha]^2. \quad (56)$$

So,

$$\begin{aligned} t' \|x\|^2 - 2t' \left\langle Jx + \frac{x^*}{2t'}, x \right\rangle &\leq t' \|x'\|^2 - 2t' \left\langle Jx + \frac{x^*}{2t'}, x' \right\rangle \\ &\quad + \lambda [\alpha' - \alpha] + t' [\alpha' - \alpha]^2. \end{aligned} \quad (57)$$

Dividing by $t' > 0$ yields

$$V\left(Jx + \frac{x^*}{2t'}; x\right) \leq V\left(Jx + \frac{x^*}{2t'}; x'\right) + \frac{\lambda}{t'} [\alpha' - \alpha] + [\alpha' - \alpha]^2. \quad (58)$$

On the other hand, we have

$$\begin{aligned} \left[\alpha' - \alpha + \frac{1}{2t'} \lambda\right]^2 &= [\alpha' - \alpha]^2 + \frac{\lambda}{t'} (\alpha' - \alpha) + \frac{\lambda^2}{4t'^2} \\ &= \frac{1}{t'} \left[t' [\alpha' - \alpha]^2 + \lambda (\alpha' - \alpha) + \frac{\lambda^2}{4t'}\right]. \end{aligned} \quad (59)$$

So,

$$t' \left[\alpha' - \alpha + \frac{1}{2t'} \lambda\right]^2 - \frac{\lambda^2}{4t'} = t' [\alpha' - \alpha]^2 + \lambda (\alpha' - \alpha). \quad (60)$$

Thus, (58) becomes

$$\begin{aligned} V\left(Jx + \frac{x^*}{2t'}; x\right) &\leq V\left(Jx + \frac{x^*}{2t'}; x'\right) \\ &\quad + \frac{1}{t'} \left[t' [\alpha' - \alpha]^2 + \lambda [\alpha' - \alpha]\right] \\ &\leq V\left(Jx + \frac{x^*}{2t'}; x'\right) + \left[\alpha' - \alpha + \frac{1}{2t'} \lambda\right]^2 - \frac{\lambda^2}{4t'^2} \end{aligned} \quad (61)$$

and so

$$V\left(Jx + \frac{x^*}{2t'}; x\right) + \frac{\lambda^2}{4t'^2} \leq V\left(Jx + \frac{x^*}{2t'}; x'\right) + \left[\alpha' - \alpha + \frac{1}{2t'} \lambda\right]^2. \quad (62)$$

Observe that

$$\begin{aligned} V_{X \times \mathbb{R}}\left(\left(Jx + \frac{x^*}{2t'}, \left(\alpha + \frac{1}{2t'} (-\lambda)\right)\right); (x, \alpha)\right) &= V\left(Jx + \frac{x^*}{2t'}; x\right) \\ &\quad + \frac{\lambda^2}{4t'^2}, \text{ and } V_{X \times \mathbb{R}}\left(\left(Jx + \frac{x^*}{2t'}, \left(\alpha + \frac{1}{2t'} (-\lambda)\right)\right); (x', \alpha')\right) \\ &= V\left(Jx + \frac{x^*}{2t'}; x'\right) + \left[\alpha' - \alpha + \frac{1}{2t'} \lambda\right]^2. \end{aligned} \quad (63)$$

Therefore,

$$\begin{aligned} V_{X \times \mathbb{R}} \left(\left(Jx + \frac{x^*}{2t'}, \left(\alpha + \frac{1}{2t'}(-\lambda) \right) \right); (x, \alpha) \right) \\ \leq V_{X \times \mathbb{R}} \left(\left(Jx + \frac{x^*}{2t'}, \left(\alpha + \frac{1}{2t'}(-\lambda) \right) \right); (x', \alpha') \right), \end{aligned} \quad (64)$$

for any $(x', \alpha') \in \text{epi } f \cap [(\bar{x}, f(\bar{x})) + \varepsilon_0 \mathbb{B}_{X \times \mathbb{R}}]$, that is,

$$(x, \alpha) \in \pi_{\text{epi } f \cap [(\bar{x}, f(\bar{x})) + \varepsilon_0 \mathbb{B}_{X \times \mathbb{R}}]} Jx + \frac{x^*}{2t'}, \left(\alpha + \frac{1}{2t'}(-\lambda) \right). \quad (65)$$

Case 2. $\lambda = 0$. In this case, we have $(x^*; 0) \in N^\pi(\text{epi } f; (x, \alpha))$ and so by Lemma 3, we obtain $(x^*, 0) \in N^\pi(\text{epi } f; (x, f(x)))$. Using the fact that $N^\pi(\text{epi } f; (x, f(x))) \subset N^F(\text{epi } f; (x, f(x)))$, we get $(x^*, 0) \in N^F(\text{epi } f; (x, f(x)))$ and so by Proposition 4, there exists a sequence $u_n \rightarrow x$ with $f(u_n) \rightarrow f(x)$, $\lambda_n \rightarrow 0^+$, and $u_n^* \in \lambda_n \partial^F f(u_n)$ (i.e., $(u_n^*/\lambda_n, -1) \in N^F \text{epi } f; (u_n, f(u_n))$) such that $\|u_n^* - x^*\|_* \rightarrow 0$. Using now Proposition 5, we choose for each $n \geq 1$

$$\begin{aligned} (x_n, \alpha_n) &\in (u_n, f(u_n)) + \frac{\lambda_n}{2} \mathbb{B}_{X \times \mathbb{R}}, (x_n, \alpha_n) \\ &\in \text{epi } f, (y_n^*, -\mu_n) \in N^\pi(\text{epi } f; (x_n, \alpha_n)), \quad (66) \\ \|y_n^* - u_n^*\| &< \frac{\lambda_n}{2} \text{ and } |\lambda_n - \mu_n| < \frac{\lambda_n}{2}. \end{aligned}$$

This ensures that $\mu_n > 0$ and so $\alpha_n = f(x_n)$. Consequently, $y_n^* \in \mu_n \partial_G^\pi f(x_n)$ with

$$\begin{aligned} \|x_n - x\| &\leq \|x_n - u_n\| + \|u_n - x\| \\ &\leq \frac{\lambda_n}{2} + \|u_n - x\| \rightarrow 0 \text{ as } n \rightarrow \infty, \\ |f(x_n) - f(x)| &\leq |\alpha_n - f(u_n)| + |f(u_n) - f(x)| \leq \frac{\lambda_n}{2} \\ &+ |f(u_n) - f(x)| \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned} \quad (67)$$

Also, we have

$$0 < \mu_n < \frac{3\lambda_n}{2} \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (68)$$

Let $x_n^* := \mu_n^{-1} y_n^*$. Then, $\|\mu_n x_n^* - x^*\| \leq \|y_n^* - u_n^*\| + \|u_n - x^*\| \rightarrow 0$ as $n \rightarrow \infty$, that is, $\mu_n x_n^* \rightarrow x^*$. For n large enough, (i.e., $\forall n \geq N_0$), we have $\|x_n - \bar{x}\| < \varepsilon_0$ and $\mu_n < 1$. Assume for a moment that $x^* \neq 0$. Let $t_n := \max \{T/2\mu_n, (\|x_n^*\|/2r\|x^*\|)\}$, $\forall n \geq N_0$, we see that

$$2t_n \geq \max \left\{ T, \frac{\|x_n^*\|}{r} \right\}, \left(\text{since } \frac{1}{\mu_n} > 1 \text{ and } \frac{1}{\|x^*\|} > 1 \right) \quad (69)$$

and hence by definition of V -p.l.n. functions with $y := x'$ and $t := 2t_n$, we obtain

$$\langle x_n^*, x' - x_n \rangle \leq f(x') - f(x_n) + t_n V(Jx_n, x'). \quad (70)$$

Multiplying this inequality by μ_n , we get

$$\langle \mu_n x_n^*, x' - x_n \rangle \leq \mu_n [f(x') - f(x_n)] + \mu_n t_n V(Jx_n, x'). \quad (71)$$

Let $\rho_n := \max \{(T/2), (\mu_n \|x_n^*\|/2r\|x^*\|)\}$. Clearly, $\rho_n \rightarrow \rho := \max \{(T/2), (1/2r)\}$ and $t_n \mu_n = \rho_n$. So, for any $n \geq N_0$, we have

$$\langle \mu_n x_n^*, x' - x_n \rangle \leq \mu_n [f(x') - f(x_n)] + \rho_n V(Jx_n, x'). \quad (72)$$

Now, taking the limit as $n \rightarrow \infty$ that yields by continuity of V and J

$$\langle x^*, x' - x \rangle \leq \rho V(Jx, x') \leq \rho V(Jx, x') + \rho [\alpha' - \alpha]^2 \quad (73)$$

and so

$$\langle x^*, x' \rangle \leq \langle x^*, x \rangle + \rho \|x\|^2 + \rho \|x'\|^2 - 2\rho \langle Jx, x' \rangle + \rho [\alpha' - \alpha]^2. \quad (74)$$

Thus,

$$\rho \|x\|^2 - \langle x^* + 2\rho Jx, x \rangle \leq \rho \|x'\|^2 - \langle x^* + 2\rho Jx, x' \rangle + \rho [\alpha' - \alpha]^2. \quad (75)$$

Dividing by $\rho > 0$ gives

$$\|x\|^2 - 2 \left\langle Jx + \frac{1}{2\rho} x^*, x \right\rangle \leq \|x'\|^2 - 2 \left\langle Jx + \frac{1}{2\rho} x^*, x' \right\rangle + [\alpha' - \alpha]^2 \quad (76)$$

and so

$$V \left(Jx + \frac{1}{2\rho} x^*, x \right) \leq V \left(Jx + \frac{1}{2\rho} x^*, x' \right) + [\alpha' - \alpha]^2. \quad (77)$$

This ensures for $\lambda = 0$

$$\begin{aligned} V_{X \times \mathbb{R}} \left(\left(Jx + \frac{x^*}{2\rho}, \left(\alpha + \frac{1}{2\rho}(-\lambda) \right) \right); (x, \alpha) \right) \\ \leq V_{X \times \mathbb{R}} \left(\left(Jx + \frac{x^*}{2\rho}, \left(\alpha + \frac{1}{2\rho}(-\lambda) \right) \right); (x', \alpha') \right), \end{aligned} \quad (78)$$

for any $(x', \alpha') \in \text{epi } f \cap [(\bar{x}, f(\bar{x})) + \varepsilon_0 \mathbb{B}_{X \times \mathbb{R}}]$, that is,

$$(x, \alpha) \in \pi_{\text{epi } f \cap [(\bar{x}, f(\bar{x})) + \varepsilon_0 \mathbb{B}_{X \times \mathbb{R}}]} Jx + \frac{x^*}{2\rho}, \left(\alpha + \frac{1}{2\rho}(-\lambda) \right). \quad (79)$$

Finally, we obtained from Case 1 and Case 2 two positive numbers ε_0 and $r_0 := 1/2\rho$ such that for any $(x, \alpha) \in \text{epi } f$ and $(x^*, -\lambda) \in N^\pi(\text{epi } f; (x, \alpha)) \cap \varepsilon_0 \mathbb{B}_{X \times \mathbb{R}}$ and any $x \in \bar{x} + \varepsilon_0 \mathbb{B}$ and $|f(\bar{x}) - \alpha| < \varepsilon_0$, we have

$$(x, \alpha) \in \pi_{\text{epi } f \cap [(\bar{x}, f(\bar{x})) + \varepsilon_0 \mathbb{B}_{X \times \mathbb{R}}]} ((Jx, \alpha) + r_0(x^*, -\lambda)). \quad (80)$$

This means by Definition 2 that the epigraph $\text{epi } f$ is generalized V -prox-regular at $(\bar{x}, f(\bar{x}))$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interests.

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